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Twisted orbital integrals and irreducible components of affine Deligne–Lusztig varieties

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We analyze the asymptotic behavior of certain twisted orbital integrals arising from the study of affine Deligne–Lusztig varieties. The main tools include the Base Change Fundamental Lemma and q-analogues of the Kostant partition functions. As an application we prove a conjecture of Miaofen Chen and Xinwen Zhu, relating the set of irreducible components of an affine Deligne–Lusztig variety modulo the action of the σ-centralizer group to the Mirković–Vilonen basis of a certain weight space of a representation of the Langlands dual group.

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1. Introduction

1.1. The main result

First introduced by Rapoport [39], affine Deligne–Lusztig varieties play an important role in arithmetic geometry and the Langlands program. One of

the main motivations to study affine Deligne–Lusztig varieties comes from the theory of p-adic uniformization, which was studied by various authors including Čerednik [6], Drinfeld [7], Rapoport–Zink [42], and more recently Howard–Pappas [21] and Kim [23]. In this theory, a p-adic formal scheme known as the Rapoport–Zink space uniformizes a tubular neighborhood in an integral model of a Shimura variety around a Newton stratum. The reduced subscheme of the Rapoport–Zink space is a special example of affine Deligne–Lusztig varieties. Thus in many cases, understanding certain cycles on Shimura varieties reduces to understanding the geometry of the affine Deligne–Lusztig variety.

In this paper, we concern the problem of parameterizing the irreducible components of affine Deligne–Lusztig varieties. We introduce some notations. Let F be a non-archimedean local field with valuation ring $\mathcal{O}_F$ and residue field $k_F = \mathbb{F}_q$. Fix a uniformizer $\pi_F \in F$. Let L be the completion of the maximal unramified extension of F, and let σ be the Frobenius automorphism of L over F. Let G be a connected reductive group scheme over $\mathcal{O}_F$. We fix $T \subset G$ to be the centralizer of a maximal $\mathcal{O}_F$-split torus, and fix a Borel subgroup $B \subset G$ containing T. For $\mu \in X_*(T)^+$ and $b \in G(L)$, the *affine Deligne–Lusztig variety* associated to (G, μ, b) is defined to be

$$X_\mu(b) = \{g \in G(L)/G(\mathcal{O}_L) \mid g^{-1}b\sigma(g) \in G(\mathcal{O}_L)\mu(\pi_F)G(\mathcal{O}_L)\}.$$

More precisely, the above set is the set of $\overline{\mathbb{F}}_q$-points of a scheme or a perfect scheme, depending on whether F has equal or mixed characteristic. See [1] and [47] for the result in mixed characteristic.

Let Σ^{top} be the set of top-dimensional irreducible components of $X_\mu(b)$. The group

$$J := J_b(F) = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}$$

naturally acts on $X_\mu(b)$. Our goal is to understand the set $J\backslash\Sigma^{\mathrm{top}}$ of J-orbits in Σ^{top}.

The motivation for studying this set is to understand cycles in the special fiber of Shimura varieties; in particular cycles arising from the basic, or supersingular locus. In [45], the authors used the description of $J\backslash\Sigma^{\mathrm{top}}$ in some special cases to prove certain cases of the Tate conjecture for Shimura varieties. In a different situation, a description of the components in the supersingular locus was used to study certain arithmetic level-raising phenomena in [33]. After the work of Xiao–Zhu [45], Miaofen Chen and Xinwen Zhu formulated a general conjecture relating $J\backslash\Sigma^{\mathrm{top}}$ to the Mirković–Vilonen cycles in the affine Grassmannian. To state the conjecture we introduce some more notations.

Let $\widehat{G}$ denote the Langlands dual group of G over $\mathbb{C}$, equipped with a Borel pair $\widehat{T} \subset \widehat{B}$, where $\widehat{T}$ is a maximal torus dual to T and equipped with an algebraic action by σ. Let $\widehat{\mathcal{S}}$ be the identity component of the σ-fixed points of $\widehat{T}$. In $X^*(\widehat{\mathcal{S}})$, there is a distinguished element λ_b, determined by b. It is the "best integral approximation" of the Newton cocharacter of b, but we omit its precise definition here (see Definition 2.6.4). The fixed μ determines a highest weight representation V_μ of $\widehat{G}$. We write $V_\mu(\lambda_b)_{\mathrm{rel}}$ for the λ_b-weight space in V_μ with respect to the action of $\widehat{\mathcal{S}}$.

Conjecture 1.1.1 (Miaofen Chen, Xinwen Zhu)**.** *There exists a natural bijection between* $J\backslash\Sigma^{\mathrm{top}}$ *and the Mirković–Vilonen basis of* $V_\mu(\lambda_b)_{\mathrm{rel}}$*. In particular,*

$$\left|J\backslash\Sigma^{\mathrm{top}}\right| = \dim V_\mu(\lambda_b)_{\mathrm{rel}}. \tag{1.1.1}$$

Our main result is the following

Theorem A (Corollary 6.3.5)**.** *Conjecture 1.1.1 holds.*

When the group G is quasi-split but not necessarily unramified, we are able to prove an analogous result, see Appendix A for the details.

1.2. Previous results

Previously, partial results towards Conjecture 1.1.1 have been obtained by Xiao–Zhu [45], Hamacher–Viehmann [16], and Nie [35], based on a common idea of reduction to the superbasic case (which goes back to [10]).

More precisely, Xiao–Zhu [45] proved the conjecture for general G, general μ, and *unramified* b, meaning that J_b and G are assumed to have equal F-rank.

Hamacher–Viehmann [16] proved the conjecture under either of the following two assumptions:

- The cocharacter μ is minuscule, and G is split over F.
- The cocharacter μ is minuscule, and b is superbasic in M, where M is the largest Levi of G inside which b is basic. (In particular if b is basic then they assume that b is superbasic).

More recently, Nie [36] proved the conjecture for arbitrary G under the assumption that μ is a sum of dominant minuscule coweights. In particular it holds when the Dynkin diagram of $G_{\overline{F}}$ only involves factors of type A. Moreover, Nie constructed a surjection from the Mirković–Vilonen basis to

the set $J\backslash\Sigma^{\mathrm{top}}$ in all cases. Thus in order to prove the conjecture, it suffices to prove the numerical relation (1.1.1) for groups besides type A.

After we finished this work, Nie uploaded online new versions of the preprint [36], in which he proved Conjecture 1.1.1 in full generality, using methods independent of ours. Our work only uses the weaker results of his as stated in the above paragraph.

1.3. Some features of the method

Our proof of Conjecture 1.1.1 is based on an approach completely different from the previous works. The key idea is to use the Lang–Weil estimate to relate the cardinality of $J\backslash\Sigma^{\mathrm{top}}$ to the asymptotic behavior of the number of points on $X_\mu(b)$ over a finite field, as the finite field grows.

We show that the number of points over a finite field, when counted suitably, is given by a twisted orbital integral. Thus we reduce the problem to the asymptotic behavior of twisted orbital integrals. We study the latter using explicit methods from local harmonic analysis and representation theory, including the Base Change Fundamental Lemma and the Kato–Lusztig formula.

In our proof, polynomials that are linear combinations of the $\mathbf{q}$-analogue of Kostant partition functions appear, and the key computation is to estimate their sizes. These polynomials (denoted by $\mathfrak{M}^0_\lambda(\mathbf{q})$ in the paper) can be viewed as a non-dominant generalization of the $\mathbf{q}$-analogue of Kostant weight multiplicity. Some properties of them are noted in [37], but beyond this there does not seem to have been a lot of study into these objects. From our proof, it seems reasonable to expect that a more thorough study of the combinatorial and geometric properties of these polynomials would shed new light on the structure of affine Deligne–Lusztig varieties, as well as the structure of twisted orbital integrals.

An interesting point in our proof is that we need to apply the Base Change Fundamental Lemma, which is only available in general for mixed characteristic local fields. In fact, the proofs of this result by Clozel [5] and Labesse [31] rely on methods only available over characteristic zero, for example the trace formula of Deligne–Kazhdan. Thus our method crucially depends on the geometric theory of mixed characteristic affine Grassmannians as in [1] and [47]. To deduce Conjecture 1.1.1 also for equal characteristic local fields, we apply results of He [18] to prove the following.

Theorem B (Theorem 3.1.1, Theorem 3.2.1)**.** *For any $Z \in \Sigma^{\mathrm{top}}$, the stabilizer $\mathrm{Stab}_Z J$ is a parahoric subgroup of J. Moreover, these parahoric subgroups, as well as the quotient set $J\backslash\Sigma^{\mathrm{top}}$, are independent of the local field*

F in a precise sense. In particular, the truth of Conjecture 1.1.1 transfers between different local fields.

Our method also allows us to show that the stabilizers of the action of J on Σ^{top} are all hyperspecial, when b is basic and unramified. This reproves a result of Xiao–Zhu [45], see Remark 1.4.3 below. For ramified b, we obtain the following result, which refines Conjecture 1.1.1 in that it provides information about the stabilizers.

Theorem C (Corollary 6.3.5 and (6.3.3))**.** *Assume* $\mathrm{char}(F) = 0$*. Assume G is F-simple and adjoint, without type A or E_6 factors. Assume b is basic and ramified. Then we have*

$$\left|J\backslash\Sigma^{\mathrm{top}}\right| \cdot \mathscr{L}_b = \sum_{Z \in J\backslash\Sigma^{\mathrm{top}}} \mathrm{vol}(\mathrm{Stab}_Z\, J)^{-1},$$

where the volumes $\mathrm{vol}(\mathrm{Stab}_Z\, J)$ *are computed with a fixed Haar measure on J, and $\mathscr{L}_b$ is a constant that depends only on b and not on μ.*

In a future work we shall explore the possibility of applying Theorems A, B, C to determine the stabilizers in general.

1.4. Overview of the proof

We now explain in more detail our proof of Conjecture 1.1.1. A standard reduction allows us to assume that b is basic, and that G is adjoint and F-simple. Throughout we also assume that G is not of type A, which is already sufficient by the work of Nie [36]. To simplify the exposition, we also assume that G is split and not of type E_6. Then $\widehat{\mathcal{S}} = \widehat{T}$, and we drop the subscript "rel" for the weight spaces in Conjecture 1.1.1.

For any $s \in \mathbb{Z}_{>0}$, we let F_s be the unramified extension of F of degree s. We denote by $\mathcal{H}_s$ the spherical Hecke algebra $\mathcal{H}(G(F_s)//G(\mathcal{O}_{F_s}))$. We may assume without loss of generality that b is s_0-decent for a fixed $s_0 \in \mathbb{N}$, meaning that $b \in G(F_{s_0})$ and

$$b\sigma(b)\cdots\sigma^{s_0-1}(b) = 1.$$

As mentioned above, our idea is to use the Lang–Weil estimate to relate the number of irreducible components to the asymptotics of twisted orbital integrals. Since $X_\mu(b)$ is only locally of (perfectly) finite type and we are only counting J-orbits of irreducible components, we need a suitable interpretation of the Lang–Weil estimate. The precise output is the following.

Proposition 1.4.1 (Proposition 4.2.4)**.** *Let $s \in s_0\mathbb{N}$. Let $f_{\mu,s} \in \mathcal{H}_s$ be the characteristic function of $G(\mathcal{O}_{F_s})\mu(\pi_F)G(\mathcal{O}_{F_s})$, and let $TO_b(f_{\mu,s})$ denote the twisted orbital integral of $f_{\mu,s}$ along $b \in G(F_s)$. We have*

(1.4.1)
$$TO_b(f_{\mu,s}) = \sum_{Z \in J\backslash\Sigma^{\mathrm{top}}} \mathrm{vol}(\mathrm{Stab}_Z\, J)^{-1} q^{s \dim X_\mu(b)} + o(q^{s \dim X_\mu(b)}), \quad s \gg 0.$$

To proceed, we apply the Base Change Fundamental Lemma to compute $TO_b(f_{\mu,s})$. There are two problems in this step. Firstly, the Base Change Fundamental Lemma can only be applied to stable twisted orbital integrals. This problem is solved because one can check that $TO_b(f_{\mu,s})$ is in fact equal to the corresponding stable twisted orbital integral. Secondly, the general Base Change Fundamental Lemma is only available for $\mathrm{char}(F) = 0$. The way to circumvent this was already discussed in §1.3 above.

We define $\delta := (\mathrm{rk}_F J_b - \mathrm{rk}_F G)/2$. Up to lower order error terms, we may combine the above-mentioned computation of $TO_b(f_{\mu,s})$ with asymptotics of the Kato–Lusztig formula [22] to rewrite the left hand side of (1.4.1), and we may use the dimension formula for $X_\mu(b)$ (by Hamacher [15] and Zhu [47]) to rewrite the right hand side of (1.4.1). The result is the following:

(1.4.2)
$$\sum_{\lambda \in X^*(\widehat{T})^+, \lambda \le \mu} \dim V_\mu(\lambda) \cdot \mathfrak{M}^0_{s\lambda}(q^{-1}) = \pm \sum_{Z \in J\backslash\Sigma^{\mathrm{top}}} \mathrm{vol}(\mathrm{Stab}_Z\, J)^{-1} q^{s\delta} + o(q^{s\delta}), \quad s \gg 0,$$

where each $\mathfrak{M}^0_{s\lambda}(q^{-1})$ is the value at $\mathbf{q} = q^{-1}$ of a polynomial $\mathfrak{M}^0_{s\lambda}(\mathbf{q}) \in \mathbb{C}[\mathbf{q}]$, given explicitly in terms of the $\mathbf{q}$-analogues of Kostant's partition functions (see Definition 5.2.7 and §5.3).

The key computation needed to further analyze (1.4.2) is summarized in the following.

Proposition 1.4.2. *Let $\lambda_b^+ \in X^*(\widehat{T})^+$ be the dominant conjugate of λ_b. For all $\lambda \in X^*(\widehat{T})^+ - \{\lambda_b^+\}$, we have*

$$\mathfrak{M}^0_{s\lambda}(q^{-1}) = o(q^{s\delta}), \ s \gg 0.$$

When G is the split adjoint E_6, we only prove a weaker form of Proposition 1.4.2, which also turns out to be sufficient for our purpose.

Proposition 1.4.2 tells us that on the left hand side of (1.4.2), only the summand indexed by $\lambda = \lambda_b^+$ has the "right size". Taking the limit, we obtain

$$\dim V_\mu(\lambda_b) \cdot \mathscr{L}_b = \sum_{Z \in J\backslash\Sigma^{\mathrm{top}}} \mathrm{vol}(\mathrm{Stab}_Z J)^{-1}, \tag{1.4.3}$$

where $\mathscr{L}_b$ is independent of μ.

In (1.4.3), we already see both the number $\dim V_\mu(\lambda_b)$ and the set $J\backslash\Sigma^{\mathrm{top}}$. In order to deduce the desired (1.1.1), one still needs some information on the volume terms $\mathrm{vol}(\mathrm{Stab}_Z J)$. It turns out that even very weak information will suffice. In §3 we show that the right hand side of (1.4.3) is equal to $R(q)$, where $R(T) \in \mathbb{Q}(T)$ is a rational function which is independent of F in a precise sense. Moreover we show that $|R(0)| = |J\backslash\Sigma^{\mathrm{top}}|$. Therefore, the desired (1.1.1) will follow from (1.4.3), if we can show that

$$\mathscr{L}_b = S(q), \quad \text{for some } S(T) \in \mathbb{Q}(T) \text{ with } |S(0)| = 1. \tag{1.4.4}$$

A remarkable feature of the formulation (1.4.4) is that it is independent of μ. We recall that in the works of Hamacher–Viehmann and Nie, special assumptions on μ are made. Hence we are able to bootstrap from known cases of Conjecture 1.1.1 (for example when $\mu = \lambda_b^+$) to establish (1.4.4), and hence to establish Conjecture 1.1.1 in general.

We end our discussion with the following remark.

Remark 1.4.3. At the moment, we are unable to directly compute the rational functions $S(T)$ appearing in (1.4.4) in general. To do this would require a much better understanding of the polynomials $\mathfrak{M}^0_{s\lambda}(\mathbf{q})$. We are however able to compute $S(T)$ in a very special case. When b is a basic unramified element in the sense of [45], we show directly that (1.4.4) is satisfied by $S(T) \equiv 1$, see §6.2. From this we deduce the conjecture for b, as well as the equality $\mathrm{vol}(\mathrm{Stab}_Z J) = 1$ for each $Z \in \Sigma^{\mathrm{top}}$. This last equality implies (according to our normalization) that $\mathrm{Stab}_Z J$ is a hyperspecial subgroup of J. This gives another proof of a result in [45], avoiding their use of Littelmann paths.

1.5. Organization of the paper

In §2, we introduce notations and state the Chen–Zhu conjecture. In §3, we study the action of J on Σ^{top}, proving Theorem B. In §4, we prove Proposition 1.4.1, and then apply the Base Change Fundamental Lemma to compute

twisted orbital integrals. In §5 we review the relationship between the coefficients of the Satake transform and the **q**-analogue of Kostant's partition functions, and draw some consequences. In §6 we state Proposition 6.3.2 as a more technical version of Proposition 1.4.2. We then deduce Conjecture 1.1.1 from Proposition 6.3.2. The proof of Proposition 6.3.2 is given in §7, §8, and §9, by analyzing the root systems case by case. In Appendix A, we generalize our main result to quasi-split groups.

1.6. Notations and conventions

We order $\mathbb{N}$ by divisibility, and write $s \gg 0$ to mean "for all sufficiently divisible $s \in \mathbb{N}$". The notation $\lim_{s\to\infty}$ will always be understood as the limit taken with respect to the divisibility on $\mathbb{N}$. If $f(s), g(s)$ are $\mathbb{C}$-valued functions defined for all sufficiently divisible $s \in \mathbb{N}$, we write

$$f(s) = o(g(s))$$

to mean that $\lim_{s\to\infty} f(s)/g(s) = 0$, where the limit is understood in the above sense. We write

$$f(s) = O(g(s))$$

to mean that

$$\exists M > 0\ \exists s_0 \in \mathbb{N}\ \forall s \in s_0\mathbb{N},\ \ |f(s)/g(s)| < M.$$

In this case we do not require $f(s)/g(s)$ to be bounded, or even defined, for all $s \in \mathbb{N}$.

For any finitely generated abelian group X, we write X_{free} for the free quotient of X. We let $\mathbb{C}[X]$ be the group algebra of X over $\mathbb{C}$, and denote by e^x the element in $\mathbb{C}[X]$ corresponding to $x \in X$.

We use $\mathbf{q}$, or $\mathbf{q}^{-1}$, or sometimes $\mathbf{q}^{-1/2}$, to denote the formal variable in a polynomial or power series ring.

The following lemma is elementary and will be used repeatedly in the paper. We omit its proof.

Lemma 1.6.1. *Let Γ be a finite group. Let X be a $\mathbb{Z}[\Gamma]$-module which is a finite free $\mathbb{Z}$-module. As usual define the norm map $\mathrm{N} : X \to X$, $x \mapsto \sum_{\gamma\in\Gamma} \gamma(x)$. Let $Y \subset X$ be a Γ-stable subgroup. Then the following statements hold.*

1. *The kernel of the map $Y \to X_{\Gamma,\mathrm{free}}$ is equal to $\{y \in Y \mid \mathrm{N}(y) = 0\}$. In particular, it is also equal to the kernel of $Y \to Y_{\Gamma,\mathrm{free}}$.*

2. *Suppose Y has a finite $\mathbb{Z}$-basis which is stable under Γ. Then the Γ-orbits in this $\mathbb{Z}$-basis define distinct elements of Y_Γ, which form a $\mathbb{Z}$-basis of Y_Γ. In particular Y_Γ is a finite free $\mathbb{Z}$-module.*
3. *The map $\mathrm{N} : X \to X$ factors through a map $X_\Gamma \to X^\Gamma$. We have a canonical isomorphism $X_\Gamma \otimes \mathbb{Q} \xrightarrow{\sim} X^\Gamma \otimes \mathbb{Q}$ given by $\frac{1}{|\Gamma|}\mathrm{N}$.* □

2. Notations and preliminaries

2.1. Basic notations

Let F be a non-archimedean local field with valuation ring $\mathcal{O}_F$ and residue field $k_F = \mathbb{F}_q$. Let $\pi_F \in F$ be a uniformizer. Let p be the characteristic of k_F. Let L be the completion of the maximal unramified extension of F, with valuation ring $\mathcal{O}_L$ and residue field $k = \overline{k_F}$. Let $\Gamma = \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group. Let σ be the Frobenius of L over F.

Let G be a connected reductive group over $\mathcal{O}_F$. In particular its generic fiber G_F is an unramified reductive group over F, i.e. is quasi-split and splits over an unramified extension of F. Then $G(\mathcal{O}_F)$ is a hyperspecial subgroup of $G(F)$. Fix a maximal $\mathcal{O}_F$-split torus A of G. Let T be the centralizer of A_F in G_F, and fix a Borel subgroup $B \subset G_F$ containing T. Then T is an unramified maximal torus in G_F. In the following we often abuse notation and simply write G for G_F.

Note that T_L is a split maximal torus in G_L. Let V be the apartment of G_L corresponding to T_L. The hyperspecial vertex $\mathfrak{s}$ corresponding to $G(\mathcal{O}_L)$ is then contained in V. We have an identification $V \cong X_*(T) \otimes \mathbb{R}$ sending $\mathfrak{s}$ to 0. Let $\mathfrak{a} \subset V$ be the alcove whose closure contains $\mathfrak{s}$, such that the image of $\mathfrak{a}$ under $V \cong X_*(T) \otimes \mathbb{R}$ is contained in the anti-dominant chamber. The action of σ induces an action on V, stabilizing both $\mathfrak{a}$ and $\mathfrak{s}$. We let $\mathcal{I}$ be the Iwahori subgroup of $G(L)$ corresponding to $\mathfrak{a}$.

2.2. The Iwahori–Weyl group

The relative Weyl group W_0 over L and the Iwahori–Weyl group W are defined by

$$W_0 = N(L)/T(L), \qquad W = N(L)/T(L) \cap \mathcal{I},$$

where N denotes the normalizer of T in G. Note that W_0 is equal to the absolute Weyl group, as T_L is split.

We have a natural exact sequence

$$1 \longrightarrow X_*(T) \longrightarrow W \longrightarrow W_0 \longrightarrow 1. \tag{2.2.1}$$

The canonical action of $N(L)$ on V factors through an action of W, and we split the above exact sequence by identifying W_0 with the subgroup of W fixing $\mathfrak{s} \in V$. See [12, Proposition 13] for more details. When considering an element $\lambda \in X_*(T)$ as an element of W, we write $t^\lambda \in W$. For any $w \in W$, we choose a representative $\dot{w} \in N(L)$.

Let W_a be the associated affine Weyl group, and $\mathbb{S}$ be the set of simple reflections associated to $\mathfrak{a}$. Since $\mathfrak{a}$ is σ-stable, there is a natural action of σ on $\mathbb{S}$. We let $\mathbb{S}_0 \subset \mathbb{S}$ be the set of simple reflections fixing $\mathfrak{s}$. Then W contains W_a as a normal subgroup, and we have a natural splitting $W = W_a \rtimes \Omega$, where Ω is the stabilizer of $\mathfrak{a}$ in W and is isomorphic to $\pi_1(G)$. The length function ℓ and the Bruhat order $\leq$ on the Coxeter group $(W_a, \mathbb{S})$ extend in a natural way to W.

For any subset P of $\mathbb{S}$, we shall write W_P for the subgroup of W generated by P.

For $w, w' \in W$ and $s \in \mathbb{S}$, we write $w \xrightarrow{s}_\sigma w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leq \ell(w)$. We write $w \to_\sigma w'$ if there is a sequence $w = w_0, w_1, \ldots, w_n = w'$ of elements in W such that for any i, $w_{i-1} \xrightarrow{s_i}_\sigma w_i$ for some $s_i \in \mathbb{S}$. Note that if moreover, $\ell(w') < \ell(w)$, then there exists i such that $\ell(w) = \ell(w_i)$ and $s_{i+1} w_i \sigma(s_{i+1}) < w_i$.

We write $w \approx_\sigma w'$ if $w \to_\sigma w'$ and $w' \to_\sigma w$. It is easy to see that $w \approx_\sigma w'$ if $w \to_\sigma w'$ and $\ell(w) = \ell(w')$. We write $w \tilde{\approx}_\sigma w'$ if there exists $\tau \in \Omega$ such that $w \approx_\sigma \tau w' \sigma(\tau)^{-1}$.

2.3. The set $B(G)$

For any $b \in G(L)$, we denote by $[b] = \{g^{-1} b \sigma(g) \mid g \in G(L)\}$ its σ-conjugacy class. Let $B(G)$ be the set of σ-conjugacy classes of $G(L)$. The σ-conjugacy classes have been classified by Kottwitz in [26] and [28], in terms of the *Newton map* $\bar{\nu}$ and the *Kottwitz map* κ. The Newton map is a map

$$\bar{\nu} : B(G) \to (X_*(T)^+_{\mathbb{Q}})^\sigma, \tag{2.3.1}$$

where $X_*(T)^+_{\mathbb{Q}}$ is the set of dominant elements in $X_*(T)_{\mathbb{Q}} := X_*(T) \otimes \mathbb{Q}$. The Kottwitz map is a map

$$\kappa = \kappa_G : B(G) \to \pi_1(G)_\Gamma.$$

By [28, §4.13], the map

$$(\bar{\nu}, \kappa) : B(G) \to (X_*(T)^+_{\mathbb{Q}})^\sigma \times \pi_1(G)_\Gamma \tag{2.3.2}$$

is injective.

Let $B(W,\sigma)$ denote the set of σ-conjugacy classes of W. The map $W \to G(L), w \mapsto \dot{w}$ induces a map

$$\Psi : B(W,\sigma) \longrightarrow B(G),$$

which is independent of the choice of the representatives $\dot{w}$. By [18], the map Ψ is surjective. We again denote by $(\bar{\nu},\kappa)$ the composition of (2.3.2) with Ψ. This composed map can be described explicitly, see [20, §1.2] for details.

The map Ψ is not injective. However, there exists a canonical lifting to the set of *straight σ-conjugacy classes*. By definition, an element $w \in W$ is called *σ-straight* if for any $n \in \mathbb{N}$,

$$\ell(w\sigma(w)\cdots\sigma^{n-1}(w)) = n\ell(w).$$

This is equivalent to the condition that $\ell(w) = \langle \bar{\nu}_w, 2\rho\rangle$, where ρ is the half sum of all positive roots. A σ-conjugacy class of W is called *straight* if it contains a σ-straight element. It is easy to see that the minimal length elements in a given straight σ-conjugacy class are exactly the σ-straight elements.

Theorem 2.3.1 ([18, Theorem 3.7])**.** *The restriction of $\Psi : B(W,\sigma) \to B(G)$ gives a bijection from the set of straight σ-conjugacy classes of W to $B(G)$.* □

2.4. The affine Deligne–Lusztig variety $X_{P,w}(b)$

Let $\mathcal{P}$ be a standard σ-invariant parahoric subgroup of $G(L)$, i.e. a σ-invariant parahoric subgroup that contains $\mathcal{I}$. In the following, we will generally abuse of notation to use the same symbol to denote a parahoric subgroup and the underlying parahoric group scheme. We denote by $P \subset \mathbb{S}$ the set of simple reflections corresponding to $\mathcal{P}$. Then $\sigma(P) = P$. We have

$$G(L) = \bigsqcup_{w \in W_P \backslash W / W_P} \mathcal{P}(\mathcal{O}_L)\dot{w}\mathcal{P}(\mathcal{O}_L).$$

For any $w \in W_P\backslash W/W_P$ and $b \in G(L)$, we set

$$X_{P,w}(b)(k) := \{g\mathcal{P}(\mathcal{O}_L) \in G(L)/\mathcal{P}(\mathcal{O}_L) \mid g^{-1}b\sigma(g) \in \mathcal{P}(\mathcal{O}_L)\dot{w}\mathcal{P}(\mathcal{O}_L)\}.$$

If $\mathcal{P} = \mathcal{I}$ (corresponding to $P = \emptyset$), we simply write $X_w(b)(k)$ for $X_{\emptyset,w}(b)(k)$.

We freely use the standard notations concerning loop groups and partial affine flag varieties, see [1, §9] or [47, §1.4]. When $\mathrm{char}(F) > 0$, it is known that $X_{P,w}(b)(k)$ could be naturally identified with the set of k-points of a locally closed sub-ind scheme $X_{P,w}(b)$ of the partial affine flag variety $\mathrm{Gr}_{\mathcal{P}}$. When $\mathrm{char}(F) = 0$, thanks to the recent breakthrough by Bhatt–Scholze [1, Corollary 9.6] (cf. also [47]), we can again identify $X_{P,w}(b)(k)$ with the k-points of a locally closed perfect sub-ind scheme $X_{P,w}(b)$ of the Witt vector partial affine flag variety $\mathrm{Gr}_{\mathcal{P}}$. In both cases, the (perfect) ind-scheme $X_{P,w}(b)$ is called an *affine Deligne–Lusztig variety*, and one could consider topological notions related to the Zariski topology on $X_{P,w}(b)$.

We are mainly interested in the case when $\mathcal{P} = G_{\mathcal{O}_L}$. In this case the corresponding set of simple reflections is $K := \mathbb{S}_0$. Recall from §2.2 that we fixed a splitting of (2.2.1) using the hyperspecial vertex $\mathfrak{s}$. According to this splitting, the subgroup W_0 of W is the same as W_K, the subgroup generated by $K = \mathbb{S}_0$. We have identifications

$$W_K \backslash W / W_K = W_0 \backslash (X_*(T) \rtimes W_0) / W_0 \cong X_*(T)/W_0 \cong X_*(T)^+.$$

For $\mu \in X_*(T)^+$, we write $X_\mu(b)$ for $X_{K,t^\mu}(b)$.

We simply write Gr_G for $\mathrm{Gr}_{G_{\mathcal{O}_L}}$. The relationship between the hyperspecial affine Deligne–Lusztig variety $X_\mu(b) \subset \mathrm{Gr}_G$ and the Iwahori affine Deligne–Lusztig varieties $X_w(b) \subset \mathrm{Gr}_{\mathcal{I}}$ is as follows. We have a projection $\pi : \mathcal{FL} \to \mathrm{Gr}_G$ which exhibits $\mathcal{FL} := \mathrm{Gr}_{\mathcal{I}}$ as an étale fibration over Gr_G. Indeed the fiber of this map is isomorphic to the fpqc quotient $L^+G/L^+\mathcal{I}$ where $L^+G, L^+\mathcal{I}$ are the positive loop groups attached to G and $\mathcal{I}$. More concretely, $L^+G/L^+\mathcal{I}$ is a finite type flag variety over k when $\mathrm{char}(F) > 0$, and is the perfection of a finite type flag variety over k when $\mathrm{char}(F) = 0$. We have

$$\pi^{-1}(X_\mu(b)) = X(\mu, b)^K := \bigcup_{w \in W_0 t^\mu W_0} X_w(b).$$

2.5. Basic information about $X_\mu(b)$

For $\lambda, \lambda' \in X_*(T)_{\mathbb{Q}}$, we write $\lambda \leq \lambda'$ if $\lambda' - \lambda$ is a non-negative rational linear combination of positive coroots. Let $\mu \in X_*(T)^+$. As in [41], we set

$$B(G, \mu) = \{[b] \in B(G) \mid \kappa([b]) = \mu^\natural, \bar{\nu}_b \leq \mu^\diamond\}.$$

Here $\mu^\natural$ denotes the image of μ in $\pi_1(G)_\Gamma$, and $\mu^\diamond \in X_*(T)_{\mathbb{Q}}$ denotes the Galois average of μ.

The following result is proved by Kottwitz [29] and Gashi [8], strengthening earlier results of Rapoport–Richartz [40], Kottwitz–Rapoport [24], and Lucarelli [34].

Theorem 2.5.1. *For $b \in G(L)$, we have $X_\mu(b) \neq \emptyset$ if and only if $[b] \in B(G,\mu)$.* □

We now let $\mu \in X_*(T)^+$ and let $b \in G(L)$ such that $[b] \in B(G,\mu)$.

Definition 2.5.2. We define $\mathrm{def}_G(b) := \mathrm{rk}_F G - \mathrm{rk}_F J_b$, called the *defect* of b.

