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Spatiotemporal Persistent Homology for Dynamic Metric Spaces

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Abstract

Characterizing the dynamics of time-evolving data within the framework of topological data analysis (TDA) has been attracting increasingly more attention. Popular instances of time-evolving data include flocking/swarming behaviors in animals and social networks in the human sphere. A natural mathematical model for such collective behaviors is a dynamic point cloud, or more generally a dynamic metric space (DMS). In this paper we extend the Rips filtration stability result for (static) metric spaces to the setting of DMSs. We do this by devising a certain three-parameter "spatiotemporal" filtration of a DMS. Applying the homology functor to this filtration gives rise to multidimensional persistence module derived from the DMS. We show that this multidimensional module enjoys stability under a suitable generalization of the Gromov–Hausdorff distance which permits metrization of the collection of all DMSs. On the other hand, it is recognized that, in general, comparing two multidimensional persistence modules leads to intractable computational problems. For the purpose of practical comparison of DMSs, we focus on both the rank invariant or the dimension function of the multidimensional persistence module that is derived from a DMS. We specifically propose to utilize a certain metric d for comparing these invariants: In our work this d is either (1) a certain generalization of the erosion distance by Patel, or (2) a specialized version of the well-known interleaving distance. In either case, the metric d can be computed in polynomial time.

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1 Introduction

Stability and Tractability of TDA for Studying Metric Spaces. Finite point clouds or finite metric spaces are amongst the most common data representations considered in topological data analysis (TDA) [13,29,33]. In particular, the stability of the Single Linkage Hierarchical Clustering (SLHC) method [16] or the stability of the persistent homology of filtered Rips complexes built on metric spaces [22,23] motivates adopting these constructions when studying metric spaces arising in applications.

Whereas there have been extensive applications of TDA to static metric data (thanks to the aforementioned theoretical underpinnings), there is not much study of *dynamic metric* data *from the TDA perspective*. Our motivation for considering dynamic metric data stems from the study and characterization of flocking/swarming behaviors of animals [5,36,37,39,53,57,63,69], convoys [41], moving clusters [43], or mobile groups [40,70]. In this paper, by extending ideas from [16,22,23,46,47], we aim at establishing a TDA framework for the study of dynamic metric spaces (DMSs) which comes together with stability theorems. We begin by describing and comparing relevant work with ours.

Lack of an Adequate Metric for DMSs. In [55], Munch considers *vineyards*—a certain notion of time-varying persistence diagrams introduced by Cohen-Steiner et al. [25]—as signatures for dynamic point clouds. Munch, in particular, shows that vineyards are stable[1] [24] under perturbations of the input dynamic point cloud [55, Thm. 17]. However, we will observe below that, for the purpose of comparing two DMSs (which we regard as models of flocking behaviors), the metrics that directly arise as the integration of the Hausdorff or Gromov–Hausdorff distance can sometimes fail to be discriminative enough (see Example 2.4 and Remark 4.6).

In [64], Halverson et al. study aggregation models for biological systems by adopting ideas from TDA. They show that topological analysis of aggregation reveals dynamical events which are not captured by classical analysis methods. Specifically, in order to extract insights about the global behavior of dynamic point clouds obtained by simulating aggregation models, they employ the so-called *CROCKER plot*.[2] This plot represents the evolution of Betti numbers of Rips complexes over the plane of *time* and *scale* parameters. In [65], Topaz et al. discretize CROCKER plots as matrices and make use of Frobenius norm for comparing any two such matrices. In [64,65], the authors do not provide stability results for CROCKER plots derived from biological aggregation models.

[1] Under a certain notion of distance arising from the integration over time of the bottleneck distance between the instantaneous persistence diagrams.

[2] Contour Realization Of Computed k-dimensional hole Evolution in the Rips complex.

Fig. 1 Fix $r > 0$. The two figures above stand for two dynamic point clouds $X_r(\cdot)$ and $Y_r(\cdot)$ in the real line each consisting of three points x_1, x_2, x_3 and y_1, y_2, y_3, respectively. Each of $X_r(\cdot)$ and $Y_r(\cdot)$ contains (1) two static points located at $-r$ and r respectively (x_1, x_3 and y_1, y_3), and (2) one dynamic point with the time-dependent coordinate either $r \sin t$ or $r\,|\sin t|$, $t \in \mathbf{R}$ (x_2 and y_2). Observe that in $X_r(\cdot)$ the unique dynamic point x_2 meets both of x_1 and x_2 periodically. On the contrary, in $Y_r(\cdot)$, the unique dynamic point y_2 meets only y_3 periodically

Motivation for Introducing a New Metric for DMSs. Consider the two dynamic point clouds $X_r(\cdot)$ and $Y_r(\cdot)$ illustrated as in Fig. 1. Let us regard them as instances of DMS with the time-dependent metrics obtained by restricting the Euclidean metric on $\mathbf{R}^2$ at each time $t \in \mathbf{R}$.

Observe that for each time $t \in \mathbf{R}$, the metric spaces $X_r(t)$ and $Y_r(t)$ are *isometric* and hence the Gromov–Hausdorff distance [12, Chap. 7] $d_{\mathrm{GH}}(X_r(t), Y_r(t))$ is *zero*. This in turn implies that the integral $\int_{t\in\mathbf{R}} d_{\mathrm{GH}}(X_r(t), Y_r(t))\, dt$ is also *zero*, implying that $X_r(\cdot)$ and $Y_r(\cdot)$ are not distinguished from each other by the *integrated* Gromov–Hausdorff distance.[3] See Remark 2.16.

However, regarding $X_r(\cdot)$ and $Y_r(\cdot)$ as models of *collective behaviors* of animals, vehicles or people, $X_r(\cdot)$ and $Y_r(\cdot)$ are clearly distinct from each other. This motivates us to seek an adequate metric that measures the difference between the *dynamics* underlying any two given DMSs. In particular, this metric should *not* be a mere sum of instantaneous differences of the given DMSs over time.

In this paper, we adopt $d_{\mathtt{dyn}}$, called the λ-*slack interleaving distance* with $\lambda = 2$ (Definition 2.10, originally introduced in [46]), as a measure of the behavioral difference between DMSs. In Sect. 4, we specifically show that the metric $d_{\mathtt{dyn}}$ returns a positive value for the pair of DMSs $X_r(\cdot)$ and $Y_r(\cdot)$ in Fig. 1, demonstrating its sensitivity.

About Stability and Tractability of $d_{\mathtt{dyn}}$. Even though the metric $d_{\mathtt{dyn}}$ is able to differentiate subtly different DMSs (Theorem 2.11), computing $d_{\mathtt{dyn}}$ is *not* tractable in general (Remark 2.13). This hinders us from utilizing $d_{\mathtt{dyn}}$ in practice. Therefore, as a pragmatic approach, we adopt the comparison of *invariants* of DMSs, rather than directly comparing DMSs. To this end,

(a) the invariants *must* be stable under perturbations of the input DMS, and
(b) the metric for comparing two invariants extracted from two DMSs *must* be efficiently computable.

[3] In [55], in order to compare two dynamic point clouds, Munch considered the *integrated* Hausdorff distance $\int d_{\mathrm{H}}$ over time. Since the metric $\int d_{\mathrm{H}}$ takes account of relative position of two dynamic point clouds inside an ambient metric space, we do not consider utilizing $\int d_{\mathrm{H}}$ for the purpose of comparing *intrinsic behaviors* of two dynamic metric data. Also, Munch considered the *integrated* bottleneck distance $\int d_{\mathrm{B}}$ by computing the Rips filtrations of dynamic point clouds at each time. However, by [22, Thm. 3.1], the metric $\int d_{\mathrm{B}}$ is upper-bounded by (twice) the integrated Gromov–Hausdorff distance, which in this case vanishes. Therefore, $\int d_{\mathrm{B}}$ does not discriminate the two dynamic point clouds given as in Fig. 1.

Contributions. In this work, we achieve both items (a) and (b) above, described as follows.

With regard to (a), we first extract invariants from a given DMS, where these invariants are in the form of 3-*dimensional* persistence modules of sets or vector spaces. These are obtained from a blend of ideas related to the Rips filtration [22,24,30], the *single linkage hierarchical clustering* (*SLHC*) method [16], and the interlevel set persistence/categorified Reeb graphs [4,9,15,26].

We are able to prove the stability of these invariants (Theorems 4.1 and 6.17) by adapting ideas from [16,22,23]. We specifically emphasize that our stability results are a generalization of the well-known stability theorems for the SLHC method [16] and the Rips filtration of a metric space [22,23]: Indeed, we show that by restricting ourselves to the class of *constant* DMSs, our results reduce to the standard stability theorems for static metric spaces in [16,22,23].

Next, in regard to item (b) above, we address the issue of computability of the metric between invariants of DMSs. In [7,8], Bjerkevik and Botnan show that computing the interleaving distance d_{I} [52] between multidimensional persistence modules can in general be NP-hard. Also, since we are not guaranteed to have interval decomposability [9,17] of the 3-dimensional modules considered in this paper, we are not in a position to utilize the bottleneck distance and relevant algorithms developed by Dey and Xin [28] instead of d_{I}.

This motivates us to further simplify our invariant M_X associated to a DMS $(X, d_X(\cdot))$, which is in the form of 3-dimensional persistence module. We focus on both the *dimension function* and the *rank function*. The dimension function $\mathrm{dm}(M_X)$ of a persistence module M_X has been studied in various contexts and with various names such as Betti curve, feature counting function, etc, [2,28,34,35,42,62]. The rank function $\mathrm{rk}(M_X)$ of M_X has also been extensively considered [17,18,51,58,59]. We observe that both of these functions (1) can themselves be computed in polynomial time, (2) can be compared to each other via the interleaving distance $d_{\mathrm{I}}^{\mathbf{Z}}$ for *integer-valued functions* (see Sect. 3.2), and (3) are stable to perturbations of $(X, d_X(\cdot))$ under $d_{\mathtt{dyn}}$ (Theorems 4.4 and 4.5). We also propose a simple algorithm for computing $d_{\mathrm{I}}^{\mathbf{Z}}$ in poly-time (Sect. 5). Therefore, we can bound the distance $d_{\mathtt{dyn}}$ in poly-time by computing $d_{\mathrm{I}}^{\mathbf{Z}}$ and either $\mathrm{dm}(\cdot)$ or $\mathrm{rk}(\cdot)$.

We in particular emphasize that our method for computing $d_{\mathrm{I}}^{\mathbf{Z}}$ provides a poly-time algorithm for bounding from below the interleaving distance between d-dimensional persistence modules M of vector spaces *without any restriction on d or on the structure of M* (even if M is not derived from a DMS).

Other Related Work. Aiming at analyzing/summarizing trajectory data such as the movement of animals, vehicles, and people, Buchin and et al. introduce the notion of *trajectory grouping structure* [11]. This is a summarization, in the form of a labeled Reeb graph, of a set of points having piecewise linear trajectories with time-stamped vertices in Euclidean space $\mathbf{R}^d$. This work was subsequently enriched in [50,66–68].

In [46,47], the thread of ideas in [11] is blended with ideas in zigzag persistence theory [14]. Specifically, particular cases of trajectory grouping structure in [11], are named *formigrams*. By clarifying the zigzag persistence structure of formigrams,

formigrams are further summarized into *barcodes*. Regarding the barcode as a signature of a set of trajectory data, the authors of [46,47] utilize these barcodes for carrying out the classification task of a family of synthetic flocking behaviors [48].

The central results in [46,47] show that barcodes or formigrams from a trajectory data are stable to perturbations of the input data [47, Thm. 5], [46, Thm. 9.21]. This work is a sequel to [46,47]. Namely, by considering Rips-like filtrations parametrized both by *time intervals* and *spatial scale*, we obtain novel stability results in every homological dimension.

Other work utilizing TDA-like ideas in the analysis of dynamic data includes: a study of time-varying merge trees or time-varying Reeb graphs [31,56]. Also, ideas of persistent homology are utilized in the study of time-varying graphs [38], discretely sampled dynamical systems [3,32] or in the study of combinatorial dynamical systems [27].

Organization. In Sect. 2 we review the notion of DMSs and the metric $d_{\mathtt{dyn}}$ on DMSs. In Sect. 3 we review the interleaving distance. In Sect. 4 we provide an overview of our new stability results about persistent homology features of DMSs. In Sect. 5 we propose and study an algorithm for computing the interleaving distance between integer-valued functions. Section 6 contains proofs of statements (theorems, etc.) from Sect. 4.

In Appendix A we describe how to analyze and compare discrete time series of metric data. In Appendix B we clarify the relationship between the rank invariants of DMSs and the CROCKER-plots of DMSs. In Appendix C we compare the interleaving distance between integer-valued functions with other relevant metrics. In Appendix D we review the stability of the single linkage hierarchical clustering (SLHC) method for static metric spaces; results in this section are generalized to those in Sect. 6.4.

2 Dynamic Metric Spaces (DMSs)

Throughout this paper, we fix a certain field $\mathbb{F}$ and only consider vector spaces over $\mathbb{F}$ whenever they arise. Any simplicial homology has coefficients in $\mathbb{F}$. By $\mathbf{Z}_+$ and $\mathbf{R}_+$, we denote the set of non-negative integers and the set of non-negative reals, respectively.

2.1 Definition of DMSs

DMSs. A DMS $\gamma_X = (X, d_X(\cdot))$ stands for a pair of finite set X with $\mathbf{R}$-parametrized metric $d_X(\cdot)\colon \mathbf{R}\times X\times X \to \mathbf{R}_+$: for each $t\in\mathbf{R}$, a certain (pseudo-)metric $d_X(t)\colon X\times X\to\mathbf{R}_+$ is obtained:

Definition 2.1 (*Dynamic metric spaces* [46]) A *dynamic metric space* is a pair $\gamma_X = (X, d_X(\cdot))$ where X is a non-empty finite set and $d_X(\cdot)\colon \mathbf{R}\times X\times X\to\mathbf{R}_+$ satisfies:

(i) For every $t\in\mathbf{R}$, $\gamma_X(t) = (X, d_X(t))$ is a pseudo-metric space.
(ii) For any $x, x'\in X$ with $x\neq x'$ the function $d_X(\cdot)(x,x')\colon \mathbf{R}\to\mathbf{R}_+$ is not identically zero.

(iii) For fixed $x, x' \in X$, $d_X(\cdot)(x, x') \colon \mathbf{R} \to \mathbf{R}_+$ is continuous.

We refer to t as the *time* parameter.

Let $(\mathscr{M}, d_{\mathrm{GH}})$ be the collection of all finite (pseudo-)metric spaces equipped with the Gromov–Hausdorff distance (Definition D.1). Any DMS $\gamma_X = (X, d_X(\cdot))$ can be seen as a continuous curve from $\mathbf{R}$ to $(\mathscr{M}, d_{\mathrm{GH}})$.

Example 2.2 [46] Examples of DMSs include:

(i) (Constant DMSs) Given a finite metric space (X, d_X), define the DMS $\gamma_X = (X, d'_X(\cdot))$ by declaring that for all $t \in \mathbf{R}$, $d'_X(t) = d_X$ as a function $X \times X \to \mathbf{R}_+$. We refer to such γ_X as a *constant* DMS and simply write $\gamma_X \equiv (X, d_X)$.
(ii) (Dynamic point clouds) A family of examples is given by n points moving continuously inside an ambient metric space (Z, d_Z) where particles are allowed to coalesce. If the n trajectories are $x_1(t), \ldots, x_n(t) \in Z$, then let $X := \{1, \ldots, n\}$ and define the DMS $\gamma_X := (X, d_X(\cdot))$ as follows: for $t \in \mathbf{R}$ and $i, j \in \{1, \ldots, n\}$, let $d_X(t)(i, j) := d_Z(x_i(t), x_j(t))$. We call γ_X a *dynamic point cloud* in Z and simply write $X(\cdot) = \{x_i(\cdot)\}_{i=1}^n$ or $X(\cdot)$.

Weak and Strong Isomorphism Between DMSs. We introduce two different notions of *isomorphism* between DMSs.

Definition 2.3 (*Isomorphism between DMSs*) Let $\gamma_X = (X, d_X(\cdot))$, $\gamma_Y = (Y, d_Y(\cdot))$ be any two DMSs.

(i) γ_X and γ_Y are *strongly isomorphic* if there exists a bijection $\varphi \colon X \to Y$ such that φ is an isometry between $\gamma_X(t) = (X, d_X(t))$ and $\gamma_Y(t) = (Y, d_Y(t))$ for all $t \in \mathbf{R}$.
(ii) γ_X and γ_Y are *weakly isomorphic* if for each $t \in \mathbf{R}$, $\gamma_X(t) = (X, d_X(t))$ is isometric to $\gamma_Y(t) = (Y, d_Y(t))$.

Any two strongly isomorphic DMSs are weakly isomorphic, but the converse is not true:

Example 2.4 (*Weakly isomorphic DMSs*) The dynamic point clouds $X_r(\cdot)$ and $Y_r(\cdot)$ described in Fig. 1 are weakly isomorphic, but not strongly isomorphic: Indeed, there is no bijection between $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ which serves as an isometry for all $t \in \mathbf{R}$.

2.2 The λ-Slack Interleaving Distance Between DMSs

We review the extended metric d_{dyn} for DMSs, which was introduced in [46, Defn. 9.13] under the name of λ-*slack interleaving distance*, for each $\lambda \in [0, \infty)$.

Definition 2.5 Let $\varepsilon \geq 0$. Given any map $d \colon X \times X \to \mathbf{R}$, by $d + \varepsilon$ we denote the map $X \times X \to \mathbf{R}$ defined as $(d + \varepsilon)(x, x') = d(x, x') + \varepsilon$ for all $(x, x') \in X \times X$.

In order to compare any two DMSs, we will utilize the notion of *tripod*:

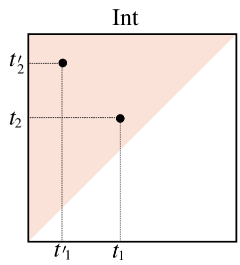

Fig. 2 The collection **Int** can be graphically represented as the upper-half plane $\{(t_1, t_2) \in \mathbf{R}^2 : t_1 \leq t_2\}$: Any closed interval $[t_1, t_2]$ of **R** is identified with the point (t_1, t_2) on $\mathbf{R}^2$. Observe that if $[t_1, t_2] \subset [t_1', t_2']$, then the point (t_1', t_2') is located at upper-left of the point (t_1, t_2) in the plane

Definition 2.6 (*Tripod*) Let X and Y be any two non-empty sets. For another set Z, any pair of *surjective* maps $R: X \overset{\varphi_X}{\twoheadleftarrow} Z \overset{\varphi_Y}{\twoheadrightarrow} Y$ is called a *tripod* between X and Y.

Given any map $d\colon X \times X \to \mathbf{R}$, let Z be any set and let $\varphi\colon Z \to X$ be any map. Then we define $(\varphi^* d)\colon Z \times Z \to \mathbf{R}$ as

$$(\varphi^* d)(z, z') := d(\varphi(z), \varphi(z')), \ (z, z') \in Z \times Z.$$

Definition 2.7 (*Comparison of metrics via tripods*) Consider any two maps $d_1\colon X \times X \to \mathbf{R}$ and $d_2\colon Y \times Y \to \mathbf{R}$. Given a tripod $R: X \overset{\varphi_X}{\twoheadleftarrow} Z \overset{\varphi_Y}{\twoheadrightarrow} Y$ between X and Y, by

$$d_1 \leq_R d_2,$$

we mean $(\varphi_X^* d_1)(z, z') \leq (\varphi_Y^* d_2)(z, z')$ for all $(z, z') \in Z \times Z$.

Let **Int** be the collection of all finite *closed* intervals of **R**. See Fig. 2.

Definition 2.8 (*Time-interlevel analysis of a DMS*) Given a DMS $\gamma_X = (X, d_X(\cdot))$, define the function $\bigvee d_X\colon \mathbf{Int} \times X \times X \to \mathbf{R}_+$ as

$$(I, x, x') \mapsto \bigvee_I d_X(x, x') := \min_{s \in I} d_X(s)(x, x').$$

In words, $\bigvee_I d_X(x, x')$ stands for the minimum distance between x and x' within the time interval I. Observe that if $I \subset I'$ are both in **Int**, then $\bigvee_{I'} d_X(x, x') \leq \bigvee_I d_X(x, x')$ for all $x, x' \in X$.

For any $t \in \mathbf{R}$, let $[t]^\varepsilon := [t - \varepsilon, t + \varepsilon] \in \mathbf{Int}$.

Definition 2.9 (*Distortion of a tripod*) Let $\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$ be any two DMSs. Let $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ be a tripod between X and Y such that

$$\text{for all } t \in \mathbf{R}, \quad \bigvee_{[t]^\varepsilon} d_X \leq_R d_Y(t) + 2\varepsilon \text{ and } \bigvee_{[t]^\varepsilon} d_Y \leq_R d_X(t) + 2\varepsilon. \tag{1}$$

We call any such R an ε-*tripod* between γ_X and γ_Y. Define the *distortion* $\mathrm{dis}^{\mathrm{dyn}}(R)$ of R to be the infimum of $\varepsilon \geq 0$ for which R is an ε-tripod.

In Definition 2.9, if R is an ε-tripod, then R is also an ε'-tripod for any $\varepsilon' \geq \varepsilon$.

Definition 2.10 (*The distance d_{dyn} between DMSs*) Given any two DMSs $\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$, we define

$$d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) := \min_R \mathrm{dis}^{\mathrm{dyn}}(R),$$

where the minimum ranges over all tripods between X and Y.

We remark that d_{dyn} is a hybrid between the Gromov–Hausdorff distance (Definition D.1) and the interleaving distance [10,21] for Reeb graphs [26]. We also remark that, in [46], d_{dyn} is introduced under the name of λ-*slack interleaving distance* for $\lambda = 2$. We use $\lambda = 2$ in this paper for ease of notation. This choice is not significant because different choices of $\lambda > 0$ yield bilipschitz equivalent metrics for DMSs [46, Prop. 11.29].

Any DMS $\gamma_X = (X, d_X(\cdot))$ is said to be *bounded* if there exists $r \in [0, \infty)$ such that for all $x, x' \in X$ and all $t \in \mathbf{R}$, $d_X(t)(x, x') \leq r$. For example, both DMSs given in Fig. 1 are bounded.

