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Constructing thin subgroups of $\mathrm{SL}(n+1,\mathbb{R})$ via bending

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We use techniques from convex projective geometry to produce many new examples of thin subgroups of lattices in special linear groups that are isomorphic to the fundamental groups of finite-volume hyperbolic manifolds. More specifically, we show that for a large class of arithmetic lattices in $\mathrm{SO}(n,1)$ it is possible to find infinitely many noncommensurable lattices in $\mathrm{SL}(n+1,\mathbb{R})$ that contain a thin subgroup isomorphic to a finite-index subgroup of the original arithmetic lattice. This class of arithmetic lattices includes all noncocompact arithmetic lattices as well as all cocompact arithmetic lattices when n is even.

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Let G be a semisimple Lie group and let $\Gamma \subset G$ be a lattice. A subgroup $\Delta \subset \Gamma$ is called a *thin group* if Δ has infinite index in Γ and is Zariski dense in G. Over the last several years, there has been a great deal of interest in thin subgroups of lattices in a variety of Lie groups; see Fuchs, Meiri and Sarnak [13; 27; 12]. Much of this interest has been motivated by work of Bourgain, Gamburd and Sarnak [9] related to expanders and "affine sieves". More generally, there is an increasingly strong sense that thin groups have many properties in common with lattices in G.

Furthermore, there is evidence that suggests that generic discrete subgroups of lattices are thin and free (see Fuchs and Rivin [12; 14]). However, there is also great interest in constructing thin groups that are not free (or even decomposable as free products). For instance, the seminal work of Kahn and Markovic [17] constructs many thin subgroups contained in any cocompact lattice of $\mathrm{SL}(2,\mathbb{C})$ that are isomorphic to the fundamental group of a closed surface. There are several generalizations of this result that exhibit thin surface groups in a variety of Lie groups. For instance, Cooper and Futer [10], and independently Kahn and Wright [18], recently proved a similar result for noncompact lattices in $\mathrm{SL}(2,\mathbb{C})$ and Kahn, Labourie and Mozes [16] proved an analogue for cocompact lattices in a large class of Lie groups.

These results naturally lead to the question of which isomorphism types of groups can occur as thin groups. Here we provide a partial answer by showing that in each

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dimension there are infinitely many finite-volume hyperbolic manifolds whose fundamental groups arise as thin subgroups of lattices in special linear groups. Our main result is:

Theorem 0.1 *If* Γ *is a cocompact* (*resp. noncocompact*) *arithmetic lattice in* $\mathrm{SO}(n,1)$ *of orthogonal type then there are infinitely many noncommensurable cocompact* (*resp. noncocompact*) *lattices in* $\mathrm{SL}(n+1,\mathbb{R})$ *that each contain a thin subgroup isomorphic to a finite-index subgroup of* Γ*.*

The definition of an arithmetic lattice of orthogonal type is given in Section 2.1. It turns out that all noncocompact arithmetic lattices in $\mathrm{SO}(n,1)$ are of orthogonal type (see the introduction of Li and Millson [20] and Morris [26, Section 6.4]), and so we have the following immediate corollary of Theorem 0.1:

Corollary 0.2 *If* Γ *is a noncocompact arithmetic lattice in* $\mathrm{SO}(n,1)$ *then there are infinitely many noncocompact lattices in* $\mathrm{SL}(n+1,\mathbb{R})$ *that contain a thin subgroup isomorphic to a finite-index subgroup of* Γ*.*

In the cocompact setting, there is another construction of arithmetic lattices in $\mathrm{SO}(n,1)$ using quaternion algebras. However, this construction only works when n is odd (again, see [20; 26, Section 6.4]), which implies:

Corollary 0.3 *If* $n \geq 3$ *is even and* Γ *is a cocompact arithmetic lattice in* $\mathrm{SO}(n,1)$ *then there are infinitely many cocompact lattices in* $\mathrm{SL}(n+1,\mathbb{R})$ *that contain a thin subgroup isomorphic to a finite-index subgroup of* Γ

Our main result generalizes several previous results regarding the existence of thin groups isomorphic to hyperbolic manifolds in low dimensions. For example, there are examples of thin surface groups in both cocompact and noncocompact lattices in $\mathrm{SL}(3,\mathbb{R})$; see Long and Reid [23; 21]. There are further examples of thin subgroups in $\mathrm{SL}(4,\mathbb{R})$ isomorphic to the fundamental groups of closed hyperbolic 3–manifolds [22] and others isomorphic to the fundamental groups of finite-volume hyperbolic 3–manifolds; see Ballas and Long [3].

Organization of the paper Section 1 provides the necessary background in convex projective geometry. Section 2 describes the relevant arithmetic lattices in both $\mathrm{SO}(n,1)$ and $\mathrm{SL}(n+1,\mathbb{R})$. Section 3 contains the construction of the thin groups in Theorem 0.1. Finally, Section 4 contains the proof that the examples constructed in Section 3 are thin.

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1 Convex projective geometry

Let $V=\mathbb{R}^{n+1}$. There is an equivalence relation on the nonzero vectors in V given by $x\sim y$ if there is $\lambda>0$ such that $\lambda x=y$. The set $S(V)$ of equivalence classes of $\sim$ is called the *projective n–sphere*. Alternatively, $S(V)$ can be regarded as the set of rays through the origin in V. Sending each equivalence class to the unique representative of length 1 gives an embedding of $S(V)$ into V as the unit n–sphere.

The group $\mathrm{GL}(V)$ acts on $S(V)$, however this action is not faithful. The kernel of this action consists of positive scalar multiples of the identity, $\mathbb{R}^+I$. Furthermore, if $A\in\mathrm{GL}(V)$ then $|\det(A)|^{-1/(n+1)}A$ has determinant ±1 and as a result we see that there is a faithful action of

$$\mathrm{SL}^{\pm}(V)=\{A\in\mathrm{GL}(V)\mid\det(A)=\pm1\}$$

on $S(V)$.

The projective sphere is a 2–fold cover of the more familiar *projective space* $P(V)$ consisting of lines through the origin in V. The covering map is given by mapping a ray through the origin to the line through the origin that contains it. There is also a 2–fold covering of Lie groups from $\mathrm{SL}^{\pm}(V)$ to $\mathrm{PGL}(V)$ that maps an element of $\mathrm{SL}^{\pm}(V)$ to its scalar class. Note that here the cover $\mathrm{SL}^{\pm}(V)$ is not connected.

Each (open) hemisphere in $S(V)$ can be identified with $\mathbb{R}^n$ via projection, in such a way that great circles on $S(V)$ are mapped to straight lines in $\mathbb{R}^n$ (see Figure 1). For this reason we refer to (open) hemispheres as *affine patches* of $S(V)$ and refer to great circles as *projective lines*. This identification allows us to define a notion of convexity for subsets of an affine patch. A set $\Omega\subset S(V)$ with nonempty interior is called *properly convex* if its closure is a convex subset of some affine patch. If in addition $\partial\Omega$ contains no nontrivial line segments then Ω is called *strictly convex*. Since Ω is convex, each point $p\in\partial\Omega$ is contained in a hyperplane disjoint from the interior of Ω. If this hyperplane is unique then p is called a C^1 *point of* $\partial\Omega$.

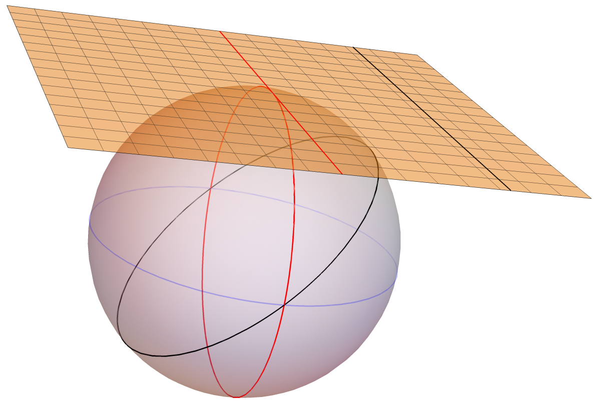

Figure 1: The projection to an affine patch.

