Packing Trees into 1-planar Graphs*

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Abstract. We introduce and study the *1-planar packing problem*: Given k graphs with n vertices $G_1, \dots, G_k$, find a 1-planar graph that contains the given graphs as edge-disjoint spanning subgraphs. We mainly focus on the case when each G_i is a tree and $k = 3$. We prove that a triple consisting of three caterpillars or of two caterpillars and a path may not admit a 1-planar packing, while two paths and a special type of caterpillar always have one. We then study 1-planar packings with few crossings and prove that three paths (resp. cycles) admit a 1-planar packing with at most seven (resp. fourteen) crossings. We finally show that a quadruple consisting of three paths and a perfect matching with $n \geq 24$ vertices admits a 1-planar packing, while such a packing does not exist if $n \leq 10$.

1 Introduction

In the *graph packing problem* we are given a collection of n-vertex graphs G_1, G_2, ..., G_k and we are requested to find a graph G that contains the given graphs as edge-disjoint spanning subgraphs. Various settings of the problem can be defined depending on the type of graphs that have to be packed and on the restrictions put on the packing graph G. The most general case is when G is the complete graph on n vertices and there is no restriction on the input graphs. Sauer and Spencer [15] prove that any two graphs with at most $n-2$ edges can

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be packed into K_n; Woźniak and Wojda [17] give sufficient conditions for the existence of a packing of three graphs.

The setting when G is K_n and each G_i is a tree ($i = 1, 2, \dots, k$) has been considered in many papers. Hedetniemi et al. [8] show that two non-star trees can always be packed into K_n. Notice that, the hypothesis that the trees are not stars is necessary for the existence of the packing because each vertex must have degree at least one in each tree, which is not possible if a vertex is adjacent to every other vertex as it is the case for a star. Wang and Sauer [16] give sufficient conditions for the existence of a packing of three trees into K_n, while Mahéo et al. [11] characterize the triples of trees that admit such a packing.

García et al. [13] consider the *planar packing problem*, that is the case when the graph G is required to be planar. They conjectured that the result of Hedetniemi et al. extends to this setting, i.e., that every pair of non-star trees can be packed into a planar graph. Notice that, when G is required to be planar, two is the maximum number of trees that we can hope to pack (because three trees have more than $3n - 6$ edges). García et al. proved their conjecture for some restricted cases, namely when one of the trees is a path and when the two trees are the same. In a series of subsequent papers the conjecture has been proved true for other pairs of trees. Oda and Ota [12] proved it when one tree is a caterpillar or it is a spider of diameter four. Frati et al. [5] extended the last result to any spider, while Frati [4] considers the case when both trees have diameter four. Geyer et al. showed that a planar packing always exists for a pair of binary trees [6] and for a pair of non-star trees [7], thus finally settling the conjecture.

In the present paper we initiate the study of the *1-planar packing problem*, i.e., the problem of packing a set of graphs into a 1-planar graph. A 1-planar graph is a graph that can be drawn so that each edge has at most one crossing [9]. 1-planar graphs have been introduced by Ringel [14] and have received increasing attention in the last years in the research area called *beyond planarity* (see, e.g., [3]). Since any two non-star trees admit a planar packing, a natural question is whether we can pack more than two trees into a 1-planar graph. On the other hand, since each 1-planar graph has at most $4n - 8$ edges [9], it is not possible to pack more than three trees into a 1-planar graph. Thus, our main question is whether any three trees with maximum vertex degree $n - 3$ admit a 1-planar packing. The restriction to trees of degree at most $n - 3$ is necessary because a vertex of degree larger than $n - 3$ in one tree cannot have degree at least one in the other two trees. The results of this paper can be listed as follows.

- We show that there are triples of structurally very simple trees that cannot be packed into a 1-planar graph (Section 3). These triples consist of three caterpillars and of two caterpillars and a path.
- Motivated by the above results, we study triples consisting of two paths and a caterpillar (Section 4). We characterize the triples consisting of two paths and a 5-legged caterpillar (a caterpillar where each vertex of the spine has no leaves attached or it has at least five) that admit such a packing. We also characterize the triples that admit a 1-planar packing and that consist of two paths and a caterpillar whose spine has exactly two vertices.