Theorem 2.11 [46, Thm. 9.14] *d_{dyn} is an* extended *metric between DMSs modulo strong isomorphism* (*Definition* 2.3 (i)). *In particular, d_{dyn} is a metric between bounded DMSs modulo strong isomorphism.*

Remark 2.12 (*d_{dyn} generalizes the Gromov–Hausdorff distance* [46, Rem. 11.28]) Given any two constant DMSs $\gamma_X \equiv (X, d_X)$ and $\gamma_Y \equiv (Y, d_Y)$, the metric d_{dyn} recovers the Gromov–Hausdorff distance between (X, d_X) and (Y, d_Y). Indeed, for any tripod R between X and Y, condition (1) reduces to

$$\left|d_X(x, x') - d_Y(y, y')\right| \leq 2\varepsilon \text{ for all } (x, y), (x', y') \in R.$$

Therefore,

$$d_{\mathrm{GH}}((X, d_X), (Y, d_Y)) = d_{\mathrm{dyn}}(\gamma_X, \gamma_Y).$$

Remark 2.13 From Remark 2.12, we conclude that the computation of d_{dyn} is in general not tractable: On the class of constant DMSs the metric d_{dyn} reduces to the Gromov–Hausdorff distance, which leads to NP-hard problems [1,60,61].

2.3 Variants of d_{dyn}

Recall that d_{dyn} denotes the λ-slack interleaving distance for $\lambda = 2$ [46, Defn. 9.13] and that this distance generalizes the Gromov–Hausdorff distance d_{GH} (Remark 2.12). In this section we discuss other natural generalizations of d_{GH}. While some of them can discriminate weakly isomorphic DMSs, others fail to do so.

We begin with a variant of the λ-slack interleaving distance which arises from a slightly different way of incorporating the λ parameter:

Definition 2.14 (*Multiplicative λ-slack interleaving distance*) For $\lambda \in (0, \infty)$, we define the *multiplicative* λ-slack interleaving distance $d_\lambda^\bullet(\gamma_X, \gamma_Y)$ between two DMSs $\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$ as the infimum ε for which there exists a tripod R between X and Y such that[4]

$$\text{for all } t \in \mathbf{R}, \quad \bigvee_{[t]^{\frac{\varepsilon}{\lambda}}} d_X \leq_R d_Y(t) + \varepsilon \quad \text{and} \quad \bigvee_{[t]^{\frac{\varepsilon}{\lambda}}} d_Y \leq_R d_X(t) + \varepsilon. \tag{2}$$

Definition 2.15 (*dyn-Gromov–Hausdorff distance between DMSs and its relation to* $d_\lambda^\bullet$) Let γ_X and γ_Y be any two DMSs and fix a tripod R between X and Y. For each $t \in \mathbf{R}$, let

$$\mathrm{dis}(R)(t) := \inf \left\{ \varepsilon \in \mathbf{R}_+ : d_X(t) \leq_R d_Y(t) + \varepsilon \text{ and } d_Y(t) \leq_R d_X(t) + \varepsilon \right\}.$$

Define

$$d_{\mathrm{GH}}^{\mathrm{dyn}}(\gamma_X, \gamma_Y) := \min_R \sup_{t \in \mathbf{R}} \, \mathrm{dis}(R)(t),$$

where the minimum is taken over all tripods R between X and Y. We call this distance the *dyn-Gromov–Hausdorff distance between* γ_X *and* γ_Y.

Note that, for the multiplicative interleaving distance $d_\lambda^\bullet$ in Definition 2.14, we have

$$\lim_{\lambda \to \infty} d_\lambda^\bullet(\gamma_X, \gamma_Y) = d_{\mathrm{GH}}^{\mathrm{dyn}}(\gamma_X, \gamma_Y).$$

Also, note that $d_{\mathrm{GH}}^{\mathrm{dyn}}$ between constant DMSs $\gamma_X \equiv (X, d_X)$ and $\gamma_Y \equiv (Y, d_Y)$ reduces to twice the Gromov–Hausdorff distance between (X, d_X) and (Y, d_Y). We

[4] In [46], the original λ-slack interleaving distance $d_\lambda(\gamma_X, \gamma_Y)$, $\lambda \in [0, \infty)$ is defined as the infimum amount of time ε for which there exists a tripod R between X and Y such that

$$\text{for all } t \in \mathbf{R}, \quad \bigvee_{[t]^{\varepsilon}} d_X \leq_R d_Y(t) + \lambda\varepsilon \quad \text{and} \quad \bigvee_{[t]^{\varepsilon}} d_Y \leq_R d_X(t) + \lambda\varepsilon.$$

In this original definition, the units of λ is (distance units)/(time units), whereas the units of λ for $d_\lambda^\bullet$ is (time units)/(distance units).

remark that $d_{\mathrm{GH}}^{\mathrm{dyn}}$ is in general *not* the supremum of the Gromov–Hausdorff distances $d_{\mathrm{GH}}(\gamma_X(t), \gamma_Y(t))$ over all times $t \in \mathbf{R}$. Specifically, we have the following inequality:

$$L_{\mathrm{GH}}^{(\infty)}(\gamma_X, \gamma_Y) := \sup_{t\in\mathbf{R}} d_{\mathrm{GH}}(\gamma_X(t), \gamma_Y(t)) = \frac{1}{2} \cdot \sup_{t\in\mathbf{R}} \min_{R} \operatorname{dis}(R)(t)$$
$$\overset{(*)}{\leq} \frac{1}{2} \cdot \min_{R} \sup_{t\in\mathbf{R}} \operatorname{dis}(R)(t) = \frac{1}{2} \cdot d_{\mathrm{GH}}^{\mathrm{dyn}}(\gamma_X, \gamma_Y).$$

The inequality denoted by $(*)$ above is often strict, as it is to be expected as a result of swapping the sup min implicit in $L_{\mathrm{GH}}^{(\infty)}$ for the min sup in the definition of $d_{\mathrm{GH}}^{\mathrm{dyn}}$.[5] For instance, for any pair γ_X, γ_Y of weakly isomorphic but not strongly isomorphic DMSs (cf. Example 2.4), one has that (1) $d_{\mathrm{GH}}(\gamma_X(t), \gamma_Y(t)) = 0$ for every $t \in \mathbf{R}$ and in turn $\sup_{t\in\mathbf{R}} d_{\mathrm{GH}}(\gamma_X(t), \gamma_Y(t)) = 0$; but in contrast (2) $d_{\mathrm{GH}}^{\mathrm{dyn}}(\gamma_X, \gamma_Y)$ is strictly positive.

It is possible to give rise to a whole family of pseudo-distances of which $L_{\mathrm{GH}}^{(\infty)}$ is a particular example.

This construction is analogous to the *integrated Hausdorff distance* between dynamic point clouds considered in [55].

Remark 2.16 (*Weak-L^p-Gromov–Hausdorff distance*) Fix any two DMSs γ_X and γ_Y. For any fully supported probability measure ζ on $\mathbf{R}$ and $p \in [1, \infty)$, define

$$L_{\mathrm{GH},\zeta}^{(p)}(\gamma_X, \gamma_Y) := \left(\int_{t\in\mathbf{R}} \big(d_{\mathrm{GH}}(\gamma_X(t), \gamma_Y(t))\big)^p d\zeta\right)^{(1/p)}.$$

It is clear that $L_{\mathrm{GH},\zeta}^{(p)}(\gamma_X, \gamma_Y)$ vanishes whenever γ_X and γ_Y are weakly isomorphic.

2.4 Persistent Homology Features of a DMS

We extend ideas from persistent homology/single linkage hierarchical clustering method for metric spaces (Appendix D) to the setting of *dynamic* metric spaces (DMSs).

Posets and Their Opposite. Given any poset $\mathbf{P} = (\mathbf{P}, \leq)$, we regard $\mathbf{P}$ as the category whose objects are the elements of $\mathbf{P}$, and for $p, q \in \mathbf{P}$, there exists a unique morphism $p \to q$ if and only if $p \leq q$. Since there exists at most one morphism between any two elements of $\mathbf{P}$, the category $\mathbf{P}$ is called *thin*. This thinness makes *every* closed diagram in $\mathbf{P}$ commute. We sometimes consider the *opposite* category of $\mathbf{P}$, which will be denoted by $\mathbf{P}^{\mathrm{op}}$. In the category $\mathbf{P}^{\mathrm{op}}$, for $p, q \in \mathbf{P}$, there exists the unique morphism $p \to q$ if and only if $p \geq q$.

Example 2.17 (**Int**) Recall the collection **Int** of all finite closed intervals of $\mathbf{R}$. We regard **Int** as poset, where the order $\leq$ is the inclusion $\subseteq$. Hence, **Int** can be seen as the category of finite closed real intervals whose morphisms are inclusions.

[5] The quantity in the LHS allows for picking a different correspondence for each time t whereas the RHS demands that a single correspondence is adequate for all times.

Product of Posets. Given any two posets $\mathbf{P}$ and $\mathbf{Q}$, we assume by default that their product $\mathbf{P} \times \mathbf{Q}$ is equipped with the partial order $\leq$ defined as $(p, q) \leq (p', q')$ if and only if $p \leq p'$ in $\mathbf{P}$ and $q \leq q'$ in $\mathbf{Q}$.

Remark 2.18 In the poset $\mathbf{Int} \times \mathbf{R}_+$, we have $(I, \delta) \leq (I', \delta')$ if and only if $I \subset I'$ and $\delta \leq \delta'$. See Fig. 3. We will regard $\mathbf{Int} \times \mathbf{R}_+$ as a subposet of the product poset $\mathbf{R}^3_\times := \mathbf{R}^{\mathrm{op}} \times \mathbf{R} \times \mathbf{R}$ via the identification $([t_1, t_2], \delta) \leftrightarrow (t_1, t_2, \delta)$. Indeed,

$$([t_1, t_2], \delta) \leq ([t_1', t_2'], \delta') \text{ in } \mathbf{Int} \times \mathbf{R}_+ \quad \text{if and only if} \quad (t_1, t_2, \delta) \leq (t_1', t_2', \delta') \text{ in } \mathbf{R}^3_\times.$$

Spatiotemporal Rips Filtration of a DMS. Let **Simp** be the category of abstract simplicial complexes with simplicial maps. By a (*simplicial*) filtration we mean a functor from a poset to **Simp**. In order to encode multiscale topological features of *DMSs* into a *single* filtration, we define the *spatiotemporal Rips filtration* of a DMS. Let us begin by recalling the *Rips complex*:

Definition 2.19 (*The Rips complex*) Let (X, d_X) be a metric space. For each $\delta \in \mathbf{R}$, by $\mathscr{R}_\delta(X, d_X)$ we mean the abstract simplicial complex on the set X where a subset $\sigma \subset X$ belongs to $\mathscr{R}_\delta(X, d_X)$ if and only if $d_X(x, x') \leq \delta$ for all $x, x' \in \sigma$. Note that if $\delta < 0$, then $\mathscr{R}_\delta(X, d_X)$ is empty.

Definition 2.20 (*The Rips filtration*) Let (X, d_X) be a metric space. The *Rips filtration of a finite metric space* (X, d_X) is the functor $\mathscr{R}_\bullet(X, d_X) \colon \mathbf{R} \to \mathbf{Simp}$ described as follows: To each $\delta \in \mathbf{R}$, the simplicial complex $\mathscr{R}_\delta(X, d_X)$ is assigned. Also, to any pair $\delta \leq \delta'$ in $\mathbf{R}$, the inclusion map $\mathscr{R}_\delta(X, d_X) \hookrightarrow \mathscr{R}_{\delta'}(X, d_X)$ is assigned.

Definition 2.21 (*The spatiotemporal Rips filtration of a DMS*) Given any DMS $\gamma_X = (X, d_X(\cdot))$, the simplicial filtration $\mathscr{R}^{\mathrm{lev}}(\gamma_X) \colon \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Simp}$ defined as in Fig. 3 is called the (*spatiotemporal*) *Rips filtration* of γ_X.

Definition 2.21 is based on a blend of ideas related to the Rips filtration [22,24,30] and the interlevel set persistence/categorified Reeb graphs [4,9,15,26]. The super-index "lev" in $\mathscr{R}^{\mathrm{lev}}(\gamma_X)$ is meant to emphasize the connection to "inter<u>lev</u>elset persistence".

Remark 2.22 (*Comprehensiveness of Definition* 2.21) We remark the following:

(i) Consider the constant DMS $\gamma_X \equiv (X, d_X)$ as in Example 2.2 (i). Then the spatiotemporal Rips filtration of γ_X amounts to the Rips filtration of (X, d_X): for all $I \in \mathbf{Int}$ and $\delta \in \mathbf{R}_+$,

$$\mathscr{R}^{\mathrm{lev}}(\gamma_X)_{(I,\delta)} = \mathscr{R}_\delta(X, d_X).$$

(ii) Let $\gamma_X = (X, d_X(\cdot))$ be a DMS. For each $t \in \mathbf{R}$, we have the Rips filtration $\mathscr{R}_\bullet(X, d_X(t)) \colon \mathbf{R}_+ \to \mathbf{Simp}$ of the metric space $(X, d_X(t))$. All those filtrations are incorporated by $\mathscr{R}^{\mathrm{lev}}(\gamma_X)$ in the following sense:

$$\mathscr{R}^{\mathrm{lev}}(\gamma_X)_{([t,t],\delta)} = \mathscr{R}_\delta(X, d_X(t)), \quad t \in \mathbf{R},\ \delta \in \mathbf{R}_+.$$

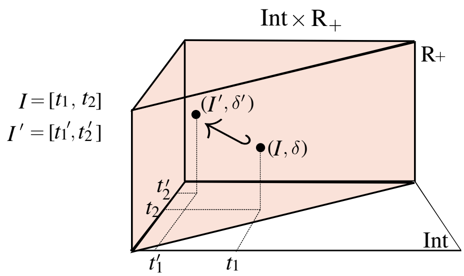

Fig. 3 To each $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$, we associate the *Rips complex* $\mathscr{R}_\delta(X, \bigvee_I d_X)$ on the metric space* $(X, \bigvee_I d_X)$. Provided another interval $I' \in \mathbf{Int}$ and scale $\delta' \in \mathbf{R}_+$ with $I \subset I'$ and $\delta \leq \delta'$, we obtain the inclusion $\mathscr{R}_\delta(X, \bigvee_I d_X) \hookrightarrow \mathscr{R}_{\delta'}(X, \bigvee_{I'} d_X)$. This construction gives rise to a 3-dimensional simplicial filtration $\mathscr{R}^{\text{lev}}(\gamma_X)$ indexed by $\mathbf{Int} \times \mathbf{R}_+$. * In fact, $\bigvee_I d_X : X \times X \to \mathbf{R}_+$ does not necessarily satisfy the triangle inequality. However, it does not prevent us from defining the Rips complex on the semi-metric space $(X, \bigvee_I d_X)$

By functoriality of the simplicial homology functor, we can define, for each $k \in \mathbf{Z}_+$, the persistence module $\mathrm{H}_k\left(\mathscr{R}^{\text{lev}}(\gamma_X)\right) := \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Vec}$.

The Rank Invariant and the Betti-0 *Function of a DMS.* We consider the rank invariant [17] of this multidimensional persistence module $\mathrm{H}_k(\mathscr{R}^{\text{lev}}(\gamma_X))$. Let

$$\underline{\mathbf{R}^6} := \{(t_1, t_2, \delta, t_1', t_2', \delta') \in \mathbf{R}^6 : [t_1, t_2] \subset [t_1', t_2'] \text{ and } \delta \leq \delta'\}. \tag{3}$$

Definition 2.23 (*The rank invariant of a DMS*) Let γ_X be any DMS. For each non-negative integer k, the k-th rank invariant of γ_X is a function $\mathrm{rk}_k(\gamma_X) : \underline{\mathbf{R}^6} \to \mathbf{Z}_+$ defined as

$$\begin{aligned}&\mathrm{rk}_k(\gamma_X)\big(t_1, t_2, \delta, t_1', t_2', \delta'\big)\\ &:= \mathrm{rank}\left(\mathrm{H}_k\left(\mathscr{R}_\delta\left(X, \bigvee_{[t_1,t_2]} d_X\right) \hookrightarrow \mathscr{R}_{\delta'}\left(X, \bigvee_{[t_1',t_2']} d_X\right)\right)\right).\end{aligned}$$

See Fig. 3.

In Appendix B we compare the rank invariant of a DMS with the *CROCKER-plots* introduced in [64].

Definition 2.24 (*The Betti*-0 *function of a DMS*) Let $\gamma_X = (X, d_X(\cdot))$ be a DMS. We define the *Betti*-0 *function* $\beta_0^{\gamma_X} : \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Z}_+$ of γ_X by sending each $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$ to the dimension of $\mathrm{H}_0\left(\mathscr{R}_\delta\left(X, \bigvee_I d_X\right)\right)$.

Example 2.25 Consider the DMSs γ_X and γ_Y given as the dynamic point clouds $X_r(\cdot)$ and $Y_r(\cdot)$ in Fig. 1 respectively. The Betti 0-functions of γ_X and γ_Y are illustrated in Fig. 4.

It is not difficult to check that if $I \subset I'$ in $\mathbf{Int}$ and $\delta \leq \delta'$ in $\mathbf{R}_+$, then $\beta_0^{\gamma_X}(I, \delta) \geq \beta_0^{\gamma_X}(I', \delta')$. This monotonicity is a special feature of Betti-0 functions, which is not

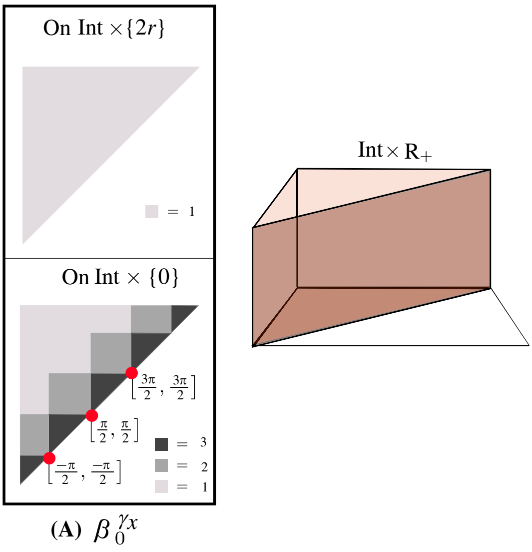

Fig. 4 (The Betti-0 functions $\beta_0^{\gamma_X}, \beta_0^{\gamma_Y}$ of the DMSs in Fig. 1) The middle figure represents the domain $\mathbf{Int} \times \mathbf{R}_+$ (Fig. 3) of $\beta_0^{\gamma_X}$ and $\beta_0^{\gamma_Y}$. (A) and (B) illustrate the value of $\beta_0^{\gamma_X}$ and $\beta_0^{\gamma_Y}$ respectively on the horizontal half-planes $\mathbf{Int} \times \{0\}$ (bottom) and $\mathbf{Int} \times \{2r\}$ (top). In particular, if $\delta \in [2r, \infty)$, $\beta_0^{\gamma_X}(I, \delta) = 1$ for all $I \in \mathbf{Int}$. The same properties hold for $\beta_0^{\gamma_Y}$

shared by *other Betti-k functions* for $k \geq 1$. We will exploit this monotonicity property to metrize the collection of Betti-0 functions and in turn to obtain a tight lower bound for d_{dyn} or d_{GH}. Also, we remark that when γ_X is a constant DMS (Example 2.2 (i)), $\beta_0^{\gamma_X}$ is constant with respect to the first factor.

3 Interleaving Distance

In this section we review the *interleaving distance* for $\mathbf{R}^d$-indexed functors [9,21,52]. In particular, the interleaving distance *between integer-valued functions* (Sect. 3.2) will be utilized for obtaining a computationally tractable lower bound for d_{dyn}.

3.1 Interleaving Distance

Natural Transformations. We recall the notion of *natural transformations* from category theory [54]: Let $\mathscr{C}$ and $\mathscr{D}$ be any categories and let $F, G : \mathscr{C} \to \mathscr{D}$ be any two functors. A natural transformation $\varphi : F \Rightarrow G$ is a collection of morphisms $\varphi_c : F_c \to G_c$ in $\mathscr{D}$ for all objects $c \in \mathscr{C}$ such that for any morphism $f : c \to c'$ in $\mathscr{C}$, the following diagram commutes:

$$\begin{array}{ccc} F_c & \xrightarrow{F(f)} & F_{c'} \\ \downarrow{\varphi_c} & & \downarrow{\varphi_{c'}} \\ G_c & \xrightarrow{G(f)} & G_{c'}. \end{array}$$

Natural transformations $\varphi\colon F \to G$ are considered as morphisms in the category $\mathscr{D}^{\mathscr{C}}$ of all functors from $\mathscr{C}$ to $\mathscr{D}$.

The Interleaving Distance Between $\mathbf{R}^d$*-Indexed Functors.* In what follows, for any $\varepsilon \in [0, \infty)$, we will denote the vector $\varepsilon(1, \ldots, 1) \in \mathbf{R}^d$ by $\boldsymbol{\varepsilon}$. The dimension d will be clearly specified in context.

Definition 3.1 ($\mathbf{v}$*-shift functor*) Let $\mathscr{C}$ be any category. For each $\mathbf{v} \in [0, \infty)^n$, the $\mathbf{v}$-shift functor $(-)(\mathbf{v})\colon \mathscr{C}^{\mathbf{R}^d} \to \mathscr{C}^{\mathbf{R}^d}$ is defined as follows:

(i) (On objects) Let $F\colon \mathbf{R}^d \to \mathscr{C}$ be any functor. Then the functor $F(\mathbf{v})\colon \mathbf{R}^d \to \mathscr{C}$ is defined as follows: For any $\mathbf{a} \in \mathbf{R}^d$,

$$F(\mathbf{v})_{\mathbf{a}} := F_{\mathbf{a}+\mathbf{v}}.$$

Also, for another $\mathbf{a}' \in \mathbf{R}^d$ such that $\mathbf{a} \leq \mathbf{a}'$ we define

$$F(\mathbf{v})(\mathbf{a} \leq \mathbf{a}') := F(\mathbf{a} + \mathbf{v} \leq \mathbf{a}' + \mathbf{v}).$$

In particular, if $\mathbf{v} = \boldsymbol{\varepsilon} \in [0, \infty)^d$, then we simply write $F(\varepsilon)$ in lieu of $F(\boldsymbol{\varepsilon})$.