Each properly convex set Ω comes equipped with a group

$$\mathrm{SL}(\Omega) = \{A \in \mathrm{SL}^{\pm}(V) \mid A(\Omega) = \Omega\}.$$

In other words, $\mathrm{SL}(\Omega)$ consists of elements of $\mathrm{SL}^{\pm}(V)$ that preserve Ω. There is a similar definition for properly convex subsets of $\mathbb{RP}^n$ and we will allow ourselves to discuss properly convex geometry in whichever setting is more convenient.

Properly convex sets also come equipped with an $\mathrm{SL}(\Omega)$–invariant metric, called the *Hilbert metric*. If $x, y \in \Omega$ then the projective line between x and y intersects $\partial\Omega$ in two points a and b (where a is chosen to be the one closer to x). In this context we define the Hilbert distance between x and y to be

$$d_\Omega(x, y) = \tfrac{1}{2}\log([a:x:y:b]),$$

where $[a:x:y:b] = |b-x||y-a|/|x-a||b-y|$ is the cross-ratio corresponding to the projective coordinate of y in the coordinate system that takes a, x and b to 0, 1 and ∞, respectively. Since projective transformations preserve cross-ratios, it follows that elements of $\mathrm{SL}(\Omega)$ are d_Ω–isometries. The presence of this metric ensures that discrete subgroups of $\mathrm{SL}(\Omega)$ act properly discontinuously on Ω.

To each properly convex $\Omega \subset S(V)$ it is possible to construct a *dual convex set* $\Omega^* \subset S(V^*)$ defined by

$$\Omega^* = \{[\phi] \in S(V^*) \mid \phi(v) > 0 \text{ for all } [v] \in \overline{\Omega}\}.$$

It is a standard fact that Ω^* is a properly convex subset of $S(V)$. For each $\gamma\in\mathrm{SL}(\Omega)$ there is a corresponding $\gamma^*\in\mathrm{SL}(\Omega^*)$ given by $\gamma^*([\phi])=[\phi\circ\gamma^{-1}]$. This map induces an isomorphism between $\mathrm{SL}(\Omega)$ and $\mathrm{SL}(\Omega^*)$. By choosing a basis for V and the corresponding dual basis for V^*, it is possible to identify $\mathrm{SL}(V^*)$ and $\mathrm{SL}(V)$, and in these coordinates the isomorphism between $\mathrm{SL}(\Omega)$ and $\mathrm{SL}(\Omega^*)$ is given by $\gamma\mapsto(\gamma^{-1})^t$.

If Ω is properly convex and $\Gamma\subset\mathrm{SL}(\Omega)$ is discrete then Ω/Γ is a *properly convex orbifold*. If Γ is torsion-free then this orbifold is a manifold. By Selberg's lemma, every properly convex orbifold is finitely covered by a properly convex manifold, and for the remainder of the paper we will almost exclusively be dealing with manifolds. Furthermore, if Ω/Γ is a properly convex manifold then there is a corresponding *dual group* $\Gamma^*\subset\mathrm{SL}(\Omega^*)$ and a corresponding *dual properly convex manifold* Ω^*/Γ^*. The manifolds Ω/Γ are diffeomorphic, but are in general not projectively equivalent.

An important example of a properly convex set is *hyperbolic n–space*, which can be constructed as follows. Let q be the quadratic form on V given by the matrix

$$J_n=\begin{pmatrix}I_n & 0\\ 0 & -1\end{pmatrix}. \tag{1}$$

This form has signature $(n,1)$, and let $\mathcal{C}_q$ be a component of the cone $\{v\in V\mid q(v)<0\}$. The image of $\mathcal{C}_q$ in $S(V)$ gives a model of hyperbolic space, called the *Klein model* of hyperbolic space, which we denote by $\mathbb{H}^n$. In this setting, d_Ω is the standard hyperbolic metric and $\mathrm{SL}(\mathbb{H}^n)$ is equal to the group $O(J_n)^+$ of elements of $\mathrm{SL}^{\pm}(V)$ that preserve both J_n and $\mathcal{C}_q$. When $\Omega=\mathbb{H}^n$ and $\Gamma\subset\mathrm{SL}(\mathbb{H}^n)$ is a discrete, torsion-free group then Ω/Γ is a *complete hyperbolic manifold*. It is a standard fact that if Ω/Γ is a complete hyperbolic manifold then the dual properly convex manifold, Ω^*/Γ^*, is projectively equivalent to Ω/Γ, with the projective equivalence being induced by the map from V to V^* induced by q.

If N is an orientable manifold then a *properly convex structure* on N is a pair $(\Omega/\Gamma,f)$ where Ω/Γ is a properly convex manifold and $f\colon N\to\Omega/\Gamma$ is a diffeomorphism. The map f induces an isomorphism $f_*\colon\pi_1N\to\Gamma$. Since $\Gamma\subset\mathrm{SL}^{\pm}(V)$, we can regard f_* as a representation from π_1N into the Lie group $\mathrm{SL}^{\pm}(V)$, which we call the *holonomy* of the structure $(\Omega/\Gamma,f)$. Since N is orientable, it is easy to show that the holonomy always has image in $\mathrm{SL}(V)$. Observe that, by definition, the holonomy is an isomorphism between π_1N and Γ, and it follows immediately that the holonomy representation is injective.

Given a properly convex structure $(\Omega/\Gamma, f)$ on N and an element $g \in \mathrm{SL}^{\pm}(V)$ it is easy to check that $g\colon \Omega \to g(\Omega)$ induces a diffeomorphism $\overline{g}\colon \Omega/\Gamma \to g(\Omega)/g\Gamma g^{-1}$ and that $(g(\Omega)/g\Gamma g^{-1}, \overline{g} \circ f)$ is also a properly convex structure on N. Furthermore, the holonomy of this new structure is obtained by postcomposing f_* with conjugation in $\mathrm{SL}^{\pm}(V)$ by g. Two properly convex structures $(\Omega/\Gamma, f)$ and $(\Omega'/\Gamma', f')$ on N are *equivalent* if there is $g \in \mathrm{SL}^{\pm}(V)$ such that $\Omega'/\Gamma' = g(\Omega)/g\Gamma g^{-1}$ and f' is isotopic to $\overline{g} \circ f$.

1.1 Generalized cusps

A generalized cusp is a certain type of properly convex manifold that generalizes a cusp in a finite-volume hyperbolic manifold. Specifically, a properly convex n–manifold $C \cong \Omega/\Gamma$ is a *generalized cusp* if Γ is a virtually abelian and $C \cong \partial C \times (0, \infty)$ with ∂C a compact strictly convex submanifold of C. In this context, ∂C being strictly convex means that for each $p \in \partial C$ there is a projective hyperplane H_p and a neighborhood U_p of p such that $H_p \cap U_p = \{p\}$. Such manifolds were recently classified by the first author, D Cooper and A Leitner [2]. One consequence of this classification is that for n–dimensional projective manifolds there are $n+1$ different *types* of generalized cusps. For the purposes of this work only two of these types (type 0 and type 1) will arise. We will also restrict to cusps with the property that ∂C is diffeomorphic to an $(n-1)$–torus. Such cusps will be called *torus cusps* and we now briefly describe these types of cusps.

