

ON THE ALMOST SURE CONVERGENCE RATE FOR A SERIES EXPANSION OF FRACTIONAL BROWNIAN MOTION

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ABSTRACT

Fractional Brownian motions (fBM) and related processes are widely used in financial modeling to capture the complicated dependence structure of the volatility. In this paper, we analyze an infinite series representation of fBM proposed in (Dzhaparidze and Van Zanten 2004), and establish an almost sure convergence rate of the series representation. The rate is also shown to be optimal. We then demonstrate how the strong convergence rate result can be applied to construct simulation algorithms with path-by-path error guarantees.

1 INTRODUCTION

The fractional Brownian motion $\{B^H(t)\}_{t \geq 0}$ with Hurst index $H \in (0, 1)$ is a centered Gaussian process with covariance function

$$\text{Cov}(B^H(s), B^H(t)) = \frac{1}{2} \cdot (|s|^{2H} + |t|^{2H} - |s - t|^{2H}). \quad (1)$$

As an extension of standard Brownian motion (BM), which corresponds to $H = 1/2$, fBM allows its disjoint increments to be correlated. In particular, when $H > 1/2$, the disjoint increments of fBM are positively correlated, while when $H < 1/2$, they are negatively correlated. Moreover, based on (1), fBM is self-similar with index H and stationary. Due to these nice properties, fBM is widely used in financial modeling, especially when long-range dependence is observed from empirical studies. For example, Comte and Renault (1998) propose to model log-volatility using fractional Brownian motion, motivated by the long memory property of the volatility process. Necula (2002) replaces BM in the Black-Scholes model with fBM and studies the corresponding option pricing problems. More recently, Gatheral, Jaisson, and Rosenbaum (2018) study the smoothness of the volatility process from high frequency data and find that log-volatility behaves essentially as a fBM with H of order 0.1. We also refer to (Hu and Øksendal 2003) and the references therein for more applications of fBM in finance.

Despite the abundant applications of fBM, the analysis of fBM based models is much more difficult than the classical BM driven processes. Analytical solutions rarely exist for those problems. One reason for such an obstacle is that fBM is not a semimartingale. Thus, most tools in stochastic analysis fail to

be applicable. Moreover, due to the complicated correlation structure, how to simulate fBM efficiently is also highly nontrivial, see, for example (Dieker 2004).

To overcome these challenges, a natural idea is to approximate fBM via some simpler stochastic processes. Over the past decades, there are numerous efforts devoted to this problem from several directions. For example, Delgado and Jolis (2000) and Li and Dai (2011) approximate fBM via Poisson processes. They construct stochastic integrals using Poisson processes and prove the convergence in law to a fBM. Similarly, Nathanael (2004) and Lindstrøm (2007) study the approximation of fBM using random walks.

Another line of research is to represent the fBM via wavelet decomposition or other series expansions. The idea is to write $B^H(t)$ in the form

$$\sum_{k=1}^{\infty} \psi_k(t) \xi_k, \quad (2)$$

where $\{\xi_k\}_{k \geq 1}$ are independent standard Gaussian random variables and $\{\psi_k(t)\}_{k \geq 1}$ are the basis functions. A pioneering work is (Meyer, Sellan, and Taqqu 1999), where the authors first use wavelet decomposition of fractional Gaussian noise, and then use integration to obtain corresponding representation for fBM. They also establish almost sure (a.s.) convergence result for the representation. However, their basis functions are hard to evaluate, which limits its application in practice. Later, Dzharidze and Van Zanten (2004) develop a series expansion of fBM as an extension of the Wiener's series representation of BM. Their basis functions are characterized by the zeros of Bessel functions, which can be calculated numerically. See also (Dzharidze and Van Zanten 2005) and (Iglói 2005) for examples of other series expansions of fBM.

For various approximations of fBM, the rate of convergence is an important measure of the quality and efficiency of the approximation. However, results on convergence rates are rather limited. When approximating fBM via Poisson processes or random walks, existing literature focuses on weak convergence and the rate of convergence is rarely obtained. For series expansions as in (2), Kühn and Linde (2002) establish the optimal rate of convergence in the L_2 norm,

$$\inf \left\{ \mathbb{E} \left[\sup_{t \in [0,1]} |B^H(t) - \sum_{k=1}^n \psi_k(t) \xi_k|^2 \right]^{1/2} : B^H(t) = \sum_{k=1}^{\infty} \psi_k(t) \xi_k \text{ a.s.} \right\} = \Omega(n^{-H} \sqrt{\log(n)}). \quad (3)$$

Most subsequent works on series expansions constructions aim to achieve this optimal rate of convergence; see, for example, (Iglói 2005) and (Dzharidze and Van Zanten 2005). Based on these works, a natural question to ask is that, can we establish the rate of convergence in a stronger almost sure sense? Specifically, we would like to show that there exist series expansions of the form (2), which satisfy

$$\sup_{t \in [0,1]} |B^H(t) - \sum_{k=1}^n \psi_k(t) \xi_k| = O(n^{-H} \sqrt{\log(n)}) \text{ a.s.} \quad (4)$$

Or equivalently, there exists a constant $C \in (0, \infty)$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{-H} \sqrt{\log(n)}} \cdot \sup_{t \in [0,1]} |B^H(t) - \sum_{k=1}^n \psi_k(t) \xi_k| \leq C \text{ a.s.}$$

Based on (3) and Fatou's Lemma, we can show that the rate of convergence in (4) is optimal. The almost sure convergence rate is valuable, not only because that it provides a stronger error guarantee, but also because that simulation algorithms that achieve path-by-path error guarantees or even exact (unbiased) simulation algorithms can be developed based on it; see, for example (Pollock, Johansen, and Roberts 2016), (Blanchet and Chen 2015) and (Blanchet, Chen, and Dong 2017) for more details of such developments.