(ii) (On morphisms) Given any natural transformation $\varphi(\mathbf{v})\colon F \Rightarrow G$, the natural transformation $\varphi(\mathbf{v})\colon F(\mathbf{v}) \Rightarrow G(\mathbf{v})$ is defined as $\varphi(\mathbf{v})_{\mathbf{a}} = \varphi_{\mathbf{a}+\mathbf{v}}\colon F(\mathbf{v})_{\mathbf{a}} \to G(\mathbf{v})_{\mathbf{a}}$ for each $\mathbf{a} \in \mathbf{R}^d$.

For any $\mathbf{v} \in [0, \infty)^d$, let $\varphi_F^{\mathbf{v}}\colon F \Rightarrow F(\mathbf{v})$ be the natural transformation whose restriction to each $F_{\mathbf{a}}$ is the morphism $F(\mathbf{a} \leq \mathbf{a} + \mathbf{v})$ in $\mathscr{C}$. When $\mathbf{v} = \boldsymbol{\varepsilon}$, we denote $\varphi_F^{\mathbf{v}}$ simply by φ_F^{ε}.

Definition 3.2 ($\mathbf{v}$*-interleaving between* $\mathbf{R}^d$*-indexed functors*) Let $\mathscr{C}$ be any category. Given any two functors $F, G\colon \mathbf{R}^d \to \mathscr{C}$, we say that they are $\mathbf{v}$*-interleaved* if there are natural transformations $f\colon F \Rightarrow G(\mathbf{v})$ and $g\colon G \Rightarrow F(\mathbf{v})$ such that

(i) $g(\mathbf{v}) \circ f = \varphi_F^{2\mathbf{v}}$,
(ii) $f(\mathbf{v}) \circ g = \varphi_G^{2\mathbf{v}}$.

In this case, we call (f, g) a $\mathbf{v}$*-interleaving pair*. When $\mathbf{v} = \varepsilon(1, \ldots, 1)$, we simply call (f, g) ε*-interleaving pair*. The interleaving distance between F and G is defined as

$$d_{\mathrm{I},d}^{\mathscr{C}}(F, G) := \inf\{\varepsilon \in [0, \infty) : F, G \text{ are } \boldsymbol{\varepsilon}\text{-interleaved}\}, \tag{4}$$

where we set $d_{\mathrm{I},d}^{\mathscr{C}}(F, G) = \infty$ if there is no ε-interleaving pair between F and G for any $\varepsilon \in [0, \infty)$. Then $d_{\mathrm{I},d}^{\mathscr{C}}$ is an extended pseudo-metric for $\mathscr{C}$-valued $\mathbf{R}^d$-indexed functors. We drop the subscript d from $d_{\mathrm{I},d}^{\mathscr{C}}$ when confusion is unlikely.

Remark 3.3 (i) Let $\mathbf{R}^{\diamondsuit}$ denote the poset either of $\mathbf{R}$ or $\mathbf{R}^{\mathrm{op}}$. The interleaving distance $d_{\mathrm{I}}^{\mathscr{C}}$ is also defined in the similar way for $\mathbf{R}^d$-indexed modules, where the poset $\mathbf{R}^d$ is equipped with the product partial order $\mathbf{R}^{\diamondsuit} \times \mathbf{R}^{\diamondsuit} \times \cdots \times \mathbf{R}^{\diamondsuit}$.

(ii) Let $\mathbf{P}$ be any non-empty *upper set* of $\mathbf{R}^d$: For every $p \in \mathbf{P}$, $U(p) := \{q \in \mathbf{R}^d : q \geq p\}$ is contained in $\mathbf{P}$. Then we can define the interleaving distance between $\mathbf{P}$-indexed modules in the manner described by Definition 3.2.

Full Interleaving. By **Sets**, we mean the category of sets with set maps as morphisms. Also, by **Vec**, we mean the category of vector spaces over a fixed field $\mathbb{F}$, with linear maps as morphisms.

Let $\mathscr{C}$ be either **Sets** or **Vec**. Given any $F, G: \mathbf{R}^d \to \mathscr{C}$, suppose that (f, g) is an ε-interleaving pair between F and G. For *each* $\mathbf{a} \in \mathbf{R}^d$, if $f_{\mathbf{a}}: F_{\mathbf{a}} \to G_{\mathbf{a}+\boldsymbol{\varepsilon}}$ and $g_{\mathbf{a}}: G_{\mathbf{a}} \to F_{\mathbf{a}+\boldsymbol{\varepsilon}}$ are surjective, then we call (f, g) a *surjective* ε-interleaving pair. If there exists a surjective ε-interleaving between F and G, we say that F and G are *fully* ε-interleaved. We define

$$\mathbf{d}^{\mathscr{C}}_{\mathrm{I},d}(F, G) := \inf \{\varepsilon \in [0, \infty) : F, G \text{ are } \textit{fully } \boldsymbol{\varepsilon}\text{-interleaved}\}.$$

We drop the subscript d from $\mathbf{d}^{\mathscr{C}}_{\mathrm{I},d}$ when confusion is unlikely. By definition, for any $F, G: \mathbf{R}^d \to \mathscr{C}$, it is clear that $d^{\mathscr{C}}_{\mathrm{I},d}(F, G) \leq \mathbf{d}^{\mathscr{C}}_{\mathrm{I},d}(F, G)$. Also, it is not difficult to check that $\mathbf{d}^{\mathscr{C}}_{\mathrm{I},d}$ is an extended pseudometric on $\mathrm{ob}(\mathscr{C}^{\mathbf{R}^d})$.

By utilizing the full interleaving distance $\mathbf{d}^{\mathscr{C}}_{\mathrm{I}}$, we obtain a lower bound for $d_{\mathtt{dyn}}$ as well as a new lower bound for the Gromov–Hausdorff distance (Theorem 4.5, Remark 4.13 and Theorem 4.14).

3.2 Interleaving Distance Between Integer-Valued Functions

In this section we consider the interleaving distance between monotonic integer-valued functions by regarding them as functors.

Poset-Valued Maps. Let $\mathbf{P}$ and $\mathbf{Q}$ be any two posets. Suppose that $f: \mathbf{P} \to \mathbf{Q}$ is any (monotonically) *increasing* map, i.e. for any $p \leq q$ in $\mathbf{P}$, $f(p) \leq f(q)$. Then by regarding $\mathbf{P}, \mathbf{Q}$ as categories, f can be regarded as a *functor*. On the other hand, suppose that $g: \mathbf{P} \to \mathbf{Q}$ is any (monotonically) *decreasing* map, i.e. for any $p \leq q$ in $\mathbf{P}$, $f(p) \geq f(q)$. Then $g: \mathbf{P} \to \mathbf{Q}^{\mathrm{op}}$ can also be called a functor.

The Interleaving Distance Between Integer-Valued Functions. Let d be a positive integer. Let $\mathbf{R}^d$ be the poset, where $\mathbf{a} = (a_1, \ldots, a_d) \leq \mathbf{b} = (b_1, \ldots, b_d)$ in $\mathbf{R}^d$ if and only if $a_i \leq b_i$ for each $i = 1, \ldots, d$. For any $\varepsilon > 0$, let $\boldsymbol{\varepsilon} := \varepsilon(1, \ldots, 1) \in \mathbf{R}^d$. Consider any non-increasing integer-valued function $F: \mathbf{R}^d \to \mathbf{Z}_+$. Note that F can be regarded as a *functor* from the poset category $\mathbf{R}^d$ to the other poset category $\mathbf{Z}_+^{\mathrm{op}}$. Since $\mathbf{Z}_+^{\mathrm{op}}$ is a *thin* category, given another functor $G: \mathbf{R}^d \to \mathbf{Z}_+^{\mathrm{op}}$, the interleaving distance (Definition 3.2) between F and G can be written a

$$d^{\mathbf{Z}_+^{\mathrm{op}}}_{\mathrm{I},d}(F, G) = \inf \left\{\varepsilon \in [0, \infty) : \forall \mathbf{a} \in \mathbf{R}^d, F_{\mathbf{a}} \geq G_{\mathbf{a}+\boldsymbol{\varepsilon}}, \text{ and } G_{\mathbf{a}} \geq F_{\mathbf{a}+\boldsymbol{\varepsilon}}\right\}.$$

The computational complexity for $d^{\mathbf{Z}_+^{\mathrm{op}}}_{\mathrm{I},d}$ is provided in Theorem 5.4. We will use $d_{\mathrm{I},d}$, or even more simply d_{I} in place of $d^{\mathbf{Z}_+^{\mathrm{op}}}_{\mathrm{I},d}$ when confusion is unlikely.

Remark 3.4 The metric d_{I} is closely related to the erosion distance [58]. See Remark 6.3.

4 Stability Theorems for Persistent Homology Features of DMSs

In this section we establish the main results of this paper: namely, stability of the rank invariant and Betti-0 function of DMSs (Sect. 4.1). We interpret these stability theorems as a generalization of the standard stability results for (static) metric spaces (Sect. 4.2).

4.1 Stability Theorems

Recall the spatiotemporal Rips filtration $\mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Simp}$ of a DMS (Definition 2.21). The poset $\mathbf{Int} \times \mathbf{R}_+$ can be regarded as an upper set of $\mathbf{R}^3_\times$ (Remarks 2.18 and 3.3 (ii)) and thus we can utilize $d_{\mathrm{I}}^{\mathbf{Vec}}$ for comparing $(\mathbf{Int} \times \mathbf{R}_+)$-indexed persistence modules.

Theorem 4.1 (Stability of spatiotemporal persistence modules induced by DMSs) *Let $\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$ be any two DMSs. Then for any $k \in \mathbf{Z}+$,*

$$d_{\mathrm{I}}^{\mathbf{Vec}} \left(\mathrm{H}_k(\mathscr{R}^{\mathrm{lev}}(\gamma_X)), \mathrm{H}_k(\mathscr{R}^{\mathrm{lev}}(\gamma_Y)) \right) \leq 2 \cdot d_{\mathtt{dyn}}(\gamma_X, \gamma_Y). \tag{5}$$

In particular, when $k = 0$, the $d_{\mathrm{I}}^{\mathbf{Vec}}$ in the LHS of the above inequality can be promoted to the full interleaving $\mathbf{d}_{\mathrm{I}}^{\mathbf{Vec}}$.

We remark that the promotion of $d_{\mathrm{I}}^{\mathbf{Vec}}$ to $\mathbf{d}_{\mathrm{I}}^{\mathbf{Vec}}$ for $k = 0$ is crucial for proving Theorem 4.5 below. See Sect. 6.2 for the proof of Theorem 4.1. This stability implies that $d_{\mathrm{I}}^{\mathbf{Vec}}$ between 3-dimensional persistence modules serves as a lower bound for $d_{\mathtt{dyn}}$. Since computing $d_{\mathrm{I}}^{\mathbf{Vec}}$ between 3-dimensional persistence modules is prohibitive [9], we make use of the *rank invariants/Betti-*0 *functions* of DMSs (Definitions 2.23 and 2.24) and the interleaving distance d_{I} between integer-valued functions (Sect. 3.2) to obtain a lower bound for $d_{\mathtt{dyn}}$ as below.

Adapted Rank Invariant of a DMS. The set $\underline{\mathbf{R}^6}$ in (3) is *not* an upper set (Remark 3.3 (ii)) of the poset

$$\mathbf{R}^6_\times := \mathbf{R} \times \mathbf{R}^{\mathrm{op}} \times \mathbf{R}^{\mathrm{op}} \times \mathbf{R}^{\mathrm{op}} \times \mathbf{R} \times \mathbf{R} \tag{6}$$

into which $(\mathbf{Int} \times \mathbf{R}_+)^{\mathrm{op}} \times (\mathbf{Int} \times \mathbf{R}_+)$ can be embedded. In order to ensure that we are in a position to utilize the metric d_{I} for comparing rank invariants of DMSs, we extend the domain of the rank invariant of a DMS to the poset $\mathbf{R}^6_\times$. Given any $(v_1, v_2, v_3) \in \mathbf{R}^3$, we write $(v_1, v_2, v_3) \in \mathbf{Int} \times \mathbf{R}_+$ if $v_1 \leq v_2$ and $v_3 \in \mathbf{R}_+$.

Any element $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbf{R}^6$, is called *admissible*, if $\mathbf{a}$ is obtained by concatenating a comparable pair of elements in $\mathbf{Int} \times \mathbf{R}_+$, i.e. both (a_1, a_2, a_3) and (a_4, a_5, a_6) belong to $\mathbf{Int} \times \mathbf{R}_+$ and $(a_1, a_2, a_3) \leq (a_4, a_5, a_6)$ in $\mathbf{Int} \times \mathbf{R}_+$. Otherwise,

$\mathbf{a}$ is called *non-admissible*. In particular, $\mathbf{a}$ is called *trivially* non-admissible, if there is no admissible $\mathbf{b} \in \mathbf{R}^6$ such that $\mathbf{b} < \mathbf{a}$ in the poset $\mathbf{R}^6_\times$ (one can check that $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbf{R}^6$ is trivially non-admissible if and only if (a_4, a_5, a_6) does not belong to $\mathbf{Int} \times \mathbf{R}_+$, i.e. $a_4 > a_5$ or $a_6 < 0$).

Definition 4.2 (*Adapted rank invariant of a DMS*) Let $\gamma_X = (X, d_X(\cdot))$ be any DMS and let $k \in \mathbf{Z}_+$. We define the map $\mathrm{rk}_k(\gamma_X) \colon \mathbf{R}^6 \to \mathbf{Z}_+ \cup \{\infty\}$, called the *k-th rank invariant of* γ_X, as follows: For $\mathbf{a} = (a_1, \ldots, a_6) \in \mathbf{R}^6$,

$$\mathrm{rk}_k(\gamma_X)(\mathbf{a}) := \begin{cases} \mathrm{rank}\left(\mathrm{H}_k\left(\mathscr{R}_\delta\left(\bigvee_I d_X\right) \hookrightarrow \mathscr{R}_{\delta'}\left(\bigvee_{I'} d_X\right)\right)\right), & \mathbf{a} \text{ is admissible,} \\ \infty, & \mathbf{a} \text{ is trivially non-admissible,} \\ 0, & \text{otherwise,} \end{cases}$$

where $I = [a_1, a_2]$, $I' = [a_4, a_5]$, $\delta = a_3$, and $\delta' = a_6$.

Note that when $\mathbf{a} \in \mathbf{R}^6$ is a concatenation of a *repeated* pair $([t_0, t_0], \delta_0), ([t_0, t_0], \delta_0) \in \mathbf{Int} \times \mathbf{R}_+$, i.e. $\mathbf{a} = (t_0, t_0, \delta_0, t_0, t_0, \delta_0)$, then

$$\mathrm{rk}_0(\gamma_X)(\mathbf{a}) = \dim\left(\mathrm{H}_0\left(\mathscr{R}_{\delta_0}(X, d_X(t_0))\right)\right) = \beta_0^{\gamma_X}(t_0, t_0, \delta_0) \text{ (Definition 2.24).}$$

We can regard $\mathrm{rk}_k(\gamma_X)$ as a *functor* $\mathbf{R}^6_\times \to (\mathbf{Z}_+ \cup \{\infty\})^{\mathrm{op}}$:

Proposition 4.3 *Let γ_X be any DMS. For any $\mathbf{a}, \mathbf{b} \in \mathbf{R}^6_\times$ with $\mathbf{a} \leq \mathbf{b}$,*

$$\mathrm{rk}_k(\gamma_X)(\mathbf{a}) \geq \mathrm{rk}_k(\gamma_X)(\mathbf{b}) \text{ in } \mathbf{Z}_+ \cup \{\infty\}.$$

See Sect. 6.3 for the proof. By virtue of Proposition 4.3, d_{I} can serve as a metric on the collection of all (adapted) rank invariants of DMSs.

By combining Theorem 4.1 with standard stability results for the rank invariant (Theorem 6.2) we arrive at:

Theorem 4.4 (Stability of the rank invariant of DMSs) *Let $\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$ be any two DMSs. For any $k \in \mathbf{Z}_+$,*

$$d_{\mathrm{I}}\left(\mathrm{rk}_k(\gamma_X), \mathrm{rk}_k(\gamma_Y)\right) \leq 2 \cdot d_{\mathtt{dyn}}(\gamma_X, \gamma_Y). \tag{7}$$

Improvement for $k = 0$. By restricting ourselves to clustering information (i.e. 0-th homology) of DMSs, we obtain a stronger lower bound for the metric $d_{\mathtt{dyn}}$. Namely, by regarding the Betti-0 function of a DMS (Definition 2.24) as a functor $\mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Z}_+^{\mathrm{op}}$, we can compare any two Betti-0 functions of DMSs via the interleaving distance d_{I} and we have:

Theorem 4.5 (Stability of the Betti-0 function) *Let γ_X and γ_Y be any two DMSs. Then*

$$d_{\mathrm{I}}\left(\beta_0^{\gamma_X}, \beta_0^{\gamma_Y}\right) \leq 2 \cdot d_{\mathtt{dyn}}(\gamma_X, \gamma_Y). \tag{8}$$

Fig. 5 The geometric realization of $\mathscr{R}_0\big(X, \bigvee_{[\frac{\pi}{2},\frac{3\pi}{2}]} d_X\big)$ and $\mathscr{R}_0\big(Y, \bigvee_{[\frac{\pi}{2},\frac{3\pi}{2}]} d_Y\big)$ for the DMSs γ_X and γ_Y in Example 2.25

We prove Theorem 4.5 in Sect. 6.4. Also, we remark that the LHSs of inequalities in (7) and (8) are computable in poly-time (Theorem 5.4) using the well-known *binary search* algorithm.

Remark 4.6 (*Sensitivity of the LHS in* (8)) Consider the DMSs γ_X and γ_Y given as in Example 2.25. The value $d_{\mathrm{I}}\left(\beta_0^{\gamma_X}, \beta_0^{\gamma_Y}\right)$ is at least r, as we will see below. This in turn implies that the metric d_{I} is sensitive enough to discriminate (the Betti-0 functions of) γ_X and γ_Y.

Details about Remark 4.6 Observe that

$$\bigvee_{[\frac{\pi}{2},\frac{3\pi}{2}]} d_X(x_i, x_j) = \begin{cases} 2, & i=1, j=3, \\ 0, & \text{otherwise}, \end{cases} \quad \bigvee_{[\frac{\pi}{2},\frac{3\pi}{2}]} d_Y(y_i, y_j) = \begin{cases} 1, & i=1, j=2, \\ 2, & i=1, j=3, \\ 0, & \text{otherwise}. \end{cases}$$

Hence, the geometric realization of Rips complexes $\mathscr{R}_0\big(X, \bigvee_{[\frac{\pi}{2},\frac{3\pi}{2}]} d_X\big)$ and $\mathscr{R}_0\big(Y, \bigvee_{[\frac{\pi}{2},\frac{3\pi}{2}]} d_Y\big)$ are illustrated in Fig. 5.

By counting the number of connected components of these complexes, we have $\beta_0^{\gamma_X}\big([\frac{\pi}{2},\frac{3\pi}{2}], 0\big) = 1$ and $\beta_0^{\gamma_Y}\big([\frac{\pi}{2},\frac{3\pi}{2}], 0\big) = 2$. Also, it is not difficult to check that for any $\varepsilon \in [0, r)$, $\mathscr{R}_\varepsilon\big(Y, \bigvee_{[\frac{\pi}{2}-\varepsilon,\frac{3\pi}{2}+\varepsilon]} d_Y\big) = \mathscr{R}_0\big(Y, \bigvee_{[\frac{\pi}{2},\frac{3\pi}{2}]} d_Y\big)$, so that

$$\beta_0^{\gamma_X}\left(\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], 0\right) = 1 < 2 = \beta_0^{\gamma_Y}\left(\left[\frac{\pi}{2} - \varepsilon, \frac{3\pi}{2} + \varepsilon\right], \varepsilon\right).$$

By the definition of d_{I}, this inequality implies that $d_{\mathrm{I}}(\beta_0^{\gamma_X}, \beta_0^{\gamma_Y})$ is at least r.

Next, we show that $d_{\mathrm{I}}(\beta_0^{\gamma_X}, \beta_0^{\gamma_Y}) \leq 2r$. For any $\varepsilon \in [2r, \infty)$ and any $I \in \mathbf{Int}$,

$$\beta_0^{\gamma_X}(I, \varepsilon) = \beta_0^{\gamma_Y}(I, \varepsilon) = 1,$$

which is illustrated in Fig. 4. Therefore, for any $([t_1, t_2], \delta) \in \mathbf{Int} \times \mathbf{R}_+$,

$$\begin{aligned} \beta_0^{\gamma_X}([t_1, t_2], \delta) &\geq \beta_0^{\gamma_Y}([t_1 - 2r, t_2 + 2r], \delta + 2r) = 1, \\ \beta_0^{\gamma_Y}([t_1, t_2], \delta) &\geq \beta_0^{\gamma_X}([t_1 - 2r, t_2 + 2r], \delta + 2r) = 1. \end{aligned}$$

Therefore, we have $d_{\mathrm{I}}(\beta_0^{\gamma_X}, \beta_0^{\gamma_Y}) \leq 2r$. □

In order to obtain a lower bound for d_{dyn} between two DMSs, computing the distance between the Betti-0 functions of the DMSs (the LHS of the inequality in (8)) is better than computing the distance between their 0-th rank invariants (the LHS of the inequality in (7)):

Proposition 4.7 *For any two DMSs $\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$,*

$$d_{\mathrm{I},6}\left(\mathrm{rk}_0(\gamma_X), \mathrm{rk}_0(\gamma_Y)\right) \leq d_{\mathrm{I},3}(\beta_0^{\gamma_X}, \beta_0^{\gamma_Y}). \tag{9}$$

Proposition 4.7 is a corollary of Proposition 6.10. The proof relies on the fact that all inner morphisms of the persistence modules $\mathrm{H}_0\left(\mathscr{R}^{\mathrm{lev}}(\gamma_X)\right)$ and $\mathrm{H}_0\left(\mathscr{R}^{\mathrm{lev}}(\gamma_Y)\right)$ are *surjective*. In Example 4.16, we consider a concrete example of the bound provided in Proposition 4.7.

4.2 Relationship with Standard Stability Theorems

The main goal of this section is to explain, when restricting ourselves to the class of constant DMSs (Example 2.2 (i)), how Theorems 4.1, 4.4 and 4.5 boil down to the well-known stability theorems for (static) metric spaces. Along the way, we also identify a new lower bound for the Gromov–Hausdorff distance, which is tighter than the bottleneck distance between the 0-th persistence diagrams of Rips filtrations (Remark 4.13 and Theorem 4.14).