– The packing technique of the bullet above is constructive and it gives rise to 1-plane graphs (i.e., 1-planar embedded graphs) with a linear number of crossings. This naturally raises the question about the number of edge crossings required by a 1-planar packing. We show that any three paths with at least six vertices can be packed into a 1-plane graph with seven edge crossings in total (Section 5). We also extend this technique to three cycles obtaining 1-plane graphs with fourteen crossings in total.
– We finally consider the 1-planar packing problem for quadruples of acyclic graphs (Section 6). Since, as already observed, four paths cannot be packed into a 1-planar graph, we consider three paths and a perfect matching. We show that when $n \geq 24$ such a quadruple admits a 1-planar packing and that when $n \leq 10$ a 1-planar packing does not exist.

Preliminary definitions are given in Section 2 and open problems are listed in Section 7. For space reasons, some proofs are sketched or moved to the appendix and the corresponding statements are marked with an asterisk.

2 Preliminaries

Given a graph G and a vertex v of G, we denote by $deg_G(v)$ the vertex degree of v in G. Let $G_1, G_2, \dots, G_k$ be k graphs with n vertices; a *packing* of $G_1, G_2, \dots, G_k$ is an n-vertex graph G that has $G_1, G_2, \dots, G_k$ as edge-disjoint spanning subgraphs. We consider the case when G is a 1-planar graph; in this case we say that G is a *1-planar packing* of $G_1, G_2, \dots, G_k$. If $G_1, G_2, \dots, G_k$ admit a (1-planar) packing G, we also say that $G_1, G_2, \dots, G_k$ *can be packed into* G. We will mainly concentrate on the case where each G_i is a tree ($1 \leq i \leq k$). In this case (and more generally when each G_i is connected), we have restrictions on the values of k and n for which a packing exists.

Property 1 (*). A 1-planar packing of k connected n-vertex graphs $G_1, \dots, G_k$ exists only if $k \leq 3$ and $n \geq 2k$. Moreover, $deg_{G_i}(v) \leq n - k$ for each vertex v.

A *caterpillar* T is a tree such that removing all the leaves results in a path called the *spine*. A *backbone* of T is a path $v_0, v_1, v_2, \dots, v_k, v_{k+1}$ of T where $v_1, v_2, \dots, v_k$ is the spine of T and v_0 and v_{k+1} are two leaves adjacent in T to v_1 and v_k, respectively. T is *h-legged* if every internal vertex of its backbone has degree either 2 or $h + 2$ in T.

3 Trees That Do Not Have a 1-planar Packing

In this section we show that there exist triples of trees that do not admit a 1-planar packing.

Theorem 1. *For every $n \geq 10$, there exists a triple of caterpillars that does not admit a 1-planar packing.*

Proof. The triple consists of three isomorphic caterpillars T_1, T_2, T_3 with $n \geq 10$ vertices. Each T_i has a backbone of length 5 and $n-5$ leaves all adjacent to the middle vertex of the spine, which we call the *center* of T_i. First, notice that each T_i satisfies Property 1, i.e., $deg_{T_i}(v) \leq n-3$. Namely, the vertex with largest degree in T_i is its center, which has degree $n-3$. Let G be any packing of T_1, T_2, and T_3 and let v_1, v_2, and v_3 be the three vertices of G where the three centers of T_1, T_2, T_3, respectively, are mapped. The three vertices v_1, v_2, and v_3 must be distinct because otherwise they would have degree larger than $n-1$ in G, which is impossible. For each v_i we have $deg_{T_i}(v_i) = n-3$ and $deg_{T_j}(v_i) \geq 1$, for $j \neq i$. This implies that $deg_G(v_i) = n-1$ for each v_i. In other words, each v_i is adjacent to all the other vertices of G. Thus, G contains $K_{3,n-3}$ as a subgraph. Since $n \geq 10$ and $K_{3,7}$ is not 1-planar [2], G is not 1-planar. □

Motivated by Theorem 1, we consider triples where one of the caterpillars is a path. Also in this case there exist triples that do not have a 1-planar packing.