Let

$$\Omega_0 = \bigl\{[x_1 : \cdots : x_{n+1}] \in P(V) \mid x_1 x_{n+1} > \tfrac{1}{2}(x_2^2 + \cdots + x_n^2)\bigr\}.$$

It is not difficult to see that Ω_0 is projectively equivalent to the Klein model for hyperbolic space. Let P_0 be the collection (of equivalence classes) of matrices with block form

$$\begin{pmatrix} 1 & v & \frac{1}{2}|v|^2 \\ 0 & I_{n-1} & v^t \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where v is a (row) vector in $\mathbb{R}^{n-1}$, I_{n-1} is the identity matrix and the zeros are blocks of the appropriate size to make (2) an $(n+1) \times (n+1)$ matrix. A simple computation shows that the elements of P_0 preserve Ω_0 (they are just the parabolic isometries of $\mathbb{H}^n$ that fix $\infty = [1 : 0 : \cdots : 0]$). There is a foliation of Ω_0 by strictly convex

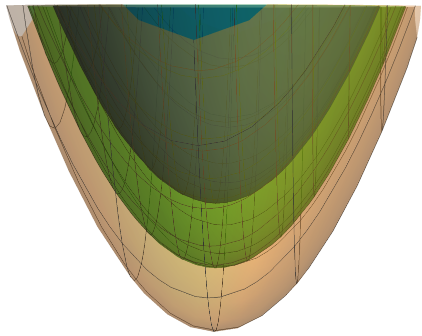

Figure 2: The domain Ω_0 and its foliation by horospheres.

hypersurfaces of the form

$$\mathcal{H}_c = \left\{[x_1 : \cdots : x_n : 1] \mid x_1 - \tfrac{1}{2}(x_2^2 + \cdots + x_n^2) = c\right\}$$

for $c > 0$ whose leaves are preserved setwise by P_0. In terms of hyperbolic geometry, the $\mathcal{H}_c$ are *horospheres* centered at ∞ and the convex hull of a leaf is a *horoball* centered at ∞. The group P_0 is isomorphic to $\mathbb{R}^{n-1}$ and so if $\Gamma \subset P_0$ is a lattice then Γ is isomorphic to $\mathbb{Z}^{n-1}$ and the quotient Ω/Γ is a *generalized* (*torus*) *cusp of type* 0.

Next, let

$$\Omega_1 = \left\{[x_1 : \cdots : x_{n+1}] \mid x_1 x_{n+1} > -\log|x_2| + \tfrac{1}{2}(x_3^2 + \cdots + x_n^2),\ x_2 x_{n+1} > 0\right\}$$

and let P_1 be the collection (of equivalence classes) of matrices of block form

$$\begin{pmatrix} 1 & 0 & v & -u + \frac{1}{2}|v|^2 \\ 0 & e^u & 0 & 0 \\ 0 & 0 & I_{n-2} & v^t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

where $u \in \mathbb{R}$, $v \in \mathbb{R}^{n-2}$, I_{n-2} is the identity matrix and the zeros are the appropriate size to make (3) an $(n+1) \times (n+1)$ matrix. Again, it is easy to check that P_1 preserves Ω_1. Elements of P_1 for which $u = 0$ are called *parabolic* and every parabolic element preserves each copy of $\mathbb{H}^{n-1}$ obtained by intersecting Ω_1 and the plane $x_2 = d$ with $d > 0$. The domain Ω_1 contains a unique line segment ℓ_∞ with endpoints q_+ and q_- in its boundary. In the coordinates we have chosen, $q_+ = [e_1]$ and $q_- = [e_2]$. These points can distinguished by the fact that q_- is a C^1 point and q_+ is not. The group P_1 preserves ℓ_∞ and the parabolic elements fix ℓ_∞ pointwise.

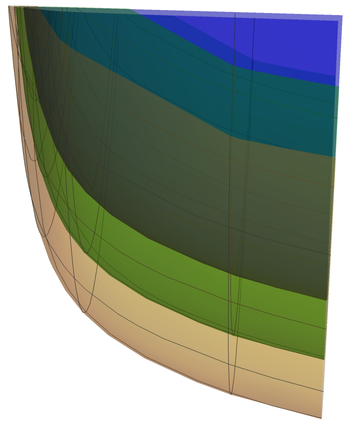

Figure 3: The domain Ω_1 and its foliation by horospheres.

Again, there is a foliation of Ω_1 by strictly convex hypersurfaces of the form

$$\mathcal{H}_c = \left\{[x_1 : \cdots : x_n : 1] \mid x_1 + \log x_2 - \tfrac{1}{2}(x_3^2 + \cdots + x_n^2) = c,\ x_2 > 0\right\}$$

for $c > 0$ that is preserved by P_1. Again, each leaf is a P_1 orbit; we call the leaves of this foliation *horospheres* and call the convex hulls of a leaves *horoballs*. Again $P_1 \cong \mathbb{R}^{n-1}$ and if $\Gamma \subset P_1$ is a lattice then $\Gamma \cong \mathbb{Z}^{n-1}$ and Ω_1/Γ is a *generalized* (*torus*) *cusp of type* 1. For the remainder of this paper, when we say generalized cusp, that will mean a generalized torus cusp of type 0 or type 1.

Generalized cusps of a fixed type are closed under two important operations: taking finite-sheeted covers and duality. If Ω/Γ is a generalized cusp then taking a finite-sheeted cover corresponds to choosing a finite-index subgroup $\Gamma' \subset \Gamma$. The group Γ' is also a lattice in P_0 or P_1 and hence Ω/Γ' is a generalized cusp. The fact that generalized cusps are closed under duality follows immediately from the observation that the group P_0^t (resp. P_1^t) obtained by taking the transpose of the elements of P_0 (resp. P_1) is conjugate to P_0 (resp. P_1).

One distinction between these two types of cusps that will be important for our purposes in Section 4 is that the group P_0 is Zariski closed, but the group P_1 is not. The Zariski closure $\overline{P}_1$ of P_1 is n–dimensional and consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & v & w \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{n-2} & v^t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

where $u \neq 0$, $w \in \mathbb{R}$ and $v \in \mathbb{R}^{n-2}$. Furthermore, we have the following lemma describing the generic orbits of $\bar{P}_1$, whose proof is a straightforward computation.

Lemma 1.1 *If* $x \notin \ker(e_2^*) \cup \ker(e_{n+1}^*)$ *then* $\bar{P}_1 \cdot x$ *is open in* $\mathbb{RP}^n$.

1.2 Bending

We now describe a construction that allows one to start with a (special) hyperbolic manifold and produce a family of inequivalent convex projective structures.

Suppose that $M = \mathbb{H}^n/\Gamma$ is a complete, finite-volume hyperbolic manifold, and suppose that M contains an embedded totally geodesic hypersurface, Σ. There is an embedding of $\mathrm{SO}(J_{n-1})$ into $\mathrm{SO}(J_n)$ via the embedding

$$\mathrm{SO}(J_{n-1}) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(J_{n-1}) \end{pmatrix}.$$

Under this embedding, the image of $\mathrm{SO}(J_{n-1})$ stabilizes a copy of $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$ and $\Sigma \cong \mathbb{H}^{n-1}/\Lambda$, where Λ is a subgroup of $\mathrm{SO}(J_{n-1}) \cap \Gamma$. For each $t \in \mathbb{R}$, the element

$$B_t = \begin{pmatrix} e^{-nt} & \\ & e^t I_n \end{pmatrix}$$

centralizes $\mathrm{SO}(J_{n-1})$ and hence centralizes Λ.

Let $N = M$ and let id: $N \to M$ be the identity; then (M, id) is a convex projective structure on N. Let $\rho\colon \pi_1 N \to \mathrm{SL}(V)$ be the holonomy of this structure. Concretely, ρ is just the inclusion of $\pi_1 N \cong \Gamma$ into $\mathrm{SL}(V)$. We now define a family, $\rho_t\colon \pi_1 N \to \mathrm{SL}(V)$, of representations such that $\rho_0 = \rho$. The construction depends on whether or not Σ is separating.