For $k \in \mathbf{Z}_+$, by post-composing the simplicial homology functor $\mathrm{H}_k : \mathbf{Simp} \to \mathbf{Vec}$ (with coefficients in the field $\mathbb{F}$) to the Rips filtration $\mathscr{R}_\bullet(X, d_X)$ of a metric space (X, d_X), we obtain the persistence module

$$\mathrm{H}_k \circ \mathscr{R}_\bullet(X, d_X) \colon \mathbf{R} \to \mathbf{Vec}.$$

Let $\mathrm{dgm}_k(\mathscr{R}_\bullet(X, d_X))$ be the k-th persistence diagram of the Rips filtration $\mathscr{R}_\bullet(X, d_X)$. Also, let d_{B} be the bottleneck distance (Definition C.1). Recall that d_{dyn} coincides with d_{GH} on the class of constant DMSs (Remark 2.12).

Remark 4.8 Consider any two constant DMSs $\gamma_X \equiv (X, d_X)$ and $\gamma_Y \equiv (Y, d_Y)$. Then, for any $k \in \mathbf{Z}_+$, inequality (5) reduces to

$$d_{\mathrm{I}}^{\mathbf{Vec}}\left(\mathrm{H}_k \circ \mathscr{R}_\bullet(X, d_X), \mathrm{H}_k \circ \mathscr{R}_\bullet(Y, d_Y)\right) \leq 2\, d_{\mathrm{GH}}((X, d_X), (Y, d_Y)), \tag{10}$$

or equivalently to

$$d_{\mathrm{B}}\left(\mathrm{dgm}_k\left(\mathscr{R}_\bullet(X, d_X)\right), \mathrm{dgm}_k\left(\mathscr{R}_\bullet(Y, d_Y)\right)\right) \leq 2 \cdot d_{\mathrm{GH}}\left((X, d_X), (Y, d_Y)\right), \tag{11}$$

which are known in [22,23]. In other words, the LHS and the RHS of inequality (5) are respectively identical to the LHS and the RHS of inequalities (10) or (11).

We define the *rank invariant* of a finite metric space as follows:

Definition 4.9 (*The rank invariant of a metric space*) Let (X, d_X) be any finite metric space and let $k \in \mathbf{Z}_+$. We define the map $\mathrm{rk}_k(X, d_X)\colon \mathbf{R}^2 \to \mathbf{Z}_+ \cup \{\infty\}$, called the *k-th rank invariant of* (X, d_X), as follows: For $\mathbf{a} = (\delta, \delta') \in \mathbf{R}^2$,

$$\mathrm{rk}_k(X, d_X)(\mathbf{a}) = \begin{cases} \mathrm{rank}\left(\mathrm{H}_k\left(\mathscr{R}_\delta(X, d_X) \hookrightarrow \mathscr{R}_{\delta'}(X, d_X)\right)\right), & \delta \leq \delta', \\ \infty, & \text{otherwise} \end{cases}$$

(cf. Definition 4.2).

In Definition 4.9, note that we can regard $\mathrm{rk}_k(X, d_X)$ as a functor $\mathbf{R}^{\mathrm{op}} \times \mathbf{R} \to (\mathbf{Z}_+ \cup \{\infty\})^{\mathrm{op}}$. Therefore, we can compare the rank invariants of any two finite metric metric spaces via the interleaving distance d_{I}.

Remark 4.10 Consider any two constant DMSs $\gamma_X \equiv (X, d_X)$ and $\gamma_Y \equiv (Y, d_Y)$. Then, for any $k \in \mathbf{Z}_+$, inequality (7) reduces to

$$d_{\mathrm{I}}(\mathrm{rk}_k(X, d_X), \mathrm{rk}_k(Y, d_Y)) \leq 2\, d_{\mathrm{GH}}((X, d_X), (Y, d_Y)). \tag{12}$$

Remark 4.11 We also remark that the LHS of (11) is greater than or equal to that of (12) by Corollary 6.4:

$$\begin{aligned} d_{\mathrm{I}}(\mathrm{rk}_k(X, d_X), \mathrm{rk}_k(Y, d_Y)) &\leq d_{\mathrm{B}}\left(\mathrm{dgm}_k\left(\mathscr{R}_\bullet(X, d_X)\right), \mathrm{dgm}_k\left(\mathscr{R}_\bullet(Y, d_Y)\right)\right) \\ &\leq 2 \cdot d_{\mathrm{GH}}((X, d_X), (Y, d_Y)). \end{aligned}$$

Definition 4.12 (*The Betti-*0 *function of a finite metric space*) Let (X, d_X) be any finite metric space. We define the *Betti-*0 *function* $\beta_0^{(X,d_X)}\colon \mathbf{R}_+ \to \mathbf{Z}_+$ of (X, d_X) by sending each $\delta \in \mathbf{R}_+$ to the dimension of $\mathrm{H}_0(\mathscr{R}_\delta(X, d_X))$ (cf. Definition 2.24).

Since $\beta_0^{(X,d_X)}$ is non-increasing function and $\mathbf{R}_+$ is an upper set of $\mathbf{R}$, we can compare any two Betti-0 functions via d_{I}.

Remark 4.13 (*Stability of the Betti-*0 *function*) Consider any two constant DMSs $\gamma_X \equiv (X, d_X)$ and $\gamma_Y \equiv (Y, d_Y)$. Then the inequality in (8) reduces to

$$d_{\mathrm{I}}\left(\beta_0^{(X,d_X)}, \beta_0^{(Y,d_Y)}\right) \leq 2\, d_{\mathrm{GH}}\left((X, d_X), (X, d_Y)\right). \tag{13}$$

In particular, as a lower bound for $2 \cdot d_{\mathrm{GH}}$, the LHS of inequality (13) is always as effective as the LHS of inequality (11) for $k = 0$:

Theorem 4.14 *For any finite metric spaces* (X, d_X) *and* (Y, d_Y),

$$d_{\mathrm{B}}\left(\mathrm{dgm}_0\left(\mathscr{R}_\bullet(X, d_X)\right), \mathrm{dgm}_0\left(\mathscr{R}_\bullet(Y, d_Y)\right)\right) \leq d_{\mathrm{I}}\left(\beta_0^{(X,d_X)}, \beta_0^{(Y,d_Y)}\right).$$

The proof is provided in Sect. 6.5. Example 4.15 below illustrates Theorem 4.14.

Example 4.15 Let $X = \{x_1, x_2\}$. For any $\varepsilon \in [0, \infty)$, we define the two metrics d_X and d_X^ε on X as

$$d_X(x_1, x_2) = 1 \text{ and } d_X^\varepsilon(x_1, x_2) = 1 + \varepsilon.$$

By definition of d_{GH} (Definitions D.1) and d_{I} (Sect. 3.2), one can check the following:

(i) $2\, d_{\mathrm{GH}}\left((X, d_X), (X, d_X^\varepsilon)\right) = \varepsilon$.

(ii) $\beta_0^{(X,d_X)}(\delta) = \begin{cases} 2, & \delta < [0, 1), \\ 1, & \delta \in [1, +\infty) \end{cases}$ and $\beta_0^{(X,d_X^\varepsilon)}(\delta) = \begin{cases} 2, & \delta < [0, 1+\varepsilon), \\ 1, & \delta \in [1+\varepsilon, +\infty). \end{cases}$

Also,

$$d_{\mathrm{I}}\left(\beta_0^{(X,d_X)}, \beta_0^{(Y,d_Y)}\right) = \varepsilon.$$

(iii) $\mathrm{dgm}_0\left(\mathscr{R}_\bullet(X, d_X)\right) = \{(0, +\infty), (0, 1)\}$, and $\mathrm{dgm}_0\left(\mathscr{R}_\bullet(X, d_X^\varepsilon)\right) = \{(0, +\infty), (0, 1+\varepsilon)\}$. Also,

$$d_{\mathrm{B}}\left(\mathrm{dgm}_0\left(\mathscr{R}_\bullet(X, d_X)\right), \mathrm{dgm}_0(\mathscr{R}_\bullet(X, d_X^\varepsilon))\right) = \min\left(\varepsilon, \frac{1+\varepsilon}{2}\right).$$

(iv) For $k \geq 1$, both $\mathrm{dgm}_k\left(\mathscr{R}_\bullet(X, d_X)\right)$ and $\mathrm{dgm}_k\left(\mathscr{R}_\bullet(Y, d_Y)\right)$ are the empty set, and thus

$$d_{\mathrm{B}}\left(\mathrm{dgm}_k\left(\mathscr{R}_\bullet(X, d_X)\right), \mathrm{dgm}_k(\mathscr{R}_\bullet(X, d_X^\varepsilon))\right) = 0.$$

Items (iii) and (iv) indicate that the best lower bound for $2\, d_{\mathrm{GH}}((X, d_X), (X, d_X^\varepsilon))$ obtained by invoking inequality (11) is $\min\left(\varepsilon, \frac{1+\varepsilon}{2}\right)$. On the other hand, from items (i) and (ii), we have

$$\varepsilon = 2\, d_{\mathrm{GH}}\left((X, d_X), (X, d_X^\varepsilon)\right) = d_{\mathrm{I}}\left(\beta_0^{(X,d_X)}, \beta_0^{(Y,d_Y)}\right),$$

which is, when $\varepsilon > 1$, strictly larger than $\min\left(\varepsilon, \frac{1+\varepsilon}{2}\right)$. This example demonstrates inequality (13) is a complement to the bottleneck stability of Rips filtration, inequality (11). Also, items (i) and (ii) show the tightness of inequality (13).

Example 4.16 Define two DMSs γ_X and γ_X' to be the *constant* DMSs which are, for every time $t \in \mathbf{R}$, isometric respectively to the metric spaces (X, d_X) and (X, d_X^ε) in Example 4.15. Then, invoking Remarks 4.10 and 4.13, one can compute:

$$d_{\mathrm{I},6}\left(\mathrm{rk}_0(\gamma_X), \mathrm{rk}_0(\gamma_X')\right) = d_{\mathrm{I},2}\left(\mathrm{rk}_0(X, d_X), \mathrm{rk}_0(X, d_X^\varepsilon)\right) = \min\left(\frac{1+\varepsilon}{2}, \varepsilon\right),$$

$$d_{\mathrm{I},3}\left(\beta_0^{\gamma_X}, \beta_0^{\gamma_X'}\right) = d_{\mathrm{I},1}\left(\beta_0^{(X,d_X)}, \beta_0^{(X,d_X^\varepsilon)}\right) = \varepsilon.$$

See below for computational details. When $\varepsilon > 1$, this example demonstrates that the RHS of inequality (9) can be strictly larger.

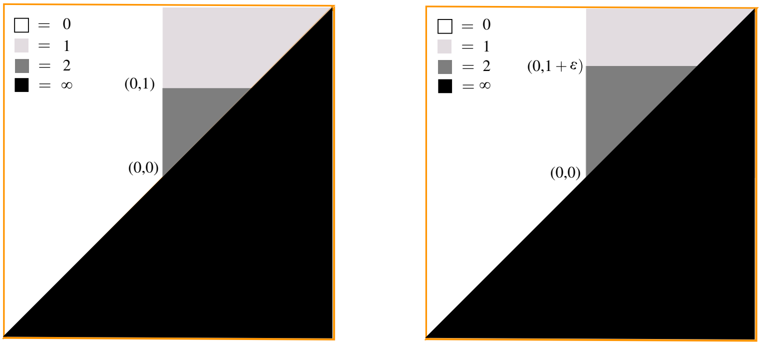

Fig. 6 The 0-th rank invariants of (X, d_X) and (X, d_X^ε) in Example 4.15

Details about Example 4.16 One can compute $\mathrm{rk}_0(X, d_X), \mathrm{rk}_0(X, d_X^\varepsilon) : \mathbf{R}^2 \to (\mathbf{Z}_+ \cup \{\infty\})^{\mathrm{op}}$ (Definition 4.9) as illustrated in Fig. 6.

From this plot, one can check that

$$d_{\mathrm{I},2}\left(\mathrm{rk}_0(X, d_X), \mathrm{rk}_0(X, d_X^\varepsilon)\right) = \begin{cases} \varepsilon, & \varepsilon \leq [0, 1], \\ \frac{1+\varepsilon}{2}, & \varepsilon \in (1, \infty), \end{cases}$$

which amounts to

$$d_{\mathrm{I},2}\left(\mathrm{rk}_0(X, d_X), \mathrm{rk}_0(X, d_X^\varepsilon)\right) = \min\left(\frac{1+\varepsilon}{2}, \varepsilon\right).$$

We already computed $\beta_0^{(X,d_X)}$ and $\beta_0^{(X,d_X^\varepsilon)}$ in Example 4.15. Observe that the value

$$\min\Big\{\alpha \in [0, \infty) : \forall \delta \in [0, \infty),\ \beta_0^{(X,d_X)}(\delta + \alpha) \leq \beta_0^{(X,d_X^\varepsilon)}(\delta),$$
$$\beta_0^{(X,d_X^\varepsilon)}(\delta + \alpha) \leq \beta_0^{(X,d_X^\varepsilon)}(\delta)\Big\}$$

is equal to ε. This implies that $d_{\mathrm{I},1}\left(\beta_0^{(X,d_X)}, \beta_0^{(X,d_X^\varepsilon)}\right) = \varepsilon$. □

5 Computing the Interleaving Distance Between Integer-Valued Functions

In this section we propose an algorithm for computing the interleaving distance between integer-valued functors based on ordinary binary search.

For $n \in \mathbf{N}$, let $[n] := \{1, \ldots, n\}$. Also, for each $d \in \mathbf{N}$, let $[n]^d \subset \mathbf{Z}^d$ be the subposet of $\mathbf{Z}^d$. Assume that $\mathbf{a} = (a_1, \ldots, a_d) \in [n]^d$. If there exists $i \in \{1, \ldots, d\}$ such that $a_i = n$, we refer to $\mathbf{a}$ as an *upper boundary point of* $[n]^d$.

Let $F\colon [n]^d \to \mathbf{Z}_+$ be any function. Then F can be regarded as an array of non-negative integers. For each $k \in \{0, \ldots, n-1\}$, the restriction $F|_{[n-k]^d}$ of F is the *lower-left block* of F. Symmetrically, we define the *upper-right block* $F|^{[n-k]^d}\colon [n-k]^d \to \mathbf{Z}_+$ of F as follows:

$$\left(F|^{[n-k]^d}\right)_{\mathbf{a}} = F_{\mathbf{a}+k(1,\ldots,1)} \text{ for } \mathbf{a} \in [n-k]^d.$$

In words, $F|^{[n-k]^d}$ is the restriction of the array F to its upper-right corner of size $(n-k)^d$ with a re-indexing (in the obvious way).

Given $F, G\colon [n]^d \to \mathbf{Z}_+$, we write $F \geq G$ if $F_{\mathbf{a}} \geq G_{\mathbf{a}}$ for all $\mathbf{a} \in [n]^d$. Let $F, G\colon [n]^d \to \mathbf{Z}_+$ be any two order-reversing functions with $0 = F_{\mathbf{a}} = G_{\mathbf{a}}$ for each upper boundary point $\mathbf{a} \in [n]^d$. For each $k \in \{0, \ldots, n-1\}$, we define the k-**test** for the pair (F, G):

Algorithm 1 k-test for $F, G\colon [n]^d \to \mathbf{Z}_+$.

if $F|_{[n-k]^d} \geq G|^{[n-k]^d}$ **and** $G|_{[n-k]^d} \geq F|^{[n-k]^d}$ **then return** Yes.
else return No.

Remark 5.1 Let $F, G\colon [n]^d \to \mathbf{Z}_+$ be any two order-reversing functions with $0 = F_{\mathbf{a}} = G_{\mathbf{a}}$ for each upper boundary point $\mathbf{a} \in [n]^d$. Fix $k \in \{0, \ldots, n-1\}$. Then

(i) suppose that the k-test for (F, G) returns "Yes". Then for any $k' \in \{k, \ldots, n-1\}$ the k'-test for (F, G) returns also "Yes",
(ii) the $(n-1)$-test for (F, G) always returns "Yes".

Example 5.2 We consider two examples.

(A) $(d = 1)$ Consider $F, G\colon [4] \to \mathbf{Z}_+$ defined as follows:

$$F := (F_1, F_2, F_3, F_4) = (5, 3, 1, 0), \quad G := (G_1, G_2, G_3, G_4) = (4, 3, 2, 0).$$

Since $F \not\geq G$ nor $G \not\geq F$, the 0-test for (F, G) returns "No". However, since

$$F|_{[3]} = (5, 3, 1) \geq (3, 2, 0) = G|^{[3]}, \text{ and } G|_{[3]} = (4, 3, 2) \geq (3, 1, 0) = F|^{[3]},$$

the 1-test for (F, G) returns "Yes". Also, one can check that for any $k \in \{2, 3\}$, the k-test returns "Yes" (cf. Remark 5.1 (i)).

(B) $(d = 2)$ Consider $F, G\colon [3]^2 \to \mathbf{Z}_+$ defined as follows:

$$F := \begin{array}{|c|c|c|}\hline F_{(1,3)} & F_{(2,3)} & F_{(3,3)} \\ \hline F_{(1,2)} & F_{(2,2)} & F_{(3,2)} \\ \hline F_{(1,1)} & F_{(1,2)} & F_{(1,3)} \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 3 & 3 & 0 \\ \hline 4 & 3 & 0 \\ \hline \end{array}, \quad G := \begin{array}{|c|c|c|}\hline G_{(1,3)} & G_{(2,3)} & G_{(3,3)} \\ \hline G_{(1,2)} & G_{(2,2)} & G_{(3,2)} \\ \hline G_{(1,1)} & G_{(1,2)} & G_{(1,3)} \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 2 & 2 & 0 \\ \hline \end{array}.$$

Since $G \not\geq F$, the 0-test for (F, G) returns "No". Also, since

$$G|_{[2]^2} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \not\geq \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 3 & 0 \\ \hline \end{array} = F|^{[2]^2},$$

the 1-test returns "No". Since $4 \geq 0$ and $2 \geq 0$, one can see that the 2-test returns "Yes".

Recall the poset category $\mathbf{Z}_+^{\mathrm{op}}$: for any $p, q \in \mathbf{Z}_+$, there exists the unique arrow $p \to q$ if and only if $p \geq q$. A *function* $\overline{F}: \mathbf{N}^d \to \mathbf{Z}_+$ can be regarded as a *functor* $\overline{F}: \mathbf{N}^d \to \mathbf{Z}_+^{\mathrm{op}}$ if and only if $\overline{F}: \mathbf{N}^d \to \mathbf{Z}_+$ is order-reversing.

By the definition of interleaving distance, we straightforwardly have:

Proposition 5.3 *For $n, d \in \mathbf{N}$, let $\overline{F}, \overline{G}: \mathbf{N}^d \to \mathbf{Z}_+^{\mathrm{op}}$ be any two functors with $0 = F_{\mathbf{a}} = G_{\mathbf{a}}$ for each upper boundary point $\mathbf{a} \in [n]^d$. Consider the restrictions $F := \overline{F}|_{[n]^d}$ and $G := \overline{G}|_{[n]^d}$. Then*

$$d_{\mathrm{I}}\left(\overline{F}, \overline{G}\right) = \min\left\{k \in \{0, 1, \ldots, n-1\} : \textit{ the } k\textit{-test for } (F, G) \textit{ returns "Yes"}\right\}.$$

Computational Complexity of Computing the Rank Invariant. Let **Vec** be the category of vector spaces over a fixed field $\mathbb{F}$ with linear maps. Let $M: [n]^d \to \mathbf{Vec}$ be a (finite) multidimensional module. Let $\mathrm{total}(M) := \sum_{\mathbf{a} \in [n]^d} \dim(M_{\mathbf{a}})$. In order to compute the rank invariant $\mathrm{rk}(M): [n]^d \to \mathbf{Z}_+$, one needs $O(\mathrm{total}(M)^{\omega})$ operations [7, Append. C], where ω is the matrix multiplication exponent.

Proposed Algorithm for Computing d_{I} and its Computational Complexity. Let $F, G: [n]^d \to \mathbf{Z}_+$ be any two order-reversing functions. Based on Proposition 5.3, in order to find the minimal $k \in \{0, \ldots, n-1\}$ for which the k-test for (F, G) (Algorithm 1) returns "Yes", we carry out **binary search**.

Let us fix $k \in \{0, \ldots, n-1\}$. For carrying out the k-test for (F, G), we compare pairs of integers from the arrays of F and G. *Assume that pairs of integers are compared one by one.* Then, notice that, depending on F and G, the number of comparisons which are necessary to complete the k-test can vary from 1 to $2(n-k)^d$. Under the assumption that the number of required comparisons is a random variable uniformly distributed in $\{1, \ldots, 2(n-d)^k\}$ one can conclude that $\frac{1+2(n-d)^k}{2} \approx (n-d)^k$ comparisons are needed on average. Under the preceeding assumptions, by results from [49, Sect. 4], we directly have:

Theorem 5.4 *The expected cost of computing $d_{\mathrm{I},d}(F, G)$ is at least $O(n^d \log n)$. Furthermore, the algorithm based on ordinary binary search has this expected cost.*

By Theorem 5.4, the expected costs of computing the LHSs of inequalities in Theorems 4.4 and 4.5, and Remarks 4.10 and 4.13 are $O(n^d \log n)$ where $d = 6, 3, 2$ and 1, respectively in order.

In Appendix C we compare $d_{\mathrm{I},d}$ with the *matching distance* [18,20,51], and with the *dimension distance* [28, Sect. 4].

6 Details About Stability Theorems

The goal of this section is to prove all theorems in Sect. 4 whose proof was not given therein.

6.1 Interleaving Stability of Rank Invariants and Dimension Functions

The Rank Invariant and Its Stability. For any persistence module $M\colon \mathbf{R}^d \to \mathbf{Vec}$, the rank invariant of M is defined as follows [17]:

Definition 6.1 (*The rank invariant*) For any $M\colon \mathbf{R}^d \to \mathbf{Vec}$, the map $\mathrm{rk}(M)\colon \mathbf{R}^{2d} \to \mathbf{Z}_+ \cup \{\infty\}$ defined as

$$\mathrm{rk}(M)(\mathbf{a},\mathbf{b}) := \begin{cases} \mathrm{rk}(\varphi_M(\mathbf{a},\mathbf{b})), & \mathbf{a} \leq \mathbf{b} \ \in \mathbf{R}^d, \\ \infty, & \text{otherwise} \end{cases}$$

is called the *rank invariant* of M.