Theorem 2. *There exists a triple consisting of a path and two caterpillars with $n = 7$ vertices that does not admit a 1-planar packing.*

Proof. Let T_i $(i = 1, 2)$ be a caterpillar with a backbone of length four whose internal vertices have degree four and three in T_i, respectively. Let G be a packing of T_1, T_2 and a path P of 7 vertices. Let v_1, v_2, v_3, and v_4 be the four vertices of G where the internal vertices of the backbones of T_1 and T_2 are mapped to. We first observe that v_1, v_2, v_3, and v_4 must be distinct. Suppose, as a contradiction, that two of them coincide, say v_1 and v_2; then $deg_{T_1}(v_1) + deg_{T_2}(v_1) \geq 6$. On the other hand $deg_P(v_1) \geq 1$, and therefore $deg_G(v_1) \geq 7$, which is impossible (since G has only 7 vertices). Denote by $G_{1,2}$ the subgraph of G containing only the edges of T_1 and T_2. Two vertices among v_1, v_2, v_3, and v_4, say v_1 and v_2, have degree 5 in $G_{1,2}$, while the other two have degree 4 in $G_{1,2}$. Consider now the edges of P. Since the maximum vertex degree in a graph of seven vertices is six, v_1 and v_2 must be the end-vertices of P, while v_3 and v_4 are internal vertices. This means that they all have degree 6 in G. The vertices distinct from v_1, v_2, v_3, and v_4 have degree 2 in $G_{1,2}$ and degree 4 in G. Thus in G there are four vertices of degree 6 and three vertices of degree 4. The only graph of seven vertices with this degree distribution is $K_7 - K_3$, which is known to be non-1-planar [10]. □

4 1-planar Packings of Two Paths and a Caterpillar

In this section we prove that a triple consisting of two paths P_1 and P_2 and a 5-legged caterpillar T with at least six vertices admits a 1-planar packing. Let P be the backbone of T and let P'_1 and P'_2 be two paths with the same length as P. We first show how to construct a 1-planar packing of P, P'_1 and P'_2. We then modify the computed packing to include the leaves of the caterpillar; this requires to transform some edges of P'_1 and P'_2 to sub-paths that pass through the added leaves. The resulting packing is a 1-planar packing of P_1, P_2 and T.

Let Γ be a 1-planar drawing, possibly with parallel edges, and let e be an edge of Γ. If e has one crossing c, then each of the two parts in which e is divided

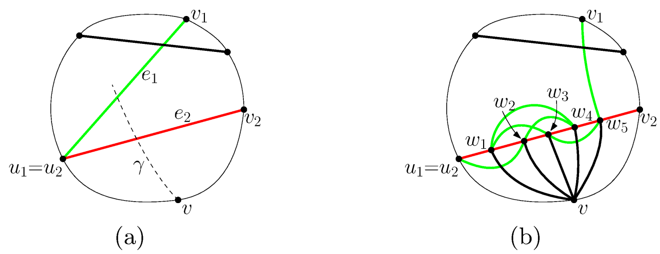


Fig. 1. 5-leaf addition operation.

by c are called *sub-edges* of e; if e has no crossing, e itself is called a *sub-edge* of e. Let v be a vertex of Γ; a *cutting curve* of v is a Jordan arc γ such that: (i) γ has v as an end-point; (ii) γ intersects two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ (possibly $u_1 = u_2$ and/or $v_1 = v_2$); (iii) γ does not intersect any other edge of Γ; (iv) e_1 and e_2 do not cross each other; (iv) if e_1 and e_2 are parallel edges (i.e. $u_1 = u_2$ and $v_1 = v_2$), they have no crossings. The *stub* of e_i with respect to γ is the sub-edge of e_i intersected by γ ($i = 1, 2$). Given a cutting curve γ of a vertex v, and an integer $k \geq 5$, a *k-leaf addition* operation adds k vertices $w_1, w_2, \ldots, w_k$ and the edges $(v, w_1), (v, w_2), \ldots, (v, w_k)$ to Γ in such a way that: (i) the added vertices subdivide the stubs of both e_1 and e_2 with respect to γ; (ii) the subgraph induced by $u_1, u_2, v_1, v_2, w_1, w_2, \ldots, w_k$ has no multiple edges (see Fig. 1 for an example). In other words, a leaf addition adds a set of vertices adjacent to v and replaces the stubs of e_1 and e_2 with two edge-disjoint paths. This operation will be used to modify the 1-planar packing of P, P'_1 and P'_2 to include the leaves of the caterpillar. When the value of k is not relevant, a k-leaf addition will be simply called a *leaf addition*.