If Σ is separating then Γ splits as an amalgamated product $\Gamma_1 *_\Lambda \Gamma_2$, where the Γ_i are the fundamental groups of the components of $M \setminus \Sigma$. Then ρ_t is defined by the property that $\rho_t(\gamma) = \rho(\gamma)$ if $\gamma \in \Gamma_1$ and $\rho_t(\gamma) = B_t \rho_0(\gamma) B_t^{-1}$ if $\gamma \in \Gamma_2$. Since B_t centralizes Λ, this gives a well-defined representation $\rho_t\colon \pi_1 N \to \mathrm{SL}(V)$.

In the separating case, $\Gamma = \Gamma' *_s$ is an HNN extension, where Γ' is the fundamental group of $M \setminus \Sigma$. In this case ρ_t is defined by the property that $\rho_t(\gamma) = \rho(\gamma)$ if $\gamma \in \Gamma'$ and $\rho_t(s) = B_t \rho(s)$. Again it is easy to see that, since B_t centralizes Λ, this gives a well-defined representation $\rho_t\colon \pi_1 N \to \mathrm{SL}(V)$.

In either case we say that the family of ρ_t is *obtained by bending* M *along* Σ. From the construction, it is not obvious that the representations ρ_t are the holonomy of a convex projective structure. However, the following theorem guarantees that this is the case:

Theorem 1.2 (see [19; 24]) *For each $t \in \mathbb{R}$ the representation ρ_t obtained by bending M along Σ is the holonomy of a properly convex projective structure on N.*

Remark 1.3 The property of being obtained from bending is closed under two important operations: taking finite-sheeted covers and duality. First, if $M = \Omega/\Gamma$ and M' is a finite-sheeted cover of M then M' is of the form Ω/Γ', where Γ' is a finite-index subgroup of Γ. If M is obtained by bending a finite-volume hyperbolic manifold N along an embedded totally geodesic hypersurface Σ then M' is obtained by simultaneously bending the cover N' of N corresponding to M' along the (possibly disjoint) totally geodesic embedded hypersurface Σ' obtained by taking the complete preimage of Σ in N'.

If $N = \mathbb{H}^n/\Gamma_0$ is a complete hyperbolic manifold containing a totally geodesic hypersurface Σ, then its dual projective manifold, N^*, is projectively equivalent to N, and hence also contains a totally geodesic hypersurface, Σ^*. If $M = \Omega/\Gamma$ is obtained from bending N along Σ, then the dual projective manifold, M^*, is obtained by bending N^* along Σ^*.

The following theorem from [4] addresses which types of cusps arise when one bends a hyperbolic manifold along a totally geodesic hypersurface.

Theorem 1.4 [4, Corollary 5.10] *Let M be a finite-volume hyperbolic manifold and let Σ be an embedded totally geodesic hypersurface. If M' is the properly convex manifold obtained by bending M along Σ then each end of M is a generalized cusp of type 0 or type 1.*

1.3 Properties of the holonomy

In this section we discuss some important properties of the holonomy representation of convex projective structures that arise from bending. A representation $\rho\colon \Gamma \to \mathrm{GL}(V)$ is called *strongly irreducible* if its restriction to any finite-index subgroup is irreducible. The main result of this section is the following:

Theorem 1.5 *Let $(\Omega/\Gamma, f)$ be a convex projective structure on M and let ρ be its holonomy. If Ω/Γ is obtained by bending a finite-volume hyperbolic manifold along an embedded totally geodesic hypersurface then ρ is strongly irreducible.*

Before proceeding with the proof of Theorem 1.5, we need a few lemmas. If P is a subset contained in some affine patch in $S(V)$ then let $\mathcal{CH}(P)$ denote the convex hull of P (note that, since P is contained in an affine patch, this is well defined).

Lemma 1.6 *Suppose that $M = \Omega/\Gamma$ is a properly convex manifold obtained from bending a finite-volume hyperbolic manifold along an embedded totally geodesic hypersurface; then $\mathcal{CH}(\Gamma\cdot p)$ has nonempty interior for any $p\in\overline{\Omega}$.*

Proof If M is closed then the result follows from [28, Proposition 3], and so we assume that M has at least 1 cusp, which by Theorem 1.4 is a generalized cusp of type 0 or type 1. Let Δ be the fundamental group of one of the generalized cusps. By [4, Lemma 5.7] we can find horoballs $\mathcal{H}$ and $\mathcal{H}'$ such that (after conjugating in $\mathrm{SL}(V)$) $\mathcal{H}\subset\Omega\subset\mathcal{H}'$. It follows that there is a unique projective hyperplane L with the property that if $p\in\overline{\Omega}\setminus L$ then $\mathcal{CH}(\Lambda\cdot p)$ contains a horoball. In particular, for such p, $\mathcal{CH}(\Gamma\cdot p)$ has nonempty interior. In the coordinates of the previous section L is the projective hyperplane coming from $\ker(e^*_{n+1})$. Furthermore, $\overline{\Omega}\cap L$ is either the point ∞ if the cusp is type 0 or the line segment ℓ_∞ from the previous section if the cusp is type 1.

In light of this, the proof will be complete if we can show that for each $p\in\overline{\Omega}$ the orbit $\Gamma\cdot p$ contains a point in $\overline{\Omega}\setminus L$. Suppose that $p\in\overline{\Omega}\cap L$. Since M is obtained by bending, it contains a subgroup Λ corresponding to the fundamental group of the totally geodesic hypersurface. This subgroup preserves a copy of $(n-1)$–dimensional hyperbolic space, $H_\Lambda\subset\Omega$, and fixes a unique point $p_\infty\in P(V)$ dual to H_Λ. It follows that if $p\neq p_\infty$ then the Λ orbit of p accumulates to any point in ∂H_Λ. Since the plane L is a supporting plane for Ω and the plane containing H_Λ meets the interior of Ω, it follows that there is a point of $\partial\Omega$ that is not contained in L and hence a $g\in\Lambda$ such that $g\cdot p\notin L$.

This leaves only the case where $p=p_\infty$. In this case let $g\in\Lambda$ be a hyperbolic isometry, let $p_0\in\partial H_\Lambda$ be its repelling fixed point, and let $h\in\Delta$ be parabolic (since Ω/Γ came from bending, such an element is guaranteed to exist). Let ℓ be the projective line connecting p_∞ and p_0. Since $p_\infty\in\ell_\infty$ and h is parabolic it follows that $h\cdot p_\infty=p_\infty$ and so ℓ and $h\cdot\ell$ are contained in a projective 2–plane, L'. Let $\Omega'=\Omega\cap L'$; then p_∞ is a C^1 point of $\partial\Omega'$. To see this, observe that unless the cusp is type 1 and $p_\infty=q_+$, p_∞ is already a C^1 point of $\partial\Omega$, and thus a C^1 point of $\partial\Omega'$. On the other hand, if $p_\infty=q_+$ then p_∞ is not a C^1 point of $\partial\Omega$; however, $\mathcal{H}\cap L'\subset\Omega'\subset\mathcal{H}'\cap L'$. Both

$\mathcal{H} \cap L'$ and $\mathcal{H}' \cap L'$ are projectively equivalent to copies of $\mathbb{H}^2$ and meet at p_∞, and so p_∞ is a C^1 point of $\partial\Omega'$.

Since $p_\infty \in \partial\mathcal{H}$ and $\mathcal{H} \subset \Omega \subset \mathcal{H}'$, it follows that the line ℓ intersects the interior of Ω. Let $x \in \ell \cap \mathrm{int}(\Omega)$ and observe that for each natural number n, $g^n \cdot x \in \ell$ and $hg^n \cdot x \in h \cdot \ell$, and both sequences limit to p_∞ as $n \to \infty$ since p_0 is the repelling fixed point of g. Since p_∞ is a C^1 point of $\partial\Omega'$, it follows from [11, Proposition 3.4(H7)] that $d_n := d_{\Omega'}(g^n \cdot x, hg^n \cdot x) \to 0$ as $n \to \infty$. However, Ω' is a totally geodesic subspace of Ω (with respect to d_Ω), and so this implies that $d_\Omega(g^n \cdot x, hg^n \cdot x) \to 0$ as $n \to \infty$. It follows that $d_\Omega(x, g^{-n}hg^n \cdot x) \to 0$ as $n \to \infty$, but this is a contradiction since the group Γ acts properly discontinuously on Ω. □

The following lemma is the basis for the proof of Theorem 1.5. The lemma and its proof are inspired by a similar result of J Vey [28, Proposition 4].