In this paper, we prove that the series expansion developed in Dzharidze and Van Zanten (2004) achieves the strong error bound as in (4). The contribution is twofold. First, similar techniques can be used to establish strong approximation results for other series expansions of fBM or other Gaussian processes.

Second, we illustrate how advanced simulation algorithms that achieves strong error guarantees can be developed based on the almost sure convergence analysis.

Similar convergence rate results for approximations of fBM have been studied in Garzón, Gorostiza, and León (2009). They establish an almost convergence rate of $O(n^{-(1-H-\delta)} \log(n)^{5/2})$ for $H > 1/2$ and $O(n^{-(H-\delta)} \log(n)^{5/2})$ for $H < 1/2$, which are weaker than ours. Recently, Chen, Dong, and Ni (2019) establish (4) for the midpoint replacement (Haar wavelet) construction of fBM. An ε -strong simulation algorithm for fBM and related processes is also developed there. However, the Haar wavelet construction has correlated coefficient terms, i.e. ξ_k 's in (2), which is more costly to generate than the independent coefficients in (2).

2 MAIN RESULT

In this section we introduce the actual series expansion that we consider and establish the almost sure convergence rate of the expansion. The proof of some of the results are delayed to Section 4.

According to Dzharidze and Van Zanten (2004), fBM $B^H(t)$ defined on $[0, 1]$ admits the following infinite series expansion almost surely,

$$B^H(t) = \sum_{n=1}^{\infty} \frac{\sqrt{2}c_H \cdot \sin(x_nt)}{x_n^{1+H} \cdot |J_{1-H}(x_n)|} \cdot W_n^+ + \sum_{n=1}^{\infty} \frac{\sqrt{2}c_H \cdot (1 - \cos(y_nt))}{y_n^{1+H} \cdot |J_{-H}(y_n)|} \cdot W_n^-. \quad (5)$$

Here $J_\nu(\cdot)$ is a Bessel function of the first kind with order ν . The sequences $x_1 < x_2 < \dots$ and $y_1 < y_2 < \dots$ are positive real zeros of J_{-H} and J_{1-H} respectively. The constant $c_H = (\pi^{-1} \Gamma(1+2H) \sin(\pi H))^{1/2}$, where $\Gamma(\cdot)$ denotes the Gamma function. W_n^+ 's and W_n^- 's are independent standard Gaussian random variables.

Note that $J_\nu(x)$ with $\nu > -1$, is the solution to the differential equation

$$x^2 \frac{d^2 J_\nu(x)}{dx^2} + x \frac{dJ_\nu(x)}{dx} + (x^2 - \nu^2) J_\nu(x) = 0.$$

It possesses an increasing infinite sequence of positive real zeros, and the n -th one is of order n , i.e. x_n (or y_n) is $O(n)$. The following lemma characterizes the decay rate of the basis functions in (5).

Lemma 1 There is a constant $C_0 \in (0, \infty)$ such that for all $n \geq 1$ and $t \in [0, 1]$,

$$\max \left\{ \frac{\sqrt{2}c_H \cdot |\sin(x_nt)|}{x_n^{1+H} \cdot |J_{1-H}(x_n)|}, \frac{\sqrt{2}c_H \cdot |1 - \cos(y_nt)|}{y_n^{1+H} \cdot |J_{-H}(y_n)|} \right\} \leq \frac{C_0}{n^{H+1/2}}.$$

For the infinite series expansion (5), we define its truncation at level n as

$$B_n^H(t) = \sum_{k=1}^n \frac{\sqrt{2}c_H \cdot \sin(x_k t)}{x_k^{1+H} \cdot |J_{1-H}(x_k)|} \cdot W_k^+ + \sum_{k=1}^n \frac{\sqrt{2}c_H \cdot (1 - \cos(y_k t))}{y_k^{1+H} \cdot |J_{-H}(y_k)|} \cdot W_k^-.$$

Then $\{B_n^H(t) : 0 \leq t \leq 1\}_{n \geq 1}$ forms a sequence of stochastic processes, and $B_n^H(t)$ converges to a fBM $B^H(t)$ in $C[0, 1]$ almost surely, where $C[0, 1]$ is the space of continuous functions endowed with uniform norm. For the truncation error, $B_n^H(t) - B^H(t)$, Dzharidze and Van Zanten (2004) establish the following bounds. There exists a constant $C \in (0, \infty)$ such that

$$\sup_{t \in [0, 1]} \sqrt{\mathbb{E}[(B^H(t) - B_n^H(t))^2]} \leq C \cdot n^{-H} \text{ and } \sqrt{\mathbb{E}\left[\int_0^1 (B^H(t) - B_n^H(t))^2 dt\right]} \leq C \cdot n^{-H}.$$

Later, Dzharidze and van Zanten (2005) extend their convergence rate results to the expected uniform norm. In particular, we summarize their result in the following lemma.

Lemma 2 There exists a constant $C_1 \in (0, \infty)$, such that for all n large enough,

$$\mathbb{E} \left[\sup_{t \in [0,1]} |B^H(t) - B_n^H(t)| \right] \leq (C_1/4) \cdot n^{-H} \sqrt{\log(n)}. \quad (6)$$

From (3), we also know that the rate of convergence in Lemma 2 is optimal. In what follows, we further strengthen the convergence rate result, and show that the series expansion (5) achieves $O(n^{-H} \sqrt{\log(n)})$ convergence rate almost surely. We next provide a formal definition of the almost sure convergence rate.