Lemma 1. *Let Γ be a 1-planar drawing possibly with parallel edges, let v be a vertex of Γ and let γ be a cutting curve of v. It is possible to execute a k-leaf addition for every $k \geq 5$ in such a way that the resulting drawing is still 1-planar.*

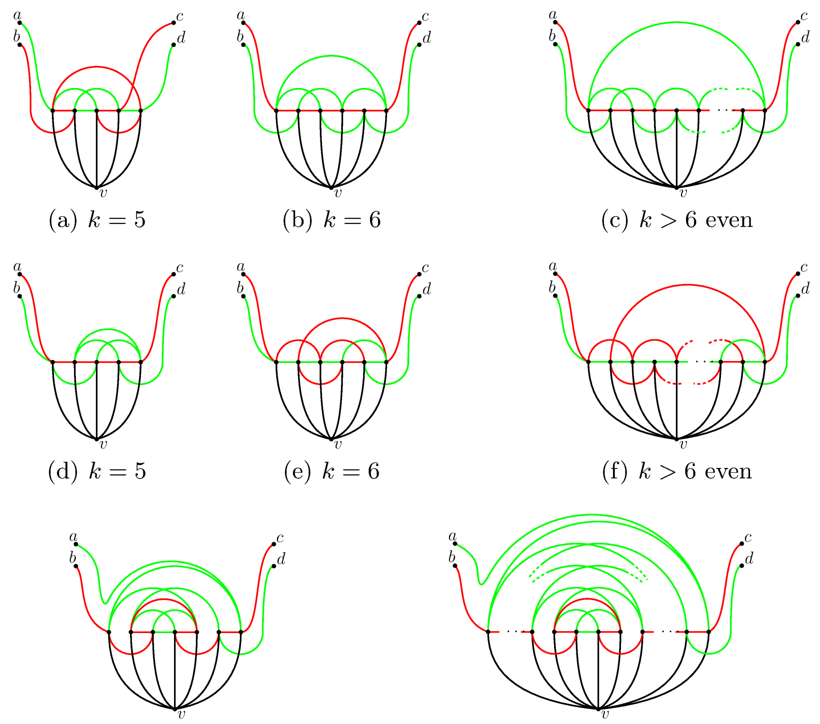
Proof. Denote by e_1 and e_2 the two edges crossed by γ. If one of them or both are crossed in Γ replace their crossing points with dummy vertices. Let e'_i be the stub of e_i with respect to γ (if e_i is not crossed in Γ, e'_i coincides with e_i). After the replacement of the crossings with the dummy vertices the two stubs e'_1 and e'_2 have no crossing. Since γ does not cross any edge distinct from e_1 and e_2, the drawing Γ' obtained by removing e'_1 and e'_2 has a face f whose boundary contains the vertex v and all the end-vertices of e'_1 and of e'_2 (these vertices are at least two and at most four). The idea now is to insert into the face f, without creating any crossing, a gadget that realizes the k-leaf addition for the desired value of $k \geq 5$. A gadget has k vertices that will be added to Γ, a vertex that will be made coincident with v, and four degree-1 vertices a, b, c, and d that will be made coincident with the end-vertices of e'_1 and e'_2. The four edges incident to a, b, c, and d, will be called *dangling edges*. In order to guarantee that the leaf addition is valid and that the drawing Γ'' obtained with the insertion of the gadget inside f is 1-planar, we have to pay attention to two aspects: (i) if a dangling edge is crossed in the gadget, then its degree-1 vertex cannot be made coincident with a dummy vertex (otherwise when we remove the dummy vertex we obtain an edge that is crossed twice); (ii) if two degree-1 vertices of the gadget have to be made coincident (because two end-vertices of e'_1 and e'_2 coincide), then the corresponding dangling edges have no vertex in common in the gadget (otherwise the leaf addition is not valid because it creates multiple edges). We use different gadgets depending on whether e_1 and e_2 are parallel edges or not. If they are parallel edges, we use the gadgets of Figs. 2(a)– 2(c) and 2(g)– 2(h). Notice that in this case, e_1 and e_2 are not crossed by definition of cutting curve. It follows that f has no dummy vertex and (i) is not a problem. On the other hand, both end-vertices of e_1 and e_2 coincide, thus we must guarantee that the dangling edges whose degree-1 vertex will coincide do not share any vertex, which is the case of the gadgets used in this case. If e_1 and e_2 are non-parallel, we use the gadgets of Figs. 2(d)– 2(f) and 2(g)– 2(h). All these gadgets have only one dangling edge that is crossed (labeled d in the figure), also, vertex d can be made coincident with vertex c without creating multiple edges. If e_1 and e_2 are non-parallel, at most two end-vertices of e'_1 and e'_2 are dummy; they cannot belong to the same stub, and they cannot coincide (because e_1 and e_2 do not cross each other). Thus we can make d coincident with a non-dummy vertex and we can make c and d coincident if needed. □