Given any $M\colon \mathbf{R}^d \to \mathbf{Vec}$, note that for any $\mathbf{a}' \leq \mathbf{a} \leq \mathbf{b} \leq \mathbf{b}'$ in $\mathbf{R}^d$,

$$\varphi_M(\mathbf{a}',\mathbf{b}') = \varphi_M(\mathbf{b},\mathbf{b}') \circ \varphi_M(\mathbf{a},\mathbf{b}) \circ \varphi_M(\mathbf{a}',\mathbf{a}).$$

Hence, we have that $\mathrm{rk}(M)(\mathbf{a}',\mathbf{b}') \leq \mathrm{rk}(M)(\mathbf{a},\mathbf{b})$. This means that $\mathrm{rk}(M)$ is a *functor* between its domain and codomain when regarded

(i) the domain $\mathbf{R}^{2d}$ as the product poset $(\mathbf{R}^d)^{\mathrm{op}} \times \mathbf{R}^d$, and
(ii) the codomain $\mathbf{Z}_+ \cup \{\infty\}$ as the poset $(\mathbf{Z}_+ \cup \{\infty\})^{\mathrm{op}}$.

We have *stability* of the rank invariant:

Theorem 6.2 (Stability of the rank invariant [58, Thm. 8.2], [59, Thm. 22]) *For any* $M, N\colon \mathbf{R}^d \to \mathbf{Vec}$,

$$d_{\mathrm{I},2d}(\mathrm{rk}(M), \mathrm{rk}(N)) \leq d_{\mathrm{I}}^{\mathbf{Vec}}(M, N). \tag{14}$$

Note that Theorem 6.2 together with Theorem 4.1 result in Theorem 4.4. Even though the proof of Theorem 6.2 is given in [58, Thm. 8.2], [59, Thm. 22] in more general setting, we provide a brief version of the proof here.

Proof Since we regard $\mathrm{rk}(M)$ as a functor from $(\mathbf{R}^d, \geq) \times (\mathbf{R}^d, \leq)$ to $(\mathbf{Z}_+ \cup \{\infty\})^{\mathrm{op}}$, for any $\varepsilon \in [0,\infty)$, the ε-shift $\mathrm{rk}(M)(\varepsilon)\colon (\mathbf{R}^d, \geq) \times (\mathbf{R}^d, \leq) \to (\mathbf{Z}_+ \cup \{\infty\})^{\mathrm{op}}$ of $\mathrm{rk}(M)$ is defined as

$$\mathrm{rk}(M)(\varepsilon)_{(\mathbf{a},\mathbf{b})} = \mathrm{rk}(M)_{(\mathbf{a}-\boldsymbol{\varepsilon},\mathbf{b}+\boldsymbol{\varepsilon})}.$$

Similarly, the ε-shift of $\mathrm{rk}(N)$ is defined.

Suppose that for some $\varepsilon \in [0,\infty)$, the pair (f,g) is an ε-interleaving pair for $M, N\colon \mathbf{R}^d \to \mathbf{Vec}$ (Definition 3.2). We show $\mathrm{rk}(N)(\varepsilon) \leq \mathrm{rk}(M)$. Pick any $(\mathbf{a},\mathbf{b}) \in \mathbf{R}^d \times \mathbf{R}^d$. If $\mathbf{a} \not\leq \mathbf{b}$ in $\mathbf{R}^d$, then $\mathrm{rk}(M)(\mathbf{a},\mathbf{b}) = \infty$, and thus we trivially have $\mathrm{rk}(N)_{(\mathbf{a}-\boldsymbol{\varepsilon},\mathbf{b}+\boldsymbol{\varepsilon})} \leq \mathrm{rk}(M)_{(\mathbf{a},\mathbf{b})}$. If $\mathbf{a} \leq \mathbf{b}$ in $\mathbf{R}^d$, then $\mathbf{a}-\boldsymbol{\varepsilon} \leq \mathbf{b}+\boldsymbol{\varepsilon}$, and since

$$\varphi_N(\mathbf{a}-\boldsymbol{\varepsilon},\mathbf{b}+\boldsymbol{\varepsilon}) = f_{\mathbf{b}} \circ \varphi_M(\mathbf{a},\mathbf{b}) \circ g_{\mathbf{a}},$$

we have $\mathrm{rk}(N)_{(\mathbf{a}-\boldsymbol{\varepsilon},\mathbf{b}+\boldsymbol{\varepsilon})} \leq \mathrm{rk}(M)_{(\mathbf{a},\mathbf{b})}$. By symmetry, we also have $\mathrm{rk}(M)(\varepsilon) \leq \mathrm{rk}(N)$, completing the proof. □

Remark 6.3 In order to compare the rank invariants, the author of [59] makes use of a generalization of the *erosion distance* in [58], which is denoted by d_{E} (see Appendix C). It can be deduced that for the LHS of inequality (14) coincides with $d_{\mathrm{E}}(\mathrm{rk}(M), \mathrm{rk}(N))$.

Given $\delta > 0$, deciding whether $d_{\mathrm{I}}^{\mathbf{Vec}}(M,N) \leq \delta$ is in general *NP-hard* [7,8].

In Theorem 6.2, substituting the comparison of M and N with that of $\mathrm{rk}(M)$ and $\mathrm{rk}(N)$ results in doubling of the underlying dimension of the interleaving distance. This increase of dimension is a price one must pay for substituting the target category $\mathbf{Vec}$ with the poset category $(\mathbf{Z}_+ \cup \{\infty\})^{\mathrm{op}}$. Despite the increase in the underlying dimension, as we show in Sect. 5, it turns out that computing d_{I} is easier than computing $d_{\mathrm{I}}^{\mathbf{Vec}}$.

For any *interval decomposable* modules $M, N\colon \mathbf{R}^d \to \mathbf{Vec}$, let $\mathscr{B}(M)$ and $\mathscr{B}(N)$ be the barcode of M and N, respectively. Then, by [9, Prop. 2.13],

$$d_{\mathrm{I}}^{\mathbf{Vec}}(M,N) \leq d_{\mathrm{B}}\left(\mathscr{B}(M), \mathscr{B}(N)\right).$$

Hence, together with Theorem 6.2 ,we straightforwardly have:

Corollary 6.4 *For any interval decomposable $M, N\colon \mathbf{R}^d \to \mathbf{Vec}$, let $\mathscr{B}(M)$ and $\mathscr{B}(N)$ be the barcode of M and N, respectively. Then*

$$d_{\mathrm{I},2d}(\mathrm{rk}(M), \mathrm{rk}(N)) \leq d_{\mathrm{B}}\left(\mathscr{B}(M), \mathscr{B}(N)\right).$$

Monotonicity and Stability of Dimension Functions for Surjective Modules

Definition 6.5 (*Surjective persistence modules*) Let $\mathscr{C}$ be either **Sets** or **Vec** and let $M\colon \mathbf{R}^d \to \mathscr{C}$ be any persistence module. We call M *surjective* if $\varphi_M(\mathbf{a},\mathbf{b})\colon M_{\mathbf{a}} \to M_{\mathbf{b}}$ is surjective for all $\mathbf{a} \leq \mathbf{b}$ in $\mathbf{R}^d$.

Example 6.6 (*The 0-th homology of the Rips filtration*) Let (X, d_X) be a metric space. By applying the 0-th (simplicial) homology functor to the *Rips filtration* of (X, d_X), we obtain surjective persistence module $\mathbf{R} \to \mathbf{Vec}$.

Definition 6.7 (*Dimension function*) Let $\mathscr{C}$ be either **Sets** or **Vec** and let $M\colon \mathbf{R}^d \to \mathscr{C}$ be any persistence module. The *dimension function* $\mathrm{dm}(M)\colon \mathbf{R}^d \to \mathbf{Z}_+$ of M is defined by sending each $\mathbf{a} \in \mathbf{R}^d$ to the cardinality of $M_{\mathbf{a}}$ (when $\mathscr{C} = \mathbf{Sets}$) or the dimension of the vector spaces $M_{\mathbf{a}}$ (when $\mathscr{C} = \mathbf{Vec}$).

Remark 6.8 In Definition 6.7, if M is a surjective persistence module, then we can regard $\mathrm{dm}(M)$ as a functor $\mathbf{R}^d \to \mathbf{Z}_+^{\mathrm{op}}$.

Proposition 6.9 (Interleaving stability of the dimension function) *Let $\mathscr{C}$ be either* **Sets** *or* **Vec** *and let $M, N\colon \mathbf{R}^d \to \mathscr{C}$ be any two* surjective *persistence modules. Then*

(i) $d_{\mathrm{I},d}\left(\mathrm{dm}(M), \mathrm{dm}(N)\right) \leq 2 \cdot d_{\mathrm{I},d}^{\mathscr{C}}(M, N)$,
(ii) $d_{\mathrm{I},d}\left(\mathrm{dm}(M), \mathrm{dm}(N)\right) \leq \mathbf{d}_{\mathrm{I},d}^{\mathscr{C}}(M, N)$.

Proof Let us assume that $\mathscr{C} = $ **Sets**. The proof for the case $\mathscr{C} = $ **Vec** is similar. We show (i). Suppose that (f, g) is an ε-interleaving pair between M and N. Pick any $\mathbf{a} \in \mathbf{R}^d$. We have $\varphi_M(\mathbf{a}, \mathbf{a}+2\boldsymbol{\varepsilon}) = g_{\mathbf{a}+\boldsymbol{\varepsilon}} \circ f_{\mathbf{a}}$. Since $\varphi_M(\mathbf{a}, \mathbf{a}+2\boldsymbol{\varepsilon})$ is surjective, we also have that $g_{\mathbf{a}+\boldsymbol{\varepsilon}}$ is surjective. Since $\varphi_N(\mathbf{a}, \mathbf{a}+\varepsilon)$ is also surjective, the composition $g_{\mathbf{a}+\boldsymbol{\varepsilon}} \circ \varphi_N(\mathbf{a}, \mathbf{a}+\boldsymbol{\varepsilon})\colon N_{\mathbf{a}} \to M_{\mathbf{a}+2\boldsymbol{\varepsilon}}$ is surjective. This implies that $\mathrm{dm}(N)_{\mathbf{a}} \geq \mathrm{dm}(M)_{\mathbf{a}+2\boldsymbol{\varepsilon}}$. By symmetry, we also have that $\mathrm{dm}(M)_{\mathbf{a}} \geq \mathrm{dm}(N)_{\mathbf{a}+2\boldsymbol{\varepsilon}}$ for each $\mathbf{a} \in \mathbf{R}^d$. Therefore, $d_{\mathrm{I}}\left(\mathrm{dm}(M), \mathrm{dm}(N)\right) \leq 2\varepsilon$, as desired.

We prove Item (ii). Suppose that there exists a *full* ε-interleaving pair between M and N. Then this directly implies that for all $\mathbf{a} \in \mathbf{R}^d$, $\mathrm{dm}(M)_{\mathbf{a}} \geq \mathrm{dm}(N)_{\mathbf{a}+\boldsymbol{\varepsilon}}$ and $\mathrm{dm}(N)_{\mathbf{a}} \geq \mathrm{dm}(M)_{\mathbf{a}+\boldsymbol{\varepsilon}}$. □

Proposition 6.10 *Let $\mathscr{C}$ be either* **Sets** *or* **Vec** *and let $M, N\colon \mathbf{R}^d \to \mathscr{C}$ be any two* surjective *persistence modules. Then*

$$d_{\mathrm{I},2d}\left(\mathrm{rk}(M), \mathrm{rk}(N)\right) \leq d_{\mathrm{I},d}\left(\mathrm{dm}(M), \mathrm{dm}(N)\right).$$

Proof Suppose that for some $\varepsilon \in [0, \infty)$, $d_{\mathrm{I}}\left(\mathrm{dm}(M), \mathrm{dm}(N)\right) < \varepsilon$. It suffices to prove that for all $\mathbf{a}, \mathbf{b} \in \mathbf{R}^d$ with $\mathbf{a} \leq \mathbf{b}$, and for all $\varepsilon' > \varepsilon$ in $[0, \infty)$,

$$\mathrm{rk}(N)(\mathbf{a} - \boldsymbol{\varepsilon}', \mathbf{b} + \boldsymbol{\varepsilon}') \leq \mathrm{rk}(M)(\mathbf{a}, \mathbf{b}).$$

Invoking that M and N are surjective, notice that $\mathrm{rk}(N)(\mathbf{a}-\boldsymbol{\varepsilon}', \mathbf{b}+\boldsymbol{\varepsilon}') = \mathrm{dm}(N)(\mathbf{b}+\boldsymbol{\varepsilon}')$ and $\mathrm{rk}(M)(\mathbf{a}, \mathbf{b}) = \mathrm{dm}(M)(\mathbf{b})$. By assumption, we readily have that $\mathrm{dm}(N)(\mathbf{b} + \boldsymbol{\varepsilon}') \leq \mathrm{dm}(M)(\mathbf{b})$, completing the proof. □

Proposition 4.7 is a corollary of Proposition 6.10.

6.2 Proof of Theorem 4.1

Before showing Theorem 4.1, we begin with the remarks below.

Remark 6.11 (*Simplicial maps between Rips complexes*) For any (semi-)metric spaces[6] (X, d_X) and (Y, d_Y), and for some $\delta, \delta' \geq 0$, consider the Rips complexes $K = \mathscr{R}_\delta(X, d_X)$ and $L = \mathscr{R}_{\delta'}(Y, d_Y)$. By the definition of Rips complex, in order to claim that a set map $p\colon X \to Y$ induces a simplicial map between (geometric realizations of) K and L, it suffices to show that whenever $x, x' \in X$ with $d_X(x, x') \leq \delta$, it holds that $d_Y(p(x), p(x')) \leq \delta'$.

[6] We call (X, d_X) a *semi-metric* space if the function $d_X\colon X \times X \to \mathbf{R}_+$ satisfies: (1) for all $x \in X$, $d_X(x, x) = 0$, and (2) for all $x, x' \in X$, $d_X(x, x') = d_X(x', x)$.

For $I = [u, u'] \in \mathbf{Int}$ and $\varepsilon \in [0, \infty)$, let $I^\varepsilon := [u - \varepsilon, u' + \varepsilon]$.

Remark 6.12 Let $\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$ be any two DMSs and let $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ be a ε-tripod between γ_X and γ_Y. Then it is not difficult to check that for any closed interval I of $\mathbf{R}$,

$$\bigvee_{I^\varepsilon} d_X \leq_R \bigvee_{I} d_Y + 2\varepsilon \;\text{ and }\; \bigvee_{I^\varepsilon} d_Y \leq_R \bigvee_{I} d_X + 2\varepsilon, \tag{15}$$

which is slightly more general than the condition in (1).

Proof of Theorem 4.1 If $d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) = \infty$, there is nothing to prove. Suppose that $d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) < \varepsilon$ for some $\varepsilon \in (0, \infty)$. Let $\mathscr{S} := \mathscr{R}^{\mathrm{lev}}(\gamma_X)$ and $\mathscr{T} := \mathscr{R}^{\mathrm{lev}}(\gamma_Y)$ (Definition 2.21). We regard $\mathbf{Int} \times \mathbf{R}_+$ as the subposet of $\mathbf{R}^{\mathrm{op}} \times \mathbf{R} \times \mathbf{R}$ (Fig. 3). Let $\mathbf{v} := \varepsilon(-1, 1, 2) \in \mathbf{R}^3$. Since $\mathbf{v} \leq \varepsilon(-2, 2, 2)$ in $\mathbf{R}^{\mathrm{op}} \times \mathbf{R} \times \mathbf{R}$, in order to prove inequality (5), it suffices to show that there are natural transformations $\mathbf{f} \colon \mathscr{S} \Rightarrow \mathscr{T}(\mathbf{v})$ and $\mathbf{g} \colon \mathscr{T} \Rightarrow \mathscr{S}(\mathbf{v})$ (between the two $\mathbf{Int} \times \mathbf{R}_+$-indexed, $\mathbf{Simp}$-valued functors) such that for each $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$, the following diagrams commute up to contiguity:

$$\begin{array}{ccc} \mathscr{S}_{(I,\delta)} & \xrightarrow{\mathscr{S}\left((I,\delta)\leq(I^{2\varepsilon},\delta+4\varepsilon)\right)} & \mathscr{S}_{(I^{2\varepsilon},\delta+4\varepsilon)} \\ & {\scriptstyle \mathbf{f}_{(I,\delta)}}\searrow \qquad \nearrow{\scriptstyle \mathbf{g}_{(I^{\varepsilon},\delta+2\varepsilon)}} & \\ & \mathscr{T}_{(I^{\varepsilon},\delta+2\varepsilon)} & \end{array} \qquad \begin{array}{ccc} & \mathscr{S}_{(I^{\varepsilon},\delta+2\varepsilon)} & \\ & {\scriptstyle \mathbf{g}_{(I,\delta)}}\nearrow \qquad \searrow{\scriptstyle \mathbf{f}_{(I^{\varepsilon},\delta+2\varepsilon)}} & \\ \mathscr{T}_{(I,\delta)} & \xrightarrow[\mathscr{T}\left((I,\delta)\leq(I^{2\varepsilon},\delta+4\varepsilon)\right)]{} & \mathscr{T}_{(I^{2\varepsilon},\delta+4\varepsilon)}. \end{array}$$

Indeed, by functoriality of homology, the existence of such pair $(\mathbf{f}, \mathbf{g})$ of natural transformations guarantees the $\mathbf{v}$-interleaving between two $(\mathbf{Int} \times \mathbf{R}_+)$-indexed modules $\mathrm{H}_k \circ \mathscr{S}$ and $\mathrm{H}_k \circ \mathscr{T}$.

Suppose that $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ is an ε-tripod between γ_X and γ_Y (Definition 2.9), which exists by the assumption $d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) < \varepsilon$. Since φ_X and φ_Y are surjective, we can take two maps $f \colon X \to Y$ and $g \colon Y \to X$ such that

$$\begin{aligned} &\{(x, f(x)) : x \in X\} \cup \{(g(y), y) : y \in Y\} \\ &\quad \subset \{(x, y) \in X \times Y : \exists z \in Z,\ x = \varphi_X(z),\ \text{and } y = \varphi_Y(z)\}. \end{aligned} \tag{16}$$

First, let us check that for any $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$, f induces a simplicial map $\mathbf{f}_{(I,\delta)} \colon \mathscr{S}(I, \delta) \to \mathscr{T}(I^\varepsilon, \delta + 2\varepsilon)$. Fix any $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$. By Remark 6.11, it suffices to show that whenever $x, x' \in X$ with $\left(\bigvee_I d_X\right)(x, x') \leq \delta$, it holds that $\left(\bigvee_{I^\varepsilon} d_Y\right)(f(x), f(x')) \leq \delta + 2\varepsilon$. This is immediate from the fact that R is an ε-tripod, and the assumption $\{(x, f(x)) : x \in X\} \subset \varphi_Y \circ \varphi_X^{-1}$, and Remark 6.12.

Furthermore, f induces a $\mathbf{v}$-morphism $\mathbf{f} : \mathscr{S} \Rightarrow \mathscr{T}(\mathbf{v})$. Indeed, because $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X$ as a set map $X \to Y$, for any $(I, \delta) \leq (J, \delta')$ in $\mathbf{Int} \times \mathbf{R}_+$, we have:

$$\mathscr{T}\left((I^\varepsilon, \delta + 2\varepsilon) \leq (J^\varepsilon, \delta' + 2\varepsilon)\right) \circ \mathbf{f}_{(I,\delta)} = \mathbf{f}_{(J,\delta')} \circ \mathscr{S}\left((I, \delta) \leq (J, \delta')\right).$$

By symmetry, it is straightforward that $g: Y \to X$ also induces a $\mathbf{v}$-morphism $\mathbf{g}: \mathscr{T} \to \mathscr{S}(\mathbf{v})$.

Next, we show that $(\mathbf{f}, \mathbf{g})$ is a $\mathbf{v}$-interleaving pair. By symmetry we only prove that for any $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$, the simplicial map $\mathbf{g}_{(I^\varepsilon, \delta+2\varepsilon)} \circ \mathbf{f}_{(I,\delta)}$ is contiguous to $\mathscr{S}\big((I, \delta) \leq (I^{2\varepsilon}, \delta + 4\varepsilon)\big)$, the simplicial map induced by the identity map on the vertex set X. Let $\sigma \subset X$ be a simplex in $\mathscr{S}(I, \delta)$. We wish to show that there is a simplex in $\mathscr{S}(I^{2\varepsilon}, \delta + 4\varepsilon)$ that contains both σ and the image $\mathrm{im}(\sigma)$ of σ by $g_{(I^\varepsilon, \delta+2\varepsilon)} \circ f_{(I,\delta)}$. To this end, we prove that the union $\sigma \cup \mathrm{im}(\sigma)$ has the diameter that is less than or equal to $\delta + 4\varepsilon$ in the (semi-)metric space $(X, \bigvee_{I^{2\varepsilon}} d_X)$. Invoking Remark 6.12, we consider the following three different cases of choosing any two elements in $\sigma \cup \mathrm{im}(\sigma)$:

(i) Take any $x, x' \in \sigma$. Since σ is a simplex in the Rips complex $\mathscr{S}(I, \delta) = \mathscr{R}_\delta\big(X, \bigvee_I d_X\big)$, we have

$$\left(\bigvee_{I^{2\varepsilon}} d_X\right)(x, x') \leq \left(\bigvee_I d_X\right)(x, x') \leq \delta < \delta + 4\varepsilon.$$

Let $\tilde{R} := \{(x, f(x)) : x \in X\} \cup \{(g(y), y) : y \in Y\}$ (see the inclusion in (16)).