Definition 1 Let $\{X_n\}_{n \geq 1}$ be a sequence of random variables and $\{a_n\}_{n \geq 1}$ a sequence of decreasing positive numbers. We say that the almost sure convergence rate of X_n is of order a_n , i.e. $X_n = O(a_n)$ a.s., if there exists a constant $C \in (0, \infty)$ such that

$$\limsup_{n \rightarrow \infty} X_n/a_n \leq C \text{ a.s..}$$

Note that the almost sure convergence rate we defined here is a very strong notion of convergence as we require the existence of finite constant C , not a path dependent random variable. We are now ready to provide the main result of this paper.

Theorem 1 For any constant $K > 2C_1$, there exist positive constants $C_2 \in (0, \infty)$, such that for n is large enough,

$$\mathbb{P} \left(\sup_{t \in [0,1]} |B^H(t) - B_n^H(t)| \geq \frac{4K}{(1-2^{-H})^2} \cdot n^{-H} \sqrt{\log(n)} \right) \leq C_2 \sqrt{\log(n)} \cdot n^{-\Lambda}, \quad (7)$$

where $\Lambda = (K - 2C_1)^2 / (8C_0^2)$.

Based on Theorem 1, we have the following corollary, which characterizes the almost sure convergence rate of $B_n^H(t)$.

Corollary 2 For $K > 4C_0 + 2C_1$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{-H} \sqrt{\log(n)}} \cdot \sup_{t \in [0,1]} |B^H(t) - B_n^H(t)| \leq \frac{4K}{(1-2^{-H})^2} \text{ a.s..} \quad (8)$$

Furthermore, the rate is optimal, i.e.

$$\inf \left\{ \sup_{t \in [0,1]} |B^H(t) - \sum_{k=1}^n \psi_k(t) \xi_k|^2 : B^H(t) = \sum_{k=1}^{\infty} \psi_k(t) \xi_k \text{ a.s.} \right\} = \Omega(n^{-H} \sqrt{\log(n)}) \text{ a.s.}$$

Proof. When $K > 4C_0 + 2C_1$, $\Lambda > 1$ and hence, the series $\sqrt{\log(n)} \cdot n^{-\Lambda} < \infty$ is summable. Using Borel-Cantelli Lemma, the events $\{\sup_{t \in [0,1]} |B^H(t) - B_n^H(t)| \geq 4K/(1-2^{-H})^2 \cdot n^{-H} \sqrt{\log(n)}\}$ happen finitely many times almost surely. Thus, we obtain (8).

We next show the optimality of the convergence rate by contradiction. Suppose there exist $\tilde{B}_n^H(t)$ and a sequence of decreasing numbers $a_n = o(n^{-H} \sqrt{\log(n)})$ such that

$$\limsup_{n \rightarrow \infty} \frac{\sup_{t \in [0,1]} |B^H(t) - \tilde{B}_n^H(t)|}{a_n} \leq C \text{ a.s..}$$

for some constant $C \in (0, \infty)$. Then by Fatou's Lemma, we have

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E} [\sup_{t \in [0,1]} |B^H(t) - \tilde{B}_n^H(t)|^2]^{1/2}}{a_n} \leq \mathbb{E} \left[\limsup_{n \rightarrow \infty} \frac{\sup_{t \in [0,1]} |B^H(t) - \tilde{B}_n^H(t)|^2}{a_n^2} \right]^{1/2} \leq C,$$

i.e., $\mathbb{E} [\sup_{t \in [0,1]} |B^H(t) - \tilde{B}_n^H(t)|^2]^{1/2} = o(n^{-H} \sqrt{\log(n)})$, which contradicts the lower bound in (3) (Kühn and Linde 2002). Hence, $n^{-H} \sqrt{\log(n)}$ is the optimal rate. $\square$

3 APPLICATION OF THE MAIN RESULT

One important application of the almost sure convergence rate analysis is to develop ε -strong simulation algorithms. Given a stochastic process $X = \{X(t) : t \in [0, 1]\}$, the ε -strong simulation algorithm aims to construct a probability space supporting both X a fully simulatable process $\hat{X}^\varepsilon = \{X^\varepsilon(t) : t \in [0, 1]\}$ such that

$$\sup_{t \in [0, 1]} |\hat{X}^\varepsilon(t) - X(t)| \leq \varepsilon \text{ a.s.}$$

for any user specified accuracy $\varepsilon > 0$.

We next introduce a strategy to construct the ε -strong simulation algorithm for fBM based on (7). The idea is to truncate the infinite series expansion up to a finite but random level such that the error of the truncated terms are suitably bounded. Specifically, the goal is to find $N(\varepsilon)$ such that

$$\sup_{t \in [0, 1]} |B_{N(\varepsilon)}^H - B^H(t)| \leq \varepsilon \text{ a.s..}$$

To achieve this goal, we use a “record-breaker” strategy introduced in (Blanchet and Sigman 2011). Specifically, we define a sequence of events (record breakers) that satisfy the following two conditions.

- C1) Beyond some random but finite level, there will be no more record breakers;
- C2) By knowing that there are no more record breakers, the contribution of the infinite “remaining” terms are well under control.