Theorem 2.5.3. *If $\mathrm{char}(F) > 0$, then $X_\mu(b)$ is a scheme locally of finite type over k. If $\mathrm{char}(F) = 0$, then $X_\mu(b)$ is a perfect scheme locally perfectly of finite type over k. In both cases the Krull dimension of $X_\mu(b)$ is equal to*

$$\langle \mu - \bar{\nu}_b, \rho \rangle - \frac{1}{2}\mathrm{def}_G(b).$$

Proof. The local (perfectly) finiteness is proved by Hamacher–Viehmann [16, Lemma 1.1], cf. [17]. The dimension formula is proved by Hamacher [15] and Xinwen Zhu [47], strengthening earlier results of Görtz–Haines–Kottwitz–Reuman [10] and Viehmann [44]. □

Definition 2.5.4. For any (perfect) scheme X, we write $\Sigma(X)$ for the set of irreducible components of X. When X is of finite Krull dimension, we write $\Sigma^{\mathrm{top}}(X)$ for the set of top dimensional irreducible components of X.

Define the group scheme J_b over F by

$$J_b(R) = \left\{ g \in G(R \otimes_F L) \mid g^{-1} b \sigma(g) = b \right\} \tag{2.5.1}$$

for any F-algebra R. Then J_b is an inner form of a Levi subgroup of G, see [42, §1.12] or [40, §1.11]. The group $J_b(F)$ acts on $X_\mu(b)$ via scheme automorphisms. In particular $J_b(F)$ acts on $\Sigma(X_\mu(b))$ and on $\Sigma^{\mathrm{top}}(X_\mu(b))$. The following finiteness result is proved in [17, Theorem 1.1].

Lemma 2.5.5. *The set $J_b(F)\backslash\Sigma(X_\mu(b))$ is finite.* □

Definition 2.5.6. We write $\mathscr{N}(\mu, b)$ for the cardinality of $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$.

2.6. The Chen–Zhu conjecture

In this paper we shall utilize the usual Langlands dual group (as a reductive group over $\mathbb{C}$ equipped with a pinned action by the Galois group), rather than the Deligne–Lusztig dual group which is used in [16]. As a result, our formulation of the Chen–Zhu conjecture below differs from [16, Conjecture 1.3, §2.1]. However it can be easily checked that the two formulations are equivalent.

The Frobenius σ acts on $X_*(T)$ via a finite-order automorphism, which we denote by θ. Let $\widehat{G}$ be the usual dual group of G over $\mathbb{C}$, which is a reductive group over $\mathbb{C}$ equipped with a Borel pair $(\widehat{B}, \widehat{T})$ and isomorphisms $X^*(\widehat{T}) \cong X_*(T), X_*(\widehat{T}) \cong X^*(T)$. These last isomorphisms, which will be regarded as equalities, identify the positive roots (resp. coroots) with the positive coroots (resp. roots). For more details on the dual group see §5.1 below.

Definition 2.6.1. Let $\widehat{\mathcal{S}}$ be the identity component of the $\hat{\theta}$-fixed points of $\widehat{T}$. Equivalently, $\widehat{\mathcal{S}}$ is the sub-torus of $\widehat{T}$ such that the map $X^*(\widehat{T}) \to X^*(\widehat{\mathcal{S}})$ is equal to the map $X^*(\widehat{T}) \longrightarrow X^*(\widehat{T})_{\hat{\theta},\mathrm{free}}$.

Definition 2.6.2. For $\mu \in X_*(T)^+ = X^*(\widehat{T})^+$, let V_μ be the highest weight representation of $\widehat{G}$ of highest weight μ. For all $\lambda' \in X^*(\widehat{T})$, we write $V_\mu(\lambda')$ for the λ'-weight space in V_μ as a representation of $\widehat{T}$. For all $\lambda \in X^*(\widehat{\mathcal{S}})$, we write $V_\mu(\lambda)_{\mathrm{rel}}$ for the λ-weight space in V_μ as a representation of $\widehat{\mathcal{S}}$.

As in §2.5, let $\mu \in X_*(T)^+$, and let $[b] \in B(G, \mu)$. By Lemma 1.6.1 (3) we identify $X_*(T)^\theta_{\mathbb{Q}}$ with

$$X_*(T)_\theta \otimes \mathbb{Q} = X_*(T)_{\theta,\mathrm{free}} \otimes \mathbb{Q} = X^*(\widehat{T})_{\hat{\theta},\mathrm{free}} \otimes \mathbb{Q} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q},$$

and we shall view $\bar{\nu}_b$ (see (2.3.1)) as an element of $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$. We also have $\kappa(b) \in \pi_1(G)_\Gamma = \pi_1(G)_\sigma$, which is equal to the image of μ.

Let $\widehat{Q}$ be the root lattice inside $X^*(\widehat{T})$. Applying Lemma 1.6.1 to $X = X^*(\widehat{T})$ and $Y = \widehat{Q}$, we obtain:

- $\widehat{Q}_{\hat{\theta}}$ is a free $\mathbb{Z}$-module. It injects into $X^*(\widehat{T})_{\hat{\theta}}$ and also injects into $X^*(\widehat{T})_{\hat{\theta},\mathrm{free}} = X^*(\widehat{\mathcal{S}})$.
- The image of the simple roots in $\widehat{Q}$ in $\widehat{Q}_{\hat{\theta}}$ (as a set) is a $\mathbb{Z}$-basis of $\widehat{Q}_{\hat{\theta}}$. We call members of this $\mathbb{Z}$-basis the *relative simple roots* in $\widehat{Q}_{\hat{\theta}}$.

Lemma 2.6.3. *There is a unique element $\tilde{\lambda}_b \in X^*(\widehat{T})_{\hat{\theta}}$ satisfying the following conditions:*

1. *The image of* $\tilde{\lambda}_b$ *in* $\pi_1(G)_\sigma$ *is equal to* $\kappa(b)$.
2. *In* $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$, *the element* $(\tilde{\lambda}_b)|_{\widehat{\mathcal{S}}} - \bar{\nu}_b$ *is equal to a linear combination of the relative simple roots in* $\widehat{Q}_{\hat{\theta}}$, *with coefficients in* $\mathbb{Q} \cap (-1, 0]$. *Here* $(\tilde{\lambda}_b)|_{\widehat{\mathcal{S}}}$ *denotes the image of* $\tilde{\lambda}_b$ *under the map* $X^*(\widehat{T})_{\hat{\theta}} \to X^*(\widehat{T})_{\hat{\theta},\mathrm{free}} = X^*(\widehat{\mathcal{S}})$.

Proof. This is just a reformulation of [16, Lemma 2.1]. □

Definition 2.6.4. Let $\tilde{\lambda}_b \in X^*(\widehat{T})|_{\hat{\theta}}$ be as in Lemma 2.6.3. We write λ_b for $(\tilde{\lambda}_b)|_{\widehat{\mathcal{S}}} \in X^*(\widehat{\mathcal{S}})$.

Conjecture 2.6.5 (Miaofen Chen, Xinwen Zhu)**.** *Let* $\mu \in X_*(T)^+$ *and let* $[b] \in B(G, \mu)$. *There exists a natural bijection between* $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$ *and the Mirković–Vilonen basis of* $V_\mu(\lambda_b)_{\mathrm{rel}}$.

Definition 2.6.6. For $\mu \in X_*(T)^+$ and $[b] \in B(G, \mu)$, we write $\mathscr{N}(\mu, b)$ for the cardinality of $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$, and write $\mathscr{M}(\mu, b)$ for $\dim V_\mu(\lambda_b)_{\mathrm{rel}}$.

Conjecture 2.6.5 has the following numerical consequence:

Conjecture 2.6.7 (Numerical Chen–Zhu)**.** *Let* $\mu \in X_*(T)^+$ *and let* $[b] \in B(G, \mu)$. *We have*

$$\mathscr{N}(\mu, b) = \mathscr{M}(\mu, b).$$

In [35], Nie obtained the following results:

Theorem 2.6.8 (Nie)**.**

1. *In order to prove Conjecture 2.6.5, it suffices to prove it when* G *is adjoint and* b *is basic.*
2. *There is a natural surjective map from the Mirković–Vilonen basis of* $V_\mu(\lambda_b)_{\mathrm{rel}}$ *to the set* $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$. *Thus in order to prove Conjecture 2.6.5, it suffices to prove Conjecture 2.6.7.*
3. *Conjecture 2.6.5 holds if* μ *is a sum of dominant minuscule elements. In particular, it holds if all absolute simple factors of* G^{ad} *are of type* A.

Remark 2.6.9. After the present paper was finished, Nie uploaded online new versions of the preprint [36], in which he proved Conjecture 2.6.5 in full generality. His methods are independent of ours. The present paper depends logically only on Nie's results stated in Theorem 2.6.8, see Remark 6.3.4 for more details.

A further standard argument, for example [20, §6], shows that one can also reduce the proof of Conjecture 2.6.7 to the case where G is F-simple. Therefore in view of Theorem 2.6.8 (1) (2), we have:

Proposition 2.6.10. *In order to prove Conjecture 2.6.5, it suffices to prove Conjecture 2.6.7 when G is adjoint, F-simple, and $b \in G(L)$ represents a basic σ-conjugacy class.* □

3. The action of $J_b(F)$

3.1. The stabilizer of a component

In this section we study the stabilizer in $J_b(F)$ of an irreducible component of $X_\mu(b)$. Here as before we let $\mu \in X_*(T)^+$ and $[b] \in B(G,\mu)$. The first main result is the following.

Theorem 3.1.1. *The stabilizer in $J_b(F)$ of each $Z \in \Sigma(X_\mu(b))$ is a parahoric subgroup of $J_b(F)$.*

We first reduce this statement to a question about the Iwahori affine Deligne–Lusztig varieties $X_w(b), w \in W_0 t^\mu W_0$. Note that $J_b(F)$ acts on each $X_w(b)$ via automorphisms.

Proposition 3.1.2. *The projection $\pi : \mathcal{FL} \to \mathrm{Gr}_G$ induces a bijection between $\Sigma(X_\mu(b))$ and $\Sigma(X(\mu,b)^K)$ compatible with the action of $J_b(F)$. Moreover, this bijection maps $\Sigma^{\mathrm{top}}(X_\mu(b))$ onto $\Sigma^{\mathrm{top}}(X(\mu,b)^K)$.*

Proof. This follows from the fact that the fiber of π is (the perfection of) a flag variety. □

In view of Proposition 3.1.2, the proof of Theorem 3.1.1 reduces to showing that the stabilizer of each irreducible component of $X(\mu,b)^K$ is a parahoric subgroup of $J_b(F)$.

Now let $Y \in \Sigma(X(\mu,b)^K)$. Then since each $X_w(b)$ is locally closed in $\mathcal{FL}$, there exists $w \in W_0 t^\mu W_0$ such that $Y \cap X_w(b)$ is open dense in Y and is an irreducible component of $X_w(b)$. Since the action of $J_b(F)$ on $X(\mu,b)^K$ preserves $X_w(b)$, it follows that $j \in J_b(F)$ stabilizes Y if and only if j stabilizes $Y \cap X_w(b)$. Hence we have reduced to showing that the stabilizer in $J_b(F)$ of any element of $\Sigma(X_w(b))$ is a parahoric subgroup. We will show that this is indeed the case in Proposition 3.1.4 below.

One important tool needed in our proof is the following result, which is [11, Corollary 2.5.3].

Proposition 3.1.3. *Let $w \in W$, and let $s \in \mathbb{S}$ be a simple reflection.*

1. *If $\ell(sw\sigma(s)) = \ell(w)$, then there exists a universal homeomorphism $X_w(b) \to X_{sw\sigma(s)}(b)$.*

2. *If $\ell(sw\sigma(s)) < \ell(w)$, then there is a decomposition $X_w(b) = X_1 \sqcup X_2$, where X_1 is closed and X_2 is open, and such that there exist morphisms $X_1 \to X_{sw\sigma(s)}(b)$ and $X_2 \to X_{sw}(b)$, each of which is the composition of a Zariski-locally trivial fiber bundle with one-dimensional fibers and a universal homeomorphism.*

Moreover the universal homeomorphism in (1) and the morphisms $X_1 \to X_{sw\sigma(s)}(b)$ and $X_2 \to X_{sw}(b)$ in (2) are all equivariant for the action of $J_b(F)$. □

Proposition 3.1.4. *Assume $X_w(b) \neq \emptyset$ and let $Z \in \Sigma(X_w(b))$. The stabilizer in $J_b(F)$ of Z is a parahoric subgroup of $J_b(F)$.*

Proof. We prove this by induction on $\ell(w)$. Assume first that $w \in W$ is of minimal length in its σ-conjugacy class. Then $X_w(b) \neq \emptyset$ implies $\Psi(w) = b$, i.e. w and b represent the same σ-conjugacy class in $B(G)$, by [18, Theorem 3.5]. In this case, by [18, Theorem 4.8] and its proof, there is an explicit description of the stabilizer of an irreducible component which we recall.

Let ${}^P W \subset W$ (resp. $W^{\sigma(P)} \subset W$) denote the set of minimal representatives of the cosets $W_P \backslash W$ (resp. $W/W_{\sigma(P)}$). Let ${}^P W^{\sigma(P)}$ be the intersection ${}^P W \cap W^{\sigma(P)}$ (cf. [18, §1.6]). By [18, Theorem 2.3], there exists $P \subset \mathbb{S}$, $x \in {}^P W^{\sigma(P)}$, and $u \in W_P$, such that:

- W_P is finite.
- x is σ-straight and $x^{-1}\sigma(P)x = P$.

In this case, there is a $J_b(F)$-equivariant universal homeomorphism between $X_w(b)$ and $X_{ux}(b)$, and we have $\Psi(ux) = \Psi(w)$, see [18, Corollary 4.4]. Hence we may assume $w = ux$. By [18, Lemma 3.2] we have $\Psi(x) = \Psi(w)$, and therefore we may assume $b = \dot{x}$. Upon replacing P, we may assume P is minimal with respect to a fixed choice of x and u satisfying the above properties.

Let $\mathcal{P}$ denote the parahoric subgroup of $G(L)$ corresponding to P. The proof of [18, Theroem 4.8] shows that

$$X_{ux}(\dot{x}) \cong J_{\dot{x}}(F) \times_{J_{\dot{x}}(F) \cap \mathcal{P}} X_{ux}^{\mathcal{P}}(\dot{x}),$$

where $X_{ux}^{\mathcal{P}}(\dot{x})$ is the reduced k-subscheme of the (perfectly) finite type scheme $L^+\mathcal{P}/L^+\mathcal{I}$ whose k-points are

$$X_{ux}^{\mathcal{P}}(\dot{x})(k) = \{g \in \mathcal{P}(\mathcal{O}_L)/\mathcal{I}(\mathcal{O}_L) \mid g^{-1}\dot{x}\sigma(g) \in \mathcal{I}(\mathcal{O}_L)\dot{u}\dot{x}\mathcal{I}(\mathcal{O}_L)\}.$$

Thus it suffices to show the stabilizer in $J_{\dot{x}}(F) \cap \mathcal{P}(\mathcal{O}_L)$ of an irreducible component of $X_{ux}^{\mathcal{P}}(\dot{x})$ is a parahoric subgroup of $J_{\dot{x}}(F)$.

Let $\overline{\mathcal{P}}$ denote the algebraic group over k, which is the reductive quotient of the special fiber of $\mathcal{P}$. Recall its Weyl group is naturally identified with W_P. Then $\mathcal{I}$ is the pre-image of a Borel subgroup $\overline{\mathcal{I}}$ of $\overline{\mathcal{P}}$ under the reduction map $\mathcal{P} \to \overline{\mathcal{P}}$. Let $\sigma_{\dot{x}}$ denote the automorphism of $\overline{\mathcal{P}}$ given by $\overline{p} \mapsto \dot{x}^{-1}\sigma(\overline{p})\dot{x}$. Then the natural map $L^+\mathcal{P}/L^+\mathcal{I} \to \overline{\mathcal{P}}/\overline{\mathcal{I}}$ induces an identification between $X^{\mathcal{P}}_{\dot{u}\dot{x}}(\dot{x})$ and (the perfection of) the finite type Deligne–Lusztig variety

$$X' = \{\overline{p} \in \overline{\mathcal{P}}/\overline{\mathcal{I}} \mid \overline{p}^{-1}\sigma_{\dot{x}}(\overline{p}) \in \overline{\mathcal{I}}\dot{u}\overline{\mathcal{I}}\}.$$

The natural projection map $\mathcal{P} \to \overline{\mathcal{P}}$ takes $J_{\dot{x}}(F) \cap \mathcal{P}(\mathcal{O}_L)$ to $\overline{\mathcal{P}}^{\sigma_{\dot{x}}}$, and the action of $J_{\dot{x}}(F) \cap \mathcal{P}(\mathcal{O}_L)$ factors through this map. Since P is minimal satisfying $u \in W_P$ and since $x^{-1}\sigma(P)x = P$, it follows that u is not contained in any $\sigma_{\dot{x}}$-stable parabolic subgroup of W_P. Therefore by [9, Corollary 1.2], X' is irreducible. It follows that the stabilizer of the irreducible component $1 \times X^{\mathcal{P}}_{ux}(\dot{x}) \subset X_{ux}(\dot{x})$ is $J_{\dot{x}}(F) \cap \mathcal{P}(\mathcal{O}_L)$, which is a parahoric of $J_{\dot{x}}(F)$. It also follows that the stabilizer of any other irreducible component of $X_{ux}(\dot{x})$ is a conjugate parahoric.

Now we assume w is not of minimal length in its σ-conjugacy class. By [19, Corollary 2.10], there exists $w' \approx_\sigma w$ and $s \in \mathbb{S}$ such that $sw'\sigma(s) < w'$. Then by Proposition 3.1.3, there is a $J_b(F)$-equivariant universal homeomorphism between $X_w(b)$ and $X_{w'}(b)$. Thus it suffices to prove the result for $X_{w'}(b)$.

Let $Z' \in \Sigma(X_{w'}(b))$, and let X_1 and X_2 be as in Proposition 3.1.3. We have either $Z' \cap X_1$ or $Z' \cap X_2$ is open dense in Z'. Assume $Z' \cap X_1$ is open dense in Z'; the other case is similar. Since $J_b(F)$ preserves X_1, it suffices to show that the stabilizer of $Z' \cap X_1$ is a parahoric. From the description of X_1, there exists an element $V \in \Sigma(X_{sw'\sigma(s)}(b))$ such that $Z' \cap X_1 \to V$ is a fibration and is $J_b(F)$-equivariant. Therefore by induction, the stabilizer of V is a parahoric of $J_b(F)$, and hence so is the stabilizer of $Z' \cap X_1$. □

3.2. Independence of F and volumes of stabilizers

The second main result of this section is that the set of $J_b(F)$-orbits of irreducible components of $X_\mu(b)$ and the volume of the stabilizer of an irreducible component depend only on the affine root system together with the action of the Frobenius. In particular, it is *independent of* F in a manner which we will now make precise. This fact is a key observation that we will need for later applications.

By [18, §6], the set of $J_b(F)$-orbits of top dimensional irreducible components of $X_w(b)$ depends only on the affine root system of G together with the

action of σ. This is proved by using the Deligne–Lusztig reduction method to relate the number of orbits to coefficients of certain *class polynomials*, which can be defined purely in terms of the affine root system for G, see *loc. cit.* for details. In view of the fibration

$$\pi : X(\mu, b)^K \to X_\mu(b)$$

it follows that the same is true for $X_\mu(b)$. In particular, the number $\mathscr{N}(\mu, b)$ depends only on the affine root system and hence does not depend on the local field F.

We will need the following stronger result. To state it, we introduce some notations. Let F' be another local field with residue field $\mathbb{F}_{q'}$. Let G' be a connected reductive group over $\mathcal{O}_{F'}$. Let $T' \subset B' \subset G'_{F'}$ be analogous to $T \subset B \subset G_F$ as in §2.1. Define the hyperspecial vertex $\mathfrak{s}'$, the apartment V', and the anti-dominant chamber $\mathfrak{a}'$ analogously to $\mathfrak{s}, V, \mathfrak{a}$. Assume there is an identification $V \cong V'$ that maps $X_*(T)^+$ into $X_*(T)^+$, maps $\mathfrak{a}$ into $\mathfrak{a}'$, maps $\mathfrak{s}$ to $\mathfrak{s}'$, and induces a σ-σ' equivariant bijection between the affine root systems. Here σ' denotes the q'-Frobenius acting on the affine roots system of G'. We fix such an identification once and for all. To the pair (μ, b), we attach a corresponding pair (μ', b') for G' as follows. The cocharacter $\mu' \in X_*(T')^+$ is defined to be the image of μ under the identification $X_*(T)^+ \cong X_*(T')^+$. To construct b', we note that since b is basic, it is represented by a unique σ-conjugacy class in Ω. The identification fixed above induces an identification of Iwahori–Weyl groups $W \cong W'$, which induces a bijection on length-zero elements. Then b' is represented by the corresponding length-zero element in W'.

By our choice of b', the affine root systems of J_b and $J_{b'}$ together with the actions of Frobenius are identified. We thus obtain a bijection between standard parahoric subgroups of J_b and those of $J_{b'}$. Let $\mathcal{J} \subset J_b(F)$ and $\mathcal{J}' \subset J_{b'}(F')$ be parahoric subgroups. We say that $\mathcal{J}$ and $\mathcal{J}'$ are *conjugate*, if the standard parahoric conjugate to $\mathcal{J}$ is sent to the standard parahoric conjugate to $\mathcal{J}'$ under the above-mentioned bijection. In the following, we write $J := J_b(F)$ and $J' := J_{b'}(F')$.

Theorem 3.2.1. *There is a bijection*

$$J\backslash\Sigma^{\mathrm{top}}(X_\mu(b)) \xrightarrow{\sim} J'\backslash\Sigma^{\mathrm{top}}(X_{\mu'}(b'))$$

with the following property. If $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$ and $Z' \in \Sigma^{\mathrm{top}}(X_{\mu'}(b'))$ are such that JZ is sent to $J'Z'$, then the parahoric subgroups $\mathrm{Stab}_Z(J) \subset J$ and $\mathrm{Stab}_{Z'}(J') \subset J'$ are conjugate.

The theorem will essentially follow from the next lemma.

Lemma 3.2.2. *Let $w' \in W'$ correspond to $w \in W$ under the identification $W \cong W'$. Then there is a bijection*

$$\Theta : J\backslash\Sigma^{\mathrm{top}}(X_w(b)) \xrightarrow{\sim} J'\backslash\Sigma^{\mathrm{top}}(X_{w'}(b'))$$

with the following property. If $Z \in \Sigma^{\mathrm{top}}(X_w(b))$ and $Z' \in \Sigma^{\mathrm{top}}(X_{w'}(b'))$ are such that $\Theta(JZ) = J'Z'$, then $\mathrm{Stab}_Z(J)$ and $\mathrm{Stab}_{Z'}(J')$ are conjugate.

Proof. We induct on $\ell(w)$. First assume w is minimal length in its σ-conjugacy class. Then by [18], $X_w(b) \neq \emptyset$ if and only if $\Psi(w) = b$, which holds if and only if $\Psi(w') = b'$, if and only if $X_{w'}(b') \neq \emptyset$. If this holds, then by [18] the group J acts transitively on $\Sigma^{\mathrm{top}}(X_w(b))$, and similarly the group J' acts transitively on $\Sigma^{\mathrm{top}}(X_{w'}(b'))$. Hence the two sets $J\backslash\Sigma^{\mathrm{top}}(X_w(b))$ and $J'\backslash\Sigma^{\mathrm{top}}(X_{w'}(b'))$ are both singletons. Let Θ be the unique map between them. The desired conjugacy of the stabilizers follows from the computation of $\mathrm{Stab}_Z(J)$ in Proposition 3.1.4.

Now assume w is not of minimal length in its σ-conjugacy class. Let $Z \in \Sigma^{\mathrm{top}}(X_w(b))$. Then as in the proof of Proposition 3.1.4, there exists $w_1 \tilde{\approx}_\sigma w$ and $s \in \mathbb{S}$ such that $sw_1\sigma(s) < w_1$. Then $X_w(b)$ is universally homeomorphic to $X_{w_1}(b)$. We fix such a universal homeomorphism and we obtain a corresponding element $Z_1 \in \Sigma^{\mathrm{top}}(X_{w_1}(b))$. By Proposition 3.1.3, there exists $U \in \Sigma^{\mathrm{top}}(X_{sw_1\sigma(s)}(b))$ or $U \in \Sigma^{\mathrm{top}}(X_{sw_1}(b))$ such that Z_1 is universally homeomorphic to a fiber bundle over U. We assume $U \in \Sigma^{\mathrm{top}}(X_{sw_1\sigma(s)}(b))$; the other case is similar. Then $\mathrm{Stab}_Z(J) = \mathrm{Stab}_U(J)$. Note that the choice of U depends on the choice of w_1 and a universal homeomorphism $X_w(b) \cong X_{w_1}$. However upon fixing these choices, the J-orbit of U is canonically associated to the J-orbit of Z.

By the induction hypothesis, we have a bijection

$$\Theta_1 : J\backslash\Sigma^{\mathrm{top}}(X_{sw_1\sigma(s)}(b)) \xrightarrow{\sim} J'\backslash\Sigma^{\mathrm{top}}(X_{s'w_1'\sigma'(s')}(b')),$$

where $s', w_1' \in W'$ correspond to s, w_1 respectively. Choose

$$U' \in \Sigma^{\mathrm{top}}(X_{s'w_1'\sigma'(s')}(b'))$$

such that $J'U' = \Theta_1(JU)$. By the induction hypothesis, $\mathrm{Stab}_U(J)$ is conjugate to $\mathrm{Stab}_{U'}(J')$. Reversing the above process we obtain $Z' \in \Sigma^{\mathrm{top}}(X_{w'}(b')))$ such that $\mathrm{Stab}_{U'}(J') = \mathrm{Stab}_{Z'}(J')$. Again the J'-orbit of Z' is canonically associated to U' upon fixing the universal homeomorphism $X_{w'(b')} \cong X_{w_1'}(b')$.

We define the map Θ to send JZ to $J'Z'$. Switching the roles of G and G', we obtain the inverse map of Θ, and so Θ is a bijection as desired. □

Proof of Theorem 3.2.1. For each $w \in W$, fix a bijection

$$\Theta : J\backslash\Sigma^{\mathrm{top}}(X_w(b)) \xrightarrow{\sim} J'\backslash\Sigma^{\mathrm{top}}(X_{w'}(b'))$$

as in Lemma 3.2.2. Let $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$. Then the pre-image $\pi^{-1}(Z)$ under the projection $\pi : X(\mu, b)^K \to X_\mu(b)$ is a top dimensional irreducible component of $X(\mu, b)^K$. Hence there exists a unique $w \in W$ such that $X_w(b) \cap \pi^{-1}(Z)$ is open dense in Z. Moreover we have $X_w(b) \cap \pi^{-1}(Z) \in \Sigma^{\mathrm{top}}(X_w(b))$. Write Y for $X_w(b) \cap \pi^{-1}(Z)$, and choose $Y' \in \Sigma^{\mathrm{top}}(X_{w'}(b'))$ such that $\Theta(JY) = J'Y'$. Then since $\dim X(\mu, b)^K = \dim X(\mu', b')^{K'}$, the closure of Y' in $X(\mu', b')^{K'}$ gives an element of $\Sigma^{\mathrm{top}}(X(\mu', b')^{K'})$, whose J'-orbit is independent of the choice of Y'. Taking the image of the last element under the projection $X(\mu', b')^{K'} \to X_{\mu'}(b')$ we obtain an element $Z' \in \Sigma^{\mathrm{top}}(X_{\mu'}(b'))$ by dimension reasons, and the orbit $J'Z'$ is independent of the choice of Y'. Moreover $\mathrm{Stab}_Z(J)$ is conjugate to $\mathrm{Stab}_{Z'}(J')$ since $\mathrm{Stab}_Y(J)$ is conjugate to $\mathrm{Stab}_{Y'}(J')$. The association $JZ \mapsto J'Z'$ gives a well-defined map

$$J\backslash\Sigma^{\mathrm{top}}(X_\mu(b)) \longrightarrow J'\backslash\Sigma^{\mathrm{top}}(X_{\mu'}(b'))$$

which satisfies the condition in the proposition. Switching the roles of G and G' we obtain the inverse map. □

For later applications we need some information on the sizes of the stabilizers appearing in Theorem 3.2.1. We now assume that b is basic, so that G and J_b are inner forms. Since b is basic we may choose a representative $\dot{\tau}$ for b where $\tau \in \Omega \subset W$. Using this one may identify the Iwahori–Weyl groups for J_b and G respecting the base alcoves. However the Frobenius action on W (or $\mathbb{S}$), defined by J_b, is given by $\tau\sigma$, where τ acts via left multiplication. See for example [20, §5] for more details. Since G and J_b are inner forms, the choice of a Haar measure on $G(F)$ determines a Haar measure on $J_b(F)$, and vice versa, see for example [27, §1].

Definition 3.2.3. We fix the Haar measure on $J_b(F)$ such that the volume of $G(\mathcal{O}_F)$ is 1. For each $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, we denote by $\mathrm{vol}(Z)$ the volume of the compact open subgroup $\mathrm{Stab}_Z(J_b(F))$ of $J_b(F)$ (see Theorem 3.1.1) under this Haar measure.

Corollary 3.2.4. *For each $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, there exists a rational function $R(t) \in \mathbb{Q}(t)$ such that*

$$\mathrm{vol}(Z) = R(q).$$

Moreover this rational function satisfies $R(0) = e(J_b)$ *and is independent of the local field* F*. Here* $e(J_b)$ *is the Kottwitz sign* $(-1)^{\mathrm{rk}_F J_b - \mathrm{rk}_F G}$*. More precisely, in the notation of Theorem 3.2.1, if* $J'Z'$ *corresponds to* JZ*, then* $\mathrm{vol}(Z') = R(q')$*.*

Proof. Since J_b splits over an unramified extension, the volume of a standard parahoric of $J_b(F)$ corresponding to any $\tau\sigma$-stable subset $K_J \subset \mathbb{S}$ can be calculated in terms of the affine root system. More precisely, let $\mathcal{K}_J$ be the corresponding parahoric subgroup of $J_b(F)$ and $\mathcal{I}_J$ be the standard Iwahori subgroup of $J_b(F)$. Then we have

$$\begin{aligned}\mathrm{vol}(\mathcal{K}_J(\mathcal{O}_F)) &= \frac{\mathrm{vol}(\mathcal{K}_J(\mathcal{O}_F))}{\mathrm{vol}(\mathcal{I}_J(\mathcal{O}_F))}.\,\mathrm{vol}(\mathcal{I}(\mathcal{O}_F)).\frac{\mathrm{vol}(\mathcal{I}_J(\mathcal{O}_F))}{\mathrm{vol}(\mathcal{I}(\mathcal{O}_F))}\\ &= \frac{[\mathcal{K}_J(\mathcal{O}_F) : \mathcal{I}_J(\mathcal{O}_F)]}{[G(\mathcal{O}_F) : \mathcal{I}(\mathcal{O}_F)]}.\frac{\mathrm{vol}(\mathcal{I}_J(\mathcal{O}_F))}{\mathrm{vol}(\mathcal{I}(\mathcal{O}_F))},\end{aligned}$$

where $\mathcal{I}$ is the standard Iwahori subgroup of $G(F)$ (whereas previously we denoted by $\mathcal{I}$ the standard Iwahori subgroup of $G(L)$). The term $[\mathcal{K}_J(\mathcal{O}_F) : \mathcal{I}_J(\mathcal{O}_F)]$ (resp. $[G(\mathcal{O}_F) : \mathcal{I}(\mathcal{O}_F)]$) is just the number of $\mathbb{F}_q$-points in the finite type full flag variety associated to the reductive quotient of the special fiber of $\mathcal{K}_J$ (resp. G).

For any connected reductive group $\overline{H}$ over $\mathbb{F}_q$ and $\overline{B}$ a Borel subgroup, let $W_{\overline{H}}$ denote the absolute Weyl group. Then we have the Bruhat decomposition

$$\overline{H}/\overline{B}(\overline{\mathbb{F}}_q) = \bigsqcup_{w \in W_{\overline{H}}} S_w.$$

We have $S_w(\mathbb{F}_q) \neq \emptyset$ if and only if $\sigma(w) = w$, in which case S_w is an affine space of dimension $\ell(w)$ defined over $\mathbb{F}_q$. In particular

$$\overline{H}/\overline{B}(\mathbb{F}_q) = \sum_{w \in W_{\overline{H}}^{\sigma}} q^{\ell(w)}.$$

It follows that $[\mathcal{K}_J(\mathcal{O}_F) : \mathcal{I}_J(\mathcal{O}_F)]$ and $[G(\mathcal{O}_F) : \mathcal{I}(\mathcal{O}_F)]$ are both polynomials in q with coefficients in $\mathbb{Z}$ and constant coefficient 1, and the polynomials depend only on the root systems of the corresponding reductive quotients of the special fiber.