(ii) Take $x \in \sigma$ and $x' \in \mathrm{im}(\sigma)$. Then $x' = g \circ f(x'')$ for some $x'' \in \sigma$. Since $(x, f(x)), (x', f(x'')), (x'', f(x'')) \in \tilde{R}$,

$$\begin{aligned}\left(\bigvee_{I^{2\varepsilon}} d_X\right)(x, x') &\leq \left(\bigvee_{I^\varepsilon} d_Y\right)(f(x), f(x'')) + 2\varepsilon \\ &\leq \left(\bigvee_I d_X\right)(x, x'') + 4\varepsilon \leq \delta + 4\varepsilon.\end{aligned}$$

(iii) Take any $x, x' \in \mathrm{im}(\sigma)$. Then there are $x'', x''' \in \sigma$ which are sent to x, x' via $g \circ f$, respectively. Since $(x, f(x'')), (x', f(x''')), (x'', f(x'')), (x''', f(x''')) \in \tilde{R}$,

$$\begin{aligned}\left(\bigvee_{I^{2\varepsilon}} d_X\right)(x, x') &\leq \left(\bigvee_{I^\varepsilon} d_Y\right)(f(x''), f(x''')) + 2\varepsilon \\ &\leq \left(\bigvee_I d_X\right)(x'', x''') + 4\varepsilon \leq \delta + 4\varepsilon.\end{aligned}$$

□

6.3 Proof of Proposition 4.3

Lemma 6.13 (Convexity of admissible vectors) *Suppose that* $\mathbf{a}, \mathbf{b} \in \mathbf{R}^6_\times$ *are admissible with* $\mathbf{a} \leq \mathbf{b}$. *Then any* $\mathbf{c} \in \mathbf{R}^6_\times$ *such that* $\mathbf{a} \leq \mathbf{c} \leq \mathbf{b}$ *is also admissible.*

Proof Let $\mathbf{a} := (a_i)_{i=1}^6$ and $\mathbf{b} := (b_i)_{i=1}^6$ and $\mathbf{c} = (c_i)_{i=1}^6$. From the assumptions that $\mathbf{a} \leq \mathbf{c} \leq \mathbf{b}$ and that $\mathbf{a}, \mathbf{b}$ are admissible, one can see that

$$b_4 \le c_4 \le a_4 \le a_1 \le c_1 \le b_1 \le b_2 \le c_2 \le a_2 \le a_5 \le c_5 \le b_5, \text{ and}$$
$$0 \le b_3 \le c_3 \le a_3 \le a_6 \le c_6 \le b_6.$$

Therefore, $\mathbf{c}$ is admissible. □

Proof of Proposition 4.3 Pick $\mathbf{a}, \mathbf{b} \in \mathbf{R}^6_\times$ such that $\mathbf{a} \le \mathbf{b}$. We consider the following cases:

(i) Both $\mathbf{a}$ and $\mathbf{b}$ are admissible.
(ii) $\mathbf{a}$ is admissible and $\mathbf{b}$ is non-admissible.
(iii) $\mathbf{a}$ is non-admissible and $\mathbf{b}$ is admissible.
(iv) Both $\mathbf{a}$ and $\mathbf{b}$ are non-admissible.

In case (i), let $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $\mathbf{b} = (b_1.b_2, b_3, b_4, b_5, b_6)$. Then we have the inclusions

$$\mathscr{R}_{b_3}\left(X, \bigvee_{[b_1,b_2]} d_X\right) \overset{i_1}{\hookrightarrow} \mathscr{R}_{a_3}\left(X, \bigvee_{[a_1,a_2]} d_X\right) \overset{i_2}{\hookrightarrow} \mathscr{R}_{a_6}\left(X, \bigvee_{[a_4,a_5]} d_X\right) \overset{i_3}{\hookrightarrow} \mathscr{R}_{b_6}\left(X, \bigvee_{[b_4,b_5]} d_X\right).$$

By applying H_k to the above inclusions, we obtain the diagram of vector spaces and linear maps

$$V_1 \xrightarrow{\mathrm{H}_k(i_1)} V_2 \xrightarrow{\mathrm{H}_k(i_2)} V_3 \xrightarrow{\mathrm{H}_k(i_3)} V_4.$$

Notice that $\mathrm{rk}_k(\mathbf{a})$ is the rank of $\mathrm{H}_k(i_2)$, whereas $\mathrm{rk}_k(\mathbf{b})$ is the rank of $\mathrm{H}_k(i_3) \circ \mathrm{H}_k(i_2) \circ \mathrm{H}_k(i_1)$. This implies that $\mathrm{rk}_k(\mathbf{a}) \ge \mathrm{rk}_k(\mathbf{b})$. In case (ii), $\mathbf{b}$ cannot be trivially non-admissible by definition. Therefore, $\mathrm{rk}_k(\gamma_X)(\mathbf{b}) = 0$. In case (iii), by Lemma 6.13, $\mathbf{a}$ must be trivially non-admissible and hence $\mathrm{rk}_k(\gamma_X)(\mathbf{a}) = \infty$. In case (iv), by the definition of trivially non-admissible, it is not possible that $\mathbf{a}$ is non-trivially non-admissible with $\mathbf{b}$ being trivially non-admissible. Therefore, we always have $\mathrm{rk}_k(\gamma_X)(\mathbf{a}) \ge \mathrm{rk}_k(\gamma_X)(\mathbf{b})$. □

6.4 Spatiotemporal Dendrogram of a DMS and Proof of Theorem 4.5

Overview of the Proof. The Betti-0 function of a DMS γ_X can be obtained by the two steps: First, adapting the ideas of the SLHC method (Appendix D.2), we induce the *spatiotemporal SLHC dendrogram* $\theta(\gamma_X)$ of γ_X. Then the dimension function $\mathrm{dm}\,(\theta(\gamma_X))$ (Definition 6.7) of $\theta(\gamma_X)$ coincides with the Betti-0 function of γ_X given in Definition 2.24. Therefore, by proving that each of the successive associations $\gamma_X \mapsto \theta(\gamma_X) \mapsto \mathrm{dm}\,(\theta(\gamma_X))$ is stable, we can show Theorem 4.5.

Partition Category and Dendrograms. Let X be a non-empty finite set. Given any two partitions P, Q of X, we write $P \le Q$ if P refines Q, i.e. for all $B \in P$, there exists a (unique) $C \in Q$ such that $B \subset C$. In this case, the surjective map $P \twoheadrightarrow Q$ sending each $B \in P$ to the unique block $C \in Q$ such that $B \subset C$ is called the *natural map* from P to Q.

Definition 6.14 (**Part**(X) *and its structure*) Let X be a non-empty finite set. By **Part**(X), we mean the subcategory of **Sets** described as follows:

(i) Objects: All partitions of X.
(ii) Morphisms: For any two partitions P, Q of X with $P \leq Q$, the unique morphism $P \twoheadrightarrow Q$ is the natural map.

We remark that any partition P of X has the corresponding equivalence relation $\sim$ on X. Namely, $P = X/\sim$, where $x \sim x'$ if and only if x, x' belong to the same block of P.

Definition 6.15 (*Dendrogram*) Let X be a non-empty finite set and let $\mathbf{P}$ be any poset. We will call any functor $\mathbf{P} \to \mathbf{Part}(X)$ a $\mathbf{P}$-*indexed dendrogram over* X or simply a *dendrogram*.

The Spatiotemporal SLHC Dendrogram of a DMS. We aim at encoding multiscale clustering features of a DMS into a *single* dendrogram (Definition 6.16). Since we take into account both temporal and spatial parameters, this dendrogram will have a *multidimensional* indexing poset, in contrast to its counterpart for a static metric space (Definition D.2). We prove that this dendrogram is stable under perturbation of the input DMS (Theorem 6.17).

Let $\gamma_X = (X, d_X(\cdot))$ be a DMS. For $I \in \mathbf{Int}$ and $\delta \in \mathbf{R}_+$, we define the equivalence relation $\sim^I_{X,\delta}$ on X as follows:

$$x \sim^I_{X,\delta} x' \;\Leftrightarrow\; \exists x = x_0, x_1, \ldots, x_n = x' \text{ in } X \text{ s.t. } \left(\bigvee_I d_X\right)(x_i, x_{i+1}) \leq \delta.$$

Observe that, for any pair $(I, \delta) \leq (J, \delta')$ in $\mathbf{Int} \times \mathbf{R}_+$, the relation $\sim^I_{X,\delta}$ is contained in $\sim^J_{X,\delta'}$ and hence

$$\left(X/\sim^I_{X,\delta}\right) \leq \left(X/\sim^J_{X,\delta'}\right). \tag{17}$$

By this monotonicity in (17), we can extend the notion of SLHC dendrogram for static metric spaces (Definition D.2) to the *spatiotemporal SLHC dendrogram* of a DMS:

Definition 6.16 (*The spatiotemporal SLHC dendrogram of a DMS*) Given any DMS $\gamma_X = (X, d_X(\cdot))$, we define the *spatiotemporal SLHC dendrogram* $\theta(\gamma_X)\colon \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Part}(X)$ of γ_X as follows:

(i) To each $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$, assign the partition $X/\sim^I_{X,\delta}$ of X.
(ii) To each pair $(I, \delta) \leq (J, \delta')$ in $\mathbf{Int} \times \mathbf{R}_+$, assign the natural map (Definition 6.14)

$$X/\sim^I_{X,\delta} \twoheadrightarrow X/\sim^J_{X,\delta'}.$$

In order to prove Theorem 4.5, we need:

Theorem 6.17 (Stability of the spatiotemporal SLHC dendrogram)

$$\mathbf{d}_{\mathrm{I}}^{\mathbf{Sets}}(\theta(\gamma_X), \theta_Y(\gamma_Y)) \leq 2 \cdot d_{\mathtt{dyn}}(\gamma_X, \gamma_Y).$$

The proof of Theorem 4.5 will be straightforward by re-interpreting Definition 2.24:

Definition 6.18 (*Another interpretation of Definition* 2.24) Let $\gamma_X = (X, d_X(\cdot))$ be a DMS. We define the Betti-0 function $\beta_0^{\gamma_X}: \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Z}_+$ of γ_X as the *dimension function* of the spatiotemporal dendrogram $\theta(\gamma_X)\colon \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Part}(X)$ of γ_X. In other words, $\beta_0^{\gamma_X}$ sends each $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$ to the number of blocks in the partition $\theta(\gamma_X)(I, \delta)$.

Proof of Theorem 4.5 Invoking that $\beta_0^{\gamma_X}$ and $\beta_0^{\gamma_Y}$ are the dimension functions of $\theta(\gamma_X)$ and $\theta(\gamma_Y)$, respectively, the proof straightforwardly follows from Proposition 6.9 and Theorem 6.17. $\square$

Proof of Theorem 6.17 Let $M := \theta(\gamma_X)\colon \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Part}(X)(\hookrightarrow \mathbf{Sets})$ and $N := \theta(\gamma_Y)\colon \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Part}(Y)(\hookrightarrow \mathbf{Sets})$. For each $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$, consider the equivalence relation $\sim_{I,\delta}^X$ on X defined, for any $x, x' \in X$, as $x \sim_{I,\delta}^X x'$ if and only if there is a sequence $x = x_0, x_1, \ldots, x_l = x'$ in X such that $\bigvee_I d_X(x_i, x_{i+1}) \leq \delta$ for each $i = 0, \ldots, l-1$. Similarly, define the equivalence relation $\sim_{I,\delta}^Y$ on Y. Note that, by definition of M and N,

$$M_{(I,\delta)} = X\big/ \sim_{I,\delta}^X \quad \text{and} \quad N_{(I,\delta)} = Y\big/ \sim_{I,\delta}^Y .$$

For $x \in X$, let $[x]_{(I,\delta)}^X$ be the block containing x in the partition $M_{(I,\delta)}$. Then, for any $(I, \delta), (J, \delta') \in \mathbf{Int} \times \mathbf{R}_+$ with $(I, \delta) \leq (J, \delta')$, the internal morphism $\varphi_M((I, \delta), (J, \delta'))$ of M sends $[x]_{(I,\delta)}^X$ to $[x]_{(J,\delta')}^X$ for each $x \in X$. We can describe the internal morphisms of N in the same way.

Suppose that $2\, d_{\mathtt{dyn}}(\gamma_X, \gamma_Y) < \varepsilon$ for some $\varepsilon \in (0, \infty)$. Then there exists an $(\varepsilon/2)$-tripod $R: X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ between γ_X and γ_Y (Definitions 2.9 and 2.10).

Since two maps $\varphi_X\colon Z \to X$ and $\varphi_Y\colon Z \to Y$ are surjective, we can take two maps $f: X \to Y$ and $g\colon Y \to X$ such that

$$\begin{aligned} &\{(x, f(x)) : x \in X\} \cup \{(g(y), y) : y \in Y\} \\ &\quad \subset \{(x, y) \in X \times Y : \exists z \in Z,\ x = \varphi_X(z),\ \text{and } y = \varphi_Y(z)\}. \end{aligned} \tag{18}$$

We will show that f, g induce a full ε-interleaving pair between M and N. For any $I = [u, u'] \in \mathbf{Int}$ and any $\alpha \in [0, \infty)$, let $I^\alpha := [u - \alpha, u' + \alpha]$. For each $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$, we define $\bar{f}_{(I,\delta)}\colon M_{(I,\delta)} \to N_{(I^\varepsilon,\delta+\varepsilon)}$ as

$$[x]_{(I,\delta)}^X \mapsto [f(x)]_{(I^\varepsilon,\delta+\varepsilon)}^Y, \quad x \in X.$$

Similarly, we define $\bar{g}_{(I,\delta)}\colon N_{(I,\delta)} \to M_{(I^\varepsilon,\delta+\varepsilon)}$. It suffices to show that for each $(I, \delta) \in \mathbf{Int} \times \mathbf{R}_+$,

(i) $\bar{f}_{(I,\delta)}$ (resp. $\bar{g}_{(I,\delta)}$) is a well-defined set map from $M_{(I,\delta)}$ to $N_{(I^\varepsilon,\delta+\varepsilon)}$ (resp. from $N_{(I,\delta)}$ to $M_{(I^\varepsilon,\delta+\varepsilon)}$),
(ii) $\bar{f}_{(I,\delta)}: M_{(I,\delta)} \to N_{(I^\varepsilon,\delta+\varepsilon)}$ and $\bar{g}_{(I,\delta)}: N_{(I,\delta)} \to M_{(I^\varepsilon,\delta+\varepsilon)}$ are surjective,
(iii) when $(I,\delta) \le (J,\delta')$ in $\mathbf{Int} \times \mathbf{R}_+$,

$$\varphi_N((I^\varepsilon,\delta+\varepsilon),(J^\varepsilon,\delta'+\varepsilon)) \circ \bar{f}_{(I,\delta)} = \bar{f}_{(J,\delta')} \circ \varphi_M((I,\delta),(J,\delta')),$$
$$\varphi_M((I^\varepsilon,\delta+\varepsilon),(J^\varepsilon,\delta'+\varepsilon)) \circ \bar{g}_{(I,\delta)} = \bar{g}_{(J,\delta')} \circ \varphi_N((I,\delta),(J,\delta')),$$

(iv) $\bar{g}_{(I^\varepsilon,\delta+\varepsilon)} \circ \bar{f}_{(I,\delta)} = \varphi_M((I,\delta),(I^{2\varepsilon},\delta+2\varepsilon))$, and $\bar{f}_{(I^\varepsilon,\delta+\varepsilon)} \circ \bar{g}_{(I,\delta)} = \varphi_N((I,\delta),(I^{2\varepsilon},\delta+2\varepsilon))$.

We prove (i). Fix $(I,\delta) \in \mathbf{Int} \times \mathbf{R}_+$. Suppose that $x' \in [x]^X_{(I,\delta)}$. It suffices to show that $f(x') \in [f(x)]^Y_{(I^\varepsilon,\delta+\varepsilon)}$. By assumption, there exist $x = x_0, \ldots, x_l = x'$ in X such that $\bigvee_I d_X(x_i, x_{i+1}) \le \delta$, $i = 1, \ldots, l-1$. Then invoking R is an $(\varepsilon/2)$-tripod between γ_X and γ_Y (see (1)), together with assumption (18) and Remark 6.12,

$$\bigvee_{I^\varepsilon} d_Y(f(x_i), f(x_{i+1})) \le \bigvee_{I^{(\varepsilon/2)}} d_Y(f(x_i), f(x_{i+1})) \le \delta + \varepsilon \text{ for } i = 1, \ldots, l-1.$$

This directly implies that $f(x') \in [f(x)]^Y_{(I^\varepsilon,\delta+\varepsilon)}$. In a similar way, it can be proved that $\bar{g}_{(I,\delta)}$ is well-defined.

Now we show (ii). Fix $(I,\delta) \in \mathbf{Int} \times \mathbf{R}_+$. We only prove that $\bar{f}_{(I,\delta)}: M_{(I,\delta)} \to N_{(I^\varepsilon,\delta+\varepsilon)}$ is surjective. Pick any $[y]^Y_{(I^\varepsilon,\delta+\varepsilon)} \in N_{(I^\varepsilon,\delta+\varepsilon)}$. Since $\varphi_Y: Z \to Y$ is surjective, there exists $z \in Z$ such that $\varphi_Y(z) = y$. Let $x := \varphi_X(z)$. Then invoking R is an $(\varepsilon/2)$-tripod between γ_X and γ_Y, together with assumption (18) and Remark 6.12,

$$\bigvee_{I^\varepsilon} d_Y(y, f(x)) \le \bigvee_{I^{\varepsilon/2}} d_Y(y, f(x)) \le \bigvee_I d_X(x,x) + \varepsilon = 0 + \varepsilon \le \delta + \varepsilon.$$

This implies that $[f(x)]^Y_{(I^\varepsilon,\delta+\varepsilon)} = [y]^Y_{(I^\varepsilon,\delta+\varepsilon)}$. Also, by definition of $\bar{f}_{(I,\delta)}$, $[x]^X_{(I,\delta)}$ is sent to $[y]^Y_{(I^\varepsilon,\delta+\varepsilon)}$ via $\bar{f}_{(I,\delta)}$. Since $[y]^Y_{(I^\varepsilon,\delta+\varepsilon)} \in N_{(I^\varepsilon,\delta+\varepsilon)}$ was arbitrary chosen, we have shown the surjectivity of $\bar{f}_{(I,\delta)}$.

Next we prove (iii). Fix $(I,\delta) \le (J,\delta')$ in $\mathbf{Int} \times \mathbf{R}_+$. We only show

$$\varphi_N((I^\varepsilon,\delta+\varepsilon),(J^\varepsilon,\delta'+\varepsilon)) \circ \bar{f}_{(I,\delta)} = \bar{f}_{(J,\delta')} \circ \varphi_M((I,\delta),(J,\delta')).$$

By the definition of maps $\varphi_M(\cdot,\cdot)$, $\varphi_N(\cdot,\cdot)$, $\bar{f}_{(\cdot,\cdot)}$ and $\bar{g}_{(\cdot,\cdot)}$, for any $[x]^X_{(I,\delta)} \in M_{(I,\delta)}$,

$$\begin{aligned}\varphi_N((I^\varepsilon,\delta+\varepsilon),(J^\varepsilon,\delta'+\varepsilon)) \circ \bar{f}_{(I,\delta)}\left([x]^X_{(I,\delta)}\right) &= \varphi_N((I^\varepsilon,\delta+\varepsilon),(J^\varepsilon,\delta'+\varepsilon))\left([f(x)]^Y_{(I^\varepsilon,\delta+\varepsilon)}\right)\\ &= [f(x)]^Y_{(J^\varepsilon,\delta'+\varepsilon)},\\ \bar{f}_{(J,\delta')} \circ \varphi_M((I,\delta),(J,\delta'))\left([x]^X_{(I,\delta)}\right) &= \bar{f}_{(J,\delta')}\left([x]^X_{(J,\delta')}\right) = [f(x)]^Y_{(J^\varepsilon,\delta'+\varepsilon)}.\end{aligned}$$

Finally, we prove (iv). Fix $(I,\delta)\in\mathbf{Int}\times\mathbf{R}_+$. We only show

$$\bar{g}_{(I^\varepsilon,\delta+\varepsilon)}\circ\bar{f}_{(I,\delta)}=\varphi_M((I,\delta),(I^{2\varepsilon},\delta+2\varepsilon)).$$

Take any $[x]^X_{(I,\delta)}\in M_{(I,\delta)}$. Then, by $\bar{g}_{(I^\varepsilon,\delta+\varepsilon)}\circ\bar{f}_{(I,\delta)}$, the block $[x]^X_{(I,\delta)}$ is sent to $[g\circ f(x)]^X_{(I^{2\varepsilon},\delta+2\varepsilon)}$. By invoking that R is an $(\varepsilon/2)$-tripod between γ_X and γ_Y and (18) and Remark 6.12, we also have

$$\begin{aligned}\bigvee_{I^{2\varepsilon}} d_X\left(x,g\circ f(x)\right) &\le \bigvee_{I^{(\varepsilon/2)}} d_X\left(x,g\circ f(x)\right)\\ &\le \bigvee_{I} d_Y(f(x),f(x))+\varepsilon=0+\varepsilon\le\delta+2\varepsilon.\end{aligned}$$

This implies that $[x]^X_{\delta+2\varepsilon}=[g\circ f(x)]^X_{\delta+2\varepsilon}$, completing the proof. □

For $t\in\mathbf{R}$, consider $[t,t]\in\mathbf{Int}$.

Remark 6.19 (*Comprehensiveness of Definition* 6.16) We remark the following (see Fig. 7):

(i) Consider the constant DMS $\gamma_X\equiv(X,d_X)$ as in Example 2.2. Then the spatiotemporal SLHC dendrogram of γ_X is amount to the SLHC dendrogram (Definition D.2) of (X,d_X): for all $I\in\mathbf{Int}$ and $\delta\in\mathbf{R}_+$,

$$\theta(\gamma_X)_{(I,\delta)}=\theta(X,d_X)_\delta.$$

(ii) Let $\gamma_X=(X,d_X(\cdot))$ be a DMS. For each $t\in\mathbf{R}$, we have the SLHC dendrogram $\theta(X,d_X(t))\colon\mathbf{R}_+\to\mathbf{Part}(X)$ of the metric space $(X,d_X(t))$ (Definition D.2). All those dendrograms are incorporated by $\theta(\gamma_X)$ in the following sense:

$$\theta_X(\gamma_X)_{([t,t],\delta)}=\theta(X,d_X(t))_\delta,\quad t\in\mathbf{R},\ \delta\in\mathbf{R}_+.$$

Remark 6.20 (*Connection to* [46]) Let $\gamma_X=(X,d_X(\cdot))$ be a DMS and fix $\delta_0\in\mathbf{R}_+$. The map $\theta^\delta_X\colon\mathbf{R}\to\mathbf{Part}(X)$ defined as

$$\theta^{\delta_0}_X(t)=X/\sim^{[t,t]}_{X,\delta_0}\quad\text{for all } t\in\mathbf{R}$$

is the *formigram* induced from γ_X with respect to the connectivity parameter δ [46].