Lemma 1.7 *Suppose that $\Omega \subset P(V)$ is properly convex and that $\Gamma \subset \mathrm{SL}(\Omega)$ is a group with the property that $\mathcal{CH}(\Gamma \cdot p)$ has nonempty interior for every $p \in \overline{\Omega}$. If L is a Γ–invariant subspace of V and $P(L) \cap \overline{\Omega} \neq \varnothing$ then $L = V$.*

Proof Let $L \subset V$ be a Γ–invariant subspace such that $P(L) \cap \overline{\Omega} \neq \varnothing$, and let p be a point in the intersection. Since $p \in \overline{\Omega}$ it follows that $\mathcal{CH}(\Gamma \cdot p)$ has nonempty interior. Furthermore, since $p \in L$ and L is both Γ–invariant and convex it follows that $\mathcal{CH}(\Gamma \cdot p) \subset P(L)$. Since $\mathcal{CH}(\Gamma \cdot p)$ has nonempty interior, so does $P(L)$. It follows that $L = V$. □

Proof of Theorem 1.5 Suppose that $L \subset V$ is a Γ–invariant subspace. First assume that $P(L) \cap \overline{\Omega} \neq \varnothing$. Combining Lemmas 1.6 and 1.7 it follows that $L = V$. On the other hand, suppose that $L \cap \overline{\Omega} = \varnothing$; then L corresponds to a nontrivial subspace $L^* \subset V^*$ such that $P(L^*) \cap \overline{\Omega^*} \neq \varnothing$. Since Ω/Γ is obtained from bending it follows that Ω^*/Γ^* is also obtained from bending a finite-volume manifold along an embedded totally geodesic hypersurface (see Remark 1.3) and so we can apply the same argument as before to show that $L^* = V^*$. It follows that $L = 0$, and so there is no proper nontrivial Γ–invariant subspace. Hence Γ acts irreducibly on V.

Finally, if Γ' is a finite-index subgroup of Γ then Ω/Γ' is a properly convex manifold that also arises from bending a finite-volume hyperbolic manifold along an embedded totally geodesic hypersurface (again, see Remark 1.3), and so by the argument above Γ' also acts irreducibly on V. □

1.4 Zariski closures and limit sets

We close this section by describing some properties of the Zariski closure of the groups obtained by bending. Before proceeding we introduce some terminology and notation. Let $g \in \mathrm{SL}(V)$; then g is *proximal* if g has a unique (counted with multiplicity) eigenvalue of maximum modulus. It follows that this eigenvalue must be real and that g is proximal if and only if g has a unique attracting fixed point for its action on $P(V)$. If G is a subgroup of $\mathrm{SL}(V)$ then G is *proximal* if it contains a proximal element.

If $G \subset \mathrm{SL}(V)$ is a group then we define the *limit set* of G, denoted by Λ_G, as

$$\Lambda_G = \overline{\{x \in P(v) \mid x \text{ a fixed point of some proximal } g \in G\}}.$$

By construction, this Λ_G is closed and if G is proximal then Λ_G is nonempty. In this generality the limit set was introduced by Goldscheid and Guivarch [15] and this construction reduces to the more familiar notion of limit set when G is a Kleinian group. The limit set has the following important properties:

Theorem 1.8 [15, Theorem 2.3] *If G is proximal and acts irreducibly on V then Λ_G is the unique minimal nonempty closed G–invariant subset of $P(V)$.*

Next, let $M = \mathbb{H}^n/\Gamma$ be a finite-volume (noncompact) hyperbolic manifold containing an embedded totally geodesic hypersurface Σ, let $\Gamma_t = \rho_t(\Gamma)$ be the group obtained by bending M along Σ, and let G_t be the Zariski closure of Γ_t. The following lemma summarizes some properties of G_t and its relation to Λ_G:

Lemma 1.9 *Let ρ_t be obtained by bending M along Σ, let $\Gamma_t = \rho_t(\Gamma)$ and let G_t be the Zariski closure of Γ_t; then:*

- *The identity component, G_t^0, of G_t is semisimple, proximal and acts irreducibly on V.*
- *$\Lambda_{G_t^0} = G_t^0 \cdot x$ for any $x \in \Lambda_{G_t^0}$.*

Proof The group G_t^0 is a finite-index subgroup of G_t and contains the group $G_t^0 \cap \Gamma_t$, which has finite index in Γ_t. By Theorem 1.5 it follows that $G_t^0 \cap \Gamma_t$ and hence G_t^0 acts irreducibly on V, and so V becomes a simple $\mathbb{R}[G_t^0]$–module. Let R_t be the unipotent radical of G_t^0 and let $V_{\mathbb{C}}$ be the complexification of V. Since R_t is unipotent and solvable, the Lie–Kolchin theorem implies that there is a nontrivial $\mathbb{C}[R_t]$–submodule, $E_{\mathbb{C}}$, of $V_{\mathbb{C}}$ consisting of simultaneous 1–eigenvectors of R_t. The submodule $E_{\mathbb{C}}$ is conjugation invariant and so there is a nontrivial $\mathbb{R}[R_t]$–submodule, $E_{\mathbb{R}}$, of V whose complexification is $E_{\mathbb{C}}$. Furthermore, since R_t is normal in G_t^0 it follows that $E_{\mathbb{R}}$

is also an $\mathbb{R}[G_t^0]$–submodule. By simplicity, it follows that $E_{\mathbb{R}}$ equals V, and so R_t acts trivially on V, and is thus trivial. Hence G_t^0 is reductive.

The group $\rho_0(\pi_1\Sigma)$ is easily seen to contain a proximal element and, by construction, $\rho_t(\pi_1\Sigma)=\rho_0(\pi_1\Sigma)$. It follows that Γ_t (and hence G_t^0) contains a proximal element g. Next, suppose that h is an element in the center of G_t^0. The element g has a 1–dimensional real eigenspace $V_g\subset V$. Since h is central it preserves V_g and thus also has a real eigenspace V_h (possibly of dimension larger than 1). However, since h is central, V_h is also G_t^0–invariant, which implies that $V_h=V$ and so h is a scalar matrix. Since $G_t^0\subset \mathrm{SL}(V)$, we must have $h=\pm I$. It follows that the center of G_t^0 is discrete. Since G_t^0 is reductive, its radical is a connected subgroup of its center, and so the radical is actually trivial. Hence G_t^0 is also semisimple.

Next, let $G_t^0=KAN$ be an Iwasawa decomposition of G_t^0. Since G_t^0 is proximal it follows from [1, Theorem 6.3] that N has a unique global fixed point $x_N\in P(V)$, which is a weight vector for the highest weight of G_t^0 with respect to this decomposition. Since A normalizes N it follows that A also preserves x_N, and so $G_t^0\cdot x_N=K\cdot x_N$ is a closed orbit (since K is compact). Furthermore, it is easy to see that $x_N\in\Lambda_{G_t^0}$ and so $G\cdot x_N$ is a closed G_t^0–invariant subset of $\Lambda_{G_t^0}$. Therefore, by Theorem 1.8, $G_t^0\cdot x_N=\Lambda_{G_t^0}$. Finally, an orbit is the orbit of any of its points and so it follows that if $x\in\Lambda_{G_t^0}$ then $\Lambda_{G_t^0}=G_t^0\cdot x$. □

2 Arithmetic lattices

Up until now we have been implicitly working over the real numbers. In this section we will have to work with other fields and rings and we would like this to be explicit in our notation. For this reason, when we discuss groups of matrices we will need to explicitly specify where the entries lie. Henceforth, we will denote $\mathrm{SO}(J_n)$ as $\mathrm{SO}(n,1)$.