Similarly the ratio $\frac{\mathrm{vol}(\mathcal{I}_J(\mathcal{O}_F))}{\mathrm{vol}(\mathcal{I}(\mathcal{O}_F))}$ can be computed as the ratio

$$\frac{\det(1 - q^{-1}\varsigma_J \mid V)}{\det(1 - q^{-1}\varsigma \mid V)} = \frac{\det(q - \varsigma_J \mid V)}{\det(q - \varsigma \mid V)}$$

where ς denotes the linear action of the Frobenius on $V = X_*(T)_{\mathbb{R}}$, and similarly for ς_J, see [27, §1]. This is also a ratio of polynomials in q with coefficients in $\mathbb{Z}$, and the ratio at $q = 0$ is equal to

$$\det(\varsigma_J)/\det(\varsigma) = (-1)^{\mathrm{rk}_F J_b - \mathrm{rk}_F G} = e(J_b).$$

Moreover the polynomials depend only on the affine root system of G and the element b. The result follows. □

Finally we record the following immediate consequence of Theorem 3.2.1.

Corollary 3.2.5. *If Conjecture 2.6.7 is true for all p-adic fields F, then it is true for all local fields F.* □

From now on we will assume that F is a p-adic field.

4. Counting points

4.1. The decent case

For each $s \in \mathbb{N}$, let F_s be the degree s unramified extension of F in L. Let $\mathcal{O}_s$ be the valuation ring of F_s, and let k_s be residue field. The number $\mathscr{N}(\mu, b)$ depends on b only via its σ-conjugacy class $[b] \in B(G)$. Recall that given $b \in G(L)$, one can associate a slope cocharacter $\nu_b \in \mathrm{Hom}_L(\mathbb{D}, G)$, where $\mathbb{D}$ is the pro-torus with character group $\mathbb{Q}$.

Definition 4.1.1. Let $s \in \mathbb{N}$. We say that an element $b \in G(L)$ is *s-decent*, if $s\nu_b$ is an integral cocharacter $\mathbb{G}_m \to G$ (as opposed to a fractional cocharacter), and

$$b\sigma(b)\cdots\sigma^{s-1}(b) = (s\nu_b)(\pi_F). \tag{4.1.1}$$

Lemma 4.1.2. *Assume $b \in G(L)$ is s-decent. Then $s\nu_b$ is defined over F_s, and b belongs to $G(F_s)$.*

Proof. The proof is identical to the proof of [42, Corollary 1.9]. □

By [26, §4.3], any class in $B(G)$ contains an element which is s-decent for some $s \in \mathbb{N}$. In the following, we hence assume without loss of generality that b is s_0-decent, for some fixed $s_0 \in \mathbb{N}$. We may and shall also assume that s_0 is divisible enough so that T is split over F_{s_0}.

Definition 4.1.3. Let $s \in s_0\mathbb{N}$. Let $G_s := \mathrm{Res}_{F_s/F} G$, so that $b \in G_s(F)$. Let Θ be the F-automorphism of G_s corresponding to the Frobenius $\sigma \in \mathrm{Gal}(F_s/F)$. Let $G_{s,b\Theta}$ be the centralizer of $b\Theta$ in G_s, which is a subgroup of G_s defined over F. Define

$$G(F_s)_{b\sigma} := \left\{ g \in G(F_s) \mid g^{-1} b\sigma(g) = b \right\}.$$

Thus $G(F_s)_{b\sigma}$ is naturally identified with $G_{s,b\Theta}(F)$.

Lemma 4.1.4. *For $s \in s_0\mathbb{N}$, there is a natural isomorphism of F-groups $J_b \cong G_{s,b\Theta}$. Moreover, $J_b(F) = G(F_s)_{b\sigma}$ as subgroups of $G(L)$.*

Proof. Let R be an F-algebra. Recall from (2.5.1) that

$$J_b(R) = \left\{ g \in G(R \otimes_F L) \mid g^{-1} b\sigma(g) = b \right\}.$$

It suffices to prove that for any $g \in J_b(R)$ we have $g \in G(R \otimes_F F_{s_0})$. Now such a g commutes with $b \rtimes \sigma$, and so it commutes with $(b \rtimes \sigma)^{s_0}$. By (4.1.1), we have $(b \rtimes \sigma)^{s_0} = (s_0\nu_b)(\pi_f) \rtimes \sigma^{s_0}$. On the other hand, by the functoriality of the association $b \mapsto \nu_b$, we know that g commutes with ν_b. It follows that g commutes with σ^{s_0}, and so $g \in G(R \otimes_F F_{s_0})$ as desired. □

4.1.5. We keep assuming that F is p-adic. In §2.4, we discussed the geometric structure on $X_\mu(b)$, as a locally closed subscheme of the Witt vector Grassmannian over $k = \overline{k_F}$. In the current setting, $X_\mu(b)$ is naturally "defined over k_{s_0}". More precisely, we can work with the version of the Witt vector affine Grassmannian as an ind-scheme over k_{s_0} rather than over k, see [1, Corollary 9.6] and cf. [47, §1.4]. Then the affine Deligne–Lusztig variety can be defined as a locally closed k_{s_0}-subscheme of the Witt vector affine Grassmannian, as in [47, §3.1.1]. The key point here is that since T is split over F_{s_0}, all the Schubert cells in the Witt vector affine Grassmannian are already defined over k_{s_0}, see [47, §1.4.3]. We denote respectively by $\mathbb{G}\mathrm{r}_G$ and $\mathbb{X}_\mu(b)$ the Witt vector affine Grassmannian and the affine Deligne–Lusztig variety over k_{s_0}, and we continue to use Gr_G and $X_\mu(b)$ to denote the corresponding objects over k.

Let us recall the moduli interpretations of $\mathbb{G}\mathrm{r}_G$ and $\mathbb{X}_\mu(b)$. For any perfect k_{s_0}-algebra R, write $W_{s_0}(R)$ for $W(R) \otimes_{W(k_{s_0})} \mathcal{O}_{s_0}$. Then $\mathbb{G}\mathrm{r}_G(R)$ is the set of pairs $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a $G_{W_{s_0}(R)}$-torsor on $W_{s_0}(R)$, and β is a trivialization of $\mathcal{E}$ on $W_{s_0}(R)[1/p]$, (see [47, Lemma 1.3]). We also have (see [47, (3.1.2)])

$$\mathbb{X}_\mu(b)(R) = \left\{ (\mathcal{E}, \beta) \in \mathbb{G}\mathrm{r}_G(R) \mid \mathrm{Inv}_x(\beta^{-1} b\sigma(\beta)) = \mu,\ \forall x \in \mathrm{spec} R \right\}.$$

Lemma 4.1.6. *For any $s \in s_0\mathbb{N}$, we have*

$$\mathbb{X}_\mu(b)(k_s) = \left\{ g \in G(F_s)/G(\mathcal{O}_s) \mid g^{-1} b\sigma(g) \in G(\mathcal{O}_s)\mu(\pi_F)G(\mathcal{O}_s) \right\}.$$

Proof. We only need to show that $\mathbb{G}\mathrm{r}_G(k_s) = G(F_s)/G(\mathcal{O}_s)$. For this it suffices to show that any $G_{\mathcal{O}_s}$-torsor over $\mathcal{O}_s$ is trivial (cf. the proof of [47, Lemma 1.3]). By smoothness this reduces to the Lang–Steinberg theorem, namely that any G_{k_s}-torsor over the finite field k_s is trivial. □

Lemma 4.1.7. *The action of $J_b(F)$ on $X_\mu(b)$ descends to a natural action on $\mathbb{X}_\mu(b)$ via k_{s_0}-automorphisms.*

Proof. By Lemma 4.1.4, $J_b(F) = G(F_{s_0})_{b\sigma}$. The group $G(F_{s_0})_{b\sigma}$ naturally acts on $\mathbb{X}_\mu(b)(R)$ by acting on the trivializations β, for each perfect k_{s_0}-algebra R. □

Lemma 4.1.8. *Up to enlarging s_0, all the irreducible components of $X_\mu(b)$ are defined over k_{s_0}, i.e., they come from base change of irreducible components of $\mathbb{X}_\mu(b)$.*

Proof. This follows from Lemma 2.5.5 and Lemma 4.1.7. □

4.2. Twisted orbital integrals and point counting

We fix $s_0 \in \mathbb{N}$ to be divisible enough so as to satisfy all the conclusions in §4.1. In particular G is split over F_{s_0} and the conclusion of Lemma 4.1.8 holds. Let $s \in s_0\mathbb{N}$.

For any $\mathbb{C}$-valued function $f \in C_c^\infty(G(F_s))$, define the twisted orbital integral

$$TO_b(f) := \int_{G(F_s)_{b\sigma}\backslash G(F_s)} f(g^{-1} b\sigma(g)) dg, \tag{4.2.1}$$

where $G(F_s)_{b\sigma}$ is equipped with an arbitrary Haar measure, and $G(F_s)$ is equipped with the Haar measure giving volume 1 to $G(\mathcal{O}_s)$. The general convergence of $TO_b(f)$ follows from the result of Ranga Rao [38]. However, in our specific case the convergence could be proved more easily. In fact, the decency equation (4.1.1) implies that $b\Theta$ is a semi-simple element of $G_s \rtimes \langle \Theta \rangle$, from which it follows that the twisted orbit is closed in $G(F_s)$. The convergence of $TO_b(f)$ then follows from the closedness of the twisted orbit, cf. [5, p. 266].

Definition 4.2.1. Let $f_{\mu,s} \in C_c^\infty(G(F_s))$ be the characteristic function of $G(\mathcal{O}_s)\mu(\pi_F)G(\mathcal{O}_s)$.

In the following we study the relationship between $TO_b(f_{\mu,s})$ and point counting on $\mathbb{X}_\mu(b)$.

Lemma 4.2.2. *Each irreducible component Z of $X_\mu(b)$ is quasi-compact, and is isomorphic to the perfection of a quasi-projective variety over k. Moreover, Z has non-empty intersection only with finitely many other irreducible components of $X_\mu(b)$.*

Proof. Since $X_\mu(b)$ is a perfect scheme by Theorem 2.5.3, the generic point η of Z and its residue field $k(\eta)$ make sense. Moreover $k(\eta)$ is a perfect field containing k. Let $(\mathcal{E}, \beta) \in X_\mu(b)(k(\eta)) \subset \mathrm{Gr}_G(k(\eta))$ correspond to η, and define $\lambda := \mathrm{Inv}(\beta) \in X_*(T)^+$. Since $\{\eta\}$ is dense in Z, it follows from [47, Lemma 1.22] that Z is contained in $\mathrm{Gr}_{G,\leq\lambda}$, the Schubert variety inside Gr_G associated to λ. On the other hand, it follows from [47, §1.4.1, Lemma 1.22] and [1, Theorem 8.3] that $\mathrm{Gr}_{G,\leq\lambda}$ is the perfection of a projective variety over k. Since Z is closed in $X_\mu(b)$ and $X_\mu(b)$ is locally closed in Gr_G, we conclude that Z is locally closed in $\mathrm{Gr}_{G,\leq\lambda}$, and hence Z is quasi-compact and isomorphic to the perfection of a quasi-projective variety over k.

Since $X_\mu(b)$ is locally perfectly of finite type (Theorem 2.5.3), each point in $X_\mu(b)$ has an open neighborhood that intersects with only finitely many irreducible components of $X_\mu(b)$. Since Z is quasi-compact, it also intersects with only finitely many irreducible components of $X_\mu(b)$. □

For each $x \in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$, we pick a representative $\tilde{x} \in \mathbb{X}_\mu(b)(k_s)$ and consider the volume of $\mathrm{Stab}_{\tilde{x}} J_b(F)$, with respect to the chosen Haar measure on $J_b(F) = G(F_s)_{b\sigma}$ (cf. Lemma 4.1.4). This volume is independent of the choice of $\tilde{x}$, and we shall denote it by vol_x.

Lemma 4.2.3. *The set $J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$ is finite. For all $\tilde{x} \in \mathbb{X}_\mu(b)(k_s)$, the stabilizer $\mathrm{Stab}_{\tilde{x}} J_b(F)$ in $J_b(F)$ is a compact open subgroup of $J_b(F)$. We have*

$$TO_b(f_{\mu,s}) = \sum_{x \in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)} \mathrm{vol}_x^{-1}.$$

Proof. Let $C = \left\{g \in G(F_s)_{b\sigma}\backslash G(F_s) \mid g^{-1}b\sigma(g) \in G(\mathcal{O}_s)\mu(\pi_F)G(\mathcal{O}_s)\right\}$. By the discussion below (4.2.1), we know that C is a compact subset of $G(F_s)_{b\sigma}\backslash G(F_s)$, as C is homeomorphic to the intersection of the compact set $G(\mathcal{O}_s)\mu(\pi_F)G(\mathcal{O}_s)$ with the closed twisted orbit of $b\sigma$. The group $G(\mathcal{O}_s)$ acts on C by right multiplication, and all the orbits under this action are open. Since C is compact, the number of orbits is finite. On the other hand, by Lemma 4.1.6 and Lemma 4.1.7, these orbits are in one-to-one correspondence with $J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$. In particular $J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$ is finite.

Now for each $x \in J_b(F)\backslash \mathbb{X}_\mu(b)(k_s)$, we denote the corresponding $G(\mathcal{O}_s)$-orbit in C by C_x. Let $\tilde{x} \in \mathbb{X}_\mu(b)(k_s)$ be a representative for x, and fix $r \in G(F_s)$ lifting $\tilde{x}$ in the sense of Lemma 4.1.6. It follows from the definition of quotient measure that the volume of C_x is the inverse of the volume of the compact open subgroup $G(F_s)_{b\sigma} \cap rG(\mathcal{O}_s)r^{-1}$ of $G(F_s)_{b\sigma}$. Since $TO_b(f_{\mu,s})$ is nothing but the volume of C, we are left to check that $G(F_s)_{b\sigma} \cap rG(\mathcal{O}_s)r^{-1} = \mathrm{Stab}_{\tilde{x}} J_b(F)$. But this follows from Lemma 4.1.4. □

Proposition 4.2.4. *Let $d = \dim X_\mu(b)$. For $s \in \mathbb{N}$ divisible by s_0, we have*

$$TO_b(f_{\mu,s}) = \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}(Z)^{-1} \, |k_s|^d + o(|k_s|^d), \quad s \gg 0.$$

Here Z runs through a set of representatives of the $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$.

Proof. In view of Lemma 2.5.5, we let $\{Z_1, \cdots, Z_M\}$ be a set of representatives of the $J_b(F)$-orbits in $\Sigma(X_\mu(b))$. For each $1 \leq i \leq M$, we write $\mathbb{J}_i$ for $\mathrm{Stab}_{Z_i}(J_b(F))$. For each $y \in \mathbb{X}_\mu(b)(k_s)$, we write $\mathbb{J}_y$ for $\mathrm{Stab}_y(J_b(F))$.

For each $1 \leq i \leq M$, we set

$$U_i := Z_i - \bigcup_{1 \leq j < i, \gamma \in J_b(F)} \gamma Z_j - \bigcup_{\gamma \in J_b(F), \gamma Z_i \neq Z_i} \gamma Z_i.$$

By Lemma 4.2.2, U_i is open dense in Z_i. By Lemma 4.1.8, we know that U_i, V_i are the base change of locally closed k_{s_0}-subschemes $\mathbb{Z}_i, \mathbb{U}_i$ of $\mathbb{X}_\mu(b)$ respectively, where $\mathbb{Z}_i, \mathbb{U}_i$ are perfections of quasi-projective varieties over k_{s_0}. Moreover, $\mathbb{Z}_i$ and $\mathbb{U}_i$ are irreducible.

We denote the natural maps

$$\coprod_{1 \leq i \leq M} \mathbb{Z}_i(k_s) \longrightarrow J_b(F)\backslash \mathbb{X}_\mu(b)(k_s)$$

and

$$\coprod_{1 \leq i \leq M} \mathbb{U}_i(k_s) \longrightarrow J_b(F)\backslash \mathbb{X}_\mu(b)(k_s)$$

by Π and π, respectively. Here we take the disjoint union of the $\mathbb{Z}_i(k_s)$ for $1 \leq i \leq M$ even though they may have non-trivial intersections in $\mathbb{X}_\mu(b)(k_s)$. Then Π is a surjection (between finite sets).

Fix an element $y \in \mathbb{Z}_i(k_s)$, and let $x = \Pi(y)$. We have

$$\begin{aligned} \mathrm{vol}_x \cdot \left|\Pi^{-1}(x)\right| &\geq \mathrm{vol}_x \cdot |\mathbb{J}_i y| = \mathrm{vol}(\mathbb{J}_y) \cdot |\mathbb{J}_i y| \\ &\geq \mathrm{vol}(\mathrm{Stab}_y(\mathbb{J}_i)) \cdot |\mathbb{J}_i y| = \mathrm{vol}(\mathbb{J}_i), \end{aligned} \tag{4.2.2}$$

where the last equality follows from the orbit-stabilizer relation.

Now fix $y \in \mathbb{U}_i(k_s)$ and let $x = \pi(y)$. We observe that $\pi^{-1}(x) = \mathbb{J}_i y$, and that $\mathbb{J}_y = \mathrm{Stab}_y(\mathbb{J}_i)$. We then have

$$\mathrm{vol}_x \cdot \left|\pi^{-1}(x)\right| = \mathrm{vol}(\mathbb{J}_y) \cdot |\mathbb{J}_i y| = \mathrm{vol}(\mathrm{Stab}_y(\mathbb{J}_i)) \cdot |\mathbb{J}_i y| = \mathrm{vol}(\mathbb{J}_i). \tag{4.2.3}$$

We now apply (4.2.2) and (4.2.3) to estimate $TO_b(f_{\mu,s})$. We have

$$\begin{aligned} TO_b(f_{\mu,s}) &= \sum_{x \in J_b(F)\backslash \mathbb{X}_\mu(b)(k_s)} \mathrm{vol}_x^{-1} && \text{(by Lemma 4.2.3)} \\ &= \sum_{i=1}^{M} \sum_{y \in \mathbb{Z}_i(k_s)} \mathrm{vol}_{\Pi(y)}^{-1} \cdot \left|\Pi^{-1}(\Pi(y))\right|^{-1} && \text{(by the surjectivity of } \Pi) \\ &\leq \sum_{i=1}^{M} \mathrm{vol}(\mathbb{J}_i)^{-1} \left|\mathbb{Z}_i(k_s)\right| && \text{(by (4.2.2)).} \end{aligned} \tag{4.2.4}$$

Similarly, we have

$$\begin{aligned} TO_b(f_{\mu,s}) &= \sum_{x \in J_b(F)\backslash \mathbb{X}_\mu(b)(k_s)} \mathrm{vol}_x^{-1} \\ &\geq \sum_{x \in \text{ image of } \pi} \mathrm{vol}_x^{-1} \\ &= \sum_{i=1}^{M} \sum_{y \in \mathbb{U}_i(k_s)} \mathrm{vol}_{\pi(y)}^{-1} \cdot \left|\pi^{-1}(\pi(y))\right|^{-1} \\ &= \sum_{i=1}^{M} \mathrm{vol}(\mathbb{J}_i)^{-1} \left|\mathbb{U}_i(k_s)\right| && \text{(by (4.2.3)).} \end{aligned} \tag{4.2.5}$$

Now let $\mathcal{U}_i$ be a quasi-projective variety whose perfection is $\mathbb{U}_i$. Then $\mathcal{U}_i$ is irreducible, and $\mathcal{U}_i(k_s) = \mathbb{U}_i(k_s)$. By the Lang–Weil bound (see [32]) applied to $\mathcal{U}_i$, we know that

$$\left|\mathbb{U}_i(k_s)\right| = |k_s|^{\dim Z_i} + o(|k_s|^{\dim Z_i}), \quad s \gg 0. \tag{4.2.6}$$

Similarly we have

$$|\mathbb{Z}_i(k_s)| = |k_s|^{\dim Z_i} + o(|k_s|^{\dim Z_i}), \quad s \gg 0. \tag{4.2.7}$$

The proposition follows from (4.2.4) (4.2.5) (4.2.6) (4.2.7). □

4.3. Applying the Base Change Fundamental Lemma in the basic case

Recall that we assumed that $[b] \in B(G, \mu)$ and b is s_0-decent. We now assume in addition that b is basic.

For $s \in \mathbb{N}$, recall from [25, §5] that the *s-th norm map* is a map

$$\begin{aligned}\mathfrak{N}_s : \{&\sigma\text{-conjugacy classes in } G(F_s)\} \\ &\longrightarrow \{\text{stable conjugacy classes in } G(F)\}.\end{aligned}$$

By [25, Proposition 5.7], two σ-conjugacy classes in $G(F_s)$ are in the same fiber of $\mathfrak{N}_s$ precisely when they are *stably σ-conjugate*, a notion that is defined in [25, §5].

Lemma 4.3.1. *Let $s \in s_0\mathbb{N}$. Then $\mathfrak{N}_s(b)$, as a stable conjugacy class in $G(F)$, consists of the single element $(s\nu_b)(\pi_F)$. Moreover, the cocharacter $s\nu_b : \mathbb{G}_m \to G$ is defined over F.*

Proof. By [25, Corollary 5.3], any element in $\mathfrak{N}_s(b)$ is $G(\overline{F})$-conjugate to

$$b\sigma(b)\cdots\sigma^{s-1}(b) \in G(F_s),$$

which is equal to $(s\nu_b)(\pi_F)$ since b is s-decent. Now since $(s\nu_b)(\pi_F)$ is central, we know that $\mathfrak{N}_s(b) = \{(s\nu_b)(\pi_F)\}$ and that $(s\nu_b)(\pi_F) \in G(F)$. It follows from the last statement that $s\nu_b$ is defined over F. □

Lemma 4.3.2. *Let $s \in s_0\mathbb{N}$. Let $b' \in G(F_s)$ be an element in the stable σ-conjugacy class of b. Then $\nu_{b'} = \nu_b$, and b' is s-decent. In particular b' is basic. Moreover, if $[b'] \in B(G, \mu)$, then b' is σ-conjugate to b in $G(F_s)$.*

Proof. By hypothesis we have $\mathfrak{N}_s(b) = \mathfrak{N}_s(b')$. By Lemma 4.3.1 applied to b, we know that the $\mathfrak{N}_s(b)$ consists of the single central element $(s\nu_b)(\pi_F) \in G(F)$. On the other hand any element of $\mathfrak{N}_s(b')$ should be $G(\overline{F})$-conjugate to $b'\sigma(b')\cdots\ \sigma^{s-1}(b')$ (by [25, Corollary 5.3]). Therefore

$$b'\sigma(b')\cdots\ \sigma^{s-1}(b') = (s\nu_b)(\pi_F).$$

By the characterization of $\nu_{b'}$ (see [26, §4.3]), the above equality implies that $\nu_{b'} = \nu_b$ and that b' is s-decent. The first part of the lemma is proved.

Now we assume $[b'] \in B(G, \mu)$. Since $B(G, \mu)$ contains a unique basic class, we have $[b'] = [b]$. Finally, by [42, Corollary 1.10], we know that b and b' must be σ-conjugate in $G(F_s)$, since they are both s-decent and represent the same class in $B(G)$. □

Let $s \in s_0\mathbb{N}$. We now consider stable twisted orbital integrals along b. By our assumption that b is s-decent and basic, we know that

$$b\sigma(b) \cdots \sigma^{s-1}(b) = (s\nu_b)(\pi_F)$$

is a central element of $G(F_s)$, and is in fact an element of $G(F)$ by Lemma 4.3.1. In particular, this element is semi-simple, and the centralizer of this element (namely G) is connected. Therefore, with the terminology of [25], an element $b' \in G(F_s)$ is stably σ-conjugate to b if and only if it is $\overline{F}$-σ-conjugate to b. This observation justifies our definition of the stable twisted orbital integral in the following, cf. [13, §5.1].

For any $\mathbb{C}$-valued function $f \in C_c^\infty(G(F_s))$, we let $STO_b(f)$ be the stable twisted orbital integral

$$STO_b(f) := \sum_{b'} e(G_{s,b'\Theta}) TO_{b'}(f),$$

where the summation is over the set of σ-conjugacy classes b' in $G(F_s)$ that are stably σ-conjugate to b, and $e(\cdot)$ denotes the Kottwitz sign. Here each $TO_{b'}$ is defined using the Haar measure on $G(F_s)$ giving volume 1 to $G(\mathcal{O}_s)$, and the Haar measure on $G(F_s)_{b'\sigma} = G_{s,b'\Theta}(F)$ that is transferred from the fixed Haar measure on $G(F_s)_{b\sigma} = G_{s,b\Theta}(F)$.

Definition 4.3.3. We denote by $\mathcal{H}_s$ the unramified Hecke algebra consisting of $G(\mathcal{O}_s)$-bi-invariant functions in $C_c^\infty(G(F_s)$, and denote by BC_s the base change map $\mathcal{H}_s \to \mathcal{H}_1$.

Definition 4.3.4. For $s \in s_0\mathbb{N}$, we write γ_s for $(s\nu_b)(\pi_F)$, and write γ_0 for γ_{s_0}. Thus γ_0 belongs to $G(F)$ (see Lemma 4.3.1) and $\gamma_s = \gamma_0^{s/s_0}$.

Proposition 4.3.5. *Assume $s \in s_0\mathbb{N}$. For any $f \in \mathcal{H}_s$, we have*

$$STO_b(f) = \mathrm{vol}(G(\mathcal{O}_F))^{-1}(\mathrm{BC}_s f)(\gamma_s),$$

where $\mathrm{vol}(G(\mathcal{O}_F))$ is defined in terms of the Haar measure on $G(F)$ transferred from the fixed Haar measure on $G_{s,b\Theta}(F)$, for the inner form $G_{s,b\Theta}$ of G.

Proof. By Lemma 4.3.1, $\mathfrak{N}_s(b)$ consists of the single central semi-simple element $\gamma_s \in G(F)$. By the Base Change Fundamental Lemma proved by Clozel [5, Theorem 7.1] and Labesse [31], we know that $STO_b(f)$ is equal to the stable orbital integral of $\mathrm{BC}_s f$ at $\mathfrak{N}_s(b)$. The latter degenerates to

$$e(G_{\gamma_s}) \cdot \frac{\mu_1}{\mu_2} \cdot (\mathrm{BC}_s f)(\gamma_s g)$$

since γ_s is central. Here μ_1 denotes the Haar measure on $G(F)$ giving volume 1 to $G(\mathcal{O}_F)$, and μ_2 denotes the Haar measure on $G_{\gamma_s}(F) = G(F)$ transferred from $G_{s,b\Theta}(F)$. The notation $\frac{\mu_1}{\mu_2}$ denotes the ratio between these two Haar measures on the same group $G(F)$. Obviously this ratio is equal to $\mathrm{vol}(G(\mathcal{O}_F))^{-1}$ as in the proposition. Finally, since $G_{\gamma_s} = G$ is quasi-split, we have $e(G_{\gamma_s}) = 1$. □

Lemma 4.3.6. *For s divisible by s_0, we have*

$$STO_b(f_{\mu,s}) = e(J_b) TO_b(f_{\mu,s}).$$

Proof. Firstly, by Lemma 4.1.4 we have $G_{s,b\Theta} = J_b$. We need to check that $TO_{b'}(f_{\mu,s}) = 0$, for any $b' \in G(F_s)$ that is stably σ-conjugate to b but not σ-conjugate to b in $G(F_s)$. Assume the contrary. Then there exists $g \in G(F_s)$ such that $f_{\mu,s}(g^{-1}b'\sigma(g)) \neq 0$, from which $g^{-1}b'\sigma(g) \in G(\mathcal{O}_s)\mu(\pi_F)G(\mathcal{O}_s)$. Hence $\kappa([b']) = \mu^\natural$ by Theorem 2.5.1. But this contradicts Lemma 4.3.2. □

Corollary 4.3.7. *Keep the notation in Proposition 4.2.4. For $s \gg 0$, we have*

$$\begin{aligned} &e(J_b)\,\mathrm{vol}(G(\mathcal{O}_F))^{-1}(\mathrm{BC}_s f_{\mu,s})(\gamma_s) \\ &\quad = \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}(Z)^{-1}\,|k_s|^d + o(|k_s|^d). \end{aligned}$$

Proof. This follows from Proposition 4.2.4, Proposition 4.3.5, and Lemma 4.3.6. □

5. Matrix coefficients for the Satake transform

5.1. General definitions and facts

In this subsection we expose general facts concerning the Satake isomorphism, for unramified reductive groups over F. The aim is to give an interpretation of the coefficients for the matrix of the inverse Satake isomorphism

in terms of a **q**-analogue of Kostant's partition function. This is well known by the work of Kato [22] in the case when G is split; we will need the case of non-split G. Our main reference is [4, §1].

Let G be an unramified reductive group over F. At this moment it is not necessary to fix a reductive model over $\mathcal{O}_F$ of G. Inside G we fix a Borel pair, namely a Borel subgroup B and a maximal torus $T \subset B$, both defined over F. In particular, T is a minimal Levi, and is split over F^{un}.

We denote by

$$\mathrm{BRD}(B,T) = (X^*(T), \Phi \supset \Delta, X_*(T), \Phi^\vee \supset \Delta^\vee)$$

the based root datum associated to (B,T). This based root datum has an automorphism θ induced by the Frobenius $\sigma \in \mathrm{Gal}(F^{\mathrm{un}}/F)$. Let $d = d_\theta < \infty$ be the order of θ.

Fix an F-pinning $(B,T,\mathbb{X}_+)$ of G. Since the Galois action on $\mathrm{BRD}(B,T)$ factors through the cyclic group generated by θ, we know that θ is a Galois-equivariant automorphism of $\mathrm{BRD}(B,T)$, and so it lifts uniquely to an F-automorphism of G preserving $(B,T,\mathbb{X}_+)$. We denote this F-automorphism of G still by θ.

Let A be the maximal split sub-torus of T. We have[1]

$$X_*(A) = X_*(T)^\theta, \quad X^*(A) = [X^*(T)/(1-\theta)X^*(T)]_{\mathrm{free}}.$$

Let ${}_F\Phi \subset X^*(A)$ be the image of $\Phi \subset X^*(T)$. The triple

$$(X^*(A), {}_F\Phi, X_*(A))$$

naturally extends to a (possibly non-reduced) root datum

$$(X^*(A), {}_F\Phi, X_*(A), {}_F\Phi^\vee),$$

see for instance [43, Theorem 15.3.8]. Elements of ${}_F\Phi$ are by definition θ-orbits in Φ. For $\alpha \in \Phi$, we write $[\alpha]$ for its θ-orbit. The θ-orbits in Δ give rise to a set of simple roots in ${}_F\Phi$, which we denote by ${}_F\Delta$. As usual, we denote the structural bijection ${}_F\Phi \xrightarrow{\sim} {}_F\Phi^\vee$ by $[\alpha] \mapsto [\alpha]^\vee$.

We let $\Phi^1 \subset {}_F\Phi$ be the subset of indivisible elements, namely, those $[\alpha] \in {}_F\Phi$ such that $\frac{1}{2}[\alpha] \notin {}_F\Phi$. The image of Φ^1 under the bijection ${}_F\Phi \xrightarrow{\sim} {}_F\Phi^\vee$ is denoted by $\Phi^{1,\vee}$. The tuple $(X^*(A), \Phi^1, X_*(A), \Phi^{1,\vee})$ has the structure of a

[1] In [4, §1.1], it is stated that $X^*(A) = X^*(T)/(1-\theta)X^*(T)$, which is not true in general.

reduced root datum. We note that ${}_F\Delta$ is also a set of simple roots in Φ^1. We henceforth also write Δ^1 for ${}_F\Delta$. For the sets $\Phi, {}_F\Phi, \Phi^1, \Phi^\vee, {}_F\Phi^\vee, \Phi^{1,\vee}$, we put a superscript $+$ to denote their respective subsets of positive elements.

As before we denote by W_0 the absolute Weyl group of G. We let $W^1 \subset W_0$ be the subgroup of elements that commute with θ. Then W^1 is a Coxeter group (see [4, §1.1]), and we denote by ℓ_1 the length function on W^1.