The key now is to define the proper sequence of record breakers. We first rewrite the infinite series expansion (5) as

$$B^H(t) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} (\lambda_{n,k}(t) \cdot V_{n,k}^+ + \gamma_{n,k}(t) \cdot V_{n,k}^-), \quad (9)$$

where

$$\lambda_{n,k}(t) = \frac{\sqrt{2}c_H \cdot \sin(x_{2^{n-1}+k-1}t)}{x_{2^{n-1}+k-1}^{1+H} \cdot |J_{1-H}(x_{2^{n-1}+k-1})|}, \quad \gamma_{n,k}(t) = \frac{\sqrt{2}c_H \cdot (1 - \cos(y_{2^{n-1}+k-1}t))}{y_{2^{n-1}+k-1}^{1+H} \cdot |J_{-H}(y_{2^{n-1}+k-1})|},$$

and $V_{n,k}^+ = W_{2^{n-1}+k-1}^+$, $V_{n,k}^- = W_{2^{n-1}+k-1}^-$. For $n \geq 1$ and $1 \leq k \leq 2^{n-1}$, we also define the partial sums

$$S_{n,k}(t) = \sum_{i=1}^k (\lambda_{n,i}(t) \cdot V_{n,i}^+ + \gamma_{n,i}(t) \cdot V_{n,i}^-),$$

and

$$M_{n,k}^\uparrow = \sup_{t \in [0, 1]} S_{n,k}(t), \quad M_{n,k}^\downarrow = \inf_{t \in [0, 1]} S_{n,k}(t).$$

Let

$$b_n = K \cdot 2^{-nH} \sqrt{n},$$

where $K > 4C_0 + 2C_1$ is a constant. Then the record breaking events can be defined as follows. We say that a record is broken at level n if

$$\mathcal{E}_n = \left\{ \max_{1 \leq k \leq 2^{n-1}} M_{n,k}^\uparrow > b_n \right\} \cup \left\{ \min_{1 \leq k \leq 2^{n-1}} M_{n,k}^\downarrow < -b_n \right\}. \quad (10)$$

Let N denote the time of the last record breaker. From Theorem 1, we can show that $\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{E}_n) < \infty$. In particular, note that, by our choice of K , we have $\Lambda > 1$ in Theorem 1. Thus, $P(N < \infty) = 1$, i.e. C1) is satisfied. We also note that for $n > N$,

$$|B^H(t) - B_n^H(t)| \leq \sum_{k=n+1}^{\infty} b_k = O(2^{-nH} \sqrt{n}).$$

Thus, C2) is also satisfied. Indeed, this truncation strategy achieves the optimal rate of convergence. We also comment that it is important to rewrite the infinite series expansion in geometrically increasing groups as in (9), and define the record breaking event for each group to achieve the optimal rate of convergence. Let $n_0(\varepsilon) = \min\{n : \sum_{k=n+1}^{\infty} b_k \leq \varepsilon\}$. If we can find N , then we can define $N(\varepsilon) = \max\{N, n_0(\varepsilon)\}$.

We remark that when defining b_n , even though any $K > 4C_0 + 2C_1$ would work, there is actually a tradeoff in the choice of K . Specifically, larger values of K lead to larger b_n 's, which are less likely to be broken. As a result, N is more likely to take smaller values. On the other hand, larger values of b_n 's lead to larger values of $n_0(\varepsilon)$. Thus, how to choose K to balance N and $n_0(\varepsilon)$ is important for the efficient implementation of the algorithm.

The remaining task is to find the last record breaking level N . This is a highly nontrivial task, as the level of the last record breaker is not a stopping time with respect to the filtration generated by the series expansion, i.e. it depends on the information of the infinitely many “future” levels. Often times, rare-event simulation techniques are employed to overcome this obstacle (see for example Blanchet, Chen, and Dong (2017)). This is because the record breaking events are often events with very small probability. More specifically, we define the sequence of record breaking times as follows. $\tau_0 = 0$ and for $i = 1, 2, \dots$, if $\tau_{i-1} < \infty$,

$$\tau_i = \inf\{n \geq \tau_{i-1} + 1 : \mathcal{E}_n \text{ happens}\};$$

otherwise, $\tau_i = \infty$. Note that $N = \sup\{\tau_i : \tau_i < \infty\}$. The key challenge is how to sample these random times efficiently. This can be achieved by combining rare-event simulation techniques with acceptance-rejection method. In particular, the idea is to construct a tilted probability measure $\mathbb{P}^*$ such that $\mathbb{P}^*(\tau_{i+1} < \infty | \tau_i = l) = 1$ and the likelihood ratio between $\mathbb{P}$ and $\mathbb{P}^*$ over the record breaking path is bounded by 1. Then given $V_{n,k}^+$'s and $V_{n,k}^-$'s for $1 \leq k \leq 2^{n-1}$ and $1 \leq n \leq \tau_i = l$, the algorithm will first generate $V_{n,k}^+$'s and $V_{n,k}^-$'s under the tilted measure for $1 \leq k \leq 2^{n-1}$ and $n = l+1, \dots, \tau_{i+1}$, and then generate a uniform random variable U independent of everything else. If $U < d\mathbb{P}/d\mathbb{P}^*$, we accept the path as the path to the next record breaking level. Otherwise, we claim that $\tau_{i+1} = \infty$, i.e. there is no more record breaker. We leave the construction of the measure $\mathbb{P}^*$ as a future task. We acknowledge that it can be a very challenging task. This is because we are dealing with uncountably many partial sums, $S_{n,k}(t)$ for $0 \leq t \leq 1$. In other words, the supremum and infimum, $M_{n,k}^\uparrow$ and $M_{n,k}^\downarrow$, are taken over continuous functions. Thus, it is very hard to locate (in terms of t) the next record breaking event.