6.5 Proof of Theorem 4.14

Proof of Theorem 4.14 We utilize $\{\!\{\cdot\}\!\}$ instead of $\{\cdot\}$ to denote *multisets*. Let $m:=|X|$, $n:=|Y|$, and without loss of generality assume that $m\le n$. Then, for some $a_1\le\cdots\le a_{m-1}$, and $b_1\le\cdots\le b_{n-1}$ in $\mathbf{R}_+$, we have

$$\begin{aligned}\mathscr{A}&:=\mathrm{dgm}_0\left(\mathscr{R}_\bullet(X,d_X)\right)\setminus\{\!\{(0,+\infty)\}\!\}=\{\!\{(0,a_i)\}\!\}_{i=1}^{m-1},\\ \mathscr{B}&:=\mathrm{dgm}_0\left(\mathscr{R}_\bullet(Y,d_Y)\right)\setminus\{\!\{(0,+\infty)\}\!\}=\{\!\{(0,b_j)\}\!\}_{j=1}^{n-1}.\end{aligned}$$

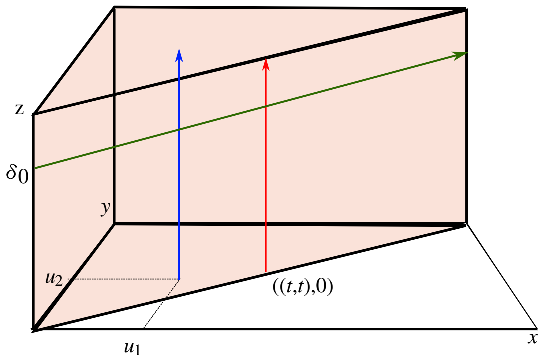

Fig. 7 Consider a DMS $\gamma_X = (X, d_X(\cdot))$. (1) If $\gamma_X \equiv (X, d_X)$, then the SLHC dendrogram $\theta(X, d_X)$ is encoded along any vertical ray, such as blue or red rays in the figure (Remark 6.19 (i)). (2) For each $t \in \mathbf{R}$, the SLHC dendrogram $\theta(X, d_X(t))$ of $(X, d_X(t))$ is recorded along the red ray (Remark 6.19(ii)) (3) Along the green horizontal line at height δ_0 over the diagonal plane $y = x$, the formigram induced from γ_X with respect to the connectivity parameter δ_0 is encoded

Then

$$d_B\left(\mathrm{dgm}_0\left(\mathscr{R}_\bullet(X, d_X)\right), \mathrm{dgm}_0\left(\mathscr{R}_\bullet(Y, d_Y)\right)\right) = d_B(\mathscr{A}, \mathscr{B}).$$

Let $A = \{a'_1, \ldots, a'_{n-1}\}$ and $B = \{b_1, \ldots, b_{n-1}\}$, where A consists of $n - m$ zeros at the beginning, followed by the sequence $a_1, a_2, \ldots, a_{m-1}$. Then notice that

$$d_B(\mathscr{A}, \mathscr{B}) \le \max_{i=1}^{n-1} \left|a'_i - b_i\right|.$$

Therefore, it suffices to show that

$$\max_{i=1}^{n-1} \left|a'_i - b_i\right| \le d_I\left(\beta_0^{(X,d_X)}, \beta_0^{(Y,d_Y)}\right). \tag{19}$$

Let $\varepsilon := d_I\left(\beta_0^{(X,d_X)}, \beta_0^{(Y,d_Y)}\right)$, i.e.,

$$\begin{aligned} \text{for all } \delta \in \mathbf{R}_+, \quad & \beta_0^{(X,d_X)}(\delta + \varepsilon) \le \beta_0^{(Y,d_Y)}(\delta), \\ & \text{and } \beta_0^{(Y,d_Y)}(\delta + \varepsilon) \le \beta_0^{(X,d_X)}(\delta). \end{aligned} \tag{20}$$

Observe the following:

(i) $\beta_0^{(X,d_X)}, \beta_0^{(Y,d_Y)}$ are monotonically decreasing as maps from $\mathbf{R}_+$ to $\mathbf{Z}_+$.
(ii) For $0 \le \delta < a_1 = a'_{n-m+1}$, we have $\beta_0^{(X,d_X)}(\delta) = m$.
(iii) For integers $k = 1, \ldots, m-1$, we have $a'_{n-k} = \min\left\{\delta \in \mathbf{R}_+ : \beta_0^{(X,d_X)}(\delta) = k\right\}$.
(iv) For integers $k = 1, \ldots, n-1$, we have $b_{n-k} = \min\left\{\delta \in \mathbf{R}_+ : \beta_0^{(Y,d_Y)}(\delta) = k\right\}$.

In order to show inequality (19), first we show that $|a'_i - b_i| \leq \varepsilon$ for $1 \leq i \leq n-m$. By construction we have $a'_1 = a'_2 = \cdots = a'_{n-m} = 0$, and thus it suffices to show that $b_i \leq \varepsilon$ for $1 \leq i \leq n-m$. By the assumption in (20) and item (ii), we have

$$\beta_0^{(Y,d_Y)}(\varepsilon) \leq \beta_0^{(X,d_X)}(0) = m.$$

Also, by items (i) and (iv), we have $b_{n-m} \leq \varepsilon$. Since $b_1 \leq b_2 \leq \cdots \leq b_{n-m-1} \leq b_{n-m}$, we have shown that $b_i \leq \varepsilon$ for $1 \leq i \leq n-m$, as desired.

Now we show that $|a'_i - b_i| \leq \varepsilon$ for $i = n-m+1, n-m+2, \ldots, n-1$. By re-indexing it suffices to prove that $|a'_{n-k} - b_{n-k}| \leq \varepsilon$ for $k = 1, \ldots, m-1$. Notice that, for $k = 1, \ldots, m-1$, by the assumption in (20) and item (iv), we have

$$\beta_0^{(X,d_X)}(b_{n-k} + \varepsilon) \leq \beta_0^{(Y,d_Y)}(b_{n-k}) = k.$$

Then by items (i) and (iii), we have that $a'_{n-k} \leq b_{n-k} + \varepsilon$. Similarly, one can prove that for $k = 1, \ldots, m-1$, it holds that $b_{n-k} \leq a'_{n-k} + \varepsilon$. Therefore, we have $|a'_{n-k} - b_{n-k}| \leq \varepsilon$ for $k = 1, \ldots, m-1$, as desired. □

7 Discussion

The primary contribution of this paper is to construct multiparameter persistent homology groups from dynamic metric data. Not only are these persistent homology groups stable to perturbations of the input, but also this stability result turns out to be a generalization of a fundamental stability theorem in topological data analysis. A second practical contribution of our paper is to propose a polynomial time algorithm that can be carried out for quantifying the behavioral difference between two dynamic metric data sets.

Appendix A: Discretization of DMSs

In order to compute the lower bound for the distance d_{dyn} given in Theorems 4.4 and 4.5 in practice, we need to *discretize* DMSs, i.e. turn DMSs into a locally constant DMSs. This discretization depends on the resolution parameter $\alpha \in (0, \infty)$, described as below. We will show that, if α is small and DMSs γ_X and γ_Y satisfy a mild assumption, then the lower bounds for $d_{\mathrm{dyn}}(\gamma_X, \gamma_Y)$ given in Theorems 4.4 and 4.5 can be well-approximated using the *α-discretized* DMSs associated to γ_X and γ_Y.

We call any map $i : \mathbf{Z}^d \to \mathbf{R}^d$ *grid-like* if i is an *strictly* injective poset morphism, i.e.

(i) for any pair $\mathbf{a} = (a_1, \ldots, a_d) < \mathbf{b} = (b_1, \ldots, b_d)$ with $a_i < b_i$, $i = 1, \ldots, d$ in $\mathbf{Z}^d$, for $i(\mathbf{a}) = (a'_1, \ldots, a'_d)$ and $i(\mathbf{b}) = (b'_1, \ldots, b'_d)$, we have $a'_i < b'_i$, $i = 1, \ldots, d$.

(ii) For all $\mathbf{c} = (c_1, \ldots, c_d) \in \mathbf{R}^d$, there are $\mathbf{a}, \mathbf{b} \in \mathbf{Z}^d$ such that $i(\mathbf{a}) \leq \mathbf{c} \leq i(\mathbf{b})$.

Given a grid-like $i\colon \mathbf{Z}^d \to \mathbf{R}^d$, for any $\mathbf{a} \in \mathbf{R}^d$, define $\lfloor \mathbf{a} \rfloor_i$ to be the maximum element in the image of $\mathbf{Z}^d$ by i which does not exceed $\mathbf{a}$.

Definition A.1 (*Discrete persistence modules*) We call a persistence module $M\colon \mathbf{R}^d \to \mathscr{C}$ *discrete* if there exists a grid-like map $i\colon \mathbf{Z}^d \to \mathbf{R}^d$ such that for each $\mathbf{a} \in \mathbf{R}^d$, the morphism $\varphi_M(\lfloor \mathbf{a} \rfloor_i, \mathbf{a})\colon M_{\lfloor \mathbf{a} \rfloor_i} \to M_{\mathbf{a}}$ is an isomorphism.

Let $\alpha \in (0, \infty)$. For any $t \in \mathbf{R}$, let $\lfloor t \rfloor_\alpha \in \alpha\mathbf{Z}$ be the greatest element in $\alpha\mathbf{Z}$ which does not exceed t. Given any DMS $\gamma_X = (X, d_X(\cdot))$, we define the *$\alpha$-discretization* of γ_X:

Definition A.2 (*Discretization of a DMS*) Let $\gamma_X = (X, d_X(\cdot))$ be any DMS and let $\alpha \in (0, \infty)$. The *α-discretization* of γ_X is the $\mathbf{R}$-parametrized family of finite (pseudo-)metric spaces $\gamma_X^\alpha := \left\{ \left(X, d_X^{\alpha\mathbf{Z}}(t)\right) : t \in \mathbf{R} \right\}$, where

$$d_X^{\alpha\mathbf{Z}}(t) := d_X(\lfloor t \rfloor_\alpha)\colon X \times X \to \mathbf{R}_+.$$

Notice that the discretization γ_X^α of γ_X does *not* necessarily satisfy Definition 2.1 (ii) and (iii) and hence γ_X^α does *not* deserve to be called a DMS. However, for convenience, we will call γ_X^α the *α-discretized DMS* of γ_X or simply the *discretized* DMS.

We can regard d_{dyn} as an extended pseudometric on a collection containing both all DMSs *and* all discretized DMSs: Indeed, items (ii) and (iii) in Definition 2.1 are not necessary to claim that d_{dyn} satisfies the triangle inequality (see the proof of [46, Thm. 9.14] in [46, Sect. 11.4.2]).

A DMS $\gamma_X = (X, d_X(\cdot))$ is said to be *l-Lipschitz* if $d_X(\cdot)(x, x')\colon \mathbf{R} \to \mathbf{R}_+$ is l-Lipschitz for every $x, x' \in X$. Assuming that γ_X is l-Lipschitz, the smaller the resolution parameter α is, the closer the discretized DMS γ_X^α to γ_X is:

Proposition A.3 *Let $\gamma_X = (X, d_X(\cdot))$ be any l-Lipschitz DMS. Then*

$$d_{\mathrm{dyn}}\left(\gamma_X, \gamma_X^\alpha\right) \le l\alpha.$$

Note that for the discretized DMS γ_X^α, we can define the rank invariant and the Betti-0 function of γ_X^α in the same way as in Definitions 2.23 and 2.24, respectively. Furthermore, in a bounded time interval $I \subset \mathbf{R}$, it is not difficult to check that both the Betti-0 function $\beta_0^{\gamma_X^\alpha}$ and the rank invariant $\mathrm{rk}_k(\gamma_X^\alpha)$, $k \in \mathbf{Z}_+$ are discrete (Definition A.1). Therefore, one can straightforwardly utilize the results in Sect. 5 for computing d_{I}.

Proposition A.4 (Approximating d_{dyn} from below with discretized DMSs) *Let $\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$ be any two l-Lipschitz DMSs.*

$$\begin{aligned} d_{\mathrm{I}}\left(\beta_0^{\gamma_X^\alpha}, \beta_0^{\gamma_Y^\alpha}\right) - 4l\alpha &\le 2 \cdot d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) \quad \textit{and} \\ d_{\mathrm{I}}\left(\mathrm{rk}_k(\gamma_X^\alpha), \mathrm{rk}_k(\gamma_Y^\alpha)\right) - 4l\alpha &\le 2 \cdot d_{\mathrm{dyn}}(\gamma_X, \gamma_Y), k \in \mathbf{Z}_+. \end{aligned}$$

Proof of Proposition A.3 For ease of notation, we prove the statement assuming that $\alpha = 1$, without loss of generality. Consider the tripod $R : X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$ (Definition 2.6). We prove that R is a l-tripod between γ_X and $\gamma_X^{\alpha\mathbf{Z}}$ (Definition 2.9). Fix $t \in \mathbf{R}$. Since $\lfloor t \rfloor \in [t-1, t+1] = [t]^1$, it is clear that $\bigvee_{[t]^1} d_X \leq_R d_X^{\alpha}(t)$ and hence $\bigvee_{[t]^1} d_X \leq_R d_X^{\alpha\mathbf{Z}}(t) + 2l$. It remains to show that $\bigvee_{[t]^1} d_X^{\alpha\mathbf{Z}} \leq_R d_X(t) + 2l$. Observe that, for any $x, x' \in X$, $\left(\bigvee_{[t]^1} d_X^{\alpha\mathbf{Z}}\right)(x, x')$ is the minimum among $d_X(\lfloor t \rfloor - 1)(x, x')$, $d_X(\lfloor t \rfloor)(x, x')$ and $d_X(\lfloor t \rfloor + 1)(x, x')$. Also, observe that all of $\lfloor t \rfloor - 1$, $\lfloor t \rfloor$, $\lfloor t \rfloor + 1$ belong to the closed interval $[t]^2 = [t-2, t+2]$. Therefore, invoking that γ_X is l-Lipschitz, for any $x, x' \in X$,

$$\left(\bigvee_{[t]^1} d_X^{\alpha\mathbf{Z}}\right)(x, x') \leq d_X(t)(x, x') + 2l.$$

This implies that $\bigvee_{[t]^1} d_X^{\mathbf{Z}} \leq_R d_X(t) + 2l$, as desired. □

Proof of Proposition A.4 We have

$$\begin{aligned} d_{\mathrm{dyn}}(\gamma_X^\alpha, \gamma_Y^\alpha) &\leq d_{\mathrm{dyn}}(\gamma_X^\alpha, \gamma_X) + d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) + d_{\mathrm{dyn}}(\gamma_Y, \gamma_Y^\alpha) \\ &\quad \text{(by the triangle inequality)}, \\ &\leq 2l\alpha + d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) \quad \text{(by Proposition A.3)}. \end{aligned}$$

Also, by Theorem 4.5, we obtain $d_{\mathrm{I}}\left(\beta_0^{\gamma_X^\alpha}, \beta_0^{\gamma_Y^\alpha}\right) \leq 2 \cdot d_{\mathrm{dyn}}(\gamma_X^\alpha, \gamma_Y^\alpha)$, and in turn the first inequality in the statement. The second inequality can be proved in a similar way. □

Appendix B: Relationship Between the Rank Invariant and CROCKER-Plot

We relate the rank invariant of a DMS to the CROCKER plot of [64]:

Definition B.1 (*The CROCKER plots of a DMS* [64]) Let $\gamma_X = (X, d_X(\cdot))$ be a DMS. For $k \in \mathbf{Z}_+$, the k-th CROCKER plot $C_k(\gamma_X)$ of γ_X is a map $\mathbf{R} \times \mathbf{R}_+ \to \mathbf{Z}_+$ sending $(t, \delta) \in \mathbf{R} \times \mathbf{R}_+$ to the dimension of the vector space $\mathrm{H}_k\left(\mathscr{R}_\delta(X, d_X(t))\right)$.

Let $\gamma_X = (X, d_X(\cdot))$ be any DMS. Note that for any time $t_0 \in \mathbf{R}$ and scale $\delta_0 \in \mathbf{R}_+$, the value of $\mathrm{rk}_k(\gamma_X)$ associated to the *repeated* pair $([t_0, t_0], \delta_0), ([t_0, t_0], \delta_0) \in \mathbf{Int} \times \mathbf{R}_+$ is identical to the dimension of the vector space $\mathrm{H}_k(\mathscr{R}_{\delta_0}(X, d_X(t_0)))$, i.e. $C_k(\gamma_X)(t_0, \delta_0)$. This implies that $\mathrm{rk}_k(\gamma_X)$ is an *enriched version* of the k-th CROCKER plot $C_k(\gamma_X)$ of γ_X.[7] Therefore, Theorem 4.4 can be interpreted somehow as establishing the stability of the CROCKER plots of a DMS.

Recall Definition 2.24, the Betti-0 function of a DMS.

[7] To illustrate this, the 0-th CROCKER plot $C_0(\gamma_X)$ is obtained by restricting $\beta_0^{\gamma_X}$ to the front diagonal vertical plane $\{[t, t] : t \in \mathbf{R}\} \times \mathbf{R}_+ \subset \mathbf{Int} \times \mathbf{R}_+$, which is colored brown in the middle picture of Fig. 4.

Remark B.2 (*Comparison between the Betti-*0 *function and the* 0*-th CROCKER plot*) Consider the DMSs γ_X and γ_Y in Fig. 1. Since the two metric spaces $\gamma_X(t)$ and $\gamma_Y(t)$ are isometric at *each* time $t \in \mathbf{R}$, the two CROCKER plots $C_0(\gamma_X)$ and $C_0(\gamma_Y)$ are identical. On the other hand, the Betti-0 function $\beta_0^{\gamma_X}$ is distinct from $\beta_0^{\gamma_Y}$ as illustrated in Fig. 4. This implies that, in comparison with the 0-th CROCKER plot, the Betti-0 function is more sensitive invariant of a DMS.

Appendix C: Other Relevant Metrics

Bottleneck Distance. Let us define:

- $\overline{\mathbf{R}} := \mathbf{R} \cup \{+\infty, -\infty\}$,
- $\mathbf{U} := \{(u_1, u_2) \in \mathbf{R}^2 : u_1 \leq u_2\}$, which is the upper-half plane above the line $y = x$ in $\mathbf{R}^2$.
- $\overline{\mathbf{U}} := \{(u_1, u_2) \in \overline{\mathbf{R}}^2 : u_1 \leq u_2\}$, which is the upper-half plane above the line $y = x$ in the extended plane $\overline{\mathbf{R}}^2$.

For $\mathbf{u} = (u_1, u_2),\ \mathbf{v} = (v_1, v_2) \in \overline{\mathbf{U}}$, let

$$\|\mathbf{u} - \mathbf{v}\|_\infty := \max\left(|u_1 - v_1|, |u_2 - v_2|\right).$$

Let X_1 and X_2 be multisets of points. Let $\alpha\colon X_1 \nrightarrow X_2$ be a matching, i.e. a partial injection. By $\mathrm{dom}(\alpha)$ and $\mathrm{im}(\alpha)$, we denote the points in X_1 and X_2 respectively, which are matched by α.

Definition C.1 (*The bottleneck distance* [24]) Let X_1, X_2 be multisets of points in $\overline{\mathbf{U}}$. Let $\alpha\colon X_1 \nrightarrow X_2$ be a matching. We call α an ε*-matching* if

(i) for all $\mathbf{u} \in \mathrm{dom}(\alpha)$, $\|\mathbf{u} - \alpha(\mathbf{u})\|_\infty \leq \varepsilon$,
(ii) for all $\mathbf{u} = (u_1, u_2) \in X_1 \setminus \mathrm{dom}(\alpha)$, $u_2 - u_1 \leq 2\varepsilon$,
(iii) for all $\mathbf{v} = (v_1, v_2) \in X_2 \setminus \mathrm{im}(\alpha)$, $v_2 - v_1 \leq 2\varepsilon$.

Their bottleneck distance $d_{\mathrm{B}}(X_1, X_2)$ is defined as the infimum of $\varepsilon \in [0, \infty)$ for which there exists an ε-matching $\alpha : X_1 \nrightarrow X_2$.

Erosion Distance. Recently, Patel generalized the notion of persistence diagrams and proposed a new metric, the *erosion distance*, for comparing generalized persistence diagrams [58]. We review a particular case of the erosion distance. Let $\mathbf{P}$ and $\mathbf{Q}$ be any two posets. Given any two maps $f, g\colon \mathbf{P} \to \mathbf{Q}$, we write $f \leq g$ if $f(p) \leq g(p)$ for all $p \in \mathbf{P}$.

Let $\mathbf{U} := \{(x, y) \in \mathbf{R}^2 : x \leq y\}$ equipped with the partial order inherited from $\mathbf{R}^{\mathrm{op}} \times \mathbf{R}$. For any $\varepsilon \in [0, \infty)$, let $\boldsymbol{\varepsilon} := (-\varepsilon, \varepsilon) \in \mathbf{U}$. Given any map $Y\colon \mathbf{U} \to \mathbf{Z}_+$ and $\varepsilon \in [0, \infty)$, define another map $\nabla_\varepsilon Y\colon \mathbf{U} \to \mathbf{Z}_+$ as $\nabla_\varepsilon Y(I) := Y(I + \boldsymbol{\varepsilon})$. If Y is order-reversing, it is clear that $\nabla_\varepsilon Y \leq Y$.

Definition C.2 (*Erosion distance* [58]) Let $Y_1, Y_2\colon \mathbf{U} \to \mathbf{Z}_+$ be any two order-reversing maps. The erosion distance between Y_1 and Y_2 is defined as

$$d_{\mathrm{E}}(Y_1, Y_2) := \inf\left\{\varepsilon \in [0, \infty) : \nabla_\varepsilon Y_i \leq Y_j,\ \text{for } i, j \in \{1, 2\}\right\},$$

with the convention that $d_{\mathrm{E}}(Y_1, Y_2) = \infty$ when there is no $\varepsilon \in [0, \infty)$ satisfying the condition in the above set.

Note that since $\mathbf{U}$ is a subposet of $\mathbf{R}^{\mathrm{op}} \times \mathbf{R}$, we can regard d_{E} is a particular case of $d_{\mathrm{I},2}$ from Sect. 3.2. The erosion distance is further generalized in [59].