The complex dual group $\widehat{G}$ of G is a connected reductive group over $\mathbb{C}$, equipped with a Borel pair $(\widehat{B}, \widehat{T})$ and an isomorphism $\mathrm{BRD}(\widehat{B}, \widehat{T}) \xrightarrow{\sim} \mathrm{BRD}(B,T)^\vee$. In particular, we have canonical identifications $X^*(\widehat{T}) \cong X_*(T)$, $X_*(\widehat{T}) \cong X^*(T)$, which we think of as equalities. We fix a pinning $(\widehat{B}, \widehat{T}, \widehat{\mathbb{X}}_+)$. The action of θ on $\mathrm{BRD}(B,T)$ translates to an action on $\mathrm{BRD}(\widehat{B}, \widehat{T})$, and the latter lifts to a unique automorphism $\hat{\theta}$ of $\widehat{G}$ that preserves $(\widehat{B}, \widehat{T}, \widehat{\mathbb{X}}_+)$. The L-group ${}^LG$ is defined as the semi-direct product $\widehat{G} \rtimes \langle \hat{\theta} \rangle$, where $\langle \hat{\theta} \rangle$ denotes the cyclic group of order d generated by $\hat{\theta}$.

We denote the group $X^*(\widehat{T})^{\hat{\theta}} = X_*(T)^\theta$ by Y^*. Let $\widehat{A}$ be the quotient torus of $\widehat{T}$ corresponding to Y^*. Then $\widehat{A}$ is also identified with the dual torus of A. Define

$$P^+ := \{\lambda \in Y^* \mid \langle \lambda, \alpha \rangle \geq 0,\ \forall \alpha \in \Delta\} = \left\{\lambda \in Y^* \mid \langle \lambda, [\alpha] \rangle \geq 0,\ \forall [\alpha] \in \Delta^1\right\},$$
$$R^+ := \text{ the } \mathbb{Z}_{\geq 0}\text{-span of } {}_F\Phi^{\vee,+} \subset Y^*.$$

The $\mathbb{C}$-vector space $\mathbb{C}[Y^*]^{W^1}$ has a basis $\{m_\mu\}_{\mu \in P^+}$, where

$$m_\mu := \sum_{\lambda \in W^1 \mu} e^\lambda. \tag{5.1.1}$$

Here $W^1\mu$ denotes the orbit of μ under W^1.

Definition 5.1.1. Let $\widehat{\mathfrak{n}}$ be the Lie algebra of the unipotent radical of $\widehat{B}$, equipped with the adjoint action by ${}^LG$ (see [4, §1.3.2]). For each $\mu \in X^*(\widehat{T})$, let $\widehat{\mathfrak{n}}(\mu)$ denote the μ-weight space in $\widehat{\mathfrak{n}}$ for the action of $\widehat{T}$. Let $w \in W^1$ and $\epsilon \in \{\pm 1\}$. We define a $\mathbb{C}[X^*(\widehat{T})]$-linear operator $E^{\epsilon w}$ on $\widehat{\mathfrak{n}} \otimes_{\mathbb{C}} \mathbb{C}[X^*(\widehat{T})]$ by letting $E^{\epsilon w}$ act on $\widehat{\mathfrak{n}}(\mu)$ via the scalar $e^{\epsilon w \mu} \in \mathbb{C}[X^*(\widehat{T})]$, for each $\mu \in X^*(\widehat{T})$. We define

$$D(E^{\epsilon w}, \mathbf{q}) := \det(1 - \mathbf{q} \cdot \hat{\theta} \cdot E^{\epsilon w},\ \widehat{\mathfrak{n}}) \in \mathbb{C}[X^*(\widehat{T})][\mathbf{q}],$$
$$P(E^{\epsilon w}, \mathbf{q}) := D(E^{\epsilon w}, \mathbf{q})^{-1} \in \mathrm{Frac}\Big(\mathbb{C}[X^*(\widehat{T})][\mathbf{q}]\Big).$$

Definition 5.1.2. Let $[\alpha] \in \Phi^1 \subset {}_F\Phi$. We say that $[\alpha]$ is of type I, if $2[\alpha] \notin {}_F\Phi$. Otherwise we say that $[\alpha]$ is of type II. For $[\alpha] \in \Phi^1$, we define

$$\mathfrak{b}([\alpha]) := \begin{cases} \#[\alpha], & \text{if } [\alpha] \text{ is of type I,} \\ \frac{1}{2}\#[\alpha] & \text{if } [\alpha] \text{ is of type II,} \end{cases}$$

where $\#[\alpha]$ denotes the size of $[\alpha]$ viewed as a θ-orbit in Φ. Then $\mathfrak{b}([\alpha]) \in \mathbb{Z}_{\geq 1}$, see [4, §1.1].

Definition 5.1.3. For any element $\beta = [\alpha]^\vee \in \Phi^{1,\vee}$ (with $[\alpha] \in \Phi^1$), we define $\mathfrak{b}(\beta)$ to be $\mathfrak{b}([\alpha])$, and we say that β is of type I or II if $[\alpha]$ is of type I or II. For any $\beta' \in {}_F\Phi^\vee$, we define $\mathfrak{b}(\beta')$ to be $\mathfrak{b}(\beta)$, where β is the element in $\Phi^{1,\vee}$ that is homothetic to β'.

Definition 5.1.4. For $\beta \in \Phi^{1,\vee}$, we define $d_\beta(\mathbf{q}) \in \mathbb{C}[Y^*][\mathbf{q}]$ as follows:

$$d_\beta(\mathbf{q}) := \begin{cases} 1 - \mathbf{q}^{\mathfrak{b}(\beta)} e^\beta, & \text{if } \beta \text{ is of type I,} \\ (1 - \mathbf{q}^{2\mathfrak{b}(\beta)} e^{\beta/2})(1 + \mathbf{q}^{\mathfrak{b}(\beta)} e^{\beta/2}), & \text{if } \beta \text{ is of type II.} \end{cases}$$

Here, when β is of type II, $\beta/2$ is always an element of ${}_F\Phi^\vee$ and in particular an element of Y^*, see [4, §1.1] or [30, §1.3].

Lemma 5.1.5. *For $\epsilon \in \{\pm 1\}$, we have*

$$D(E^\epsilon, \mathbf{q}) = \prod_{\epsilon\beta \in \Phi^{1,\vee,+}} d_\beta(\mathbf{q}), \qquad P(E^\epsilon, \mathbf{q}) = \prod_{\epsilon\beta \in \Phi^{1,\vee,+}} d_\beta(\mathbf{q})^{-1}.$$

Proof. The case $\epsilon = 1$ is [4, Lemma 1.3.7]. The case $\epsilon = -1$ is proved in the same way, by switching the roles of positive elements and negative elements in $\Phi^{1,\vee}$. □

Definition 5.1.6. For each $\lambda \in Y^*$, we define $\mathcal{P}(\lambda, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$ as follows. In view of Definition 5.1.4 and Lemma 5.1.5, we have an expansion

$$P(E^{-1}, \mathbf{q}) = \sum_{\lambda \in R^+} \mathcal{P}(\lambda, \mathbf{q}) e^{-\lambda}, \tag{5.1.2}$$

with each $\mathcal{P}(\lambda, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$. We set $\mathcal{P}(\lambda, \mathbf{q}) := 0$ for all $\lambda \in Y^* - R^+$.

Corollary 5.1.7. *For $\lambda \in R^+ - \{0\}$, the constant term $\mathcal{P}(\lambda, 0)$ of $\mathcal{P}(\lambda, \mathbf{q})$ is 0.*

Proof. This immediately follows from Lemma 5.1.5 and Definition 5.1.6. □

Definition 5.1.8. Let $\rho^\vee \in X_*(T) \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$ be the half sum of elements in $\Phi^{\vee,+}$, and let $\rho \in X^*(T) \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$ be the half sum of elements in Φ^+. Then $\rho^\vee$ in fact lies in $Y^* \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$, and is equal to the half sum of elements in $\Phi^{1,\vee,+}$, see [4, §1.2]. For $w \in W^1$ and $\mu \in Y^*$, we let

$$w \bullet \mu := w(\mu + \rho^\vee) - \rho^\vee \in Y^*.$$

We also write $w\bullet(\cdot)$ for the induced action of w on $\mathbb{C}[Y^*]$. Define the operator

$$J : \mathbb{C}[Y^*] \longrightarrow \mathbb{C}[Y^*], \quad f \longmapsto \sum_{w \in W^1} (-1)^{\ell_1(w)} w \bullet f.$$

Definition 5.1.9. For $\lambda \in P^+$ and for a formal variable $\mathbf{q}$, we define

$$\tau_\lambda(\mathbf{q}) := J(e^\lambda) P(E^{-1}, \mathbf{q}) = \sum_{\mu \in R^+} J(e^\lambda)\, \mathcal{P}(\mu, \mathbf{q}) e^{-\mu} \in \mathbb{C}[Y^*][[\mathbf{q}]]. \tag{5.1.3}$$

Definition 5.1.10. For any $\lambda \in P^+ \subset Y^* = X^*(\widehat{T})^{\hat{\theta}}$, we define V_λ to be the irreducible representation of $\widehat{G}$ of highest weight λ.

Theorem 5.1.11 (Weyl character formula, [4, Theorem 1.4.1])**.** *Let $\lambda \in P^+$. Then $\tau_\lambda(1) \in \mathbb{C}[Y^*]^{W^1}$. The character of V_λ, as a function on $\widehat{T}$, descends to the function on $\widehat{A}$ given by $\tau_\lambda(1)$.* □

Definition 5.1.12. For any $\lambda \in P^+$, we simply write τ_λ for the element $\tau_\lambda(1) \in \mathbb{C}[Y^*]^{W^1}$.

5.2. Matrix coefficients

We now fix a reductive model of G over $\mathcal{O}_F$ as in §2.1. As before we denote by $\mathcal{H}_1$ the spherical Hecke algebra $\mathcal{H}(G(F)//G(\mathcal{O}_F))$. For each $\mu \in X_*(A)$, we let $f_\mu \in \mathcal{H}_1$ be the characteristic function of $G(\mathcal{O}_F)\mu(\pi_F)G(\mathcal{O}_F)$. Then the $\mathbb{C}$-vector space $\mathcal{H}_1$ has a basis given by f_μ, for $\mu \in P^+ \subset X_*(A)$.

Recall that the *Satake isomorphism* is a canonical $\mathbb{C}$-algebra isomorphism

$$\mathrm{Sat} : \mathcal{H}_1 \xrightarrow{\sim} \mathbb{C}[Y^*]^{W^1},$$

see for instance [4, §1.5]. In the following, we simply write f_μ for $\mathrm{Sat}(f_\mu)$, which shall cause no confusion. At this point we have introduced three bases of the $\mathbb{C}$-vector space $\mathbb{C}[Y^*]^{W^1}$, namely $\{m_\mu\}, \{\tau_\mu\}, \{f_\mu\}$, all indexed by

$\mu \in P^+$ (see (5.1.1) and Definition 5.1.12 for m_μ and τ_μ). We denote some of the transition matrices between these bases as follows:

$$m_\mu = \sum_\lambda n_\mu^\lambda \tau_\lambda, \qquad \tau_\mu = \sum_\lambda t_\mu^\lambda f_\lambda, \qquad m_\mu = \sum_\lambda \mathfrak{M}_\mu^\lambda f_\lambda.$$

In the following we deduce a formula for $\mathfrak{M}_\mu^\lambda$ from known formulas for n_μ^λ and t_μ^λ.

Definition 5.2.1. As in [4, §1.7], we have a partition of Y^* into the following subsets:

$$Y_0^* := \left\{\lambda \in Y^* \mid \exists w \in W^1, w \text{ is a reflection, } w \bullet \lambda = \lambda\right\},$$
$$Y_w^* := \left\{\lambda \in Y^* \mid w \bullet \lambda \in P^+\right\},\ w \in W^1.$$

For each $x \in W^1 \sqcup \{0\}$ we let $e_x : Y^* \to \{0, 1\}$ be the characteristic function of Y_x^*.

Theorem 5.2.2 (van Leeuwen's formula, [4, Lemma 1.7.4])**.** *For $\mu, \lambda \in P^+$, we have*

$$n_\mu^\lambda = \sum_{w' \in W^1/W_\mu^1} \sum_{w \in W^1} (-1)^{\ell_1(w)} e_w(w'\mu)\delta(w \bullet (w'\mu), \lambda).$$

Here $\delta(\cdot, \cdot)$ is the Kronecker delta, and W_μ^1 is the subgroup of W^1 generated by the reflections attached to those $[\alpha] \in \Delta^1$ such that $\langle \mu, [\alpha]\rangle = 0$. □

Definition 5.2.3. For $\lambda, \lambda' \in P^+$, we define

$$K_{\lambda',\lambda}(\mathbf{q}) := \sum_{w \in W^1} (-1)^{\ell_1(w)} \mathcal{P}(w \bullet \lambda' - \lambda, \mathbf{q}). \tag{5.2.1}$$

Remark 5.2.4. The notation $K_{\lambda',\lambda}$ in Definition 5.2.3 is compatible with [22] when G is split.

Theorem 5.2.5 (Kato–Lusztig formula, [4, Theorem 1.9.1])**.** *For $\mu, \lambda \in P^+$, we have*

$$t_\mu^\lambda = K_{\mu,\lambda}(|k_F|^{-1})\, |k_F|^{-\langle \lambda, \rho\rangle}.$$ □

Corollary 5.2.6. *Write q for $|k_F|$. For $\mu, \lambda \in P^+$, we have*

$$\mathfrak{M}_\mu^\lambda = q^{-\langle \lambda, \rho\rangle} \sum_{w' \in W^1/W_\mu^1} \sum_{w \in W^1} (-1)^{\ell_1(w)} \left(1 - e_0(w'\mu)\right) \mathcal{P}\left(w \bullet (w'\mu) - \lambda, q^{-1}\right).$$

Proof. We compute

$$\begin{aligned}\mathfrak{M}_\mu^\lambda &= \sum_{\lambda'\in P^+} n_\mu^{\lambda'} t_{\lambda'}^\lambda \\ &= \sum_{\substack{\lambda'\in P^+\\ w'\in W^1/W_\mu^1\\ w''\in W^1}} (-1)^{\ell_1(w'')} e_{w''}(w'\mu)\delta(w''\bullet(w'\mu),\lambda')K_{\lambda',\lambda}(q^{-1})q^{-\langle\lambda,\rho\rangle} \\ &= \sum_{\substack{w'\in W^1/W_\mu^1\\ w''\in W^1\\ w\in W^1}} (-1)^{\ell_1(w'')} e_{w''}(w'\mu)q^{-\langle\lambda,\rho\rangle}(-1)^{\ell_1(w)}\,\mathcal{P}\left((ww'')\bullet(w'\mu)-\lambda,q^{-1}\right).\end{aligned}$$

where the second equality is by Theorems 5.2.2, 5.2.5, and the third equality is by (5.2.1). Under the substitution $ww'' \mapsto w$, the above is equal to

$$q^{-\langle\lambda,\rho\rangle}\sum_{w'\in W^1/W_\mu^1}\sum_{w''\in W^1}\sum_{w\in W^1}(-1)^{\ell_1(w)}e_{w''}(w'\mu)\,\mathcal{P}\left(w\bullet(w'\mu)-\lambda,q^{-1}\right).$$

Since $\sum_{w''\in W^1} e_{w''}(\cdot) = 1 - e_0(\cdot)$, the proof is finished. □

Motivated from Corollary 5.2.6, we make the following definition.

Definition 5.2.7. For $\mu,\lambda\in P^+$ and a formal variable $\mathbf{q}^{-1/2}$, we define

$$\begin{aligned}\mathfrak{M}_\mu^\lambda(\mathbf{q}^{-1}) :=& \mathbf{q}^{-\langle\lambda,\rho\rangle}\sum_{w'\in W^1/W_\mu^1}\sum_{w\in W^1}(-1)^{\ell_1(w)}\left(1-e_0(w'\mu)\right)\cdot \\ &\cdot\mathcal{P}\left(w\bullet(w'\mu)-\lambda,\mathbf{q}^{-1}\right)\in\mathbb{C}[Y^*][\mathbf{q}^{-1/2}].\end{aligned}$$

As a special case, we define

$$\begin{aligned}\mathfrak{M}_\mu^0(\mathbf{q}^{-1}) :=& \sum_{w'\in W^1/W_\mu^1}\sum_{w\in W^1}(-1)^{\ell_1(w)}\left(1-e_0(w'\mu)\right)\cdot \\ &\cdot\mathcal{P}\left(w\bullet(w'\mu),\mathbf{q}^{-1}\right)\in\mathbb{C}[Y^*][\mathbf{q}^{-1}].\end{aligned}\tag{5.2.2}$$

Lemma 5.2.8. *Let $\mu,\lambda\in P^+$. Let $\nu\in X_*(A\cap Z_G)$. Then $\mathfrak{M}_\mu^\lambda(\mathbf{q}^{-1}) = \mathfrak{M}_{\mu-\nu}^{\lambda-\nu}(\mathbf{q}^{-1})$.*

Proof. In fact, we have $\langle\lambda,\rho\rangle = \langle\lambda-\nu,\rho\rangle$, $W_\mu^1 = W_{\mu-\nu}^1$, $e_0(w'\mu) = e_0(w'(\mu-\nu))$, and $w\bullet(w'\mu)-\lambda = w\bullet(w'(\mu-\nu))-(\lambda-\nu)$, for all $w'\in W^1/W_\mu^1$ and $w\in W^1$. □

5.3. Interpretation in terms of Kostant partitions

In certain cases the polynomial $\mathcal{P}(\lambda, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$ in Definition 5.1.6 has a concrete description as a $\mathbf{q}$-analogue of Kostant's partition function, which we now explain. Let $\mathbb{P}$ be the set of all functions ${}_F\Phi^{\vee,+} \to \mathbb{Z}_{\geq 0}$. We shall typically denote an element of $\mathbb{P}$ by $\underline{m}$, and denote its value at any $\beta \in {}_F\Phi^{\vee,+}$ by $m(\beta)$. For $\underline{m} \in \mathbb{P}$, we define

$$\Sigma(\underline{m}) := \sum_{\beta \in {}_F\Phi^{\vee,+}} m(\beta)\beta \in R^+ \subset Y^*,$$
$$|\underline{m}| := \sum_{\beta \in {}_F\Phi^{\vee,+}} m(\beta)\mathfrak{b}(\beta) \in \mathbb{Z}_{\geq 0}.$$

Here $\mathfrak{b}(\beta)$ is as in Definition 5.1.3.

For all $\lambda \in Y^*$, we define $\mathbb{P}(\lambda)$ to be the set of $\underline{m} \in \mathbb{P}$ such that $\Sigma(\underline{m}) = \lambda$. Thus $\mathbb{P}(\lambda)$ is empty unless $\lambda \in R^+$. Elements of $\mathbb{P}(\lambda)$ are called *Kostant partitions* of λ. For any $L \in \mathbb{Z}_{\geq 0}$, we define $\mathbb{P}(\lambda)_L$ to be the set of $\underline{m} \in \mathbb{P}(\lambda)$ such that $|\underline{m}| = L$. For $\lambda \in Y^*$, we define

$$\mathcal{P}_{\mathrm{Kos}}(\lambda, \mathbf{q}) := \sum_{\underline{m} \in \mathbb{P}(\lambda)} \mathbf{q}^{|\underline{m}|} \in \mathbb{C}[\mathbf{q}].$$

This is known in the literature as the $\mathbf{q}$-*analogue of Kostant's partition function*, at least when G is split.

Proposition 5.3.1. *The following statements hold.*

1. *Assume ${}_F\Phi = \Phi^1$. For all $\lambda \in Y^*$ we have $\mathcal{P}(\lambda, \mathbf{q}) = \mathcal{P}_{\mathrm{Kos}}(\lambda, \mathbf{q})$.*
2. *In general, to each $\underline{m} \in \mathbb{P}$, we can attach a polynomial $\mathcal{Q}(\underline{m}, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$, with the following properties:*
   (a) *For all $0 < x < 1$, we have $|\mathcal{Q}(\underline{m}, x)| \leq 1$.*
   (b) *For any $\lambda \in Y^*$ we have*

$$\mathcal{P}(\lambda, \mathbf{q}) = \sum_{\underline{m} \in \mathbb{P}(\lambda)} \mathcal{Q}(\underline{m}, \mathbf{q})\mathbf{q}^{|\underline{m}|}.$$

Proof. Part (1) immediately follows from Definitions 5.1.4, 5.1.6. For part (2), we note that if $\beta \in \Phi^{1,\vee}$ is of type II, then $\beta' := \beta/2$ is an element of

${}_F\Phi^\vee$, and we have

$$\begin{aligned} d_\beta(\mathbf{q})^{-1} &= \left[\sum_{i=0}^{\infty}(\mathbf{q}^{2\mathfrak{b}(\beta)}e^{\beta/2})^i\right]\left[\sum_{i=0}^{\infty}(-\mathbf{q}^{\mathfrak{b}(\beta)}e^{\beta/2})^i\right] \\ &= \left[\sum_{i=0}^{\infty}(\mathbf{q}^{2\mathfrak{b}(\beta')}e^{\beta'})^i\right]\left[\sum_{i=0}^{\infty}(-\mathbf{q}^{\mathfrak{b}(\beta')}e^{\beta'})^i\right] \\ &= \sum_{n=0}^{\infty}\mathcal{R}_{\beta,n}(\mathbf{q})(\mathbf{q}^{\mathfrak{b}(\beta')}e^{\beta'})^n, \end{aligned}$$

with

$$\mathcal{R}_{\beta,n}(\mathbf{q}) = \sum_{i=0}^{n}(-1)^{n-i}\mathbf{q}^{i\mathfrak{b}(\beta')} \in \mathbb{C}[\mathbf{q}], \quad \forall n \in \mathbb{Z}_{\geq 0}.$$

We observe that for all $0 < x < 1$ we have

$$|\mathcal{R}_{\beta,n}(x)| \leq 1. \tag{5.3.1}$$

Now for each $\beta' \in {}_F\Phi^\vee$ and each $n \geq 0$, define

$$\mathcal{Q}_{\beta',n}(\mathbf{q}) := \begin{cases} \mathcal{R}_{2\beta',n}(\mathbf{q}), & \text{if } 2\beta' \in \Phi^{1,\vee}, \\ 1, & \text{if } 2\beta' \notin \Phi^{1,\vee}. \end{cases}$$

We take

$$\mathcal{Q}(\underline{m},\mathbf{q}) := \prod_{\beta' \in {}_F\Phi^\vee}\mathcal{Q}_{\beta',m(\beta')}(\mathbf{q}).$$

Then condition (a) follows from the construction and the observation (5.3.1). Condition (b) follows from Lemma 5.1.5 and Definition 5.1.6. □

5.4. Computation with the base change

We keep the setting of §4.3 and §5.1. We assume that s_0 is divisible by the order d of θ, and consider $s \in s_0\mathbb{N}$.

The Satake isomorphism for $\mathcal{H}_s$ is

$$\mathrm{Sat} : \mathcal{H}_s \xrightarrow{\sim} \mathbb{C}[X^*(\widehat{T})]^{W_0}.$$

For each $\mu \in X^*(\widehat{T})^+$, let τ'_μ be the character of the highest weight representation V_μ of $\widehat{G}$ of highest weight μ. Then

$$\{\tau'_\mu\}_{\mu \in X^*(\widehat{T})^+}$$

is a basis of $\mathbb{C}[X^*(\widehat{T})]^{W_0}$. This basis is the absolute analogue of the basis $\{\tau_\mu\}_{\mu\in P^+}$ of $\mathbb{C}[Y^*]^{W^1}$ (i.e., they are the same if $\theta = 1$).

Recall from §2.6 and §5.1 that we have

$$Y^* = X_*(A) = X^*(\widehat{T})^{\hat{\theta}}, \quad X^*(\widehat{\mathcal{S}}) = X^*(\widehat{T})_{\hat{\theta},\mathrm{free}}.$$

By Lemma 1.6.1 (3), the composition

$$Y^* \otimes \mathbb{Q} \to X^*(\widehat{T}) \otimes \mathbb{Q} \to X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$$

is invertible. We denote its inverse map by $\lambda \mapsto \lambda^{(1)}$. For all $\lambda \in X^*(\widehat{\mathcal{S}})$, we define $\lambda^{(s)}$ to be $s\lambda^{(1)}$, which lies in Y^* since s is divisible by d. Thus we have a map

$$X^*(\widehat{\mathcal{S}}) \longrightarrow Y^*, \quad \lambda \longmapsto \lambda^{(s)}, \tag{5.4.1}$$

which is an isomorphism after $\otimes\mathbb{Q}$. In the case $\theta = 1$, this is none other than the multiplication-by-s map from Y^* to itself. In general, we denote by $X^*(\widehat{\mathcal{S}})^+ \subset X^*(\widehat{\mathcal{S}})$ the natural image of $X^*(\widehat{T})^+$. Then (5.4.1) maps $X^*(\widehat{\mathcal{S}})^+$ into $P^+ \subset Y^*$. Moreover, the action of W^1 on $X^*(\widehat{T})$ induces an action of W^1 on $X^*(\widehat{\mathcal{S}})$, and the map (5.4.1) is W^1-equivariant.

Proposition 5.4.1. *Under the Satake isomorphisms, the base change map* $\mathrm{BC}_s : \mathcal{H}_s \to \mathcal{H}_1$ *becomes*

$$\begin{aligned} &\mathrm{BC}_s : \mathbb{C}[X^*(\widehat{T})]^{W_0} \longrightarrow \mathbb{C}[Y^*]^{W^1} \\ &\forall \mu \in X^*(\widehat{T})^+, \ \tau'_\mu \longmapsto \sum_{\lambda\in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}} \cdot m_{\lambda^{(s)}}. \end{aligned}$$

Proof. To simplify notation we write X^* for $X^*(\widehat{T})$. To compute BC_s as a map $\mathbb{C}[X^*]^{W_0} \to \mathbb{C}[Y^*]^{W^1}$, it suffices to compose the map with the natural inclusion $\mathbb{C}[Y^*]^{W^1} \subset \mathbb{C}[X^*]$. For each $\mu \in X^{*,+}$, let

$$m'_\mu := \sum_{\lambda\in W_0(\mu)} e^\lambda.$$

Then $\{m'_\mu\}_{\mu\in X^{*,+}}$ is a basis of $\mathbb{C}[X^*]^{W_0}$. This basis is just the absolute analogue of the basis $\{m_\mu\}_{\mu\in P^+}$ of $\mathbb{C}[Y^*]^{W^1}$. It easily follows from definitions (see for example [2]) that BC_s as a map $\mathbb{C}[X^*]^{W_0} \to \mathbb{C}[X^*]$ sends each m'_μ

to

$$\sum_{\lambda \in W_0(\mu)} e^{\lambda + \hat{\theta}\lambda + \cdots + \hat{\theta}^{s-1}\lambda}.$$

It follows that for all $\mu \in X^{*,+}$, we have

$$\mathrm{BC}_s\, \tau'_\mu = \sum_{\lambda \in X^*} \dim V_\mu(\lambda) e^{\lambda + \hat{\theta}\lambda + \cdots + \hat{\theta}^{s-1}\lambda}. \tag{5.4.2}$$

Here the summation is over X^* and not over $X^{*,+}$. For each $\lambda \in X^*$, the element

$$\lambda + \hat{\theta}\lambda + \cdots + \hat{\theta}^{s-1}\lambda \in X^*$$

lies in $Y^* \subset X^*$, and its image under the natural map

$$Y^* = (X^*)^{\hat{\theta}} \longrightarrow X^*(\widehat{\mathcal{S}}) = (X^*)_{\hat{\theta},\mathrm{free}}$$

is equal to the image of $s\lambda \in X^*$ under the natural map $X^* \to X^*(\widehat{\mathcal{S}})$. In other words, we have

$$\lambda + \hat{\theta}\lambda + \cdots + \hat{\theta}^{s-1}\lambda = (\lambda|_{\widehat{\mathcal{S}}})^{(s)},$$

where $\lambda|_{\widehat{\mathcal{S}}} \in X^*(\widehat{\mathcal{S}})$ denotes the image of λ under $X^* \to X^*(\widehat{\mathcal{S}})$. Hence by (5.4.2) we have

$$\mathrm{BC}_s\, \tau'_\mu = \sum_{\lambda \in X^*} \dim V_\mu(\lambda) e^{(\lambda|_{\widehat{\mathcal{S}}})^{(s)}},$$

which is easily seen to be equal to

$$\sum_{\lambda \in X^*(\widehat{\mathcal{S}})} \dim V_\mu(\lambda)_{\mathrm{rel}}\; e^{\lambda^{(s)}} = \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\; m_{\lambda^{(s)}}. \qquad \square$$

Since b is basic and s_0-decent, and since s is divisible by s_0, by Lemma 4.3.1 the cocharacter $s\nu_b : \mathbb{G}_m \to G$ is a cocharacter of Z_G defined over F. In particular we may view $s\nu_b \in X_*(A) = Y^*$.

Corollary 5.4.2. *For $\mu \in X^*(\widehat{T})^+$, we have*

$$(\mathrm{BC}_s\, \tau'_\mu)(\gamma_s) = \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\; \mathfrak{M}^0_{\lambda^{(s)} - s\nu_b}(|k_F|^{-1}).$$

Proof. By Proposition 5.4.1 we have

$$(\mathrm{BC}_s\, \tau'_\mu)(\gamma_s) = \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ m_{\lambda^{(s)}}(\gamma_s).$$

Recall from Lemma 4.3.1 that $s\nu_b$ is a central cocharacter of G defined over F. By Corollary 5.2.6 and Definition 5.2.7, each $m_{\lambda^{(s)}}(\gamma_s)$ is equal to $\mathfrak{M}^{s\nu_b}_{\lambda^{(s)}}(|k_F|^{-1})$, which by Lemma 5.2.8 is equal to $\mathfrak{M}^0_{\lambda^{(s)}-s\nu_b}(|k_F|^{-1})$. □

5.5. Some inductive relations

We keep the setting and notation of §2.6 and §5.1. We assume in addition that G is adjoint, and that G is F-simple. To emphasize the group G we write $\mathfrak{M}^0_{\lambda,G}(\mathbf{q}^{-1})$ for the polynomial $\mathfrak{M}^0_\lambda(\mathbf{q}^{-1})$ in Definition 5.2.7. In the following we discuss how to reduce the understanding of these polynomials to the case where G is absolutely simple.

We write Dynk_G for the Dynkin diagram of (G, B, T). By our assumption that G is adjoint and F-simple, the action of $\langle\theta\rangle$ on Dynk_G is transitive on the connected components. Let d_0 be the number of connected components of Dynk_G. Fix one connected component Dynk^+_G of Dynk_G once and for all. The connected Dynkin diagram Dynk^+_G, together with the automorphism θ^{d_0}, determines an unramified, adjoint, absolutely simple group G' over F, equipped with an F-pinning $(B', T', \mathbb{X}'_+)$. We apply the constructions in §5.1 to G'. We shall add an apostrophe in the notation when we denote an object associated to G', e.g., $A', (Y^*)'$.

We have natural identifications

$$\begin{aligned}
&(X^*(A), {}_F\Phi, X_*(A), {}_F\Phi^\vee) \cong (X^*(A'), ({}_F\Phi)', X_*(A'), ({}_F\Phi^\vee)'),\\
&\Phi^1 \cong (\Phi^1)', \qquad \Phi^{1,\vee} \cong (\Phi^{1,\vee})',\\
&Y^* \cong (Y^*)', \qquad W^1 \cong (W^1)'.
\end{aligned}$$

To be more precise, all the above identifications are derived from an identification

$$X_*(T) \cong \bigoplus_{i=0}^{d_0-1} X_*(T'), \tag{5.5.1}$$

under which the automorphism θ on the left hand side translates to the following automorphism on the right hand side:

$$(\chi_0, \chi_1, \cdots, \chi_{d_0-1}) \longmapsto (\theta'\chi_{d_0-1}, \chi_0, \chi_1, \cdots, \chi_{d_0-2}).$$

In particular, the identification $(Y^*)' \cong Y^*$, when composed with $Y^* = X_*(A) \subset X_*(T)$ and with (5.5.1), is the diagonal map

$$(Y^*)' \longrightarrow \bigoplus_{i=0}^{d_0-1} X_*(T'), \quad \chi' \longmapsto (\chi', \cdots, \chi'). \tag{5.5.2}$$

Proposition 5.5.1. *For $\lambda \in Y^*$ and $\lambda' \in (Y^*)'$ that correspond to each other, we have*

$$\mathfrak{M}^0_{\lambda,G}(\mathbf{q}^{-1}) = \mathfrak{M}^0_{\lambda',G'}(\mathbf{q}^{-d_0}),$$

as an element of $\mathbb{C}[Y^*][\mathbf{q}^{-1}] \cong \mathbb{C}[(Y^*)'][\mathbf{q}^{-1}]$.