4 PROOF OF THE MAIN RESULT

In this section, we provide the proof of Lemma 1 and Theorem 1. Lemma 1 is a direct result of the fact that the n -th zero of Bessel function is of order n . Its proof is based on first order expansion of the Bessel function.

The proof of Theorem 1 is more involved. Following the record breaker construction in Section 3, we first rearrange the series expansion into groups. This allows us to bound the probability of the event

$$\sup_{0 \leq t \leq 1} |B^H(t) - B_n^H(t)| \geq 4K/(1 - 2^{-H})^2 \cdot n^{-H} \sqrt{\log(n)}$$

in Theorem 1 by the probability of the record breakers. Then we develop bounds for the probability of the record breaking events. There, we construct a proper nonnegative submartingale, which allows us to

¹Here we are using $\frac{d\mathbb{P}}{d\mathbb{P}^*}$ to denote the likelihood ratio of the generated path.

apply Doob's martingale inequality with respect to the level index. We also apply Borel-TIS inequality with respect to the time index, utilizing the fact that $B_n^H(t)$ is a centered Gaussian process.

Proof of Lemma 1. We begin the proof by presenting some properties of Bessel function. Consider a Bessel function of the first kind: $J_\nu(x)$ with order $\nu > -1$. It is well-known that $J_\nu(x)$ has the decomposition

$$J_\nu(x) = \frac{1}{2}(H_\nu^{(1)}(x) + H_\nu^{(2)}(x)), \quad (11)$$

where $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ denote the Hankel functions of the first and second kind. Furthermore, the first order expansions of $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are

$$\begin{aligned} H_\nu^{(1)}(x) &= \sqrt{\frac{2}{\pi x}} \cdot \exp\{i(x - \nu\pi/2 - \pi/4)\} \cdot (1 + R_1(x)), \\ H_\nu^{(2)}(x) &= \sqrt{\frac{2}{\pi x}} \cdot \exp\{-i(x - \nu\pi/2 - \pi/4)\} \cdot (1 + R_2(x)), \end{aligned} \quad (12)$$

where the reminders $R_1(x)$ and $R_2(x)$ satisfy the following estimation ((Olver 1997)),

$$|R_i(x)| \leq \frac{4\nu^2 - 1}{4} \cdot \exp\{|\nu^2 - 1/4| \cdot |x|^{-1}\} \cdot |x|^{-1}, \quad i = 1, 2. \quad (13)$$

Combining (11)-(13), we obtain

$$|J_\nu(x)| \geq \sqrt{\frac{2}{\pi x}} \cdot \left(|\cos(x - \nu\pi/2 - \pi/4)| - (4\nu^2 - 1)/4 \cdot \exp\{|\nu^2 - 1/4| \cdot |x|^{-1}\} \cdot |x|^{-1} \right).$$

In the following, we use $j_{\nu,k}$ to denote the k -th positive zero of $J_\nu(x)$ and let

$$\beta_{\nu,k} = (k + \nu/2 - 1/4)\pi.$$

Then using the McMahon's asymptotic representation of the zeros of Bessel function (McMahon 1894), we have

$$j_{\nu,k} = \beta_{\nu,k} - \frac{4\nu^2 - 1}{8\beta_{\nu,k}} - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{544\beta_{\nu,k}^3} + O(\beta_{\nu,k}^{-5}).$$

Note that for any $k \in \mathbb{Z}$, $|\cos(\beta_{1-H,k} + H/2 \cdot \pi - \pi/4)| = 1$. Thus when n is large enough,

$$|J_{-H}(y_n)| > \sqrt{(2\pi y_n)^{-1}}. \quad (14)$$

Similarly, we can also prove that, for n large enough,

$$|J_{1-H}(x_n)| > \sqrt{(2\pi x_n)^{-1}}. \quad (15)$$

To continue our proof, we need two results about the sizes of $\{j_{\nu,k}\}_{k \geq 1}$ and $\{\beta_{\nu,k}\}_{k \geq 1}$.

1) The interlacing of zeros of Bessel functions (Watson 1995). Let $j_{\nu,1} < j_{\nu,2} < \dots$ be the positive real zeros of $J_\nu(x)$, arranged in ascending order of magnitude, then for $\nu > -1$,

$$0 < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < \dots$$

2) Lower bound for the real zeros of Bessel functions (Gatteschi and Giordano 2000). For $-1/2 \leq v \leq 1/2$ and $k \geq 1$, we have $j_{v,k} \geq \beta_{v,k}$.

Based on above results, by setting $v = -H$ and $1 - H$, we obtain the following lower bounds for x_n and y_n ,

$$\begin{aligned} x_n = j_{-H,n} &> j_{1-H,n-1} > \beta_{1-H,n-1} = (n + (1-H)/2 - 5/4) \cdot \pi, \\ y_n = j_{1-H,n} &\geq \beta_{1-H,n-1} = (n + (1-H)/2 - 1/4) \cdot \pi. \end{aligned} \quad (16)$$