Matching Distance [18,51]. In brief, the matching distance d_{match} compares rank invariants via one-dimensional reduction along lines. Namely, for any $M, N \colon \mathbf{R}^d \to$ **Vec**, the matching distance between $\mathrm{rk}(M)$ and $\mathrm{rk}(N)$ is defined as

$$d_{\mathrm{match}}(\mathrm{rk}(M), \mathrm{rk}(N)) := \sup_{L: u = s\boldsymbol{m} + b} m^* d_{\mathrm{B}}(\mathscr{B}(M|_L), \mathscr{B}(N|_L)), \tag{21}$$

where L varies in the set of all the lines parametrized by $u = s\boldsymbol{m} + b$, with $m^* := \min_i m_i > 0$, $\max_i m_i = 1$, $\sum_i^n b_i = 0$. Specifically, d_{match} is upper bounded by $d_{\mathrm{I}}^{\mathbf{Vec}}$ [51]. We briefly discuss about the algorithms for d_{match} and their computational cost:

- For $d = 1$, the RHS of equation (21) reduces to the bottleneck distance between the barcodes of M and N. The bottleneck distance can be computed in time $O(n^{1.5} \log n)$ where n is the total cardinality of the two barcodes [45]. See also [19].
- For $d = 2$, d_{match} can be computed exactly in time $O(n^{11})$ where n is the size of *finite presentations* of M and N [44].
- For $d \geq 2$, algorithms for approximating d_{match} within any threshold $\varepsilon > 0$ are proposed in [6,20]. In particular, for the case $d \geq 3$ which is of our interest, the running time for the proposed algorithm is proportional to $\left(\frac{d}{\varepsilon}\right)^d$ in the worst case [20, Sect. 3.1].

Dimension Distance [28, Sect. 4]. Let $M, N \colon \mathbf{R}^d \to$ **Vec** be any two persistence modules. If M, N are *nice*[8], then the *dimension distance* d_0 between $\mathrm{dm}(M)$ and $\mathrm{dm}(N)$ serves as a lower bound for $d_{\mathrm{I}}^{\mathbf{Vec}}(M, N)$ [28, Thm. 39]. A strength of d_0 is the computational efficiency. Let $M', N' \colon [n]^d \to$ **Vec** be any two finite persistence modules. The entire computation for $d_0(\mathrm{dm}(M'), \mathrm{dm}(N'))$ takes only $O(n^2 \log n)$ [28, Sect. 4.2].

If a persistence module M is obtained by applying the 0-th homology functor to the spatiotemporal Rips filtration of a DMS γ_X (Definition 2.21), then every internal morphim $\varphi_M(\cdot, \cdot)$ is surjective, and hence M is nice. Specifically, $\mathrm{dm}(M)$ coincides with the Betti-0 function $\beta_0^{\gamma_X}$ (Definition 2.24). Therefore, one can utilize d_0 for comparing Betti-0 functions of DMSs and for obtaining a lower bound of $d_{\mathtt{dyn}}$ (by virtue of Theorem 4.1).

On the other hand, for $k \geq 1$, a persistence module M obtained by applying the k-th homology functor to the spatiotemporal Rips filtration of a DMS does not necessarily satisfy the "nice" condition. This prevents us from freely utilizing d_0 in order to obtain a lower bound for $d_{\mathtt{dyn}}$.

[8] A persistence module $M \colon \mathbf{R}^d \to$ **Vec** is nice if there exists a value $\varepsilon_0 \in \mathbf{R}_+$ such that for every $\varepsilon < \varepsilon_0$, each internal morphism $\varphi_M(\mathbf{a}, \mathbf{a} + \boldsymbol{\varepsilon})$ is either injective or surjective (or both).

Appendix D: Stability of the Single Linkage Hierarchical Clustering Method

We review the single linkage hierarchical clustering (SLHC) method and its stability under the Gromov–Hausdorff distance. We begin by reviewing the Gromov–Hausdorff distance.

Appendix D.1: The Gromov–Hausdorff Distance

The Gromov–Hausdorff distance d_{GH} (Definition D.1) measures how far two metric spaces are from being isometric.

Let (X, d_X) and (Y, d_Y) be any two metric spaces and let $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ be a tripod between X and Y. Then the *distortion* of R is defined as

$$\mathrm{dis}(R) := \sup_{z,z' \in Z} \left| d_X\left(\varphi_X(z), \varphi_X(z')\right) - d_Y\left(\varphi_Y(z), \varphi_Y(z')\right) \right|.$$

Definition D.1 (*Gromov–Hausdorff distance* [12, Sect. 7.3.3]) Let (X, d_X) and (Y, d_Y) be any two metric spaces. Then

$$d_{\mathrm{GH}}\left((X, d_X), (Y, d_Y)\right) = \frac{1}{2} \inf_R \ \mathrm{dis}(R),$$

where the infimum is taken over all tripods R between X and Y. In particular, any tripod R between X and Y with $\mathrm{dis}(R) \leq \varepsilon$ is said to be an *ε-tripod* between (X, d_X) and (Y, d_Y).

The computation cost of d_{GH} leads to NP-hard problem, even for metric spaces of simple structure [1,61]. Therefore, one of practical approaches for estimating d_{GH} is to search for tractable lower bounds.

Appendix D.2: Single Linkage Hierarchical Clustering (SLHC) Method

Let (X, d_X) be a finite metric space. For each $\delta \in \mathbf{R}_+$, we define the equivalence relation $\sim_\delta$ on X as

$$x \sim_\delta x' \text{ if and only if } \exists x = x_0, \ldots, x_n \text{ in } X \text{ s.t. } d_X(x_i, x_{i+1}) \leq \delta.$$

Observe that for any $\delta \leq \delta'$ in $\mathbf{R}_+$, the inclusion $\sim_\delta \subset \sim_{\delta'}$ holds, leading to $(X/\sim_\delta) \leq (X/\sim_{\delta'})$ in $\mathbf{Part}(X)$ (Definition 6.14).

Definition D.2 (*The dendrogram from the SLHC*) Let (X, d_X) be a finite metric space. The dendrogram $\theta(X, d_X)\colon \mathbf{R}_+ \to \mathbf{Part}(X)$ defined by sending $\delta \in \mathbf{R}_+$ to $X/\sim_\delta$ is called the *SLHC dendrogram* of (X, d_X). □

The Ultrametric Induced by the Single Linkage Hierarchical Clustering Method [16]. An ultrametric space (X, u_X) is a metric space satisfying the *strong triangle inequality*: for all $x, x', x'' \in X$, $u_X(x, x') \leq \max\left\{u_X(x, x''), u_X(x'', x')\right\}$.

Let (X, d_X) be a finite metric space and consider its SLHC dendrogram $\theta(X, d_X)\colon \mathbf{R}_+ \to \mathbf{Part}(X)$. For any $x, x' \in X$, define

$$u_X(x, x') := \min\{\delta \in [0, \infty) : \ x, x' \text{ belong to the same block of } X/\sim_\delta\}.$$

It is not difficult to check that $u_X : X \times X \to \mathbf{R}_+$ is a ultrametric and that $u_X(x, x') \leq d(x, x')$, for all $x, x' \in X$.

Definition D.3 (*The ultrametrics induced by the single linkage hierarchical clustering* [16]) Given any finite metric space (X, d_X), the ultrametric space (X, u_X) defined as above is said to be the *ultrametric space induced by the SLHC on* (X, d_X) and we write $(X, u_X) = \mathscr{H}^{\mathrm{SL}}(X, d_X)$.

The assignment $(X, d_X) \mapsto \mathscr{H}^{\mathrm{SL}}(X, d_X)$ is known to be 1-Lipschitz with respect to the Gromov–Hausdorff distance:

Theorem D.4 (Stability of the SLHC [16]) *For any two finite metric spaces* (X, d_X) *and* (Y, d_Y)*, let* (X, u_X) *and* (Y, u_Y) *be the ultrametric spaces induced from* (X, d_X) *and* (Y, d_Y) *by the SLHC method. Then*

$$d_{\mathrm{GH}}((X, u_X), (Y, u_Y)) \leq d_{\mathrm{GH}}((X, d_X), (Y, d_Y)). \tag{22}$$

Remark D.5 The term $d_{\mathrm{GH}}((X, u_X), (Y, u_Y))$ in (22) cannot be approximated within any factor less than 3 in polynomial time, unless P = NP [47, Thm. 3]. Therefore, in a practical viewpoint, it is desirable to find another lower bound for d_{GH}.

The Gromov–Hausdorff distance can be bounded from below by the bottleneck distance between persistence diagrams associated to Rips filtrations: see inequality (11). Computing the LHS of inequality (11) can be carried out in polynomial time [45].

Remark D.6 Observe that both of the LHSs of the inequalities in (22) and (11) with $k = 0$ measure the difference between clustering features of (X, d_X) and (Y, d_Y). In fact, for any two finite metric spaces (X, d_X) and (Y, d_Y), the persistence modules $\mathrm{H}_0\left(\mathscr{R}_\bullet(X, d_X)\right)$ and $\mathrm{H}_0\left(\mathscr{R}_\bullet(Y, d_Y)\right)$ are isomorphic to $\mathrm{H}_0\left(\mathscr{R}_\bullet(X, u_X)\right)$ and $\mathrm{H}_0\left(\mathscr{R}_\bullet(Y, u_Y)\right)$, respectively. Therefore,

$$\begin{aligned} &d_{\mathrm{B}}\left(\mathrm{dgm}_0\left(\mathscr{R}_\bullet(X, d_X)\right), \mathrm{dgm}_0\left(\mathscr{R}_\bullet(Y, d_Y)\right)\right) \\ &\quad \leq d_{\mathrm{GH}}\left((X, u_X), (Y, u_Y)\right) \leq 2 \cdot d_{\mathrm{GH}}\left((X, d_X), (Y, d_Y)\right). \end{aligned}$$

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References

1. Agarwal, P.K., Fox, K., Nath, A., Sidiropoulos, A., Wang, Y.: Computing the Gromov–Hausdorff distance for metric trees. International Symposium on Algorithms and Computation. Lecture Notes in Computer Science, vol. 9472, pp. 529–540. Springer, Heidelberg (2015)

2. Babichev, A., Morozov, D., Dabaghian, Y.: Robust spatial memory maps encoded by networks with transient connections. PLoS Comput. Biol. **14**(9), e1006433 (2018)
3. Bauer, U., Edelsbrunner, H., Jablonski, G., Mrozek, M.: Persistence in sampled dynamical systems faster. arXiv preprint arXiv:1709.04068 (2017)
4. Bendich, P., Edelsbrunner, H., Morozov, D., Patel, A.: Homology and robustness of level and interlevel sets. Homology Homotopy Appl. **15**(1), 51–72 (2013)
5. Benkert, M., Gudmundsson, J., Hübner, F., Wolle, T.: Reporting flock patterns. Comput. Geom. **41**(3), 111–125 (2008)
6. Biasotti, S., Cerri, A., Frosini, P., Giorgi, D.: A new algorithm for computing the 2-dimensional matching distance between size functions. Pattern Recognit. Lett. **32**(14), 1735–1746 (2011)
7. Bjerkevik, H.B., Botnan, M.B.: Computational complexity of the interleaving distance. In: Proceedings of the 34th International Symposium on Computational Geometry (SoCG 2018), pp. 13:1–13:15 (2018)
8. Bjerkevik, H.B., Botnan, M.B., Kerber, M.: Computing the interleaving distance is NP-hard. Found. Comput. Math. (2019). https://doi.org/10.1007/s10208-019-09442-y
9. Botnan, M., Lesnick, M.: Algebraic stability of zigzag persistence modules. Algebra. Geom. Topol. **18**(6), 3133–3204 (2018)
10. Bubenik, P., Scott, J.A.: Categorification of persistent homology. Discrete Comput. Geom. **51**(3), 600–627 (2014)
11. Buchin, K., Buchin, M., van Kreveld, M.J., Speckmann, B., Staals, F.: Trajectory grouping structure. JoCG **6**(1), 75–98 (2015)
12. Burago, D., Burago, Yu., Ivanov, S.: A Course in Metric Geometry, vol. 33. American Mathematical Society, Providence (2001)
13. Carlsson, G.: Topology and data. Bull. Am. Math. Soc. **46**, 255–308 (2009)
14. Carlsson, G., de Silva, V.: Zigzag persistence. Found. Comput. Math. **10**(4), 367–405 (2010)
15. Carlsson, G., de Silva, V., Morozov, D.: Zigzag persistent homology and real-valued functions. In: Proceedings of the 25th Annual Symposium on Computational Geometry, pp. 247–256. ACM (2009)
16. Carlsson, G., Mémoli, F.: Characterization, stability and convergence of hierarchical clustering methods. J. Mach. Learn. Res. **11**, 1425–1470 (2010)
17. Carlsson, G., Zomorodian, A.: The theory of multidimensional persistence. Discrete Comput. Geom. **42**(1), 71–93 (2009)
18. Cerri, A., Di Fabio, B., Ferri, M., Frosini, P., Landi, C.: Betti numbers in multidimensional persistent homology are stable functions. Math. Methods Appl. Sci. **36**(12), 1543–1557 (2013)
19. Cerri, A., Di Fabio, B., Jabłoński, G., Medri, F.: Comparing shapes through multi-scale approximations of the matching distance. Comput. Vis. Image Understand. **121**, 43–56 (2014)
20. Cerri, A., Frosini, P.: A new approximation algorithm for the matching distance in multidimensional persistence. (2011)
21. Chazal, F., Cohen-Steiner, D., Glisse, M., Guibas, L.J., Oudot, S.: Proximity of persistence modules and their diagrams. In: Proceeding of 25th ACM Symposium on Computational Geometry, pp. 237–246 (2009)
22. Chazal, F., Cohen-Steiner, D., Guibas, L.J., Mémoli, F., Oudot, S.Y.: Gromov–Hausdorff stable signatures for shapes using persistence. In: Proceedings of SGP (2009)
23. Chazal, F., De Silva, V., Oudot, S.: Persistence stability for geometric complexes. Geom. Dedicata **173**(1), 193–214 (2014)
24. Cohen-Steiner, D., Edelsbrunner, H., Harer, J.: Stability of persistence diagrams. Discrete Comput. Geom. **37**(1), 103–120 (2007)
25. Cohen-Steiner, D., Edelsbrunner, H., Morozov, D.: Vines and vineyards by updating persistence in linear time. In: Proceedings of the 22nd Annual Symposium on Computational Geometry, pp. 119–126. ACM (2006)
26. De Silva, V., Munch, E., Patel, A.: Categorified Reeb graphs. Discrete Comput. Geom. **55**(4), 854–906 (2016)
27. Dey, T.K., Juda, M., Kapela, T., Kubica, J., Lipiński, M., Mrozek, M.: Persistent homology of morse decompositions in combinatorial dynamics. SIAM J. Appl. Dyn. Syst. **18**(1), 510–530 (2019)
28. Dey, T.K., Xin, C.: Computing bottleneck distance for 2-D interval decomposable modules. In: Proceedings of the Thirty-Fourth International Symposium on Computational Geometry (SoCG 2018), pp. 32:1–32:15 (2018)
29. Edelsbrunner, H., Harer, J.: Persistent homology: a survey. Contemp. Math. **453**, 257–282 (2008)

30. Edelsbrunner, H., Harer, J.: Computational Topology: An Introduction. American Mathematical Society, Providence (2010)
31. Edelsbrunner, H., Harer, J., Mascarenhas, A., Pascucci, V., Snoeyink, J.: Time-varying Reeb graphs for continuous space-time data. Comput. Geom. **41**(3), 149–166 (2008)
32. Edelsbrunner, H., Jabłoński, G., Mrozek, M.: The persistent homology of a self-map. Found. Comput. Math. **15**(5), 1213–1244 (2015)
33. Ghrist, R.: Barcodes: the persistent topology of data. Bull. Am. Math. Soc. **45**(1), 61–75 (2008)
34. Giusti, C., Ghrist, R., Bassett, D.S.: Two's company, three (or more) is a simplex. J. Comput. Neurosci. **41**(1), 1–14 (2016)
35. Giusti, C., Pastalkova, E., Curto, C., Itskov, V.: Clique topology reveals intrinsic geometric structure in neural correlations. Proc. Natl. Acad. Sci. **112**(44), 13455–13460 (2015)
36. Gudmundsson, J., van Kreveld, M.: Computing longest duration flocks in trajectory data. In: Proceedings of the 14th Annual ACM International Symposium on Advances in Geographic Information Systems, pp. 35–42. ACM (2006)
37. Gudmundsson, J., van Kreveld, M., Speckmann, B.: Efficient detection of patterns in 2d trajectories of moving points. Geoinformatica **11**(2), 195–215 (2007)
38. Hajij, M., Wang, B., Scheidegger, C., Rosen, P.: Visual detection of structural changes in time-varying graphs using persistent homology. In: Pacific Visualization Symposium (PacificVis), 2018 IEEE, pp. 125–134. IEEE (2018)
39. Huang, Y., Chen, C., Dong, P.: Modeling herds and their evolvements from trajectory data. International Conference on Geographic Information Science, pp. 90–105. Springer, New York (2008)
40. Hwang, S.-Y., Liu, Y.-H., Chiu, J.-K., Lim, E.-P.: Mining mobile group patterns: a trajectory-based approach. PAKDD, vol. 3518, pp. 713–718. Springer, New York (2005)
41. Jeung, H., Yiu, M.L., Zhou, X., Jensen, C.S., Shen, H.T.: Discovery of convoys in trajectory databases. Proc. VLDB Endow. **1**(1), 1068–1080 (2008)
42. Kahle, M., Meckes, E.: Limit the theorems for Betti numbers of random simplicial complexes. Homology Homotopy Appl. **15**(1), 343–374 (2013)
43. Kalnis, P., Mamoulis, N., Bakiras, S.: On discovering moving clusters in spatio-temporal data. SSTD, vol. 3633, pp. 364–381. Springer, Berlin (2005)
44. Kerber, M., Lesnick, M., Oudot, S.: Exact computation of the matching distance on 2-parameter persistence modules. In: Proceedings of the 35th International Symposium on Computational Geometry, pp. 46:1–46:15 (2019)
45. Kerber, M., Morozov, D., Nigmetov, A.: Geometry helps to compare persistence diagrams. J. Exp. Algorithm. **22**, 1–4 (2017)
46. Kim, W., Mémoli, F.: Stable signatures for dynamic graphs and dynamic metric spaces via zigzag persistence. arXiv preprint arXiv:1712.04064 (2017)
47. Kim, W., Mémoli, F.: Formigrams: Clustering summaries of dynamic data. In: Proceedings of 30th Canadian Conference on Computational Geometry (CCCG18) (2018)
48. Kim, W., Mémoli, F., Smith, Z.: https://research.math.osu.edu/networks/formigrams
49. Knight, W.J.: Search in an ordered array having variable probe cost. SIAM J. Comput. **17**(6), 1203–1214 (1988)
50. Kostitsyna, I., van Kreveld, M.J., Löffler, M., Speckmann, B., Staals, F.: Trajectory grouping structure under geodesic distance. In: 31st International Symposium on Computational Geometry, SoCG 2015, June 22–25, 2015, Eindhoven, The Netherlands, pp. 674–688 (2015)
51. Landi, C.: The rank invariant stability via interleavings. Research in Computational Topology, pp. 1–10. Springer, Berlin (2018)
52. Lesnick, M.: The theory of the interleaving distance on multidimensional persistence modules. Found. Comput. Math. **15**(3), 613–650 (2015)
53. Li, Z., Ding, B., Han, J., Kays, R.: Swarm: mining relaxed temporal moving object clusters. Proc. VLDB Endow. **3**(1–2), 723–734 (2010)
54. Mac Lane, S.: Categories for the Working Mathematician. Graduate Texts in Mathematics, vol. 5. Springer, New York (2013)
55. Munch, E.: Applications of persistent homology to time varying systems. PhD thesis (2013)
56. Oesterling, P., Heine, C., Weber, G.H., Morozov, D., Scheuermann, G.: Computing and visualizing time-varying merge trees for high-dimensional data. Topological Methods in Data Analysis and Visualization, pp. 87–101. Springer, Berlin (2015)

57. Parrish, J .K., Hamner, W M e: Animal Groups in Three Dimensions: How Species Aggregate. Cambridge University Press, Cambridge (1997)
58. Patel, A.: Generalized persistence diagrams. J. Appl. Comput. Topol. **1**, 397–419 (2018)
59. Puuska, V.: Erosion distance for generalized persistence modules. arXiv preprint arXiv:1710.01577 (2017)
60. Schmiedl, F.: Shape Matching and Mesh Segmentation. PhD thesis, Technische Universität München (2014)
61. Schmiedl, F.: Computational aspects of the Gromov–Hausdorff distance and its application in non-rigid shape matching. Discrete Comput. Geom. **57**(4), 854–880 (2017)
62. Scolamiero, M., Chachólski, W., Lundman, A., Ramanujam, R., Öberg, S.: Multidimensional persistence and noise. Found. Comput. Math. **17**(6), 1367–1406 (2017)
63. Sumpter, D.J.: Collective Animal Bbehavior. Princeton University Press, Princeton (2010)
64. Topaz, C.M., Ziegelmeier, L., Halverson, T.: Topological data analysis of biological aggregation models. PloS ONE **10**(5), e0126383 (2015)
65. Ulmer, M., Ziegelmeier, L., Topaz, C.M.: Assessing biological models using topological data analysis. arXiv preprint arXiv:1811.04827 (2018)
66. van Goethem, A., van Kreveld, M.J., Löffler, M., Speckmann, B., Staals, F.: Grouping time-varying data for interactive exploration. In: 32nd International Symposium on Computational Geometry, SoCG 2016, June 14–18, 2016, Boston, MA, USA, pp. 61:1–61:16 (2016)
67. van Kreveld, M.J., Löffler, M., Staals, F.: Central trajectories. J. Comput. Geom. **8**(1), 366–386 (2017)
68. van Kreveld, M.J., Löffler, M., Staals, F., Wiratma, L.: A refined definition for groups of moving entities and its computation. In: 27th International Symposium on Algorithms and Computation, ISAAC 2016, December 12–14, 2016, Sydney, Australia, pp. 48:1–48:12 (2016)
69. Vieira, M.R., Bakalov, P., Tsotras, V.J.: On-line discovery of flock patterns in spatio-temporal data. In: Proceedings of the 17th ACM SIGSPATIAL International Conference on Advances in Geographic Information Systems, pp. 286–295. ACM (2009)
70. Wang, Y., Lim, E.-P., Hwang, S.-Y.: Efficient algorithms for mining maximal valid groups. VLDB J. **17**(3), 515–535 (2008)