Proof. When $\beta \in \Phi^{1,\vee}$ corresponds to $\beta' \in (\Phi^{1,\vee})'$, we know that β is of the same type (I or II) as β', and we have $\mathfrak{b}(\beta) = d_0\mathfrak{b}'(\beta')$. It follows from Lemma 5.1.5 and Definition 5.1.6 that $\mathcal{P}(\lambda, \mathbf{q}) = \mathcal{P}'(\lambda', \mathbf{q}^{d_0})$ for all $\lambda \in Y^*$ and $\lambda' \in (Y^*)'$ that correspond to each other. The proposition then follows from Definition 5.2.7. □

Next we deduce a relation between the construction of λ_b in §2.6 for G and for G'. Denote by $\widehat{\mathcal{S}}'$ the counterpart of $\widehat{\mathcal{S}}$ for G'. Since G (resp. G') is adjoint, we know that $X^*(\widehat{T})$ (resp. $X^*(\widehat{T}')$) has a $\mathbb{Z}$-basis consisting of the fundamental weights. It then easily follows from Lemma 1.6.1 (2) that we have

$$X^*(\widehat{T})_{\hat{\theta}} = X^*(\widehat{T})_{\hat{\theta},\mathrm{free}} = X^*(\widehat{\mathcal{S}}),$$
$$X^*(\widehat{T}')_{\hat{\theta}'} = X^*(\widehat{T}')_{\hat{\theta}',\mathrm{free}} = X^*(\widehat{\mathcal{S}}'),$$

and we have natural identifications

$$X^*(\widehat{\mathcal{S}}) \cong X^*(\widehat{\mathcal{S}}'), \quad \widehat{Q}_{\hat{\theta}} \cong \widehat{Q}'_{\hat{\theta}'}, \quad \pi_1(G)_\sigma \cong \pi_1(G')_\sigma.$$

Fix an arbitrary $\mu \in X_*(T)$. Choose $\mu' \in X_*(T')$, such that the image of μ' in $X^*(\widehat{\mathcal{S}}')$ corresponds to the image of μ in $X^*(\widehat{\mathcal{S}})$. Such μ' always exists because the map $X_*(T') = X^*(\widehat{T}') \to X^*(\widehat{\mathcal{S}}')$ is surjective. It then follows that the image $\mu^\natural \in \pi_1(G)_\sigma$ of μ and the image $(\mu')^\natural \in \pi_1(G')_\sigma$ of μ' correspond to each other. Let $[b]$ (resp. $[b']$) be the unique basic element of $B(G, \mu)$ (resp. $B(G', \mu')$).

Proposition 5.5.2. *In the above setting, the elements $\lambda_b \in X^*(\widehat{\mathcal{S}})$ and $\lambda_{b'} \in X^*(\widehat{\mathcal{S}}')$ correspond to each other, under the identification $X^*(\widehat{\mathcal{S}}) \cong X^*(\widehat{\mathcal{S}}')$.*

Proof. This immediately follows from the uniqueness in Lemma 2.6.3. □

6. The main result

6.1. The number of irreducible components in terms of combinatorial data

We keep the setting of §2.6 and §4.1. Thus we fix a reductive group scheme G over $\mathcal{O}_F$, an element $\mu \in X_*(T)^+$, and a basic class $[b] \in B(G,\mu)$. In this section, we relate the number of irreducible components $\mathscr{N}(\mu,b)$ to some combinatorial data.

As in §4.1, we fix $s_0 \in \mathbb{N}$ such that b is s_0-decent. As in §5.4 we assume s_0 is divisible by the order d of θ, and various natural numbers $s \in \mathbb{N}$ that are divisible by s_0. In particular, G will always be split over the extension F_s of F. We shall write

$$q_s := |k_s| = |k_F|^s .$$

By Corollary 4.3.7, we have

$$e(J_b)\operatorname{vol}(G(\mathcal{O}_F))^{-1}(\mathrm{BC}_s f_{\mu,s})(\gamma_s) = \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X_\mu(b))} \operatorname{vol}(Z)^{-1} q_s^{\dim X_\mu(b)} + o(q_s^{\dim X_\mu(b)}). \tag{6.1.1}$$

By the dimension formula in Theorem 2.5.3, we have

$$\dim X_\mu(b) = \langle \mu, \rho \rangle - \frac{1}{2}\mathrm{def}_G(b) \tag{6.1.2}$$

(since $\bar{\nu}_b$ is central). In particular, from (6.1.1) we get

$$(\mathrm{BC}_s f_{\mu,s})(\gamma_s) = O(q_s^{\langle \mu,\rho\rangle - \frac{1}{2}\mathrm{def}_G(b)}). \tag{6.1.3}$$

Proposition 6.1.1. *With the notation in §5.4, we have*

$$\mathrm{BC}_s(\tau'_\mu)(\gamma_s) = q_s^{-\langle \mu,\rho\rangle}(\mathrm{BC}_s f_{\mu,s})(\gamma_s) + o(q_s^{-\frac{1}{2}\mathrm{def}_G(b)}).$$

Proof. For λ running over $X^*(\widehat{T})^+$, the Satake transforms of $f_{\lambda,s}$, which we still denote by $f_{\lambda,s}$, form a basis of $\mathbb{C}[X^*(\widehat{T})]^{W_0}$. By the split case of Theorem 5.2.5, we have

$$\tau'_\mu = \sum_{\lambda \in X^*(\widehat{T})^+} K'_{\mu,\lambda}(q_s^{-1}) q_s^{-\langle \lambda,\rho\rangle} f_{\lambda,s},$$

where $K'_{\mu,\lambda}(\cdot)$ is the absolute analogue of (5.2.1), i.e., it is defined by (5.2.1) with θ replaced by 1. By Definition 5.1.6, Corollary 5.1.7, and (5.2.1), we have

$$K'_{\mu,\lambda}(q_s^{-1}) = \begin{cases} 1 + O(q_s^{-1}), & \lambda = \mu, \\ O(q_s^{-1}), & \lambda < \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\tau'_\mu = q_s^{-\langle \mu,\rho\rangle} f_{\mu,s} + \sum_{\lambda \in X^*(\widehat{T})^+,\ \lambda \le \mu} O(q_s^{-1-\langle \lambda,\rho\rangle}) f_{\lambda,s} \tag{6.1.4}$$

Note that (6.1.3) is valid with μ replaced by each $\lambda \in X^*(\widehat{T})^+, \lambda \le \mu$, because we still have $[b] \in B(G,\lambda)$. The proposition then follows from (6.1.4) and the above-mentioned bounds provided by (6.1.3) with μ replaced by each $\lambda \le \mu$. □

Corollary 6.1.2. *We have*

$$\mathrm{BC}_s(\tau'_\mu)(\gamma_s) = e(J_b) \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X)} \mathrm{vol}(Z)^{-1} q_s^{-\frac{1}{2}\mathrm{def}_G(b)} + o(q_s^{-\frac{1}{2}\mathrm{def}_G(b)}). \tag{6.1.5}$$

Proof. This follows by combining (6.1.1), (6.1.2), and Proposition 6.1.1. □

Theorem 6.1.3. *Assume the Haar measures are normalized such that $G(\mathcal{O}_F)$ has volume 1. There exists a rational function $S_{\mu,b}(t) \in \mathbb{Q}(t)$ that is independent of the local field F (in the same sense as Corollary 3.2.4), such that*

$$S_{\mu,b}(0) = \mathscr{N}(\mu, b), \tag{6.1.6}$$

$$S_{\mu,b}(q_1) = e(J_b) \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}(Z)^{-1}, \tag{6.1.7}$$

and such that

$$S_{\mu,b}(q_1) q_s^{-\frac{1}{2}\mathrm{def}_G(b)} = \sum_{\lambda \in X^*(\widehat{S})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ \mathfrak{M}^0_{\lambda^{(s)} - s\nu_b}(q_1^{-1}) + o(q_s^{-\frac{1}{2}\mathrm{def}_G(b)}). \tag{6.1.8}$$

In particular

$$S_{\mu,b}(q_1) = \lim_{s\to\infty} q_s^{\frac{1}{2}\mathrm{def}_G(b)} \sum_{\lambda\in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ \mathfrak{M}^0_{\lambda^{(s)}-s\nu_b}(q_1^{-1}). \tag{6.1.9}$$

Proof. Fix a set of representatives $\{Z_i \mid 1 \le i \le \mathscr{N}(\mu,b)\}$ for the $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$. For each Z_i, let $R_{Z_i}(t) \in \mathbb{Q}(t)$ be the rational function associated to Z_i as in Corollary 3.2.4. Let

$$S_{\mu,b}(t) := e(J_b) \sum_{i=1}^{\mathscr{N}(\mu,b)} R_{Z_i}(t)^{-1}.$$

Then $S_{\mu,b}(t)$ belongs to $\mathbb{Q}(t)$ and it satisfies (6.1.6), (6.1.7). It follows from Corollary 3.2.4 and (6.1.5) that

$$\mathrm{BC}_s(\tau'_\mu)(\gamma_s) = S_{\mu,b}(q_1) q_s^{-\frac{1}{2}\mathrm{def}_G(b)} + o(q_s^{-\frac{1}{2}\mathrm{def}_G(b)}).$$

Comparing this with Corollary 5.4.2, we obtain (6.1.8). □

The upshot of this theorem is that the right hand side of (6.1.9) is purely combinatorial and can be computed (at least in certain instances) using Kostant's partition function $\mathcal{P}_{\mathrm{Kos}}(\lambda, \mathbf{q})$. Moreover the fact that $S_{\mu,b}(t)$ is a rational function independent of the local field F, means that it is in principle determined by its values $S_{\mu,b}(q_1)$ for infinitely many choices of q_1. Once $S_{\mu,b}(t)$ is determined, the number $\mathscr{N}(\mu,b)$ can be read off from (6.1.6).

6.2. The case of unramified elements

In this subsection we apply Theorem 6.1.3 to prove Conjecture 2.6.5 for *unramified* and basic b. This is a new proof of the result [45, Theorem 4.4.14].

We keep the setting of §6.1. Assume in addition that b is *unramified*, in the sense of [45, §4.2]. Then we have $J_b \cong G$, and hence $\mathrm{def}_G(b) = 0, e(J_b) = 1$. In view of Theorem 6.1.3, we would like to compute

$$\lim_{s\to\infty} \sum_{\lambda\in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ \mathfrak{M}^0_{\lambda^{(s)}-s\nu_b}(q_1^{-1}).$$

Proposition 6.2.1. *Let $\lambda \in X^*(\widehat{\mathcal{S}})^+$ and $s \in s_0\mathbb{N}$. We have*

$$\mathfrak{M}^0_{\lambda^{(s)}-s\nu_b}(q_1^{-1}) = \begin{cases} 1, & \text{if } \lambda = \lambda_b, \\ O(q_1^{-as}) \text{ for some } a \in \mathbb{R}_{>0}, & \text{otherwise.} \end{cases}$$

Proof. Firstly, by [45, Lemma 4.2.3], $\lambda_b \in X^*(\widehat{\mathcal{S}})$ is the unique element such that $\lambda_b^{(s)} = s\nu_b$ for one (and hence all) $s \in s_0\mathbb{N}$. (In particular, $\lambda_b \in X^*(\widehat{\mathcal{S}})^+$ as ν_b is central.) Thus for $\lambda \in X^*(\widehat{\mathcal{S}})^+$, we have $\lambda = \lambda_b$ if and only if $\lambda^{(s)} - s\nu_b = 0$ for one (and hence all) $s \in s_0\mathbb{N}$.

Let $w \in W^1$. Since $w\rho^\vee - \rho^\vee$ is not in R^+ for $w \neq 1$, it follows from Definition 5.1.6 and Definition 5.2.7 that $\mathfrak{M}^0_0(q^{-1}) = 1 \in \mathbb{C}[q^{-1}]$. This proves the case $\lambda^{(s)} = s\nu_b$.

Now assume $\lambda^{(s)} \neq s\nu_b$ (for all s). Fix $w, w' \in W^1$. We write $\mu_s := \lambda^{(s)} - s\nu_b$ and $\psi_s := w \bullet (w'\mu_s)$. By the formula (5.2.2), it suffices to show that

$$\exists a > 0,\ \mathcal{P}(\psi_s, q_1^{-1}) = O(q_1^{-as}). \tag{6.2.1}$$

If $\psi_s \notin R^+$ for some value of s, then by definition $\mathcal{P}(\psi_s, q^{-1}) = 0 \in \mathbb{C}[q^{-1}]$. Hence we may ignore these values of s. On the other hand, if $\psi_s \in R^+$ for some $s \in s_0\mathbb{N}$, then it is easy to see that $\psi_{ns} \in R^+$ for all $n \in \mathbb{N}$. We thus assume that $\psi_s \in R^+$ for all sufficiently divisible s. Then for such s we have

$$ww'\mu_s \in \mathrm{span}_{\mathbb{Q}}({}_F\Phi^\vee) - \{0\} \subset Y^* \otimes \mathbb{Q}.$$

Hence there exists a non-zero $\mathbb{Q}$-linear functional f_0 on $\mathrm{span}_{\mathbb{Q}}({}_F\Phi^\vee)$, and a constant $c_0 \in \mathbb{Q} - \{0\}$, such that

$$f_0(ww'\mu_s) = s \cdot c_0,\ \forall s \gg 0. \tag{6.2.2}$$

We define

$$A := \left|f_0(w\rho^\vee - \rho^\vee)\right| \in \mathbb{R}_{\geq 0}, \qquad B := \max_{\beta \in {}_F\Phi^{\vee,+}} |f_0(\beta)| + 1 \in \mathbb{R}_{>0}.$$

Then for $s \gg 0$ we have

$$\min_{\underline{m} \in \mathbb{P}(\psi_s)} |\underline{m}| \geq B^{-1} \cdot |f_0(\psi_s)| \geq B^{-1}(\left|f_0(ww'\mu_s)\right| - A) = B^{-1}(s\,|c_0| - A),$$

where the last equality follows from (6.2.2). Since B^{-1} and c_0 are both non-zero, there is a constant $N_0 > 0$ such that

$$\min_{\underline{m} \in \mathbb{P}(\psi_s)} |\underline{m}| > N_0 \cdot s, \ \forall s \gg 0. \tag{6.2.3}$$

Combining (6.2.3) and Proposition 5.3.1 (2), we have

$$\left|\mathcal{P}(\psi_s, q_1^{-1})\right| \leq \#\mathbb{P}(\psi_s) \cdot q_1^{-N_0 s}. \tag{6.2.4}$$

On the other hand, note that for any $\mathbb{Q}$-linear functional f on $\mathrm{span}_{\mathbb{Q}}({}_F\Phi^\vee)$, the function $s \mapsto f(\psi_s)$ is an affine function in s. Hence there exists a constant $L > 0$ such that

$$\forall s \gg 0, \ \forall \underline{m} \in \mathbb{P}(\psi_s), \ \ |\underline{m}| \leq Ls.$$

It follows that

$$\#\mathbb{P}(\psi_s) = \sum_{l=1}^{Ls} \#\mathbb{P}(\psi_s)_l \leq \sum_{l=1}^{Ls} l^{\#({}_F\Phi^{\vee,+})} \leq (Ls)^M \tag{6.2.5}$$

for some constant $M > 0$. The desired estimate (6.2.1) then follows from (6.2.4) and (6.2.5). □

Theorem 6.2.2. *Keep the setting of §6.1. Assume in addition that b is unramified. Then $\mathscr{N}(\mu, b) = \mathscr{M}(\mu, b)$. Moreover, for any $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, the group $\mathrm{Stab}_Z(J_b(F))$ is a hyperspecial subgroup of $J_b(F) = G(F)$.*

Proof. Let $S_{\mu,b}(t) \in \mathbb{Z}(t)$ be as in Theorem 6.1.3. By (6.1.9) and Proposition 6.2.1, we have

$$S_{\mu,b}(q_1) = \dim V_\mu(\lambda)_{\mathrm{rel}},$$

where λ is the unique element of $X^*(\widehat{\mathcal{S}})^+$ such that $\lambda^{(s_0)} = s_0\nu_b$. By varying the local field F, we see that $S_{\mu,b}(t)$ is the constant $\dim V_\mu(\lambda_b)_{\mathrm{rel}} = \mathscr{M}(\mu, b)$. In particular

$$\mathscr{N}(\mu, b) = S_{\mu,b}(0) = \mathscr{M}(\mu, b).$$

For the second part, by our normalization we have $\mathrm{vol}(Z) \leq 1$ for each element $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, where equality holds if and only if $\mathrm{Stab}_Z(J_b(F))$ is hyperspecial. On the other hand, combining (6.1.6) and (6.1.7) and the fact that $S_{\mu,b}(t)$ is constant, we have

$$\mathscr{N}(\mu, b) = \sum_{Z \in J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}(Z)^{-1}.$$

It follows that each $\mathrm{vol}(Z)$ must be 1, and that $\mathrm{Stab}_Z(J_b(F))$ is hyperspecial. □

6.3. The general case

We now prove the general case of Conjecture 2.6.5. By Proposition 2.6.10, there is no loss of generality in assuming that G is adjoint and F-simple, and that $[b] \in B(G)$ is basic. In particular $\bar{\nu}_b = 0$. In the following, we fix such G and $[b]$, and freely use the notation from §6.1. Note that we only fix $[b]$ and do *not* fix a prescribed $\mu \in X_*(T)^+$ such that $[b] \in B(G, \mu)$.

As in Definition 2.6.4, we have $\lambda_b \in X^*(\widehat{\mathcal{S}})$. Denote by λ_b^+ the unique element in the W^1-orbit of λ_b that lies in $X^*(\widehat{\mathcal{S}})^+$. We define (cf. the discussion above Lemma 2.6.3)

$$\Lambda(b) := \left\{ \lambda \in X^*(\widehat{\mathcal{S}})^+ \mid \lambda \neq \lambda_b^+,\ \lambda - \lambda_b \in \widehat{Q}_{\hat{\theta}} \right\}.$$

The following lemma is the motivation for introducing $\Lambda(b)$.

Lemma 6.3.1. *Let $\lambda \in X^*(\widehat{\mathcal{S}})^+$ and let $\mu \in X_*(T)^+$. Assume that $[b] \in B(G, \mu)$. If $V_\mu(\lambda)_{\mathrm{rel}} \neq 0$, then $\lambda \in \Lambda(b) \sqcup \{\lambda_b^+\}$.*

Proof. Assume that $V_\mu(\lambda)_{\mathrm{rel}} \neq 0$. Then there exists $\lambda' \in X^*(\widehat{T})$ lifting λ, such that $V_\mu(\lambda') \neq 0$. It follows that μ and λ' have the same image in $\pi_1(G)$, and hence the same image in $\pi_1(G)_\sigma$. Let λ'' be the image of λ' in $X^*(\widehat{T})_{\hat{\theta}}$. Then the image of λ'' in $\pi_1(G)_\sigma$ is equal to that of μ, namely $\kappa(b)$. Hence λ'' and $\tilde{\lambda}_b$ have the same image in $\pi_1(G)_\sigma$. By the exact sequence

$$\widehat{Q}_{\hat{\theta}} \to X^*(\widehat{T})_{\hat{\theta}} \to \pi_1(G)_\sigma$$

we know that $\tilde{\lambda}_b - \lambda''$ lies in $\widehat{Q}_{\hat{\theta}}$. Now recall from the discussion above Lemma 2.6.3 that $\widehat{Q}_{\hat{\theta}}$ injects into both $X^*(\widehat{T})_{\hat{\theta}}$ and $X^*(\widehat{\mathcal{S}})$. Hence $\lambda_b - \lambda$ belongs to $\widehat{Q}_{\hat{\theta}}$ viewed as a subgroup of $X^*(\widehat{\mathcal{S}})$. This shows that $\lambda \in \Lambda(b) \sqcup \{\lambda_b^+\}$. □

Since G is F-simple, all the simple factors of $G_{\overline{F}}$ have the same Dynkin type. We shall call this type the type of G. The following proposition is the key result towards the proof of Conjecture 2.6.7.

Proposition 6.3.2 (Key estimate)**.** *Assume G is adjoint, F-simple, and not of type A. Let $[b] \in B(G)$ be a basic class. Assume $[b]$ is not unramified.*

1. *Assume G is not a Weil restriction of the split adjoint E_6. For all $\lambda \in \Lambda(b)$, there exists $a > 0$, such that*

$$\mathfrak{M}^0_{\lambda^{(s)}}(q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)}). \tag{6.3.1}$$

   Moreover, there exists $\mu_1 \in X^*(\widehat{T})^+$ that is minuscule, such that $[b] \in B(G, \mu_1)$ and $\mathscr{M}(\mu_1, b) := \dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = 1$.
2. *Assume G is a Weil restriction of the split adjoint E_6 (necessarily along an unramified extension of F). Then there is an element $\lambda_{\mathrm{bad}} \in \Lambda(b)$ with the following properties:*
   - *For all $\lambda \in \Lambda(b) - \{\lambda_{\mathrm{bad}}\}$, there exists $a > 0$, such that*

$$\mathfrak{M}^0_{\lambda^{(s)}}(q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)}). \tag{6.3.2}$$

   - *There exist $\mu_1, \mu_2 \in X^*(\widehat{T})^+$, such that μ_1 is minuscule and μ_2 is a sum of dominant minuscule elements, such that $b \in B(G, \mu_1) \cap B(G, \mu_2)$, and such that*

$$\mathscr{M}(\mu_1, b) := \dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = 1,$$

     and

$$V_{\mu_1}(\lambda_{\mathrm{bad}})_{\mathrm{rel}} = 0, \quad V_{\mu_2}(\lambda_{\mathrm{bad}})_{\mathrm{rel}} \neq 0.$$

The proof of Proposition 6.3.2 will occupy §7, §8, §9 below. We now admit this proposition.

Theorem 6.3.3. *Conjecture 2.6.7 holds for G adjoint, F-simple, not of type A, and for $[b] \in B(G)$ basic.*

Remark 6.3.4. Here is the logical dependence of our proof of Theorem 6.3.3 on the previous work of other authors:

- If $[b]$ is unramified, then our proof is logically independent of the approaches in [45], [16], or [36].
- If $[b]$ is not unramified, and if we are in the situation of Proposition 6.3.2 (1), then our proof depends on results from [16].
- If $[b]$ is not unramified, and if we are in the situation of Proposition 6.3.2 (2), then our proof depends on Nie's result Theorem 2.6.8 (3) applied to G and μ_1, μ_2.

Proof of Theorem 6.3.3. If $[b]$ is unramified, then the present theorem is just Theorem 6.2.2 (which is also valid for type A). From now on we assume $[b]$ is not unramified. To simplify notation, we write def for $\mathrm{def}_G(b)$.

Assume we are in the situation of Proposition 6.3.2 (1). By Theorem 6.1.3, Lemma 6.3.1, and Proposition 6.3.2 (1), for all $\mu \in X_*(T)^+$ such that $[b] \in B(G,\mu)$, we have

$$S_{\mu,b}(q_1) = \mathscr{M}(\mu,b) \lim_{s\to\infty} q_s^{\frac{1}{2}\mathrm{def}} \mathfrak{M}^0_{\lambda_b^{+,(s)}}(q_1^{-1}). \tag{6.3.3}$$

In particular, we have

$$S_{\mu,b}(q_1) = \frac{\mathscr{M}(\mu,b)}{\mathscr{M}(\mu_1,b)} S_{\mu_1,b}(q_1) = \mathscr{M}(\mu,b) S_{\mu_1,b}(q_1).$$

By varying the local field F (whilst preserving the affine root system of G) we conclude that

$$S_{\mu,b}(t) = \mathscr{M}(\mu,b) S_{\mu_1,b}(t) \in \mathbb{Q}(t). \tag{6.3.4}$$

Evaluating at 0, we have

$$\mathscr{N}(\mu,b) = S_{\mu,b}(0) = \mathscr{M}(\mu,b) S_{\mu_1,b}(0) = \mathscr{M}(\mu,b)\mathscr{N}(\mu_1,b).$$

On the other hand, as μ_1 is minuscule, it is shown in [16, Theorem 1.4] that $\mathscr{N}(\mu_1,b) \le \mathscr{M}(\mu_1,b)$. Since $\mathscr{N}(\mu_1,b)$ is a positive natural number and $\mathscr{M}(\mu_1,b) = 1$, we have $\mathscr{N}(\mu_1,b) = 1$. Thus $\mathscr{N}(\mu,b) = \mathscr{M}(\mu,b)$, as desired.

Now assume we are in the situation of Proposition 6.3.2 (2). For each $\mu \in X_*(T)^+$, we write d_μ for $\dim V_\mu(\lambda_{\mathrm{bad}})_{\mathrm{rel}}$. By Theorem 6.1.3, Lemma 6.3.1, and Proposition 6.3.2 (2), for all $\mu \in X_*(T)^+$ such that $[b] \in B(G,\mu)$, we have

$$S_{\mu,b}(q_1) = \lim_{s\to\infty} \left[\mathscr{M}(\mu,b) q_s^{\frac{1}{2}\mathrm{def}} \mathfrak{M}^0_{\lambda_b^{+,(s)}}(q_1^{-1}) + d_\mu q_s^{\frac{1}{2}\mathrm{def}} \mathfrak{M}^0_{\lambda_{\mathrm{bad}}^{(s)}}(q_1^{-1}) \right]. \tag{6.3.5}$$

In particular, taking μ to be μ_1 and μ_2 respectively, we obtain

$$S_{\mu_1,b}(q_1) = \lim_{s\to\infty} q_s^{\frac{1}{2}\mathrm{def}} \mathfrak{M}^0_{\lambda_b^{+,(s)}}(q_1^{-1}), \tag{6.3.6}$$

$$S_{\mu_2,b}(q_1) = \lim_{s\to\infty} \left[\mathscr{M}(\mu_2,b) q_s^{\frac{1}{2}\mathrm{def}} \mathfrak{M}^0_{\lambda_b^{+,(s)}}(q_1^{-1}) + d_{\mu_2} q_s^{\frac{1}{2}\mathrm{def}_G(b)} \mathfrak{M}^0_{\lambda_{\mathrm{bad}}^{(s)}}(q_1^{-1}) \right]. \tag{6.3.7}$$

Comparing (6.3.5) (6.3.6) (6.3.7) and using $d_{\mu_2} \neq 0$, we obtain

$$S_{\mu,b}(q_1) = \mathscr{M}(\mu, b) S_{\mu_1,b}(q_1) + \frac{d_\mu}{d_{\mu_2}} (S_{\mu_2,b}(q_1) - \mathscr{M}(\mu_2, b) S_{\mu_1,b}(q_1)).$$

By varying F, we obtain

$$S_{\mu,b}(t) = \mathscr{M}(\mu, b) S_{\mu_1,b}(t) + \frac{d_\mu}{d_{\mu_2}} (S_{\mu_2,b}(t) - \mathscr{M}(\mu_2, b) S_{\mu_1,b}(t)), \tag{6.3.8}$$

as an equality in $\mathbb{Q}(t)$. Since μ_1, μ_2 are sums of dominant minuscule elements, Theorem 2.6.8 (3) implies that

$$\mathscr{N}(\mu_1, b) = \mathscr{M}(\mu_1, b) = 1, \text{ and } \mathscr{N}(\mu_2, b) = \mathscr{M}(\mu_2, b).$$

Consequently we have $S_{\mu_1,b}(0) = 1$ and $S_{\mu_2,b}(0) = \mathscr{M}(\mu_2, b)$. Evaluating (6.3.8) at $t = 0$, we obtain

$$\mathscr{N}(\mu, b) = S_{\mu,b}(0) = \mathscr{M}(\mu, b) + \frac{d_\mu}{d_{\mu_2}} (\mathscr{M}(\mu_2, b) - \mathscr{M}(\mu_2, b)) = \mathscr{M}(\mu, b)$$

as desired. □

Corollary 6.3.5. *Conjecture 2.6.7 is true in full generality.*

Proof. By Proposition 2.6.10, we reduce to the case where G is adjoint and F-simple, and $[b]$ is basic. If G is not of type A, the conjecture is proved in Theorem 6.3.3. If G is of type A, the conjecture follows from Theorem 2.6.8 (3). □

The rest of the paper is devoted to the proof of Proposition 6.3.2.

6.4. Reduction to the absolutely simple case

Lemma 6.4.1. *Proposition 6.3.2 holds true if it holds for all G that are absolutely simple and adjoint, not of type A.*

Proof. Let G be as in Proposition 6.3.2, not necessarily absolutely simple. Fix a basic $[b] \in B(G)$ as in Proposition 6.3.2. Let G' be the auxiliary absolutely simple and adjoint group over F, constructed in §5.5. We keep the notation established there. Note that $[b]$ is completely determined by $\kappa_G(b) \in \pi_1(G)_\sigma$. We construct a basic $[b'] \in B(G')$ as in §5.5, such that $\kappa_G(b)$ and $\kappa_{G'}(b')$ correspond to each other under the identification $\pi_1(G)_\sigma \cong \pi_1(G')_\sigma$.

We write EDynk_G for the extended Dynkin diagram of G and write $\mathrm{Aut}(\mathrm{EDynk}_G)$ for its automorphism group. We write $|\mathrm{EDynk}_G|$ for the set of nodes in EDynk_G. Similarly for G'.

We claim that

$$\mathrm{def}_G(b) = \mathrm{def}_{G'}(b'). \tag{6.4.1}$$

In fact, there is a natural embedding $\pi_1(G) \rtimes \langle\theta\rangle \hookrightarrow \mathrm{Aut}(\mathrm{EDynk}_G)$ given by the identification of $\pi_1(G)$ with the stabilizer Ω in W of the base alcove, and the natural faithful action of Ω on EDynk_G. The number $\mathrm{def}_G(b)$ is computed as the number of θ-orbits minus the number of $[\mu] \rtimes \theta$-orbits in $|\mathrm{EDynk}_G|$, where $[\mu] \in \pi_1(G)$ is any lift of $\kappa_G(b) \in \pi_1(G)_\sigma$. Similarly, choosing a lift $[\mu'] \in \pi_1(G')$ of $\kappa_{G'}(b')$, we compute $\mathrm{def}_{G'}(b')$ as the number of θ'-orbits minus the number of $[\mu'] \rtimes \theta'$-orbits in $|\mathrm{EDynk}_{G'}|$. Now by construction, $\mathrm{EDynk}_{G'}$ is identified with a particular connected component of EDynk_G. We may thus embed $\mathrm{Aut}(\mathrm{EDynk}_{G'})$ into $\mathrm{Aut}(\mathrm{EDynk}_G)$ by extending the action trivially to other connected components. Then inside $\mathrm{Aut}(\mathrm{EDynk}_G)$ we have the following relations:

$$\theta' = \theta^{d_0}, \quad \pi_1(G) = \bigoplus_{i=0}^{d_0-1} \theta^i \pi_1(G') \theta^{-i}.$$

In particular, we have an embedding $\pi_1(G') = \theta^0 \pi_1(G') \theta^0 \hookrightarrow \pi_1(G)$. We may arrange that $[\mu]$ is the image of $[\mu']$ under this embedding. Then we have

$$\begin{aligned} \#\{\theta\text{-orbits in } |\mathrm{EDynk}_G|\} &= \#\left\{\theta'\text{-orbits in } |\mathrm{EDynk}_{G'}|\right\}, \\ \#\{[\mu] \rtimes \theta\text{-orbits in } |\mathrm{EDynk}_G|\} &= \#\left\{[\mu'] \rtimes \theta'\text{-orbits in } |\mathrm{EDynk}_{G'}|\right\}. \end{aligned}$$

The claim follows.

Next, we naturally identify $X^*(\widehat{\mathcal{S}})$ with $X^*(\widehat{\mathcal{S}}')$. Then it is easy to see that λ_b corresponds to $\lambda_{b'}$ under this identification. For clarity, we denote the analogue of the map (5.4.1) for G' as:

$$X^*(\widehat{\mathcal{S}}') \longrightarrow (Y^*)', \quad \lambda \longmapsto \lambda^{((s))}.$$

The target of the above map is identified with Y^*. Then since the identification $Y^* \cong (Y^*)'$ amounts to the diagonal map (5.5.2), we see that

$$\lambda^{(d_0 s)} = \lambda^{((s))}, \ \forall \lambda \in X^*(\widehat{\mathcal{S}}). \tag{6.4.2}$$

Combining (6.4.1), (6.4.2) with Propositions 5.5.1, 5.5.2, we see that the bounds (6.3.1) and (6.3.2) in Proposition 6.3.2 for $(G, b, s := d_0 s')$ reduce to the corresponding bounds for (G', b', s'). In the situation of Proposition 6.3.2 (2), we define λ_{bad} for (G, b) to be equal to that for (G', b'), under the identification $\Lambda(b) \cong \Lambda(b')$.