Combining inequalities (14)-(16), we have

$$\begin{aligned} x_n^{1+H} |J_{1-H}(x_n)| &> \sqrt{(2\pi)^{-1}} \cdot \pi^{H+1/2} \cdot (n + (1-H)/2 - 5/4)^{H+1/2} \text{ and} \\ y_n^{1+H} |J_{-H}(y_n)| &> \sqrt{(2\pi)^{-1}} \cdot \pi^{H+1/2} \cdot (n + (1-H)/2 - 1/4)^{H+1/2}, \end{aligned}$$

which imply that there exists a positive constant C_0 such that for all $n \geq 1$ and $t \in [0, 1]$,

$$\max \left\{ \frac{\sqrt{2}c_H \cdot |\sin(x_n t)|}{x_n^{1+H} \cdot |J_{1-H}(x_n)|}, \frac{\sqrt{2}c_H \cdot |1 - \cos(y_n t)|}{y_n^{1+H} \cdot |J_{-H}(y_n)|} \right\} \leq \frac{C_0}{n^{H+1/2}}.$$

Hence, we finish the proof of Lemma 1. $\square$

Proof of Theorem 1. Similar to the definition of the record breakers in Section 3, we rearrange the elements of the infinite series expansion (5) by groups, where the n -th group contains 2^{n-1} elements. Then we can rewrite the truncation error $B^H(t) - B_n^H(t)$ as

$$B^H(t) - B_n^H(t) = \sum_{m=\ell_n}^{\infty} \sum_{i=t_m}^{2^{m-1}} (\lambda_{m,i}(t) \cdot V_{m,i}^+ + \gamma_{m,i}(t) \cdot V_{m,i}^-),$$

where $\ell_n = \lceil \log_2(n+1) \rceil$, $t_m = 1\{m > \ell_n\} + (n+1 - 2^{\ell_n-1}) \cdot 1\{m = \ell_n\}$. Note that ℓ_n is the “group” that n belongs to. t_{ℓ_n} the position of n in the ℓ_n -th group. $t_m = 1$ for $m > \ell_n$. Based on these notations, we define, with a slight modification to our definitions in Section 3,

$$\begin{aligned} S_{m,k}(t) &= \sum_{i=t_m}^k (\lambda_{m,i}(t) \cdot V_{m,i}^+ + \gamma_{m,i}(t) \cdot V_{m,i}^-) \\ M_{m,k}^{\uparrow} &= \sup_{0 \leq t \leq 1} S_{m,k}(t), \quad M_{m,k}^{\downarrow} = \inf_{0 \leq t \leq 1} S_{m,k}(t). \end{aligned}$$

Now for $b_m = K \cdot 2^{-mH} \sqrt{m}$ with $K > 2C_1$, the record breaking event is defined as

$$\mathcal{E}_m = \left\{ \max_{t_m \leq k \leq 2^{m-1}} M_{m,k}^{\uparrow} > b_m \right\} \cup \left\{ \min_{t_m \leq k \leq 2^{m-1}} M_{m,k}^{\downarrow} < -b_m \right\},$$

We first note that

$$\sup_{t \in [0,1]} |B^H(t) - B_n^H(t)| \leq \sum_{m=\ell_n}^{\infty} \sup_{t \in [0,1]} |S_{m,2^{m-1}}(t)| \leq \sum_{m=\ell_n}^{\infty} \max \{ |M_{m,2^{m-1}}^{\uparrow}|, |M_{m,2^{m-1}}^{\downarrow}| \}.$$

Hence, if none of the events $\{\mathcal{E}_m\}_{m \geq \ell_n}$ happens, we have

$$\sup_{t \in [0,1]} |B^H(t) - B_n^H(t)| \leq \sum_{m=\ell_n}^{\infty} b_m.$$

Let $\zeta = \sum_{m=\ell_n}^{\infty} 2^{-mH} \sqrt{m}$. Then

$$(1 - 2^{-H}) \cdot \zeta = 2^{-\ell_n H} \sqrt{\ell_n} + \sum_{m=\ell_n+1}^{\infty} 2^{-mH} \cdot (\sqrt{m} - \sqrt{m-1}) \leq 2^{-\ell_n H} \sqrt{\ell_n} + \frac{2^{-\ell_n H}}{1 - 2^{-H}}.$$

When n is large enough, since $\ell_n = \lceil \log_2(n+1) \rceil$,

$$\sum_{m=\ell_n}^{\infty} b_m = K \cdot \zeta \leq \frac{2K}{(1 - 2^{-H})^2} \cdot 2^{-\ell_n H} \sqrt{\ell_n} \leq \frac{4K}{(1 - 2^{-H})^2} \cdot n^{-H} \sqrt{\log(n)}.$$

As a result, we have

$$\left\{ \sup_{t \in [0,1]} |B^H(t) - B_n^H(t)| \geq \frac{4K}{(1 - 2^{-H})^2} \cdot n^{-H} \sqrt{\log(n)} \right\} \subseteq \bigcup_{m=\ell_n}^{\infty} \mathcal{E}_m.$$

In the following, we focus on controlling the probability $\mathbb{P}(\mathcal{E}_m)$. For each n and $k = t_m \cdots, 2^{m-1}$, we use

$$\mathcal{F}_k^m = \sigma\{(V_{m,t_m}^+, V_{m,t_m}^-), \dots, (V_{m,k}^+, V_{m,k}^-)\}$$

to denote the natural filtration of stochastic process $S_{m,k}(t)$.