Let F be a number field and recall that F is *totally real* if every embedding $\sigma\colon F\to\mathbb{C}$ has the property that $\sigma(F)\subset\mathbb{R}\subset\mathbb{C}$. By choosing one of these embeddings we will regard F as a subfield of $\mathbb{R}$. If $\alpha\neq 0$ is an element of a totally real field then define $s(\alpha)$ to be the number of nonidentity embeddings $\sigma\colon F\to\mathbb{R}$ for which $\sigma(\alpha)>0$.

2.1 Lattices in $\mathrm{SO}(n,1)$

There are multiple constructions that give rise to different classes of arithmetic lattices in $\mathrm{SO}(n,1)$. We now explain the simplest of these constructions and the only one that will be relevant for our purposes.

Let F be a totally real number field, let $\mathcal{O}_F$ be its ring of integers and suppose we have chosen $\alpha_1,\dots,\alpha_n$ to be positive elements of $\mathcal{O}_F$ such that $s(\alpha_i)=0$ (ie the α_i are negative under all other embeddings of F). Let $\vec{\alpha}=(\alpha_1,\dots,\alpha_n)$ and define $J^{\vec{\alpha}}=\operatorname{diag}(\alpha_1,\dots,\alpha_n,-1)$. Next, let $\mathbb{X}\in\{\mathbb{R},F,\mathcal{O}_F\}$ and define the groups $\mathrm{SO}(J^{\vec{\alpha}},\mathbb{X})=\{A\in\mathrm{SL}(n+1,\mathbb{X})\mid A^tJ^{\vec{\alpha}}A=J^{\vec{\alpha}}\}$. It is well known that $\mathrm{SO}(J^{\vec{\alpha}},\mathcal{O}_F)$ is a lattice in $\mathrm{SO}(J^{\vec{\alpha}},\mathbb{R})$ (see [26, Section 6.4], particularly Proposition 6.4.4, for a detailed explanation). Furthermore, the forms $J^{\vec{\alpha}}$ and J_n are $\mathbb{R}$–equivalent and so $\mathrm{SO}(J^{\vec{\alpha}},\mathbb{R})$ and $\mathrm{SO}(n,1)$ are conjugate Lie groups and so $\mathrm{SO}(J^{\vec{\alpha}},\mathcal{O}_F)$ is conjugate to a lattice in $\mathrm{SO}(n,1)$. Hence we can regard $\mathbb{H}^n/\mathrm{SO}(J^{\vec{\alpha}},\mathcal{O}_F)$ as a hyperbolic orbifold. The lattices constructed in this fashion are cocompact if and only if $F\neq\mathbb{Q}$. A lattice in $\mathrm{SO}(n,1)$ that is commensurable with $\mathrm{SO}(J^{\vec{\alpha}},\mathcal{O}_F)$ for some choice of F and $\vec{\alpha}$ is called an *arithmetic lattice of orthogonal type*.

If $\widetilde{\Gamma}=\mathrm{SO}(J^{\vec{\alpha}},\mathcal{O}_F)$ as constructed above, then $O=\mathbb{H}^n/\widetilde{\Gamma}$ will contain several immersed totally geodesic hypersurfaces, and we now describe one of them and show how it can be promoted to an embedded totally geodesic hypersurface with nice intersection properties in a finite-sheeted manifold cover of O. Specifically, let $\vec{\alpha}_1=(\alpha_2,\dots,\alpha_n)$; then $\widetilde{\Gamma}_1=\mathrm{SO}(J^{\vec{\alpha}_1},\mathcal{O}_F)$ embeds reducibly in $\mathrm{SO}(J^{\vec{\alpha}},\mathcal{O}_F)$ via

$$\mathrm{SO}(J^{\vec{\alpha}_1},\mathcal{O}_F)\hookrightarrow\begin{pmatrix}1 & \\ & \mathrm{SO}(J^{\vec{\alpha}_1},\mathcal{O}_F)\end{pmatrix}.$$

Furthermore, $\widetilde{\Gamma}_1$ is (commensurable with) a lattice in $\mathrm{SO}(n-1,1)$. The obvious embedding of $\widetilde{\Gamma}_1$ into $\widetilde{\Gamma}$ induces an immersion of $\mathbb{H}^{n-1}/\widetilde{\Gamma}_1$ in $\mathbb{H}^n/\widetilde{\Gamma}$. By combining results of Bergeron [8] and Selberg's lemma, we can find finite-index subgroups Γ (resp. Γ_1) such that $M=\mathbb{H}^n/\Gamma$ (resp. $M_1=\mathbb{H}^{n-1}/\Gamma_1$) is a manifold and M_1 is an embedded totally geodesic hypersurface in M. Furthermore, if M is noncompact, then by using the argument from [4, Theorem 7.1] it is possible pass to a further finite cover of M where all the cusps are torus cusps and the intersection of M_1 with one of the cusps is connected. Shortly we will bend M along M_1 in order to produce thin subgroups in lattices in $\mathrm{SL}(n+1,\mathbb{R})$.

2.2 Lattices in $\mathrm{SL}(n+1,\mathbb{R})$

Next, we describe the lattices in $\mathrm{SL}(n+1,\mathbb{R})$ in which we will construct thin subgroups. The construction is similar to the one in the previous section, and can be thought of as its "unitary" analogue.

Again, let F be a totally real number field, let $\mathcal{O}_F$ be its ring of integers, and suppose we have chosen $\alpha_1, \dots, \alpha_n$ to be positive elements of $\mathcal{O}_F$ such that $s(\alpha_i) = 0$. Next, let L be a real quadratic extension of F and let $\mathcal{O}_L$ be the ring of integers of this number field. L is a quadratic extension of F and so there is a unique nontrivial Galois automorphism of L over F, which we denote by $\tau\colon L \to L$.

If M is a matrix with entries in L then the *conjugate transpose of* M (*over* L), denoted by M^*, is the matrix obtained by taking the transpose of M and applying τ to its entries. A matrix M is called τ*–Hermitian* if it has entries in L and is equal to its conjugate transpose. Observe that the matrix $J^{\vec{\alpha}}$ is diagonal with entries in F, and so $J^{\vec{\alpha}}$ is τ–Hermitian. Furthermore, it is a standard result (see [26, Section 6.8], for example) that $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau) := \{A \in \mathrm{SL}(n+1, \mathcal{O}_L) \mid A^* J^{\vec{\alpha}} A = J^{\vec{\alpha}}\}$ is an arithmetic lattice in $\mathrm{SL}(n+1, \mathbb{R})$ that is cocompact if and only if $F \neq \mathbb{Q}$.

3 The construction

In this section we describe the construction of the thin groups in Theorem 0.1. Recall that F is a totally real number field and $\alpha_1, \dots, \alpha_n$ are positive elements of F such that $s(\alpha_i) = 0$.

Next, we construct a certain real quadratic extension of L. In order to proceed with the construction, we require the following:

Lemma 3.1 *Let F be any totally real field and $N > 0$; then F contains infinitely many units u with the properties that*:

1. *At the identity embedding of F, $u > N$.*
2. *At all the other embeddings $\sigma\colon F \to \mathbb{R}$ one has $0 < \sigma(u) < 1$.*

Proof Suppose that $[F : \mathbb{Q}] = k + 1$ and let $v_1, \dots, v_k$ be generators of the unit group, $\mathcal{O}_F^\times$, as determined by Dirichlet's unit theorem.

There is an embedding $\sigma\colon F \to \mathbb{R}^{k+1}$ given by $\sigma(x) = (\sigma_1(x), \dots, \sigma_{k+1}(x))$, where the σ_i are all the embeddings of F into $\mathbb{R}$, chosen so that σ_1 is the identity. By replacing each v_i with its square we can suppose that $\sigma(v_i)$ is contained in the positive orthant of $\mathbb{R}^{k+1}$. This will replace $\mathcal{O}_F^\times$ with a subgroup of finite index in $\mathcal{O}_F^\times$.