Finally, by hypothesis the desired μ_1' or $\{\mu_1', \mu_2'\}$ are already defined for (G', b'), as in Proposition 6.3.2. Under the identification (5.5.1) we define $\mu_i \in X_*(T)^+$ to be $(\mu_i', 0, \cdots, 0)$ for $i = 1, 2$. □

6.5. Strategy of proving Proposition 6.3.2 in the absolutely simple case

In Lemma 6.4.1, we already reduced the proof of Proposition 6.3.2 to the absolutely simple case. **From now on until the end of the paper, we assume that G is an absolutely simple adjoint group over F which is not of type A.**

As in the proof of Lemma 6.4.1, we denote by EDynk_G the extended Dynkin diagram of G, denote by $\mathrm{Aut}(\mathrm{EDynk}_G)$ its automorphism group, and denote by $|\mathrm{EDynk}_G|$ the set of nodes. To prove Proposition 6.3.2, consider a basic class $[b] \in B(G)$ which is not unramified. Since b is not unramified, we have $\kappa_G(b) \neq 0$, and in particular the groups $\pi_1(G)$ and $\mathrm{Aut}(\mathrm{EDynk}_G)$ are non-trivial. By our assumptions on G, we see that the following are the only possibilities for Dynk_G and θ (viewed as an automorphism of Dynk_G):

1. Type $B_n, n \geq 2, \theta = \mathrm{id}$.
2. Type $C_n, n \geq 3, \theta = \mathrm{id}$.
3. Type $D_n, n \geq 4, \theta = \mathrm{id}$.
4. Type $D_n, n \geq 5, \theta$ has order 2.
5. Type D_4, θ has order 2.
6. Type D_4, θ has order 3.
7. Type $E_6, \theta = \mathrm{id}$.
8. Type E_6, θ has order 2.
9. Type $E_7, \theta = \mathrm{id}$.

In fact, the above are the only cases (apart from type A) where EDynk_G has non-trivial automorphisms. Our proof of Proposition 6.3.2 will be based on this classification.

In §7 below, we prove many cases of the estimates (6.3.1) and (6.3.2). To be precise, we define an explicit subset $\Lambda(b)_{\mathrm{good}}$ of $\Lambda(b)$, and prove (6.3.1) or (6.3.2) for all $\lambda \in \Lambda(b)_{\mathrm{good}}$. These shall be done in a case-by-case manner according to the previous classification of $(\mathrm{Dynk}_G, \theta)$.

In §8, we finish the proof of (6.3.1) for all $\lambda \in \Lambda(b)$. We also define λ_{bad} in Proposition 6.3.2 (2), and finish the proof of (6.3.2) for all $\lambda \in \Lambda(b) - \{\lambda_{\mathrm{bad}}\}$. In view of what we have done in §7, we only need to analyze elements

$\lambda \in \Lambda(b) - \Lambda(b)_{\mathrm{good}}$. If we are in the situation of Proposition 6.3.2 (1), then we show (6.3.1) for all $\lambda \in \Lambda(b) - \Lambda(b)_{\mathrm{good}}$. If we are in the situation of Proposition 6.3.2 (2), then we define a distinguished element $\lambda_{\mathrm{bad}} \in \Lambda(b)$ and show (6.3.2) for all $\lambda \in \Lambda(b) - \Lambda(b)_{\mathrm{good}} \cup \{\lambda_{\mathrm{bad}}\}$.

In §9, we construct the element μ_1 in Proposition 6.3.2 (1), and the elements μ_1, μ_2 in Proposition 6.3.2 (2). We then check that they satisfy the desired properties. The proof of Proposition 6.3.2 is then finished.

7. Proof of the key estimate, Part I

The goal of this section is to define an explicit subset $\Lambda(b)_{\mathrm{good}}$ of $\Lambda(b)$, and to prove (6.3.1) or (6.3.2) for all $\lambda \in \Lambda(b)_{\mathrm{good}}$.

7.1. Types $B, C, D, \theta = \mathrm{id}$

7.1.1. The norm method. We follow [3, Chapitre VI §4] for the presentation of the root systems of types B_n, C_n, D_n, and for the choice of simple roots. The root systems will be embedded in a vector space $E = \mathbb{R}^n$, with standard basis $e_1, \cdots, e_n$, and standard inner product $\langle e_i, e_j \rangle = \delta_{ij}$ so that we may identify the coroots and coweights with subsets of the same vector space. Following *loc. cit.*, we define the following lattices in E:

$$L_0 := \{(\xi_1, \cdots, \xi_n) \in E \mid \xi_i \in \mathbb{Z}\},$$

$$L_1 := \left\{(\xi_1, \cdots, \xi_n) \in L_0 \mid \sum_{i=1}^{n} \xi_i \in 2\mathbb{Z}\right\},$$

$$L_2 := L_0 + \mathbb{Z}(\frac{1}{2}\sum_{i=1}^{n} e_i).$$

We assume $\theta = \mathrm{id}$, so that $T = A$ and $\widehat{\mathcal{S}} = \widehat{T} = \widehat{A}$. The cocharacter lattice $X_*(T)$ is identified with the coweight lattice in E. Moreover $\pi_1(G)$ is equal to the quotient of the coweight lattice modulo the coroot lattice in E.

Since $[b] \in B(G)$ is basic, it is uniquely determined by $\kappa_G(b) \in \pi_1(G)_\sigma = \pi_1(G)$. The defect $\mathrm{def}_G(b)$ of b is computed in the way indicated in the proof of Lemma 6.4.1.

For any $v = (\xi_1, \cdots, \xi_n) \in E$, we write

$$|v| := |\xi_1| + \cdots + |\xi_n|. \tag{7.1.1}$$

It is easy to verify the following three facts.

1. $|\cdot|$ is a norm on E.
2. $|wv| = |v|$ for any $w \in W_0$ and $v \in E$.
3. For any coroot $\alpha^\vee \in \Phi^\vee$, we have $|\alpha^\vee| \leq \delta$, where $\delta = 2$.

Now given any subset S of $\Lambda(b)$, we define $\mathscr{D}(S) := \min_{\lambda \in S} |\lambda|$. (The minimum obviously exists.) In the following, we will specify a subset $\Lambda(b)_{\mathrm{good}}$ of $\Lambda(b)$, satisfying

$$\mathscr{D}(\Lambda(b)_{\mathrm{good}}) > \delta \cdot \mathrm{def}_G(b)/2. \tag{7.1.2}$$

We show how to get the bound (6.3.1) for all $\lambda \in \Lambda(b)_{\mathrm{good}}$, from (7.1.2).

Let $\lambda \in \Lambda(b)_{\mathrm{good}}$. Fix $w, w' \in W^1$. We write $\psi_s := w \bullet (w'\lambda^{(s)})$. By the formula (5.2.2), it suffices to show that

$$\mathcal{P}(\psi_s, q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)}) \tag{7.1.3}$$

for some $a > 0$.

By Proposition 5.3.1 (2), we have the bound

$$\left|\mathcal{P}(\psi_s, q_1^{-1})\right| \leq \#\mathbb{P}(\psi_s) \cdot q_1^{-N_s}, \quad N_s := \min_{\underline{m} \in \mathbb{P}(\psi_s)} |\underline{m}|. \tag{7.1.4}$$

Suppose $\underline{m} \in \mathbb{P}(\psi_s)$. Then

$$\begin{aligned}\delta \sum_{\beta \in {}_F\Phi^{\vee,+}} m(\beta) &\geq \left| \sum_{\beta \in {}_F\Phi^{\vee,+}} m(\beta)\beta \right| = |\psi_s| \\ &\geq \left|ww'\lambda^{(s)}\right| - \left|w\rho^\vee - \rho^\vee\right| = s\,|\lambda| - C \geq s \cdot \mathscr{D}(\Lambda(b)_{\mathrm{good}}) - C,\end{aligned}$$

where C is a constant independent of s, and $|\cdot|$ is the norm defined in (7.1.1). Since $\theta = \mathrm{id}$, we have ${}_F\Phi^{\vee,+} = \Phi^{\vee,+}$, and $\mathfrak{b}(\beta) = 1$ for all $\beta \in \Phi^{\vee,+}$. Hence the leftmost term in the above inequalities is none other than $\delta\,|\underline{m}|$. It follows that

$$N_s \geq (s\mathscr{D}(\Lambda(b)_{\mathrm{good}}) - C)\delta^{-1}.$$

By the above estimate and (7.1.2), we have

$$q_1^{-N_s} = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a')}) \tag{7.1.5}$$

for some $a' > 0$.

On the other hand, by the same argument as in the proof of Proposition 6.2.1, we have

$$\#\mathbb{P}(\psi_s) \leq (Ls)^M \tag{7.1.6}$$

for some constants $L, M > 0$. The desired estimate (7.1.3) then follows from (7.1.4) (7.1.5) (7.1.6).

In the following we specify the definition of $\Lambda(b)_{\mathrm{good}}$ satisfying (7.1.2), for types B, C, D with $\theta = \mathrm{id}$.

7.1.2. Type $B_n, n \geq 2, \theta = \mathrm{id}$. The simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = e_n$. The simple coroots are $\alpha_i^\vee = \alpha_i$ for $1 \leq i \leq n-1$, and $\alpha_n^\vee = 2e_n$. The fundamental weights are $\varpi_i = e_1 + \cdots + e_i$ for $1 \leq i \leq n-1$, and $\varpi_n = \frac{1}{2}(e_1 + \cdots + e_n)$. The coroot lattice is L_1, and the coweight lattice is L_0. We have

$$P^+ = \{(\xi_1, \cdots, \xi_n) \mid \xi_i \in \mathbb{Z}, \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \geq 0\}.$$

We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element is represented by $e_1 \in L_0$. Recall that we assumed that $\kappa_G(b)$ is non-trivial, so there is only one choice of $\kappa_G(b)$ (and hence only one choice of the basic $b \in B(G, \mu)$). We have $\lambda_b = -e_n$, and $\lambda_b^+ = e_1$. Since $\kappa_G(b)$ acts on EDynk_G via its unique non-trivial automorphism, we easily see (both for $n = 2$ and for $n \geq 3$) that $\mathrm{def}_G(b) = 1$. We take $\Lambda(b)_{\mathrm{good}} := \Lambda(b)$, and we have $\mathscr{D}(\Lambda(b)) = 2$. The inequality (7.1.2) is satisfied.

7.1.3. Type $C_n, n \geq 3, \theta = \mathrm{id}$. The simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = 2e_n$. The simple coroots are $\alpha_i^\vee = \alpha_i$ for $1 \leq i \leq n-1$, and $\alpha_n^\vee = e_n$. The fundamental weights are $\varpi_i = e_1 + \cdots + e_i$ for $1 \leq i \leq n$. The coroot lattice is L_0, the coweight lattice is L_2. We have

$$P^+ = \{(\xi_1, \cdots, \xi_n) \in L_2 \mid \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \geq 0\}.$$

We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element is represented by $(\frac{1}{2}, \cdots, \frac{1}{2}) \in L_2$. Since $\kappa_G(b)$ is non-trivial, we have

$$\lambda_b = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \cdots, (-1)^n \frac{1}{2}), \quad \lambda_b^+ = (\frac{1}{2}, \cdots, \frac{1}{2}).$$

Since $\kappa_G(b)$ acts on the EDynk_G via its unique non-trivial automorphism, we easily see that $\mathrm{def}_G(b) = \lceil \frac{n}{2} \rceil$ (i.e. the smallest integer $\geq n/2$). We take $\Lambda(b)_{\mathrm{good}} := \Lambda(b)$, and we have $\mathscr{D}(\Lambda(b)) = (n+2)/2$. The inequality (7.1.2) is satisfied.

7.1.4. Type $D_n, n \geq 4, \theta = \mathrm{id}$. The simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-2$, and $\alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n$. The simple coroots are $\alpha_i^\vee = \alpha_i$. The fundamental weights are

$$\begin{aligned}\varpi_i &= e_1 + \cdots + e_i,\ 1 \leq i \leq n-2,\\ \varpi_{n-1} &= \frac{1}{2}(e_1 + e_2 + \cdots + e_{n-1} - e_n),\\ \varpi_n &= \frac{1}{2}(e_1 + e_2 + \cdots + e_n).\end{aligned}$$

The coroot lattice is L_1, the coweight lattice is L_2. We have

$$P^+ = \{(\xi_1, \cdots, \xi_n) \in L_2 \mid \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq |\xi_n|\}.$$

Case: n is odd. We have $\pi_1(G) \cong \mathbb{Z}/4\mathbb{Z}$, and a generator is represented by $(\frac{1}{2}, \cdots, \frac{1}{2}) \in L_2$. For $i = 1, 2, 3$, we let $b_i \in B(G)$ correspond to the image of $i(\frac{1}{2}, \cdots, \frac{1}{2})$ in $\pi_1(G)$. Then

$$\begin{aligned}\lambda_{b_1} &= \sum_{i=1}^{n-2} \frac{(-1)^i}{2} e_i - \frac{1}{2} e_{n-1} + \frac{(-1)^{(n+1)/2}}{2} e_n, & \lambda_{b_1}^+ &= (\frac{1}{2}, \cdots, \frac{1}{2}),\\ \lambda_{b_2} &= -e_{n-1}, & \lambda_{b_2}^+ &= e_1,\\ \lambda_{b_3} &= \sum_{i=1}^{n-2} \frac{(-1)^i}{2} e_i - \frac{1}{2} e_{n-1} + \frac{(-1)^{(n-1)/2}}{2} e_n, & \lambda_{b_3}^+ &= (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}).\end{aligned}$$

Since up to automorphisms of $\mathbb{Z}/4\mathbb{Z}$, there is only one way that $\mathbb{Z}/4\mathbb{Z}$ could act on EDynk_G, we easily see that

$$\mathrm{def}_G(b_1) = \mathrm{def}_G(b_3) = \frac{n+3}{2},\ \mathrm{def}_G(b_2) = 2.$$

Let

$$\lambda_{1,\mathrm{bad}} := (\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}), \quad \lambda_{3,\mathrm{bad}} := (\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{1}{2}).$$

For $i = 1, 3$, we obviously have $\lambda_{i,\mathrm{bad}} \in \Lambda(b_i)$. We take

$$\Lambda(b_i)_{\mathrm{good}} := \Lambda(b_i) - \{\lambda_{i,\mathrm{bad}}\}. \tag{7.1.7}$$

Then $\mathscr{D}(\Lambda(b_i)_{\mathrm{good}}) = \frac{n+4}{2}$ and the inequality (7.1.2) is satisfied.

For $i = 2$, we take $\Lambda(b_2)_{\mathrm{good}} := \Lambda(b_2)$, and we have $\mathscr{D}(\Lambda(b_2)) = 3$. The inequality (7.1.2) is satisfied.

Case: n is even. We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The three non-trivial elements are represented by

$$(\frac{1}{2}, \cdots, \frac{1}{2}), \quad e_1, \quad (\frac{1}{2}, \cdots, \frac{1}{2}) + e_1 \in L_2.$$

Correspondingly we have

$$\begin{aligned}
\lambda_{b_1} &= \sum_{i=1}^{n-2} \frac{(-1)^i}{2} e_i - \frac{1}{2} e_{n-1} + \frac{(-1)^{n/2}}{2} e_n, & \lambda_{b_1}^+ &= (\frac{1}{2}, \cdots, \frac{1}{2}), \\
\lambda_{b_2} &= -e_{n-1}, & \lambda_{b_2}^+ &= e_1, \\
\lambda_{b_3} &= \sum_{i=1}^{n-2} \frac{(-1)^i}{2} e_i - \frac{1}{2} e_{n-1} + \frac{(-1)^{n/2+1}}{2} e_n, & \lambda_{b_3}^+ &= (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}).
\end{aligned}$$

Since $\kappa_G(b_1)$ and $\kappa_G(b_3)$ are related to each other by the automorphism of the based root system $e_n \mapsto -e_n$, it is clear that they correspond to the two horizontal symmetries of order two of EDynk_G. On the other hand, the action of $\kappa_G(b_2)$ on EDynk_G is of order two, is distinct from the two horizontal symmetries, and commutes with the two horizontal symmetries. Hence this must correspond to the vertical symmetry of EDynk_G that has precisely two orbits of size two and fixes all the other nodes. Thus we have

$$\mathrm{def}_G(b_1) = \mathrm{def}_G(b_3) = \frac{n}{2}, \quad \mathrm{def}_G(b_2) = 2.$$

For $i = 1, 2, 3$ we take $\Lambda(b_i)_{\mathrm{good}} := \Lambda(b_i)$. Then we have $\mathscr{D}(\Lambda(b_1)) = \mathscr{D}(\Lambda(b_3)) = (n+2)/2$, and $\mathscr{D}(\Lambda(b_2)) = 3$. The inequality (7.1.2) is satisfied.

7.2. Type $D_n, n \geq 5, \theta$ has order 2

The simple (absolute) roots and coroots are the same as in §7.1.4, embedded in $E = \mathbb{R}^n$. We identify E with $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Then θ acts on E by

$$(\xi_1, \cdots, \xi_n) \mapsto (\xi_1, \cdots, \xi_{n-1}, -\xi_n).$$

The subgroup $X_*(A) \subset X_*(T) = L_2$ is given by $\{(\xi_1, \cdots, \xi_n) \in L_2 \mid \xi_n = 0\}$. Let $L_2' \subset \mathbb{Q}^{n-1}$ be the analogue of L_2, namely $L_2' = \mathbb{Z}^{n-1} + \mathbb{Z}(\frac{1}{2}, \cdots, \frac{1}{2})$.

The quotient $X_*(T) = L_2 \to X^*(\widehat{\mathcal{S}})$ is the same as

$$L_2 \longrightarrow L_2',\ (\xi_1, \cdots, \xi_n) \mapsto (\xi_1, \cdots, \xi_{n-1}).$$

The map

$$(s) : X^*(\widehat{\mathcal{S}}) \longrightarrow X_*(A),\ \lambda \longmapsto \lambda^{(s)}$$

(for $s \in 2\mathbb{Z}_{\geq 1}$) is given by

$$L_2' \longrightarrow X_*(A), \quad (\xi_1, \cdots, \xi_{n-1}) \longmapsto (s\xi_1, \cdots, s\xi_{n-1}, 0). \tag{7.2.1}$$

The set ${}_F\Phi^{\vee,+}$, as a subset of $X_*(A)$, is equal to

$$\{e_i \pm e_j \mid 1 \leq i < j \leq n-1\} \cup \{2e_i \mid 1 \leq i \leq n-1\}.$$

We have

$$\mathtt{b}(e_i \pm e_j) = 1,\ 1 \leq i < j \leq n-1; \qquad \mathtt{b}(2e_i) = 2,\ 1 \leq i \leq n-1.$$

Moreover ${}_F\Phi^{\vee}$ is reduced. We have

$$P^+ = \{(\xi_1, \cdots, \xi_n) \in L_2 \mid \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq \xi_n = 0\},$$

$$X^*(\widehat{\mathcal{S}})^+ = \left\{(\xi_1, \cdots, \xi_{n-1}) \in X^*(\widehat{\mathcal{S}}) = L_2' \mid \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq 0\right\}.$$

We again write $e_1, \cdots, e_{n-1}$ for the standard basis of $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q} = \mathbb{Q}^{n-1}$. The relative simple roots in $\widehat{Q}_{\hat{\theta}} \subset X^*(\widehat{\mathcal{S}})$ are:

$$e_1 - e_2, e_2 - e_3, \cdots, e_{n-2} - e_{n-1}, e_{n-1}$$

(i.e., the same as type B_{n-1}.)

Case: n is odd. We have $\pi_1(G) \cong \mathbb{Z}/4\mathbb{Z}$, and σ acts on $\pi_1(G)$ by the unique non-trivial automorphism of $\pi_1(G)$. Hence $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element is represented by

$$-\frac{1}{2}(e_1 - e_2) - \frac{1}{2}(e_3 - e_4) \cdots - \frac{1}{2}(e_{n-2} - e_{n-1}) + \frac{1}{2}e_n \in L_2 = X_*(T).$$

The image of the above element in $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q} = \mathbb{Q}^{n-1}$ is obviously equal to a linear combination of the relative simple roots in $\widehat{Q}_{\hat{\theta}}$ with coefficients in

$\mathbb{Q} \cap (-1, 0]$. Hence this image is λ_b, and so

$$\lambda_b = (-\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{2}, \frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q},$$
$$\lambda_b^+ = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}.$$

If γ is any generator of $\pi_1(G) \cong \mathbb{Z}/4\mathbb{Z}$, then the number of orbits of $\gamma \rtimes \theta$ in $|\mathrm{EDynk}_G|$ is $2 + \frac{n-3}{2}$, while the number of orbits of θ in $|\mathrm{EDynk}_G|$ is n. Hence

$$\mathrm{def}_G(b) = \frac{n-1}{2}. \tag{7.2.2}$$

We have

(7.2.3)
$$\Lambda(b) = \left\{ (\xi_1 + \frac{1}{2}, \cdots, \xi_{n-1} + \frac{1}{2}) \in L_2' \mid \xi_1 > 0, \forall i, \xi_i \in \mathbb{Z}, \xi_i \geq \xi_{i+1} \right\}.$$

We take $\Lambda(b)_{\mathrm{good}} := \Lambda(b)$. In the following we show (6.3.1) for all $\lambda \in \Lambda(b)$. The proof will be similar to the arguments in §7.1.1.

Fix $w, w' \in W^1$. We write $\psi_s := w \bullet (w' \lambda^{(s)})$. By the formula (5.2.2), to show (6.3.1) it suffices to show that

$$\exists a > 0,\ \mathcal{P}(\psi_s, q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)}). \tag{7.2.4}$$

Again we have the bounds (7.1.4) and (7.1.6), so it suffices to find a suitable lower bound of $|\underline{m}|$, for $\underline{m} \in \mathbb{P}(\psi_s)$. We keep the definition (7.1.1) of the norm $|\cdot|$ on $E = \mathbb{R}^n$. For $\underline{m} \in \mathbb{P}(\psi_s)$, we have

$$\begin{aligned} 2\,|\underline{m}| &:= 2\sum_\beta m(\beta)\mathfrak{b}(\beta) \geq 2\sum_\beta m(\beta) \geq \left|\sum_\beta m(\beta)\beta\right| \\ &= |\psi_s| \geq \left|ww'\lambda^{(s)}\right| - \left|w\rho^\vee - \rho^\vee\right| = \left|\lambda^{(s)}\right| - C, \end{aligned}$$

where the sums are over $\beta \in {}_F\Phi^{\vee,+}$, and C is a constant independent of s. On the other hand, by (7.2.1) and (7.2.3), we have $\left|\lambda^{(s)}\right| \geq s(n+1)/2$. In conclusion we have

$$2\,|\underline{m}| \geq s \cdot \frac{n+1}{2} + C. \tag{7.2.5}$$

Combining (7.2.2) (7.2.5) with (7.1.4) (7.1.6), we obtain the desired (7.2.4). Note that in the above proof, we only used the fact that $(n+1)/2 > \mathrm{def}_G(b)$.

Case: n is even. We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The action of σ on $\pi_1(G)$ swaps the classes represented by $(\frac{1}{2}, \cdots, \frac{1}{2})$ and $(\frac{1}{2}, \cdots, \frac{1}{2}) + e_1 \in L_2$, and fixes the class represented by e_1. Hence $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element is represented by

$$-\frac{1}{2}(e_1 - e_2) - \frac{1}{2}(e_3 - e_4) \cdots - \frac{1}{2}(e_{n-1} - e_n) \in L_2 = X_*(T).$$

The image of the above element in $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q} = \mathbb{Q}^{n-1}$ is obviously equal to a linear combination of the relative simple roots in $\widehat{Q}_{\hat{\theta}}$ with coefficients in $\mathbb{Q} \cap (-1, 0]$. Hence this image is λ_b, and so

$$\lambda_b = (-\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q},$$
$$\lambda_b^+ = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}.$$

Let $\gamma \in \pi_1(G)$ be the class of $(\frac{1}{2}, \cdots, \frac{1}{2}) \in L_2$. We have seen in §7.1.4 that γ acts on EDynk_G via one of the two order-two horizontal symmetries of EDynk_G. Hence $\gamma \rtimes \sigma$ acts on EDynk_G via one of the two order-four horizontal symmetries of EDynk_G, and the number of orbits is $1 + \frac{n-2}{2} = \frac{n}{2}$. On the other hand the number of orbits of θ in $|\mathrm{EDynk}_G|$ is n. Hence $\mathrm{def}_G(b) = n/2$. The set $\Lambda(b)$ is again given by (7.2.3). We take $\Lambda(b)_{\mathrm{good}} := \Lambda(b)$. The proof of (6.3.1) for all $\lambda \in \Lambda(b)$ is exactly the same as in the odd case, using the fact that $(n+1)/2 > \mathrm{def}_G(b)$.

7.3. Type D_4, θ has order 2

The difference between this case and §7.2 is that the D_4 Dynkin diagram has three (rather than one) automorphisms of order two. However we explain why the proof of (6.3.1) for all $\lambda \in \Lambda(b)_{\mathrm{good}} := \Lambda(b)$ is the same. In fact, there exists a permutation τ of $\{1, 3, 4\}$, such that the root system can be embedded into $\mathbb{R}^4$ with simple roots:

$$\alpha_{\tau(1)} = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_{\tau(3)} = e_3 - e_4,\ \alpha_{\tau(4)} = e_3 + e_4,$$

and such that θ acts on $\mathbb{R}^4$ by $e_4 \mapsto -e_4$.

If $\tau = 1$, then the extra node in EDynk_G is given by $\alpha_0 = -e_1 - e_2$, and the proof is exactly the same as §7.2. For general τ, we still have $\pi_1(G)_\sigma \cong$

$\mathbb{Z}/2\mathbb{Z}$ and hence a unique choice of b, and the only place in the proof in §7.2 that could change is the computation of $\mathrm{def}_G(b)$, as the extra node in EDynk_G is no longer given by $-e_1 - e_2$. However, it is still true that as long as $\kappa_G(b)$ is the non-trivial element in $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, we have $\mathrm{def}_G(b) = 2$, which is $n/2$ for $n = 4$. In fact, for any order-two element $\gamma \in \pi_1(G) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ which is not fixed by σ, the action of $\gamma \rtimes \theta$ on $|\mathrm{EDynk}_G|$ must be of order four and have two orbits. The computation of $\mathrm{def}_G(b)$ easily follows.

7.4. Type D_4, θ has order 3

In this case $\pi_1(G) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We know that θ acts on $\pi_1(G)$ by an order-three permutation of the three non-trivial elements. Thus $\pi_1(G)_\sigma = 0$ and any basic b is unramified.

7.5. Type $E_6, \theta = \mathrm{id}$

We consider the root system E_6 embedded in $\mathbb{R}^9$, which we will consider as $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. The set of roots is given by the 18 elements consisting of permutations of

$$(1,-1,0;0,0,0;0,0,0), \quad (0,0,0;1,-1,0;0,0,0), \quad (0,0,0;0,0,0;1,-1,0)$$

under the group $S_3 \times S_3 \times S_3$, together with the 54 elements given by the permutations of

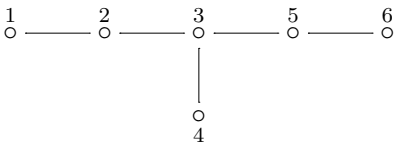
$$(\frac{2}{3},-\frac{1}{3},-\frac{1}{3};\frac{2}{3},-\frac{1}{3},-\frac{1}{3}:\frac{2}{3},-\frac{1}{3},-\frac{1}{3}), \quad (-\frac{2}{3},\frac{1}{3},\frac{1}{3};-\frac{2}{3},\frac{1}{3},\frac{1}{3}:-\frac{2}{3},\frac{1}{3},\frac{1}{3})$$

under the same group. We will call the first set of roots *type A roots*, and the second set *type B roots*. A type A root is positive if and only if the coordinate 1 appears to the left of the -1. A type B root is positive if and only if the first coordinate is positive.

A choice of simple roots is given by

$$\begin{aligned}
&\alpha_1 = (0,0,0;0,1,-1;0,0,0), && \alpha_2 = (0,0,0;1,-1,0;0,0,0),\\
&\alpha_3 = (\frac{1}{3},-\frac{2}{3},\frac{1}{3};-\frac{2}{3},\frac{1}{3},\frac{1}{3};-\frac{2}{3},\frac{1}{3},\frac{1}{3}), && \alpha_4 = (0,1,-1;0,0,0;0,0,0),\\
&\alpha_5 = (0,0,0;0,0,0;1,-1,0), && \alpha_6 = (0,0,0;0,0,0;0,1,-1).
\end{aligned}$$

The corresponding Dynkin diagram is



Under the standard pairing of $\mathbb{R}^9$ with itself, each root is equal to its own corresponding coroots. We therefore identify $\mathbb{R}^9$ with its dual and do not distinguish between roots and coroots. The subspace of $\mathbb{R}^9$ generated by the roots is given by the equations

$$x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = x_7 + x_8 + x_9 = 0 \tag{7.5.1}$$

where x_i are the standard coordinates. The fundamental weights are given by

$$\begin{aligned}
&\varpi_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 0, 0, 0), && \varpi_2 = (\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; 0, 0, 0),\\
&\varpi_3 = (2, -1, -1; 0, 0, 0; 0, 0, 0), && \varpi_4 = (1, 0, -1; 0, 0, 0; 0, 0, 0),\\
&\varpi_5 = (\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 0, 0, 0; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), && \varpi_6 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; 0, 0, 0; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}).
\end{aligned}$$

For an element $\lambda = \sum_{i=1}^{6} a_i \varpi_i$ with $a_i \in \mathbb{Z}$, we have λ lies in the root lattice if and only

$$a_5 - a_6 - a_2 + a_1 \equiv 0 \mod 3. \tag{7.5.2}$$

We have $\pi_1(G) \cong \mathbb{Z}/3\mathbb{Z}$, with the isomorphism being given by

$$\lambda = \sum_{i=1}^{6} a_i \varpi_i \longmapsto a_5 - a_6 - a_2 + a_1 \mod 3.$$

Moreover λ is dominant if and only $a_i \geq 0$ for $i = 1, \ldots, 6$.

We let b_i, $i = 1, 2$ denote the non-trivial elements in $\pi_1(G)$. We have $\lambda_{b_1}^+ = \varpi_1$ and $\lambda_{b_2}^+ = \varpi_6$. We set

$$\begin{aligned}
\Lambda(b_1)_{\text{good}} &:= \Lambda(b_1) - \{\varpi_5, \varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\},\\
\Lambda(b_2)_{\text{good}} &:= \Lambda(b_2) - \{\varpi_2, \varpi_4 + \varpi_6, \varpi_5 + \varpi_1, 2\varpi_1\}.
\end{aligned}$$

We let $|\cdot|$ be the standard Euclidean norm on $\mathbb{R}^9$. Then $|\cdot|$ is W_0-invariant, and we have $|\alpha^\vee| \leq \delta := \sqrt{2}$, for all $\alpha^\vee \in \Phi^\vee$. Given any subset S of $\Lambda(b_i)$, we define $\mathscr{D}(S) := \min_{\lambda \in S} |\lambda|$. We claim that

$$\mathscr{D}(\Lambda(b_1)_{\text{good}}) > \sqrt{8}, \quad \mathscr{D}(\Lambda(b_2)_{\text{good}}) > \sqrt{8}.$$

Since $\text{def}_G(b) = 4$ (which we know by counting orbits of the unique non-trivial symmetry of EDynk_G) and $\delta = \sqrt{2}$, the claim will imply the inequality (7.1.2), and by exactly the same argument as in §7.1.1, we conclude that (6.3.1) holds for all $\lambda \in \Lambda(b_i)_{\text{good}}$.

We now prove the claim. By the obvious symmetry (16)(25)(3)(4) of the Dynkin diagram, it suffices to only discuss $\Lambda(b_1)_{\text{good}}$.