First, we notice that $\{M_{m,k}^{\uparrow}\}$ is a submartingale and $\{M_{m,k}^{\downarrow}\}$ is a supermartingale with respect to filtration $\{\mathcal{F}_k^m\}$, since

$$\mathbb{E}[M_{m,k+1}^{\uparrow} | \mathcal{F}_k^m] = \mathbb{E}\left[\sup_{t \in [0,1]} S_{m,k+1}(t) | \mathcal{F}_k^m\right] \geq \sup_{t \in [0,1]} \mathbb{E}[S_{m,k+1}(t) | \mathcal{F}_k^m] = \sup_{t \in [0,1]} S_{m,k}(t) = M_{m,k}^{\uparrow},$$

and

$$\mathbb{E}[M_{m,k+1}^{\downarrow} | \mathcal{F}_k^m] = \mathbb{E}\left[\inf_{t \in [0,1]} S_{m,k+1}(t) | \mathcal{F}_k^m\right] \leq \inf_{t \in [0,1]} \mathbb{E}[S_{m,k+1}(t) | \mathcal{F}_k^m] = \inf_{t \in [0,1]} S_{m,k}(t) = M_{m,k}^{\downarrow}.$$

Now fix a constant $\theta > 0$. $e^{\theta x}$ is a strictly increasing convex function. Hence,

$$\mathbb{E}[\exp\{\theta M_{m,k+1}^{\uparrow}\} | \mathcal{F}_k^m] \geq \exp\{\theta \mathbb{E}[M_{m,k+1}^{\uparrow} | \mathcal{F}_k^m]\} \geq \exp\{\theta M_{m,k}^{\uparrow}\},$$

where the first inequality follows from conditional Jensen's inequality and the second inequality follows from the fact that $e^{\theta x}$ is increasing and $\{M_{m,k}^{\uparrow}\}$ is a submartingale. This implies that $\{\exp\{\theta M_{m,k}^{\uparrow}\}\}$ is a nonnegative submartingale with respect to $\{\mathcal{F}_k^m\}$. Applying Doob's martingale inequality, we have that for all $\theta > 0$,

$$\mathbb{P}\left(\max_{t_m \leq k \leq 2^{m-1}} M_{m,k}^{\uparrow} \geq b_m\right) = \mathbb{P}\left(\max_{t_m \leq k \leq 2^{m-1}} \exp\{\theta M_{m,k}^{\uparrow}\} \geq \exp\{\theta b_m\}\right) \leq \mathbb{E}[\exp\{\theta M_{m,2^{m-1}}^{\uparrow}\}] / \exp\{\theta b_m\}.$$

Let $\mathcal{M}_m^+ := M_{m,2^{m-1}}^{\uparrow} = \sup_{t \in [0,1]} S_{m,2^{m-1}}(t)$. Then

$$\mathcal{M}_m^+ \leq \sup_{t \in [0,1]} |S_{m,2^{m-1}}(t)| \leq \sup_{t \in [0,1]} |B^H(t) - B_{2^{m-1}-1+t_m}^H(t)| + \sup_{t \in [0,1]} |B^H(t) - B_{2^m-1}^H(t)|$$

Using Lemma 2, when m is large enough, we have

$$\begin{aligned} \mathbb{E}[\mathcal{M}_m^+] &\leq C_1/4 \cdot (2^{m-1} - 1 + t_m)^{-H} \sqrt{\log(2^{m-1} - 1 + t_m)} + C_1/4 \cdot (2^m - 1)^{-H} \sqrt{\log(2^m - 1)} \\ &\leq C_1 \cdot 2^{-mH} \sqrt{m}. \end{aligned}$$

On the other hand, for the variance of $S_{m,2^{m-1}}(t)$, using Lemma 1, we obtain that for any $t \in [0, 1]$ and m large enough,

$$\mathbb{E}[S_{m,2^{m-1}}(t)^2] \leq 2C_0^2 \cdot \sum_{k=t_m}^{2^{m-1}} (2^{m-1} + k - 1)^{-(2H+1)} \leq 2C_0^2 \cdot \sum_{k=1}^{2^{m-1}} (2^{m-1} + k - 1)^{-(2H+1)} \leq 4C_0^2 \cdot 2^{-2mH}.$$

Furthermore,

$$\sup_{t \in [0,1]} \mathbb{E}[S_{m,2^{m-1}}(t)^2] \leq 4C_0^2 \cdot 2^{-2mH}.$$

In what follows, we use μ_m and σ_m^2 to denote $\mathbb{E}[\mathcal{M}_m^+]$ and $\sup_{t \in [0,1]} \mathbb{E}[S_{m,2^{m-1}}(t)^2]$, respectively. Then by Borel-TIS inequality, for any $x > 0$, we have

$$\mathbb{P}(\mathcal{M}_m^+ - \mu_m > x) \leq \exp\{-x^2/(2\sigma_m^2)\}. \quad (17)$$

Thus,

$$\begin{aligned} \mathbb{E}[\exp\{\theta \mathcal{M}_m^+\}] &= \mathbb{E}\left[\exp\left\{\theta \mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ \leq \mu_m\} + \theta \mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ > \mu_m\}\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{\theta \mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ \leq \mu_m\}\right\} \cdot \exp\left\{\theta \mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ > \mu_m\}\right\}\right] \\ &\leq \exp\{\theta \mu_m\} \cdot \mathbb{E}\left[\exp\left\{\theta \mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ > \mu_m\}\right\}\right], \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbb{E}\left[\exp\left\{\theta \mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ > \mu_m\}\right\}\right] &= \mathbb{E}\left[\int_0^\infty 1\left\{\exp\left\{\theta \mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ > \mu_m\}\right\} > t\right\} dt\right] \\ &= \int_0^\infty \mathbb{P}(\mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ > \mu_m\} > \log t / \theta) dt \\ &= \int_0^1 1 dt + \int_1^{e^{\theta \mu_m}} \mathbb{P}(\mathcal{M}_m^+ > \mu_m) dt + \int_{\exp\{\theta \mu_m\}}^\infty \mathbb{P}(\mathcal{M}_m^+ > \log t / \theta) dt \\ &\leq \exp\{\theta \mu_m\} + \int_{\exp\{\theta \mu_m\}}^\infty \exp\left\{-(\log t / \theta - \mu_m)^2 / 2\sigma_m^2\right\} dt, \end{aligned}$$