Taking componentwise logarithms gives a map, log: $\mathbb{R}_+^{k+1} \to \mathbb{R}^{k+1}$, where $\mathbb{R}_+^{k+1}$ is the positive orthant in $\mathbb{R}^{k+1}$. Furthermore, since each v_i is a unit, it follows that

$\log(\sigma(v_i))$ lies in the hyperplane where the sum of the coordinates is equal to zero. Dirichlet's unit theorem implies that the set $B = \{\log(\sigma(v_1)), \dots, \log(\sigma(v_k))\}$ is a basis for this hyperplane, so there is a linear combination of their images which yields the vector $\vec{a} = (1, -1/k, -1/k, \dots, -1/k)$, with respect to the basis B, hence there is a rational linear combination giving a vector very close to $\vec{a}$. By scaling to clear denominators, one obtains an *integer* linear combination with the property that the last k coordinates are negative and the first coordinate is positive. After possibly taking further powers (to arrange $u > N$) and exponentiating one obtains a unit with the required properties. □

Remark 3.2 Once a unit u satisfies the above conditions, so do all its powers.

Next, let u be one of the units guaranteed by Lemma 3.1 for $N > 2$. Note that by construction, $u^2-4>0$ and $\sigma(u^2-4)=\sigma(u)^2-4<0$ for all nonidentity embeddings of F. In particular, this implies that u is not a square. Let s be a root of the polynomial $p_u(x) = x^2 - ux + 1$ and let $L = F(s)$. Note that $u^2 - 4$ is the discriminant of this polynomial. By construction, L is a real quadratic extension of F. Furthermore, since $\sigma(u^2-4) < 0$ for all nonidentity embeddings, L has exactly 2 real places. Let $\tau\colon L \to L$ be the unique nontrivial Galois automorphism of L over F. By construction, $s \in \mathcal{O}_L$ and since $\tau(s)$ is the other root of $p_u(x)$, a simple computation shows that $\tau(s) = 1/s$, and so $s \in \mathcal{O}_L^\times$. With this in mind, we henceforth call elements $u \in L$ such that $\tau(u) = 1/u$ *τ–unitary*, or just *unitary* if τ is clear from context. Note that τ–unitary elements in $\mathcal{O}_L$ are all units.

Every power of s (and indeed $-s$) is also unitary. Furthermore, we note that these are the only possible unitary elements of $\mathcal{O}_L^\times$. The reason is this: notice that the rank of the unit group of $\mathcal{O}_F$ is $[F:\mathbb{Q}]-1$. Also, $F(s)$ has two real embeddings (coming from s and $1/s$) and all the other embeddings lie on the unit circle (in other words, s is a so-called *Salem number*) since we required the other embeddings of u to be less than 2 in absolute value. So, by Dirichlet's theorem, the unit group of $\mathcal{O}_L$ has rank

$$2 + \tfrac{1}{2}(2[F:\mathbb{Q}]-2) - 1 = [F:\mathbb{Q}],$$

which is 1 larger than the rank of $\mathcal{O}_F^\times$. Since τ induces an automorphism of the unit group that fixes $\mathcal{O}_F^\times$, the possibilities for are all accounted for by s and its powers.

From the discussion of the previous section we can find torsion-free subgroups Γ (resp. Γ_1) commensurable with $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ (resp. $\mathrm{SO}(J^{\vec{\alpha}_i}, \mathcal{O}_F)$) such that $M_1 := \mathbb{H}^{n-1}/\Gamma_1$ is an embedded submanifold of $M := \mathbb{H}^n/\Gamma$. As previously mentioned, we can regard (M, id) as a complete hyperbolic (and hence convex projective) structure on M whose holonomy ρ is the inclusion of Γ into $\mathrm{SL}(n+1,\mathbb{R})$. Since M contains an embedded totally geodesic hypersurface, M_1, it is possible to bend M along M_1 to produce a family of representations $\rho_t\colon \Gamma \to \mathrm{SL}(n+1,\mathbb{R})$. We now show that for various special values of the parameter t, the group $\rho_t(\Gamma)$ will be a thin group inside a lattice in $\mathrm{SL}(n+1,\mathbb{R})$. These special values turn out to be logarithms of unitary elements of $\mathcal{O}_L$.

The main goal of the remainder of this section is to prove the following theorem:

Theorem 3.3 *If $u \in \mathcal{O}_L$ is unitary and $t = \log|u|$ then $\rho_t(\Gamma) \subset \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.*

In order to prove Theorem 3.3 we need a preliminary lemma. Recall that in Section 1.2 we defined for each $t \in \mathbb{R}$ the matrix

$$B_t = \begin{pmatrix} e^{-nt} & \\ & e^t I_n \end{pmatrix}.$$

Lemma 3.4 *If $u \in \mathcal{O}_L$ is unitary and $t = \log|u|$ then:*

- $B_t \in \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.
- B_t *centralizes* Γ_1.

Proof If $u \in \mathcal{O}_L$ is unitary then so is $-u$, and so without loss of generality we assume that $u > 0$. Since u is unitary we have

$$B_t^* J^{\vec{\alpha}} B_t = \begin{pmatrix} u^{-n} & 0 \\ 0 & uI_n \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & J^{\vec{\alpha}_1} \end{pmatrix} \begin{pmatrix} u^n & 0 \\ 0 & u^{-1} I_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & J^{\vec{\alpha}_1} \end{pmatrix} = J^{\vec{\alpha}},$$

which proves that $B_t \in \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.

For the second point, let $\{e_1, \dots, e_{n+1}\}$ be the standard basis for $\mathbb{R}^{n+1}$ and let $\{e_1^*, \dots, e_{n+1}^*\}$ be the corresponding dual basis. For each t, B_t acts trivially on the projective spaces corresponding to $\langle e_1 \rangle$ and $\ker(e_1^*)$. By construction Γ_1 preserves both of these subspaces, and so B_t centralizes Γ_1. □

Proof of Theorem 3.3 First, observe that $\Gamma \subset \mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F) \subset \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ for any $L = F(s)$. There are now two cases. If $M \setminus M_1$ is separating then, as described in Section 1.2, Γ splits as an amalgamated product $G_1 *_{\Gamma_1} G_2$, and ρ_t is defined by the property that $\rho_t(\gamma) = \rho_0(\gamma)$ if $\gamma \in G_1$ and $\rho_t(\gamma) = B_t \rho_0(\gamma) B_t^{-1}$ if $\gamma \in G_2$. By the previous observation, $\rho_0(\gamma) \in \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ for any $\gamma \in \Gamma$ and, by Lemma 3.4, $B_t \in \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$. It follows that $\rho_t(\Gamma) \le \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.

The separating case is similar. In this case, $\Gamma = \Gamma' *_s$ is an HNN extension, where $\Gamma' = \pi_1(M \setminus M_1)$ and ρ_t is defined by the property that $\rho_t(\gamma) = \rho_0(\gamma)$ if $\gamma \in \Gamma'$ and $\rho_t(s) = B_t \rho_0(s)$. Using a similar argument as before it follows that $\rho_t(\Gamma) \le \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$. □

4 Certifying thinness

The goal of this section is to certify the thinness of the examples produced in the previous section. Before proceeding we recall some notation. Γ and Γ_1 are finite-index subgroups of $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ and $\mathrm{SO}(J^{\vec{\alpha}_1}, \mathcal{O}_F)$ such that $M = \mathbb{H}^n/\Gamma$ is a manifold and $M_1 = \mathbb{H}^{n-1}/\Gamma_1$ is an embedded totally geodesic submanifold. Furthermore, if M is noncompact then all of the cusps are torus cusps and the intersection of M_1 with one of these cusps is connected. Let ρ_t be obtained by bending M along M_1; let $\Gamma_t = \rho_t(\Gamma)$. By Theorems 1.2 and 1.4 there is a properly convex set Ω_t such that $M_t := \Omega_t/\Gamma_t$ is a properly convex manifold that is diffeomorphic to M. Furthermore, if M is noncompact then M_t has generalized cusp ends.