Let $\lambda = \sum_{i=1}^6 a_i \varpi_i \in \Lambda(b_1)$, with $a_i \in \mathbb{Z}_{\geq 0}$, and suppose $|\lambda| \leq \sqrt{8}$. We will show

$$|\lambda| \in \{\varpi_5, \varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\}.$$

Since $\lambda \in \Lambda(b_1)$, we have by (7.5.2) that

$$a_5 - a_6 - a_2 + a_1 \equiv 1 \mod 3. \tag{7.5.3}$$

By looking at the first three coordinates of λ and using the triangle inequality, we easily obtain the inequalities

$$a_1 \leq 3,\ a_2 \leq 1,\ a_3 \leq 1,\ a_4 \leq 3,\ a_5 \leq 1,\ a_6 \leq 3.$$

If $a_3 = 1$, then we have $a_i > 0$ for some $i \neq 3$ since $\lambda \in \Lambda(b_1)$, hence $a_3 = 0$.

If $a_2 = 1$, we have $a_5 = 0$ and $a_1, a_4, a_6 \leq 1$ (by looking at the first 3 coordinates). We check each case and see that only $\lambda = \varpi_2 + \varpi_6$ is possible.

If $a_5 = 1$, we similarly obtain that $\lambda = \varpi_5$ is the only possibility using (7.5.3).

The only cases left are when the only non-zero coefficients are a_1, a_4, a_6. Again by looking at the first three coordinates, we see that $a_1 + a_4 + a_6 \leq 3$. We check each case and see that the only possibilities are $\lambda = \varpi_1 + \varpi_4$ and $\lambda = 2\varpi_6$.

7.6. Type E_6, θ has order 2

We keep the notation §7.5. Then θ acts on the root system via the action on $\mathbb{R}^9$ given by

$$(x_1, x_2, x_3; x_4, x_5, x_6; x_7, x_8, x_9) \longmapsto (x_1, x_2, x_3; x_7, x_8, x_9; x_4, x_5, x_6).$$

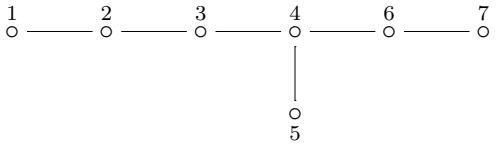
It therefore acts on $\pi_1(G)$ by switching the two non-trivial elements. Hence $\pi_1(G)_\sigma = 0$ and all basic elements are unramified.

7.7. Type $E_7, \theta = \mathrm{id}$

We consider the root system E_7 as a subset of $\mathbb{R}^8$. The set of roots is given by the 56 permutations of $(1, -1, 0, 0, 0, 0, 0, 0)$ and the $\binom{8}{4}$ permutations of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. A set of simple roots is given by

$$\alpha_1 = (0,0,0,0,0,0,-1,1), \qquad \alpha_2 = (0,0,0,0,0,-1,1,0),$$
$$\alpha_3 = (0,0,0,0,-1,1,0,0), \qquad \alpha_4 = (0,0,0,-1,1,0,0,0),$$
$$\alpha_5 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}), \qquad \alpha_6 = (0,0,-1,1,0,0,0,0),$$
$$\alpha_7 = (0,-1,1,0,0,0,0,0).$$

The corresponding Dynkin diagram is



Under the standard pairing of $\mathbb{R}^8$ with itself, roots correspond to coroots and we therefore do not distinguish between them. The subspace of $\mathbb{R}^8$ generated by the roots is the hyperplane given by the equation $\sum_{i=1}^{8} x_i = 0$.

The corresponding fundamental weights are given by

$$\varpi_1 = (\frac{3}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{3}{4}),$$
$$\varpi_2 = (\frac{3}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}),$$
$$\varpi_3 = (\frac{9}{4},-\frac{3}{4},-\frac{3}{4},-\frac{3}{4},-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}),$$
$$\varpi_4 = (3,-1,-1,-1,0,0,0,0),$$
$$\varpi_5 = (\frac{7}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4}),$$
$$\varpi_6 = (2,-1,-1,0,0,0,0,0),$$
$$\varpi_7 = (1,-1,0,0,0,0,0,0).$$

For an element $\lambda = \sum_{i=1}^{7} a_i \varpi_i, a_i \in \mathbb{Z}$, we know λ lies in the root lattice if and only if $a_1 + a_3 + a_5 \equiv 0 \mod 2$. We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$. By assumption $\kappa_G(b) \in \pi_1(G)$ is the non-trivial element. Then we have $\lambda_b^+ = \varpi_1$. We set $\Lambda(b)_{\mathrm{good}} := \Lambda(b) - \{\varpi_5\}$.

We let $|\cdot|$ be the standard Euclidean norm on $\mathbb{R}^8$. Then $|\cdot|$ is W_0-invariant, and we have $|\alpha^\vee| \leq \delta := \sqrt{2}$, for all $\alpha^\vee \in \Phi^\vee$. Given any subset S of $\Lambda(b)$, we define $\mathscr{D}(S) := \min_{\lambda \in S} |\lambda|$. We claim that

$$\mathscr{D}(\Lambda(b)_{\mathrm{good}}) \geq \frac{\sqrt{22}}{2}.$$

Since $\mathrm{def}_G(b) = 3$ and $\delta = \sqrt{2}$, the claim will imply the inequality (7.1.2), and by exactly the same argument as in §7.1.1, we conclude that (6.3.1) holds for all $\lambda \in \Lambda(b)_{\mathrm{good}}$.

We now prove the claim. Suppose $\lambda = \sum_{i=1}^{7} a_i \lambda_i \in \Lambda(b)$ with $a_i \in \mathbb{Z}_{\geq 0}$, and $|\lambda| \leq \sqrt{22}/2$. We will show that $\lambda = \varpi_5$. By looking at the first four coordinates, we obtain the trivial (in)equalities:

$$\varpi_1 \leq 2,\ \varpi_2 \leq 1,\ \varpi_3 = 0,\ \varpi_4 = 0,\ \varpi_5 \leq 1,\ \varpi_6 \leq 1,\ \varpi_7 \leq 1.$$

We also obtain $\sum_{i=1}^{7} a_i \leq 2$. It is not hard to see that $\lambda = \varpi_5$ is the only possibility.

8. Proof of the key estimate, Part II

The goals of this section include:

- to finish the proof of (6.3.1) in Proposition 6.3.2 (1);
- to define λ_{bad} and to prove (6.3.2) in Proposition 6.3.2 (2).

In §7, we already proved (6.3.1) and (6.3.2) for all $\lambda \in \Lambda(b)_{\mathrm{good}}$. Moreover the subset $\Lambda(b)_{\mathrm{good}} \subset \Lambda(b)$ is proper only in the following three cases:

Proper-I Type $D_n, n \geq 5, n$ is odd, $\theta = \mathrm{id}$, $b = b_1$ or b_3. See §7.1.4.
Proper-II Type $E_6, \theta = \mathrm{id}, b = b_1$ or b_2. See §7.5.
Proper-III Type $E_7, \theta = \mathrm{id}$. See §7.7.

8.1. Combinatorics for D_n

In order to treat the case **Proper-I**, we need some combinatorics for the type D_n root system. The material in this subsection is only needed in the proof of Proposition 8.2.1 below.

Let n be an integer ≥ 5. We keep the presentation of the type D_n root system in a vector space $\mathbb{R}^n$, as in §7.1.4. In particular we keep the choice of positive roots. We do not distinguish between roots and coroots. Let Φ_{D_n} be the set of roots and let $\Phi^+_{D_n}$ be the set of positive roots. Thus

$$\Phi^+_{D_n} = \{e_i \pm e_j \mid 1 \leq i < j \leq n\}.$$

If $m > n$ is another integer, we embed $\mathbb{R}^n$ into $\mathbb{R}^m$ via the inclusion of the standard bases $\{e_1, \cdots, e_n\} \hookrightarrow \{e_1, \cdots, e_m\}$. In this way we view Φ_{D_n} (resp. $\Phi^+_{D_n}$) as a natural subset of Φ_{D_m} (resp. $\Phi^+_{D_m}$).

In the following we fix an odd integer $n \geq 5$. We keep the notation in §5.3, with respect to ${}_F\Phi^{\vee,+} = \Phi^+_{D_n}$ and $\mathfrak{b} \equiv 1$. The goal of this subsection is to prove the following:

Proposition 8.1.1. *Let* $(\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$. *For* $t \in \mathbb{N}$, *let*

$$\lambda_t := (6t, 2t\nu_2, 2t\nu_3 \cdots, 2t\nu_n).$$

When $t \in \mathbb{N}$ *is sufficiently large, for any integer* L *in the interval* $[0, (n + 3.5)t]$, *we have*

$$\sum_{S \subset \Phi^+_{D_n}} (-1)^{|S|} \#\mathbb{P}(\lambda_t - \sum_{\beta \in S} \beta)_L = 0. \tag{8.1.1}$$

In order to prove Proposition 8.1.1, we shall use some graph theory to facilitate the computation of the left hand side of (8.1.1). We first recall some standard terminology from graph theory. Recall that a *graph* consists of vertices and edges, such that each edge links two distinct vertices. A *path* on a graph is a sequence $(v_1, E_1, \cdots, v_k, E_k, v_{k+1})$, where v_i and v_{i+1} are the two distinct vertices linked by the edge E_i, for each i, and such that $E_1, \cdots, E_k$ are all distinct. A *tree* is a graph, in which any two vertices are connected by exactly one path. In particular, on a tree we may represent each path $(v_1, E_1, \cdots, v_k, E_k, v_{k+1})$ unambiguously by the sequence $(v_1, v_2, \cdots, v_{k+1})$, and we know that $v_1, \cdots, v_{k+1}$ are all distinct. A *rooted tree* is a pair $(\mathscr{T}, O)$, where $\mathscr{T}$ is a tree and O is a distinguished vertex of $\mathscr{T}$, called the *root*.

In a rooted tree $(\mathscr{T}, O)$, every vertex $v \neq O$ has a unique *parent*, that is, the vertex w such that the unique path from O to v passes through w immediately before reaching v. In this situation we call v a *child* of w. A vertex of $(\mathscr{T}, O)$ that does not have children is called an *end vertex*. We denote by $|\mathscr{T}|$ the set of vertices, and denote by $|\mathscr{T}|_{\text{end}}$ the set of end vertices. By a *complete family line* on $(\mathscr{T}, O)$, we shall mean a path $(v_0, v_1, \cdots, v_k)$

such that $v_0 = O$ and $v_k \in |\mathscr{T}|_{\mathrm{end}}$. Note that in this case each v_i is the parent of v_{i+1}.

In the following, we fix an abelian group E, and fix subsets U, V of E, such that U is finite and $0 \notin V$. In applications, we shall take $E = \mathbb{R}^n$, $U = \{e_1 + \nu_j e_j \mid 2 \leq j \leq n\}$ for some $(\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$, and $V = \Phi^+_{D_n}$.

Definition 8.1.2. By a (U, V)-*tree*, we mean a tuple $(\mathscr{T}, O, \phi, \psi)$, where $(\mathscr{T}, O)$ is a finite rooted tree, and ϕ, ψ are maps

$$\phi : |\mathscr{T}| - \{O\} \longrightarrow U,$$
$$\psi : |\mathscr{T}| - |\mathscr{T}|_{\mathrm{end}} \longrightarrow V.$$

We shall graphically represent a (U, V)-tree $(\mathscr{T}, O, \phi, \psi)$ by marking

$$\boxed{\phi(v) \parallel \psi(v)}$$

at each vertex v. If $\phi(v)$ or $\psi(v)$ is not defined, we leave it as blank. (Thus O could be recognized as the unique vertex where $\phi(v)$ is left as blank.)

Definition 8.1.3. We say that a (U, V)-tree $(\mathscr{T}, O, \phi, \psi)$ is *admissible*, if the following conditions are satisfied:

1. Each $w \in |\mathscr{T}| - |\mathscr{T}|_{\mathrm{end}}$ has precisely two children $v, \bar{v}$. Moreover, we have $\phi(v) - \phi(\bar{v}) \in \{\psi(w), -\psi(w)\}$.
2. Every complete family line $(v_0, v_1, \cdots, v_k)$ on $(\mathscr{T}, O)$ satisfies $k = |U| - 1$. Moreover, $\phi(v_i)$ are distinct from each other for $1 \leq i \leq k$, and $\psi(v_i)$ are distinct from each other for $0 \leq i \leq k-1$.

Definition 8.1.4. Let $(\mathscr{T}, O, \phi, \psi)$ be an admissible (U, V)-tree. Let $w \in |\mathscr{T}| - |\mathscr{T}|_{\mathrm{end}}$. Since $\psi(w) \neq 0$, there is a unique ordering of the two children $v, \bar{v}$ of w, such that $\phi(v) - \phi(\bar{v}) = \psi(w)$. We call v a *positive vertex* and call $\bar{v}$ a *negative vertex*.

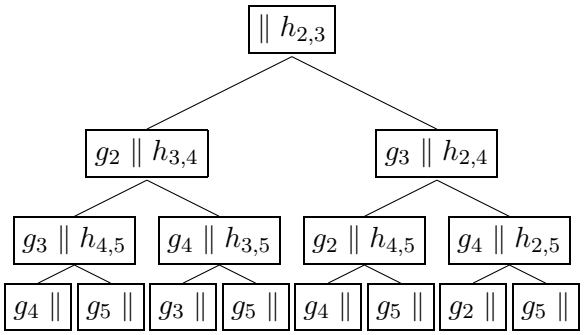
Proposition 8.1.5. *For each odd integer $n \geq 5$ and each $(\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$, we let $E := \mathbb{R}^n$, $U := \{e_1 + \nu_j e_j \mid 2 \leq j \leq n\}$, and $V := \Phi^+_{D_n}$. Then there exists an admissible (U, V)-tree.*

Proof. We prove this by induction on n. To simplify notation, when n and $(\nu_2, \cdots, \nu_n)$ are fixed, we define

$$g_j := e_1 + \nu_j e_j, \quad h_{i,j} := e_i - \nu_i \nu_j e_j,$$

for $i, j \in \{2, \cdots, n\}$.

The base chase is $n = 5$. One checks that the (U, V)-tree represented by the following diagram is admissible:

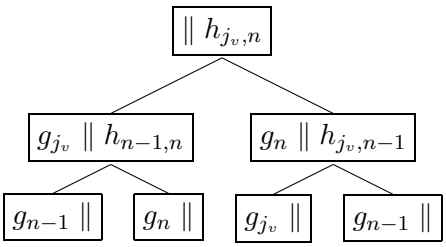


Now assuming the proposition is proved for $n-2$, we prove it for n (with $n \geq 7$). Let $(\nu_2, \cdots, \nu_n)$, U, and V be as in the proposition. Let $U' := \{g_j \mid 2 \leq j \leq n-2\}$, and $V' := \Phi^+_{D_{n-2}}$. By induction hypothesis there exists an admissible (U', V')-tree $(\mathscr{T}', O, \phi', \psi')$. We shall construct an admissible (U, V)-tree based on this.

First, to each end vertex v of $\mathscr{T}'$, we associate a (U, V)-tree $(\mathscr{T}_v, O_v, \phi_v, \psi_v)$ as follows. Let $(O, v_1, \cdots, v_k = v)$ be the complete family line on $\mathscr{T}'$ from O to v. Since $(\mathscr{T}', O, \phi', \psi')$ is admissible, we have $k = n-4$, and there is a unique index $j_v \in \{2, \cdots, n-2\}$ such that

$$\left\{\phi'(v_i) \mid 1 \leq i \leq k\right\} = \{g_j \mid 2 \leq j \leq n-2, j \neq j_v\}.$$

We define $(\mathscr{T}_v, O_v, \phi_v, \psi_v)$ to be:



Now for each end vertex v of $\mathscr{T}'$, we glue $\mathscr{T}_v$ to $\mathscr{T}'$ by identifying the root O_v of $\mathscr{T}_v$ with v. We also combine the marking $\boxed{\phi'(v) \|}$ at v with the marking $\boxed{\| \psi_v(O_v)}$ at O_v to get the new marking $\boxed{\phi'(v) \| \psi_v(O_v)}$. In this way we obtain a larger tree $\mathscr{T}$ containing $\mathscr{T}'$, and a (U, V)-tree $(\mathscr{T}, O, \phi, \psi)$.

It is then an elementary exercise to check that the (U,V)-tree $(\mathscr{T},O,\phi,\psi)$ thus obtained is admissible. As an example, we check that for each complete family line $(v_0,v_1,\cdots,v_k)$ on $(\mathscr{T},O)$, the elements $\psi(v_i)$ are distinct for $0\le i\le k-1$. The other desired conditions can all be checked similarly. Note that $(v_0,v_1,\cdots,v_{k-2})$ is a complete family line on $(\mathscr{T}',O)$. For $0\le i\le k-3$, we have $\psi(v_i)=\psi'(v_i)$, and these are distinct from each other by induction hypothesis. Now writing v for v_{k-2}, we have $\psi(v_{k-2})=\psi_v(O_v)$, and $\psi(v_{k-1})=\psi_v(v_{k-1})$. By the definition of ψ_v, we see that the set formed by $\psi(v_{k-2})$ and $\psi(v_{k-1})$ is either $\{h_{j_v,n},h_{n-1,n}\}$ or $\{h_{j_v,n},h_{j_v,n-1}\}$. Thus these two elements are distinct from each other, and being in $V-V'$, they are distinct from $\psi(v_i)$ for $0\le i\le k-3$, since the latter elements are in V'. Thus we have shown that $\psi(v_i)$ are distinct for $0\le i\le k-1$. □

Fix an odd integer $n\ge 5$, and fix $(\nu_2,\cdots,\nu_n)\in\{\pm1\}^{n-1}$. Let U,V be as in Proposition 8.1.5, and fix an admissible (U,V)-tree $(\mathscr{T},O,\phi,\psi)$ as in that proposition. Let 2^V denote the power set of V.

For each $v\in|\mathscr{T}|-\{O\}$, we define a subset $C_v\subset 2^V$ as follows. Let w be the parent of v. We define

$$C_v:=\begin{cases}\{S\in 2^V\mid \psi(w)\notin S\}, & \text{if } v \text{ is a positive vertex,}\\ \{S\in 2^V\mid \psi(w)\in S\}, & \text{if } v \text{ is a negative vertex.}\end{cases}$$

Now let $(O,v_1,\cdots,v_k=v)$ be the unique path on $\mathscr{T}$ from O to v. We define

$$D_v:=\bigcap_{i=1}^{k}C_{v_i}\subset 2^V.$$

We also define $D_O:=2^V$. It is easy to see that if $v,\bar v$ are the two children of any vertex w, with v positive, then

$$\begin{aligned}&D_v=\{S\in D_w\mid \psi(w)\notin S\},\quad D_{\bar v}=\{S\in D_w\mid \psi(w)\in S\},\\ &D_w=D_v\sqcup D_{\bar v}.\end{aligned}\tag{8.1.2}$$

In this situation, there is a bijection

$$D_v\xrightarrow{\sim}D_{\bar v},\quad S\longmapsto S\cup\{\psi(w)\}.\tag{8.1.3}$$

Next, for any λ in the root lattice $\mathrm{span}_{\mathbb{Z}}(\Phi_{D_n})$, any $L\in\mathbb{Z}_{\ge 0}$, and any $v\in|\mathscr{T}|$, we define a subset $\mathbb{P}(\lambda)^v_L$ of $\mathbb{P}(\lambda)_L$. (See §5.3 for the definition of

$\mathbb{P}(\lambda)_L$.) If $v \neq O$, let $(O, v_1, \cdots, v_k = v)$ be the unique path from O to v. We define

$$\mathbb{P}(\lambda)_L^v := \{\underline{m} \in \mathbb{P}(\lambda)_L \mid m(\phi(v_i)) = 0,\ \forall 1 \leq i \leq k\}.$$

If $v = O$, we define

$$\mathbb{P}(\lambda)_L^O := \mathbb{P}(\lambda)_L.$$

For any two $v_1, v_2 \in |\mathscr{T}|$, we define

$$\mathbb{P}(\lambda)_L^{v_1 - v_2} := \mathbb{P}(\lambda)_L^{v_1} - \mathbb{P}(\lambda)_L^{v_2}.$$

Lemma 8.1.6. *Let λ be an element in the root lattice $\mathrm{span}_{\mathbb{Z}}(\Phi_{D_n})$. For each subset $S \subset \Phi_{D_n}^+$, let $\lambda_S := \lambda - \sum_{\beta \in S} \beta$. Let $w \in |\mathscr{T}| - |\mathscr{T}|_{\mathrm{end}}$, and let $v, \bar{v}$ be its two children. Then*

$$\sum_{S \in D_w} (-1)^{|S|} \#\mathbb{P}(\lambda_S)_L^w = \sum_{S \in D_v} (-1)^{|S|} \#\mathbb{P}(\lambda_S)_L^v + \sum_{S \in D_{\bar{v}}} (-1)^{|S|} \#\mathbb{P}(\lambda_S)_L^{\bar{v}}.$$

Proof. We may assume v is positive. In view of (8.1.2) and the bijection (8.1.3), it suffices to show that for each $S \in D_v$ we have

$$\#\mathbb{P}(\lambda_S)_L^{w-v} = \#\mathbb{P}(\lambda_S - \psi(w))_L^{w-\bar{v}}. \tag{8.1.4}$$

To show this we consider the map

$$f : \mathbb{P}(\lambda_S)_L^{w-v} \longrightarrow \mathbb{P}(\lambda_S - \psi(w))_L^{w-\bar{v}}$$

defined by

$$f(\underline{m})(\beta) := \begin{cases} m(\beta) - 1, & \text{if } \beta = \phi(v), \\ m(\beta) + 1, & \text{if } \beta = \phi(\bar{v}), \\ m(\beta), & \text{else.} \end{cases}$$

One easily checks that f is well-defined and bijective, using the admissibility of $(\mathscr{T}, O, \phi, \psi)$. This proves (8.1.4), and hence the lemma. □

Lemma 8.1.7. *Let $|\cdot|$ be the norm on $\mathbb{R}^n$ as in (7.1.1). Fix a real number $M > n/2$. Let $(\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$, and $t \in \mathbb{N}$. Let $\lambda \in \mathrm{span}_{\mathbb{Z}}(\Phi_{D_n})$, such that*

$$|\lambda - (6t, 2t\nu_2, 2t\nu_3 \cdots, 2t\nu_n)| < t/M. \tag{8.1.5}$$

Let $\underline{m} \in \mathbb{P}(\lambda)$. *Assume there is a subset* $I \subset \{2, \cdots, n\}$ *of cardinality* $n-2$, *such that* $m(e_1 + \nu_i e_i) = 0$ *for all* $i \in I$. *Then*

$$|\underline{m}| \geq (n + 4 - \frac{n}{2M})t.$$

Proof. Assume the contrary. Let j_0 be the unique element of $\{2, \cdots, n\} - I$. Define $\underline{m}' \in \mathbb{P}$ by:

$$\forall \beta \in \Phi^+_{D_n},\ m'(\beta) := \begin{cases} 0, & \text{if } \beta \in \{e_1 \pm e_i \mid 2 \leq i \leq n\}, \\ m(\beta), & \text{else.} \end{cases}$$

Write $\lambda = \lambda_1 e_1 + \sum_{i=2}^n \lambda_i \nu_i e_i$, with $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$. From our assumption (8.1.5), it easily follows that each $\lambda_i > 0$. We have

$$\Sigma(\underline{m}') = \sum_{i \in I} (\lambda_i + m(e_1 - \nu_i e_i))\nu_i e_i + \lambda'_{j_0} \nu_{j_0} e_{j_0},$$

with some $\lambda'_{j_0} \in \mathbb{R}$. Obviously $|\underline{m}'| = |\underline{m}| - \lambda_1$, so we have

$$2\left|\underline{m}'\right| = 2\left|\underline{m}\right| - 2\lambda_1 \geq \left|\Sigma(\underline{m}')\right| \geq \sum_{i \in I} \lambda_i + m(e_1 - \nu_i e_i),$$

from which we get:

$$\begin{aligned} 0 \leq & \sum_{i \in I} m(e_1 - \nu_i e_i) \leq 2\left|\underline{m}\right| - 2\lambda_1 - \sum_{i \in I} \lambda_i \\ & < 2(n + 4 - \frac{n}{2M})t - 2(6t - t/M) - |I|\,(2t - t/M) = 0, \end{aligned}$$

a contradiction. □

We are now ready to prove Proposition 8.1.1.

Proof of Proposition 8.1.1. As in the proposition, we fix $(\nu_2, \cdots, \nu_n)$. We let $\lambda_S := \lambda_t - \sum_{\beta \in S} \beta$, for any subset $S \subset \Phi^+_{D_n}$. Let $U := \{e_1 + \nu_j e_j \mid 2 \leq j \leq n\}$, and let $V := \Phi^+_{D_n}$. Fix an admissible (U, V)-tree $(\mathscr{T}, O, \phi, \psi)$ as in Proposition 8.1.5. By repeatedly applying Lemma 8.1.6 (with respect to $\lambda := \lambda_t$),

we obtain

$$\sum_{S\subset\Phi^+_{D_n}}(-1)^{|S|}\#\mathbb{P}(\lambda_S)_L=\sum_{S\in D_O}(-1)^{|S|}\#\mathbb{P}(\lambda_S)^O_L$$
$$=\sum_{v\in|\mathscr{T}|_{\mathrm{end}}}\sum_{S\in D_v}(-1)^{|S|}\#\mathbb{P}(\lambda_S)^v_L.$$

Hence the proposition is proved once we show the following claim:

Claim. When t is sufficiently large, for each $L\in\mathbb{Z}\cap[0,3.5t]$ and for each $v\in|\mathscr{T}|_{\mathrm{end}}$, we have

$$\sum_{S\in D_v}(-1)^{|S|}\#\mathbb{P}(\lambda_S)^v_L=0.$$

To prove the claim, we fix a real number $M>n$. Let $|\cdot|$ be the norm on $\mathbb{R}^n$ as in (7.1.1). When t is sufficiently large, we have

$$\max_{S\subset\Phi^+_{D_n}}\left|\sum_{\beta\in S}\beta\right|<t/M.$$

Thus we can apply Lemma 8.1.7 to $\lambda:=\lambda_S$ for each $S\subset\Phi^+_{D_n}$. Using the admissibility of $(\mathscr{T},O,\phi,\psi)$, we know that any $\underline{m}\in\mathbb{P}(\lambda_S)^v_L$ satisfies the hypothesis in Lemma 8.1.7 about the vanishing of $m(e_1+\nu_ie_i)$. Thus by that lemma, $\mathbb{P}(\lambda_S)^v_L$ is non-empty only if

$$L\geq(n+4-\frac{n}{2M})t>(n+3.5)t.$$

This proves the claim. The proof of the proposition is complete. □

8.2. The case Proper-I

We treat type D_n with $n\geq5$ odd, and $\theta=\mathrm{id}$, $b=b_1$ or b_3. See §7.1.4. By symmetry we only need to consider $b=b_1$. Recall in this case $\Lambda(b_1)-\Lambda(b_1)_{\mathrm{good}}=\{\lambda_{1,\mathrm{bad}}\}$, where

$$\lambda_{1,\mathrm{bad}}=(\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},-\frac{1}{2}),$$

and we have $\mathrm{def}_G(b_1)=\frac{n+3}{2}$.

Proposition 8.2.1. *The bound (6.3.1) holds for* $\lambda = \lambda_{1,\mathrm{bad}}$.

Proof. The proof uses the results from §8.1. By the formula (5.2.2), it suffices to show for each $w'' \in W^1 = W_0$, that

(8.2.1)
$$\sum_{w \in W^1 = W_0} (-1)^{\ell_1(w)} \mathcal{P}(w''\lambda_{1,\mathrm{bad}}^{(s)} + w\rho^\vee - \rho^\vee, q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)})$$

for some $a > 0$. Here we have made the change of variable $ww' \mapsto w''$ in (5.2.2), and have used the fact that $e_0(w'\lambda_{1,\mathrm{bad}}^{(s)}) \equiv 0$ for all $w' \in W^1$, as long as $s \gg 0$.

Fix w'', and write $\zeta_s := w''\lambda_{1,\mathrm{bad}}^{(s)}$. Let $|\cdot|$ be the norm on $\mathbb{R}^n$ defined in (7.1.1). Since $W_0 \subset \{\pm 1\}^n \rtimes S_n$, there exist $1 \leq j \leq n$, $\varepsilon \in \{\pm 1\}$, and $\underline{\nu} = (\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$, such that

$$\zeta_s = (\frac{1}{2}s\nu_2, \frac{1}{2}s\nu_3, \cdots, \frac{1}{2}s\nu_j, \frac{3}{2}s\varepsilon, \frac{1}{2}s\nu_{j+1}, \cdots, \frac{1}{2}s\nu_n),$$

where $\frac{3}{2}s\varepsilon$ is at the j-th place.

Assume either $j \neq 1$ or $\varepsilon = -1$. Then for $s \gg 0$ and all $w \in W_0$, we have $\zeta_s + w\rho^\vee - \rho^\vee \notin R^+$, and so $\mathcal{P}(\zeta_s + w\rho^\vee - \rho^\vee, q_1^{-1}) = 0$. We are done in this case.

Hence we assume $j = 1$ and $\varepsilon = 1$. Assume without loss of generality that $s = 4t$ for $t \in \mathbb{N}$. As an easy application of the Weyl character formula, we know that any function $\mathscr{P}$ from Y^* to any abelian group satisfies

$$\sum_{w \in W_0} (-1)^{\ell_1(w)} \mathscr{P}(w\rho^\vee - \rho^\vee) = \sum_{S \subset \Phi^{\vee,+}} (-1)^{|S|} \mathscr{P}(-\sum_{\beta \in S} \beta).$$

In particular, the left hand side of (8.2.1) is equal to

$$\sum_{S \subset \Phi^{\vee,+}} (-1)^{|S|} \mathcal{P}(\zeta_s - \sum_{\beta \in S} \beta, q_1^{-1}).$$

By Proposition 5.3.1 (1) and Proposition 8.1.1, the above is equal to

(8.2.2)
$$\sum_{L \in \mathbb{Z},\ L > (n+3.5)t} q_1^{-L} \sum_{S \subset \Phi^{\vee,+}} (-1)^{|S|} \#\mathbb{P}(\zeta_s - \sum_{\beta \in S} \beta)_L.$$

By the same argument as in the proof of Proposition 6.2.1, the expression

$$\left| \sum_{L \in \mathbb{Z}_{\geq 0}} \sum_{S \subset \Phi^{\vee,+}} (-1)^{|S|} \#\mathbb{P}(\zeta_s - \sum_{\beta \subset S} \beta)_L \right|$$

is of polynomial growth in s (or in t). Hence (8.2.2) is bounded by $O(q_1^{-(n+3.4)t})$. Since $s \cdot \mathrm{def}_G(b)/2 = (n+3)t$, the desired bound (8.2.1) follows. □

8.3. The case Proper-II

We now treat type $E_6, \theta = \mathrm{id}, b = b_1$ or b_2. See §7.5. By symmetry, it suffices to treat the case of b_1. Recall in this case

$$\Lambda(b_1) - \Lambda(b_1)_{\mathrm{good}} = \{\varpi_5, \varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\},$$

and we have $\mathrm{def}_G(b_1) = 4$.

Proposition 8.3.1. *The bound (6.3.2) holds for all $\lambda \in \{\varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\}$. In other words, in Proposition 6.3.2 (2) (for $b = b_1$) we may take λ_{bad} to be ϖ_5.*

Proof. We define a function $|\cdot|' : \mathbb{R}^9 \to \mathbb{R}_{\geq 0}$ in the following way. For any $v = \sum_{i=1}^9 x_i e_i \in \mathbb{R}^9$, we define

$$|v|' := \max_{\substack{i,j \in \{0,1,2\}, i \neq j; \\ k,l \in \{1,2,3\}}} |x_{3i+k} - x_{3j+l}|.$$

In other words, we think of $\mathbb{R}^9$ as $(\mathbb{R}^3)^3$, and we take the largest difference between a coordinate in one factor of $\mathbb{R}^3$ and a coordinate in a different factor. Then $|\cdot|'$ is a semi-norm, i.e., it is compatible with scalar multiplication by $\mathbb{R}$ and satisfies the triangle inequality. Note that $|\cdot|'$ is not W_0-invariant. By the explicit description of the roots, we have $|\alpha|' = 1$ for all positive roots α.

Claim. $|\mu|' \geq 7/3$ for all $\mu \in W_0\lambda$.