where the last inequality follows from Borel-TIS inequality (17) by setting $x = \log t / \theta - \mu_m > 0$. Note also that

$$\begin{aligned} \int_{\exp\{\theta \mu_m\}}^\infty \exp\left\{-(\log t / \theta - \mu_m)^2 / 2\sigma_m^2\right\} dt &= \int_0^\infty \theta \exp\left\{-x^2 / (2\sigma_m^2) + \theta(x + \mu_m)\right\} dx \\ &= \theta \exp\{\theta \mu_m + \theta^2 \sigma_m^2 / 2\} \cdot \sqrt{2\pi \sigma_m^2} \cdot \int_0^\infty \frac{1}{\sqrt{2\pi \sigma_m^2}} \cdot e^{-\frac{(x - \sigma_m^2 \theta)^2}{2\sigma_m^2}} dx \\ &\leq \theta \exp\{\theta \mu_m + \theta^2 \sigma_m^2 / 2\} \cdot \sqrt{2\pi \sigma_m^2}. \end{aligned}$$

Above all, we have

$$\begin{aligned} \mathbb{P}\left(\max_{t_m \leq k \leq 2^{m-1}} M_{m,k}^+ \geq b_m\right) &\leq \mathbb{E}[\exp\{\theta \mathcal{M}_m^+\}] / \exp\{\theta b_m\} \\ &\leq \exp\{\theta \mu_m\} \cdot \mathbb{E}\left[\exp\left\{\theta \mathcal{M}_m^+ \cdot 1\{\mathcal{M}_m^+ > \mu_m\}\right\}\right] / \exp\{\theta b_m\} \\ &\leq \exp\{\theta(\mu_m - b_m)\} \cdot (\exp\{\theta \mu_m\} + \theta \exp\{\theta \mu_m + \theta^2 \sigma_m^2 / 2\} \cdot \sqrt{2\pi \sigma_m^2}) \\ &= \exp\{\theta(2\mu_m - b_m)\} + \theta \sqrt{2\pi \sigma_m^2} \cdot \exp\{\theta(2\mu_m - b_m) + \sigma_m^2 \theta^2 / 2\}. \end{aligned}$$

Recall that we have proved that $\mu_m \leq C_1 \cdot 2^{-mH} \sqrt{m}$ and $\sigma_m^2 \leq 4C_0^2 \cdot 2^{-2mH}$. Also recall that we choose $b_m = K2^{-mH} \sqrt{m}$ for $K > 2C_1$. Then we have

$$\begin{aligned} \mathbb{P}\left(\max_{t_m \leq k \leq 2^{m-1}} M_{m,k}^\uparrow \geq b_m\right) &\leq \exp\{\theta(2C_1 - K)2^{-mH} \sqrt{m}\} \\ &\quad + 2C_0 \theta \sqrt{2\pi} \cdot 2^{-mH} \exp\{\theta(2C_1 - K)2^{-mH} \sqrt{m} + 2^{-2mH} \theta^2 \cdot 2C_0^2\}. \end{aligned}$$

This inequality holds for any $\theta > 0$, and when we choose $\theta = (K - 2C_1)2^{mH} \sqrt{m}/(4C_0^2) > 0$, we obtain

$$\begin{aligned} \mathbb{P}\left(\max_{t_m \leq k \leq 2^{m-1}} M_{m,k}^\uparrow \geq b_m\right) &\leq \exp\{-(K - 2C_1)^2 m/(4C_0^2)\} \\ &\quad + (K - 2C_1) \cdot \sqrt{2\pi m}/(2C_0) \cdot \exp\{-(K - 2C_1)^2 m/(8C_0^2)\}. \end{aligned}$$

Similarly, we can build the same upper bounds for the probability of the downward crossing events, i.e., $\{\min_{t_m \leq k \leq 2^{m-1}} M_{m,k}^\downarrow \leq -b_m\}$.

Lastly, we obtain that there exist some positive constant C_2 and $\Lambda = (K - 2C_1)^2 n/(8C_0^2)$ such that for n large enough,

$$\begin{aligned} \mathbb{P}\left(\sup_{t \in [0,1]} |B^H(t) - B_n^H(t)| \geq \frac{4K}{(1 - 2^{-H})^2} \cdot n^{-H} \sqrt{\log(n)}\right) \\ \leq \sum_{m=\ell_n}^{\infty} \mathbb{P}(\mathcal{E}_m) \leq \sum_{m=\ell_n}^{\infty} \mathbb{P}\left(\max_{t_m \leq k \leq 2^{m-1}} M_{m,k}^\uparrow \geq b_m\right) + \sum_{m=\ell_n}^{\infty} \mathbb{P}\left(\min_{t_m \leq k \leq 2^{m-1}} M_{m,k}^\downarrow \leq -b_m\right) \\ \leq C_2 n^{-\Lambda} \sqrt{\log(n)}. \end{aligned}$$

This concludes the proof of Theorem 1. □

We conclude this section by commenting that our proof only depends on the decay rate of basis functions and the fact that the coefficients are centered Gaussian random variables. It does not rely on the specific form of basis functions. Hence, the analysis is potentially applicable to other infinite series expansions of fBM or infinite series expansions of other Gaussian processes.

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