The main theorem is a corollary of the following result:

Proposition 4.1 *Suppose that* ρ_t *is obtained by bending* M *along* M_1; *then*:

(1) *For every* t, ρ_t *is injective.*

(2) *If* $u \in \mathcal{O}_L$ *is unitary with* $t = \log|u|$, *then* $\rho_t(\Gamma)$ *is of infinite index in* $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.

(3) *For any* $t \neq 0$, $\rho_t(\Gamma)$ *is Zariski dense in* $\mathrm{SL}(n+1,\mathbb{R})$.

In particular, $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ *contains a thin group isomorphic to* $\pi_1 M$.

Proof The first two points are simple. For (1) observe that by Section 1.2, ρ_t is the holonomy of a convex projective structure on M.

Let $\Gamma_t = \rho_t(\Gamma)$. For (2), we can use the fact that the manifold $\mathbb{H}^n/\Gamma$ contains an embedded hypersurface, as we observed earlier. It follows from [25] that the group Γ virtually surjects onto $\mathbb{Z}$ and thus has infinite abelianization. More precisely, one can pass to a finite cover, M', of $\mathbb{H}^n/\Gamma$ that contains an embedded nonseparating hypersurface, Σ'. There is a nontrivial cohomology class in $H^1(M',\mathbb{Z})$ that is Poincaré dual to Σ', which gives the virtual surjection onto $\mathbb{Z}$. Since $\mathrm{SU}(J^{\vec{\alpha}},\mathcal{O}_L,\tau)$ is a lattice in a high-rank Lie group, it follows that it has property (T) (see [26, Proposition 13.4.1]). Furthermore, any finite-index subgroup of $\mathrm{SU}(J^{\vec{\alpha}},\mathcal{O}_L,\tau)$ will also have property (T) and thus will have finite abelianization (see [26, Corollary 13.1.5]). Since the groups Γ_t are all abstractly isomorphic, it follows that Γ_t is not a lattice; this implies (2).

The third point breaks into two cases depending on whether or not Γ is a cocompact lattice in $\mathrm{SO}(n,1)$. We treat the cocompact case first. By Theorem 1.2, it follows that Γ_t acts cocompactly on a properly convex set Ω_t. Since Γ is a cocompact lattice in $\mathrm{SO}(n,1)$, the group Γ is word hyperbolic and it follows from work of Benoist [6] that for each t the domain Ω_t is strictly convex. Hence Ω_t cannot be written as a nontrivial product of properly convex sets. Applying [6, Theorem 1.1] it follows that Γ_t is either Zariski dense or Ω_t is the projectivization of an irreducible symmetric convex cone. Suppose we are in the latter case. Irreducible symmetric convex cones were classified by Koecher (see [7, Fact 1.3] for a precise statement) and since Ω_t is strictly convex it follows that $\Omega_t \cong \mathbb{H}^n$. It follows that Γ_t is conjugate to a lattice in $\mathrm{SO}(n,1)$, which by Mostow rigidity must be Γ. However, bending in this context never produces conjugate representations, since any such conjugacy would centralize the subgroup corresponding to the complement of the bending hypersurface. However, this subgroup is nonelementary and this is a contradiction. Therefore, Γ_t is Zariski dense if $t \neq 0$, which concludes the cocompact case.

The noncocompact case is an immediate corollary of the following proposition, whose proof occupies the remainder of this section. □

Proposition 4.2 *If M is noncompact, ρ_t is obtained by bending M along M_1, and $\Gamma_t = \rho_t(\Gamma)$, then Γ_t is Zariski dense.*

The strategy for proving Proposition 4.2 is to apply the following two results from [5]:

Theorem 4.3 [5, Lemma 3.9] *Suppose that $G \subset \mathrm{SL}(V)$ is a connected, semisimple, proximal Lie subgroup acting irreducibly on V. If G acts transitively on $P(V)$ then either $V = \mathbb{R}^n$ and $G = \mathrm{SL}(n,\mathbb{R})$, or $V = \mathbb{R}^{2n}$ and $G = \mathrm{Sp}(2n,\mathbb{R})$.*

The next theorem allows us to rule out the second possibility in our case of interest.

Theorem 4.4 [5, Corollary 3.5] *If $\Gamma \subset \mathrm{SL}(V)$ acts strongly irreducibly on V and preserves an open properly convex subset then Γ does not preserve a symplectic form.*

Proof of Proposition 4.2 Let G_t be the Zariski closure of Γ_t and let G_t^0 be the identity component of G_t. We now show that $G_t^0 = \mathrm{SL}(n+1,\mathbb{R})$. By applying Lemma 1.9 we see that G_t^0 satisfies all of the hypotheses of Theorem 4.3 except for transitivity.

Since the intersection of M_1 with one of the cusps of M is connected we can apply [4, Theorem 6.1] to conclude that M_t has at least one type 1 cusp. It follows that (after possibly conjugating) G_t^0 contains the Zariski closure of P_1. Since Γ_t acts irreducibly on V it is not the case that $\Lambda_{G_t^0}$ is contained in $\ker(e_2^*) \cup \ker(e_{n+1}^*)$, and so by Lemma 1.1 we can choose a point $x \in \Lambda_{G_t^0}$ such that $\overline{P}_1 \cdot x$ is open in $P(V)$. It follows that $G_t^0 \cdot x$ has nonempty interior and is hence open. Finally, by Lemma 1.9, $G_t^0 \cdot x = \Lambda_{G_t^0}$, which is closed, hence G_t^0 acts transitively on $P(V)$.

Finally, by Theorem 4.4, Γ_t does not preserve a symplectic form and hence neither does G_t^0. Applying Theorem 4.3 it follows that $G_t^0 = \mathrm{SL}(n+1,\mathbb{R})$. □

Proof of Theorem 0.1 Since Γ is an arithmetic group of orthogonal type in $\mathrm{SO}(n,1)$ there is a totally real number field F with ring of integers $\mathcal{O}_F$ as well as $\vec{\alpha} = (\alpha_1,\dots,\alpha_n)$ such that Γ is commensurable with $\mathrm{SO}(J^{\vec{\alpha}},\mathcal{O}_F)$. The group Γ is cocompact if and only if $F \neq \mathbb{Q}$.

By using standard separability arguments, we can pass to a finite-index subgroup Γ' such that $M = \mathbb{H}^n/\Gamma'$ contains an embedded totally geodesic hypersurface M_1 with the property that if M is noncompact then it has only torus cusps and is such that M_1 has connected intersection with at least one of the cusps.

Let ρ_t be obtained by bending M along M_1. Let $v \in \mathcal{O}_F^\times$ be an element guaranteed by Lemma 3.1 and let $L = F(s)$, where s is a root of $p_v(x)$, and let τ be the nontrivial Galois automorphism of L over F. Next, let $u = s^n$ be a τ–unit in $\mathcal{O}_F^\times$. If $t = \log|u|$ then by Theorem 3.3 it follows that $\rho_t(\Gamma') \subset \mathrm{SU}(J^{\vec{\alpha}},\mathcal{O}_L,\tau)$. Furthermore, by Proposition 4.1, $\rho_t(\Gamma')$ is a thin subgroup of $\mathrm{SU}(J^{\vec{\alpha}},\mathcal{O}_L,\tau)$. Again, $\mathrm{SU}(J^{\vec{\alpha}},\mathcal{O}_L,\tau)$ is cocompact if and only if $F \neq \mathbb{Q}$ and by varying v and $\vec{\alpha}$ it is possible to produce infinitely many noncommensurable lattices. □

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