We prove the claim. We first record explicitly the W_0-orbit of λ. To state it we need some notation. Let C_3 be the cyclic group of order 3 with a fixed generator c. We let C_3 act on $S_3 \times S_3 \times S_3$ via

$$c : (\sigma_1, \sigma_2, \sigma_3) \longmapsto (\sigma_2, \sigma_3, \sigma_1).$$

Let H denote the semi-direct product $(S_3 \times S_3 \times S_3) \rtimes C_3$. Then we have an action of H on $\mathbb{R}^9$, where $S_3 \times S_3 \times S_3$ acts naturally on the coordinate indices, and $c \in C_3$ acts via

$$c : (x_1, x_2, x_3; x_4, x_5, x_6; x_7, x_8, x_9) \longmapsto (x_4, x_5, x_6; x_7, x_8, x_9; x_1, x_2, x_3).$$

For $\lambda = \varpi_2 + \varpi_6$, its W_0-orbit is given by the union of the H-orbits of the following 7 vectors:

$$(2, -1, -1; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}), \qquad (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 3, 3, -6),$$
$$(\frac{4}{3}, \frac{1}{3}, -\frac{5}{3}; 1, 0, -1; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), \qquad (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; 1, 0, -1; \frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}),$$
$$(\frac{4}{3}, \frac{1}{3}, -\frac{5}{3}; 0, 0, 0; \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}), \qquad (\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 0, 0, 0; \frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}),$$
$$(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 1, 0, -1; \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}).$$

For $\lambda = \varpi_1 + \varpi_4$, its W_0-orbit is the union of the H-orbits of the following 4 vectors:

$$(\frac{4}{3}, -\frac{1}{3}, -\frac{5}{3}; 0, 0, 0; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), \qquad (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 0, 0, 0; \frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}),$$
$$(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 1, 0, -1; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), \qquad (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 1, 0, -1; \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}).$$

For $\lambda = 2\varpi_6$, its W_0-orbit is the H-orbit of

$$(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 0, 0, 0; \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}).$$

One sees easily that $|\cdot|'$ of all the above vectors are $\geq 7/3$. Since $|\cdot|'$ is invariant under the action of H, it follows that $|\mu|' \geq 7/3$ holds for all $\mu \in W_0\lambda$. The claim is proved.

Based on the claim, we prove (6.3.2) for $\lambda \in \{\varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\}$, using an argument similar to §7.1.1. By the formula (5.2.2), it suffices to show for each $w, w' \in W_0$ that

$$\mathcal{P}(\psi_s, q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b_1)+a)})$$

for some $a > 0$, where $\psi_s := w \bullet (w'\lambda^{(s)})$. By the same argument as in §7.1.1, we easily reduce to proving: For some constant $a > 0$,

$$\min\{|\underline{m}| : \underline{m} \in \mathbb{P}(\psi_s)\} \geq s(\frac{1}{2}\mathrm{def}_G(b_1) + a) = s(2 + a), \tag{8.3.1}$$

for all $s \gg 0$. By the previous claim, for all $\underline{m} \in \mathbb{P}(\psi_s)$ we have

$$1 \cdot \sum_{\beta \in \Phi^{\vee,+}} m(\beta) \geq \left| \sum_{\beta \in \Phi^{\vee,+}} m(\beta)\beta \right|' = |\psi_s|' \geq \left| ww'\lambda^{(s)} \right|' - \left| w\rho^\vee - \rho^\vee \right|'$$
$$= s \left| ww'\lambda \right|' - C \geq s \cdot \frac{7}{3} - C,$$

where C is a constant independent of s. Here the number 1 appearing in the leftmost term is equal to $\min_{\beta \in \Phi^{\vee,+}} |\beta|'$. Since $\theta = \mathrm{id}$, the leftmost term in the above inequalities is none other than $|\underline{m}|$. The desired (8.3.1) follows. □

8.4. The case Proper-III

We now treat type $E_7, \theta = \mathrm{id}$, and $[b] \in B(G)$ being the unique basic class such that $\kappa_G(b)$ is the non-trivial element of $\pi_1(G) = \mathbb{Z}/2\mathbb{Z}$. See §7.7. Recall in this case $\Lambda(b) - \Lambda(b)_{\mathrm{good}} = \{\varpi_5\}$, and $\mathrm{def}_G(b) = 3$.

Proposition 8.4.1. *The bound (6.3.1) holds for $\lambda = \varpi_5$.*

Proof. Firstly, the W_0-orbit of λ is given by all permutations under S_8 of the following vectors:

$$\lambda_1 = (\frac{7}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}),$$
$$\lambda_2 = (\frac{5}{4}, -\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}),$$
$$\lambda_3 = (-\frac{7}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}),$$
$$\lambda_4 = (-\frac{5}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}).$$

Indeed it is easy to see that all these elements lie in $W_0\lambda$ (using the fact that the W_0 contains the copy of S_8), and one easily computes the size of $W_0\lambda$ to prove that these are all the elements of $W_0\lambda$.

For $1 \leq i \leq 8$ and $\tau \in S_8$, we define functions

$$|\cdot|_i : \mathbb{R}^8 \to \mathbb{R}_{\geq 0}, \quad |\cdot|_\tau : \mathbb{R}^8 \to \mathbb{R}_{\geq 0}$$

in the following way. For any $v = \sum_{i=1}^8 x_i e_i \in \mathbb{R}^8$, we define

$$|v|_i := |x_i|, \quad |v|_\tau := \left|x_{\tau(1)}\right| + \left|x_{\tau(2)}\right| + \left|x_{\tau(3)}\right| + \left|x_{\tau(4)}\right|.$$

Then $|\cdot|_i, |\cdot|_\tau$ are all semi-norms on $\mathbb{R}^8$.

Note that in the proof of Proposition 8.3.1, we reduced to proving (8.3.1) for **each fixed pair** $(w, w') \in W_0 \times W_0$. During the proof of (8.3.1) for the fixed (w, w'), we only needed to apply the semi-norm $|\cdot|'$ to $ww'\lambda$, and not to any other element of $W_0\lambda$. Hence for each element in $W_0\lambda$, we could use a semi-norm, which is specifically designed for that element, to finish the proof. In the current case, we reduce to proving that each $\mu \in W_0\lambda$ satisfies at least one of the following inequalities:

$$|\mu|_i > \frac{\mathrm{def}_G(b)}{2} \min_{\beta \in \Phi^{\vee,+}} |\beta|_i = \frac{3}{2} \min_{\beta \in \Phi^{\vee,+}} |\beta|_i, \tag{8.4.1}$$

$$|\mu|_\tau > \frac{\mathrm{def}_G(b)}{2} \min_{\beta \in \Phi^{\vee,+}} |\beta|_\tau = \frac{3}{2} \min_{\beta \in \Phi^{\vee,+}} |\beta|_\tau \tag{8.4.2}$$

for some $1 \le i \le 8$ or for some $\tau \in S_8$.

When μ is an S_8-permutation of λ_1 or λ_3, assume the i_0-th coordinate of μ is $\pm 7/4$. Then μ satisfies the inequality (8.4.1) indexed by i_0. In fact, any $\beta \in \Phi^{\vee,+}$ satisfies $|\beta|_{i_0} \le 1$, and we have $|\mu|_{i_0} = 7/4$.

When μ is an S_8-permutation of λ_2 or λ_4, there exists $\tau \in S_8$ such that

$$|\mu|_\tau = \frac{5}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{7}{2}.$$

On the other hand $|\beta|_\tau \le 2$ for all $\beta \in \Phi^{\vee,+}$ and all $\tau \in S_8$. Therefore (8.4.2) holds for some τ. □

9. Proof of the key estimate, Part III

We have proved all the statements in Proposition 6.3.2, except the existence of μ_1 in Proposition 6.3.2 (1), and the existence of μ_1, μ_2 in Proposition 6.3.2 (2). In this section we construct them.

First assume that G is not of type E_6, and that $\theta = \mathrm{id}$. We easily examine all such cases in §7 and see that $\lambda_b^+ \in X^*(\widehat{T})^+$ is always minuscule. Hence we may take $\mu_1 := \lambda_b^+$. Since G is adjoint and $\theta = \mathrm{id}$, the condition that $b \in B(G, \mu_1)$ is equivalent to the condition that b and μ_1 have the same image in $\pi_1(G)_\sigma = \pi_1(G)$, which is true by construction. Moreover, we have $\dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = \dim V_{\lambda_b^+}(\lambda_b^+) = 1$. The proof of Proposition 6.3.2 is complete in these cases.

The only remaining cases are the following:

Nonsplit-I Type $D_n, n \ge 5, \theta$ has order 2. See §7.2.

Nonsplit-II Type D_4, θ has order 2. See §7.3.
Split-E_6 Type $E_6, \theta = \mathrm{id}$. See §7.5

In fact, in all the other cases listed in §6.5 where θ is non-trivial, namely cases (6) and (8), we have shown in §7.4 and §7.6 that any basic $[b] \in B(G)$ is unramified, so we do not need to consider these cases.

9.1. The case Nonsplit-I

As we showed in §7.2, we have $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, and there is a unique choice of basic $[b] \in B(G)$ corresponding to the non-trivial element in $\pi_1(G)_\sigma$. Moreover we have

$$\lambda_b^+ = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}.$$

Recall that $X_*(T) = X^*(\widehat{T}) = L_2 \subset \mathbb{R}^n$. We take

$$\mu_1 := (\frac{1}{2}, \cdots, \frac{1}{2}) \in L_2 = X_*(T) = X^*(\widehat{T}).$$

Then μ_1 is in $X^*(\widehat{T})^+$ and is minuscule. From the description of the action of σ on $\pi_1(G)$ in §7.2, the image of μ_1 in $\pi_1(G)_\sigma$ is the non-trivial element, and hence $[b] \in B(G, \mu_1)$. Finally, the only weights in $X^*(\widehat{T})$ of V_{μ_1} are the elements of $W_0\mu_1$. Among all these weights, there is precisely one that restricts to $\lambda_b^+ \in X^*(\widehat{\mathcal{S}})$, namely μ_1. Hence we have

$$\dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = \dim V_{\mu_1}(\lambda_b^+)_{\mathrm{rel}} = \dim V_{\mu_1}(\mu_1) = 1.$$

9.2. The case Nonsplit-II

We keep the notation of §7.3. Note that what τ is does not affect the subset $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of $\mathbb{R}^4$. Nor does it affect the coroot lattice and the coweight lattice. Moreover, no matter what τ is, the quotient map $X_*(T) \to X^*(\widehat{\mathcal{S}})$ is always the same as

$$\begin{aligned} L_2 &\longrightarrow L_2' \\ (\xi_1, \xi_2, \xi_3, \xi_4) &\longmapsto (\xi_1, \xi_2, \xi_3), \end{aligned}$$

where $L_2 = \mathbb{Z}^4 + \mathbb{Z}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $L_2' = \mathbb{Z}^4 + \mathbb{Z}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence the situation is precisely the same as the case **Nonsplit-I** discussed in §9.1. More precisely, for the unique basic $[b] \in B(G)$ that maps to the non-trivial element in $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, we have $\lambda_b = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), \lambda_b^+ = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and we take $\mu_1 := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

9.3. The case Split-E_6

We keep the notation of §7.5. To finish the proof of Proposition 6.3.2 (2), we need to construct μ_1 and μ_2. By symmetry we only need to consider b_1, among b_1, b_2. Recall from §7.5 that $\lambda_{b_1}^+ = \varpi_1$. Recall from Proposition 8.3.1 that the distinguished element λ_{bad} in $\Lambda(b_1)$ is ϖ_5. Since $\theta = \mathrm{id}$, we have $\widehat{\mathcal{S}} = \widehat{T}$.

Note that $\lambda_{b_1}^+ = \varpi_1$ is minuscule. We take $\mu_1 := \varpi_1$. Then the only weight of V_{μ_1} in $X^*(\widehat{\mathcal{S}})^+ = X^*(\widehat{T})^+$ is $\mu_1 = \lambda_{b_1}^+$, and $\dim V_{\mu_1}(\lambda_{b_1})_{\mathrm{rel}} = \dim V_{\mu_1}(\mu_1) = 1$. We have $b_1 \in B(G, \mu_1)$, because the image of $\mu_1 = \lambda_{b_1}^+$ in $\pi_1(G)_\sigma = \pi_1(G)$ is the same as that of λ_{b_1}, which is the same as $\kappa(b_1)$.

Note that ϖ_6 is also minuscule. We take $\mu_2 := 2\varpi_1 + \varpi_6$. Then μ_2 is a sum of dominant minuscule coweights. By (7.5.2) we know that $\mu_2 - \varpi_1$ is in the coroot lattice. Hence μ_2 represents the same element in $\pi_1(G)_\sigma = \pi_1(G)$ as ϖ_1, and in particular $b_1 \in B(G, \mu_2)$. We are left to check that $V_{\mu_2}(\lambda_{\mathrm{bad}})_{\mathrm{rel}}$, which is $V_{\mu_2}(\varpi_5)$, is non-zero. One computes that $\dim V_{\mu_2}(\varpi_5) = 14$, (see for example *LiE online service*, http://young.sp2mi.univ-poitiers.fr/cgi-bin/form-prep/marc/dom_char.act?x1=2&x2=0&x3=0&x4=0&x5=0&x6=1&rank=6&group=E6).[2] The proof of Proposition 6.3.2 is complete.

Appendix A. Irreducible components for quasi-split groups

We explain in this appendix how we can use our results combined with [18] to obtain a description of the number of $J_b(F)$-orbits of irreducible components of affine Deligne–Lusztig varieties associated to a group which is quasi-split but not necessarily unramified. The main result Theorem A.3.1 is a generalization of Conjecture 2.6.7.

A.1. Basic definitions

We extend the notations introduced in §2. We let F, L, k_F, k, σ, Γ be as in §2. However now we only assume that G is a quasi-split reductive group over F. Let $T \subset G$ be the centralizer of a maximal F-split torus in G. Then T is a maximal torus in G since G is quasi-split. We fix B to be a Borel subgroup of G (over F) containing T. Let $\breve{A} \subset T_L$ be the maximal L-split sub-torus of T_L. Note that T_L is a minimal Levi subgroup of G_L, so $\breve{A}$ is a maximal L-split torus in G_L. Let $N \subset G_L$ denote the normalizer of $\breve{A}$. Let V be the apartment of G_L corresponding to $\breve{A}$. Let $\mathfrak{a}$ be a σ-stable alcove in

[2] Note that in LiE, our $\alpha_2, \alpha_3, \alpha_4$ are indexed by $3, 4, 2$ respectively.

V, and let $\mathfrak{s}$ be a σ-stable special vertex lying in the closure of $\mathfrak{a}$. Denote by $\mathcal{I}$ the Iwahori group scheme over $\mathcal{O}_F$ associated to $\mathfrak{a}$, and denote by $\mathcal{K}$ the special parahoric group scheme over $\mathcal{O}_F$ associated to $\mathfrak{s}$. Let $\Gamma_0 \subset \Gamma$ denote the inertia subgroup, which is also identified with $\mathrm{Gal}(\overline{L}/L)$. The choice of $\mathfrak{s}$ gives an identification $V \cong X_*(T)_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R}$, sending $\mathfrak{s}$ to 0. In the following we freely use the identification in Lemma 1.6.1 (3). We assume that under the identification

$$V \cong X_*(T)_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R} \cong X_*(T)^{\Gamma_0}_{\mathbb{R}}, \tag{A.1.1}$$

the image of $\mathfrak{a}$ is contained in the anti-dominant chamber $-X_*(T)^+_{\mathbb{R}}$.

The Iwahori–Weyl group is defined to be $W := N(L)/(T(L) \cap \mathcal{I}(\mathcal{O}_L))$. For any $w \in W$ we choose a representative $\dot{w} \in N(L)$. We write $W_0 := N(L)/T(L)$ for the relative Weyl group of G over L. Then we have a natural exact sequence:

$$1 \longrightarrow X_*(T)_{\Gamma_0} \longrightarrow W \longrightarrow W_0 \longrightarrow 1.$$

Similar to §2.2, the canonical action of $N(L)$ on V factors through an action of W, and we split the above exact sequence by identifying W_0 with the subgroup of W fixing $\mathfrak{s} \in V$. For $\underline{\mu} \in X_*(T)_{\Gamma_0}$ we write $t^{\underline{\mu}}$ for the corresponding element in W. The Frobenius σ induces an action on W which stabilizes the set $\mathbb{S}$ of simple reflections. See [12] for more details.

Let $B(G)$ (resp. $B(W,\sigma)$) denote the set of σ-conjugacy classes of $G(L)$ (resp. W). Let $X_*(T)^+_{\Gamma_0,\mathbb{Q}}$ denote the intersection of $X_*(T)_{\Gamma_0} \otimes \mathbb{Q} \cong X_*(T)^{\Gamma_0}_{\mathbb{Q}}$ with $X_*(T)^+_{\mathbb{Q}}$. Similar to §2.3, we have a commutative diagram

$$\begin{array}{ccc} B(W,\sigma) & \xrightarrow{\quad\Psi\quad} & B(G), \\ & {\scriptstyle(\bar{\nu},\kappa)} \searrow \quad \swarrow {\scriptstyle(\bar{\nu},\kappa)} & \\ & (X_*(T)^+_{\Gamma_0,\mathbb{Q}})^\sigma \times \pi_1(G)_\Gamma & \end{array} \tag{A.1.2}$$

where Ψ is a surjection, and the map $(\bar{\nu},\kappa)$ on $B(W,\sigma)$ can be described explicitly. These statements are proved in [18], see [20, §1.2] for an exposition.

Let $w \in W$ and $b \in G(L)$. We define the affine Deligne–Lusztig variety $X_w(b)$ as follows:

$$X_w(b) := \{g\mathcal{I}(\mathcal{O}_L) \in G(L)/\mathcal{I}(\mathcal{O}_L) \mid g^{-1}b\sigma(g) \in \mathcal{I}(\mathcal{O}_L)\dot{w}\mathcal{I}(\mathcal{O}_L)\}.$$

Now let $\underline{\mu} \in X_*(T)_{\Gamma_0}$ be the image of an element $\mu \in X_*(T)^+$. Similarly we define the affine Deligne–Lusztig variety $X_{\underline{\mu},K}(b)$ as follows:

$$X_{\underline{\mu},K}(b) := \{g\mathcal{K}(\mathcal{O}_L) \in G(L)/\mathcal{K}(\mathcal{O}_L) \mid g^{-1}b\sigma(g) \in \mathcal{K}(\mathcal{O}_L)\dot{t}^{\underline{\mu}}\mathcal{K}(\mathcal{O}_L)\}.$$

When F is of equal (resp. mixed) characteristic, $X_w(b)$ and $X_{\underline{\mu},K}(b)$ are schemes (resp. perfect schemes) locally of finite type (resp. locally perfectly of finite type) over k, see [17].

We define the set

$$B(G,\underline{\mu}) := \{[b] \in B(G) \mid \kappa([b]) = \mu^\natural, \bar{\nu}_b \leq \mu^\diamond\}.$$

Here $\mu^\natural$ denotes the image of μ in $\pi_1(G)_\Gamma$, and $\mu^\diamond \in (X_*(T)^+_{\Gamma_0,\mathbb{Q}})^\sigma$ denotes the average over the σ-orbit of $\underline{\mu} \in X_*(T)_{\Gamma_0}$. Note that both $\mu^\natural$ and $\mu^\diamond$ depend only on $\underline{\mu}$, which justifies the notation $B(G,\underline{\mu})$. The set $B(G,\underline{\mu})$ controls the non-emptiness pattern of $X_{\underline{\mu},K}(b)$.

Theorem A.1.1. *We have $X_{\underline{\mu},K}(b) \neq \emptyset$ if and only if $[b] \in B(G,\underline{\mu})$.*

Proof. This is proved in [18, Theorem 7.1] assuming $\mathrm{char}(F) > 0$. The same proof extends to the case $\mathrm{char}(F) = 0$. □

For $b \in G(L)$, the group $J_b(F)$ acts on $X_{\underline{\mu},K}(b)$ via scheme automorphisms. Our goal is to understand the cardinality

$$\mathscr{N}(\underline{\mu},b) := \#\Big(J_b(F)\backslash\Sigma^{\mathrm{top}}(X_{\underline{\mu},K}(b))\Big). \tag{A.1.3}$$

For simplicity, from now on we assume that G is adjoint. The general case reduces to this case by a standard argument.

A.2. A dual group construction

The desired formula for (A.1.3) will be expressed in terms of a canonical reductive subgroup of the dual group $\widehat{G}$. We keep the assumption that G is adjoint.

As in §5.1, we let

$$\mathrm{BRD}(B,T) = (X^*(T), \Phi \supset \Delta, X_*(T), \Phi^\vee \supset \Delta^\vee)$$

be the based root datum associated to (B,T), equipped with an action by Γ. Let $\widehat{G}$ be the dual group of G over $\mathbb{C}$, which is equipped with a Borel

pair $(\widehat{B},\widehat{T})$ and an isomorphism $\mathrm{BRD}(\widehat{B},\widehat{T}) \xrightarrow{\sim} \mathrm{BRD}(B,T)^\vee$. We fix a pinning $(\widehat{B},\widehat{T},\widehat{\mathbb{X}}_+)$. The action of Γ on $\mathrm{BRD}(B,T)$ translates to an action on $\mathrm{BRD}(\widehat{B},\widehat{T})$, and the latter lifts to a unique action of Γ on $\widehat{G}$ via algebraic automorphisms that preserve $(\widehat{B},\widehat{T},\widehat{\mathbb{X}}_+)$.

We define

$$\widehat{H} := \widehat{G}^{\Gamma_0,0}, \tag{A.2.1}$$

namely the identity component of the Γ_0-fixed points of $\widehat{G}$. This construction was also considered by Zhu [46] and Haines [14]. By [14, Proposition 5.1], $\widehat{H}$ is a reductive subgroup of $\widehat{G}$, and it has a pinning of the form $(\widehat{B}^{\Gamma_0,0},\widehat{T}^{\Gamma_0,0},\widehat{\mathbb{X}}'_+)$. Moreover, the induced action of the Frobenius $\sigma \in \Gamma/\Gamma_0$ on $\widehat{H}$ preserves this pinning. We write $\widehat{B}_H := \widehat{B}^{\Gamma_0,0}$ and $\widehat{T}_H := \widehat{T}^{\Gamma_0,0}$. Let $\hat{\theta}$ denote the automorphism of $\widehat{H}$ given by σ. We define

$$\widehat{\mathcal{S}} := (\widehat{T}_H)^{\hat{\theta},0}.$$

Note that since G is adjoint, the fundamental coweights of G form a Γ-stable $\mathbb{Z}$-basis of $X_*(T)$. It then follows from Lemma 1.6.1 (2) that $X_*(T)_{\Gamma_0}$ and $X_*(T)_\Gamma$ are both free. Hence we in fact have $\widehat{T}_H = \widehat{T}^{\Gamma_0}$ and $\widehat{\mathcal{S}} = \widehat{T}^{\Gamma}$. This observation will simplify our exposition.

Lemma A.2.1. *Let $b \in G(L)$. There is a unique element $\lambda_b \in X^*(\widehat{\mathcal{S}})$ satisfying the following conditions:*

1. *The image of λ_b under $X^*(\widehat{\mathcal{S}}) = X_*(T)_\Gamma \to \pi_1(G)_\Gamma$ is equal to $\kappa(b)$.*
2. *In $X^*(\widehat{\mathcal{S}})_{\mathbb{Q}} = X^*(\widehat{T})_\Gamma \otimes \mathbb{Q} = (X_*(T)_{\Gamma_0})_\sigma \otimes \mathbb{Q} \cong (X_*(T)_{\Gamma_0,\mathbb{Q}})^\sigma$, the element $\lambda_b - \bar{\nu}_b$ is equal to a linear combination of the restrictions to $\widehat{\mathcal{S}}$ of the simple roots in $\Phi^\vee \subset X^*(\widehat{T})$, with coefficients in $\mathbb{Q} \cap (-1,0]$.*

Proof. The proof is the same as Lemma 2.6.3. □

A.3. The main result

Assume that G is adjoint and quasi-split over F. Let $\underline{\mu} \in X_*(T)_{\Gamma_0}$ be the image of an element $\mu \in X_*(T)^+$. Let $b \in G(L)$. Define $\widehat{H}$ as in (A.2.1). Let $V_{\underline{\mu}}^{\widehat{H}}$ denote the highest weight representation of $\widehat{H}$ of highest weight $\underline{\mu} \in X^*(\widehat{T}_H)^+$. Let $\lambda_b \in X^*(\widehat{\mathcal{S}})$ be as in Lemma A.2.1, and let $V_{\underline{\mu}}^{\widehat{H}}(\lambda_b)_{\mathrm{rel}}$ be the λ_b-weight space in $V_{\underline{\mu}}^{\widehat{H}}$, for the action of $\widehat{\mathcal{S}}$.

Theorem A.3.1. *Assume that* $[b] \in B(G, \underline{\mu})$. *Then*

$$\mathscr{N}(\underline{\mu}, b) = \dim V_{\underline{\mu}}^{\widehat{H}}(\lambda_b)_{\mathrm{rel}}.$$

Remark A.3.2. The appearance of the representation $V_{\underline{\mu}}^{\widehat{H}}$ of the subgroup $\widehat{H}$ of $\widehat{G}$ in Theorem A.3.1 is compatible with the ramified geometric Satake in [46].

Proof of Theorem A.3.1. The idea of the proof is to reduce to the unramified case. For this we first construct an auxiliary unramified reductive group over F.

From $\widehat{H}$ and its pinned automorphism $\hat{\theta}$, we obtain an unramified reductive group H over F, whose dual group is $\widehat{H}$. By definition H is equipped with a Borel pair (B_H, T_H), and a σ-equivariant isomorphism of based root data $\mathrm{BDR}(B_H, T_H) \xrightarrow{\sim} \mathrm{BDR}(\widehat{B}_H, \widehat{T}_H)^\vee$. We write

$$\mathrm{BDR}(B_H, T_H) = (X^*(T_H), \Phi_H, X_*(T_H), \Phi_H^\vee).$$

Then we have canonical σ-equivariant identifications

$$X^*(T_H) \cong X_*(\widehat{T}_H) \cong X_*(\widehat{T})^{\Gamma_0} \cong X^*(T)^{\Gamma_0}$$

and

$$X_*(T_H) \cong X^*(\widehat{T}_H) \cong X^*(\widehat{T})_{\Gamma_0} \cong X_*(T)_{\Gamma_0},$$

which we shall treat as identities. Here $X_*(T)_{\Gamma_0}$ is free, as we have noted before.

Note that $T_{H,L}$ is a maximal split torus in H_L. Let V_H be the corresponding apartment, and fix a hyperspecial vertex $\mathfrak{s}_H$ in V_H (coming from the apartment of H corresponding to the maximal F-split sub-torus of T_H). We fix a σ-stable alcove $\mathfrak{a}_H \subset V_H$ whose closure contains $\mathfrak{s}_H$. We identify

$$V_H \cong X_*(T_H) \otimes \mathbb{R}, \tag{A.3.1}$$

sending $\mathfrak{s}_H$ to 0, such that the image of $\mathfrak{a}_H$ is in the anti-dominant chamber.

Since $X_*(T_H) = X_*(T)_{\Gamma_0}$, the two identifications (A.1.1) and (A.3.1) give rise to a σ-equivariant identification $V \cong V_H$ which maps $\mathfrak{a}$ onto $\mathfrak{a}_H$, and maps $\mathfrak{s}$ to $\mathfrak{s}_H$.

By [14, Corollary 5.3], the set of coroots $\Phi_H^\vee \subset X_*(T_H) = X_*(T)_{\Gamma_0}$ is given by $\breve{\Sigma}^\vee$, where $\breve{\Sigma}$ is the échelonnage root system of Bruhat–Tits, see [14, §4.3]. In particular, the coroot lattice in $X_*(T_H)$ is isomorphic to the

Γ_0-coinvariants of the coroot lattice in $X_*(T)$. Moreover from $\Phi_H^\vee = \breve{\Sigma}^\vee$ we know that the affine Weyl group of G and the affine Weyl group of H are equal, under the identification $V \cong V_H$. See [12] for more details. Since the translation groups $X_*(T_H)$ and $X_*(T)_{\Gamma_0}$ are also identified, we have an identification between the Iwahori–Weyl group W of G and the Iwahori–Weyl group W_H of H. This identification is σ-equivariant.

Note that the bottom group in the diagram (A.1.2) and its analogue for H are identified. Using the identification of W and W_H, and using the surjectivity of the map $\Psi : B(W, \sigma) \to B(G)$ and its analogue $\Psi_H : B(W_H, \sigma) \to B(H)$, we construct $[b_H] \in B(H)$ whose invariants are the same as those of $[b]$. Since the set $B(G, \underline{\mu})$ is defined in terms of the invariants $(\bar{\nu}, \kappa)$ and ditto for $B(H, \underline{\mu})$, we see that $[b] \in B(G, \underline{\mu})$ if and only if $[b_H] \in B(H, \underline{\mu})$. Here in writing $B(H, \underline{\mu})$ we view $\underline{\mu}$ as an element of $X_*(T_H)^+$.

To relate the geometry of $X_{\underline{\mu},K}(b)$ with the geometry of $X_{\underline{\mu}}(b_H)$, we use the class polynomials in [18]. For each $w \in W$ and each σ-conjugacy class $\mathcal{O}$ in W, we let

$$f_{w,\mathcal{O}} \in \mathbb{Z}[v - v^{-1}]$$

denote the class polynomial defined in [18, §2.3].

Using the fibration

$$\bigcup_{w \in W_0 t^{\underline{\mu}} W_0} X_w(b) \longrightarrow X_{\underline{\mu},K}(b),$$

we have an identification

$$J_b(F)\backslash\Sigma^{\mathrm{top}}\left(\bigcup_{w \in W_0 t^{\underline{\mu}} W_0} X_w(b)\right) \cong J_b(F)\backslash\Sigma^{\mathrm{top}}(X_{\underline{\mu},K}(b)). \tag{A.3.2}$$

By [18, Theorem 6.1], we have the formula

$$\dim X_w(b) = \max_{\mathcal{O}} \frac{1}{2}(\ell(w) + \ell(\mathcal{O}) + \deg f_{w,\mathcal{O}})) - \langle \overline{\nu}_b, 2\rho \rangle.$$

where $\mathcal{O}$ runs through σ-conjugacy classes in W such that $(\overline{\nu}, \kappa)(\mathcal{O}) = (\overline{\nu}, \kappa)(b)$ and where $\ell(\mathcal{O})$ denotes the length of a minimal length element in $\mathcal{O}$. Moreover the proof of [18, Theorem 6.1] also shows that the cardinality of $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_w(b))$ is equal to the leading coefficient of $\sum_{\mathcal{O}} v^{\ell(w)+\ell(\mathcal{O})} f_{w,\mathcal{O}}$. Since each $X_w(b)$ is locally closed in the union $\bigcup_{w \in W_0 t^{\underline{\mu}} W_0} X_w(b)$, any top dimensional irreducible component in the union is the closure of a top dimensional irreducible component in $X_w(b)$ for a unique w. It follows that

the cardinality of

$$J_b(F)\backslash\Sigma^{\mathrm{top}}\left(\bigcup_{w\in W_0t^{\underline{\mu}}W_0} X_w(b)\right)$$

is equal to the leading coefficient of

$$\sum_{w\in W_0t^{\underline{\mu}}W_0}\sum_{\mathcal{O}} v^{l(w)+l(\mathcal{O})}f_{w,\mathcal{O}}. \tag{A.3.3}$$

By (A.3.2), this number is just $\mathscr{N}(\underline{\mu}, b)$. Since the term (A.3.3) only depends on the quadruple $(W, \sigma, \underline{\mu}, (\overline{\nu}, \kappa)(b))$, the same is true for $\mathscr{N}(\underline{\mu}, b)$.

Applying the same argument to H, we see that $\mathscr{N}(\underline{\mu}, b_H)$ only depends on the quadruple $(W_H, \sigma, \underline{\mu}, (\overline{\nu}, \kappa)(b_H))$. Now since the quadruples $(W, \sigma, \underline{\mu}, (\overline{\nu}, \kappa)(b))$ and $(W_H, \sigma, \underline{\mu}, (\overline{\nu}, \kappa)(b_H))$ are identified, we have $\mathscr{N}(\underline{\mu}, b) = \mathscr{N}(\underline{\mu}, b_H)$. It thus remains to check

$$\mathscr{N}(\underline{\mu}, b_H) = \dim V_{\underline{\mu}}^{\widehat{H}}(\lambda_b)_{\mathrm{rel}}. \tag{A.3.4}$$

By assumption $[b] \in B(G, \underline{\mu})$, and so $[b_H] \in B(H, \underline{\mu})$. The right hand side of (A.3.4) is easily seen to be equal to $\mathscr{M}(\underline{\mu}, b_H)$ in Conjecture 2.6.7, for the group H. Hence the desired (A.3.4) follows from the main result Corollary 6.3.5 of the paper. □

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