We are ready to describe our construction of a 1-planar packing of P_1, P_2, and T. We use different techniques for different lengths of the backbone of T.

Lemma 2. *Two paths and a 5-legged caterpillar whose backbone contains $n' \geq 6$ vertices admit a 1-planar packing.*

Proof. We start with the construction of a 1-planar packing of the three paths P'_1, P'_2 and P. Let n' be the number of vertices of P'_1, P'_2 and P, assume first that $n' \geq 8$ and $n' \equiv 0 \pmod 4$. A 1-planar packing of P'_1, P'_2 and P for this case is shown in Fig. 3(a) for $n' = 16$ and it is easy to see that it can be extended to any n' multiple of four. Assume that the backbone P of T is the path shown in black in Fig. 3(a). To add the leaves of T to the construction we define a cutting curve for each vertex v that has some leaves attached; we then execute a leaf addition operation for each such vertex. By Lemma 1 it is possible to execute each leaf addition so to guarantee the 1-planarity of the resulting drawing. The cutting curve for each internal vertex of P is shown in Fig. 3(a)

Suppose now that $n' \geq 8$ and $n' \not\equiv 0 \pmod 4$. In this case we first construct a 1-planar packing of three paths with $n'' = 4k$ vertices (with $k = \lfloor \frac{n'}{4} \rfloor$) with the same construction as in the previous case and then we add one, two or three vertices as shown in Figs. 3(b), 3(c), and 3(d), where we also show the cutting curves for each internal vertex of P. If n' is equal to 6 or 7, we use the same approach, the only difference is in the construction of the 1-planar packing of P'_1, P'_2 and P. The construction for such a packing and the cutting curves for the internal vertices of P are shown in Figs. 4(a) and 4(b). □



(a) $k = 5$ (b) $k = 6$ (c) $k > 6$ even

(d) $k = 5$ (e) $k = 6$ (f) $k > 6$ even

(g) $k = 7$ (h) $k > 7$ odd

Fig. 2. Gadgets for the proof of Lemma 1. (a)–(c) are used for parallel edges; (d)–(f) are used for non-parallel edges; (g)–(h) are used for parallel and non-parallel edges.

Lemma 3 (*). *Two paths and a 5-legged caterpillar T whose backbone contains $n' = 5$ vertices admit a 1-planar packing, unless T is a path.*

Proof. If T is a path, then P_1, P_2 and T are all paths of length five, and by Property 1 a 1-planar packing of P_1, P_2 and T does not exist. Suppose therefore that at least one internal vertex of the backbone P of T has some leaves attached. In this case we use an approach similar to the one described in the proof of Lemma 2. However, as we have just explained, a 1-planar packing of P'_1, P'_2 and P cannot exist in this case. We start with a 1-planar packing with two pairs of parallel edges. For each pair, one of the two parallel edges belongs to P'_1 and the other one to P'_2. We will remove the parallel edges by performing the leaf addition operations. To this aim we must guarantee that there is a cutting curve for each pair of parallel edges. The 1-planar packing P'_1, P'_2 and P and the cutting curves for the internal vertices of P are shown in Fig. 4(c), for the case when we have at least two vertices with leaves attached. Indeed, if only two vertices have leaves attached, they are either consecutive along the backbone or not. In the first case, the cutting curves of the vertices labeled a and b will remove the

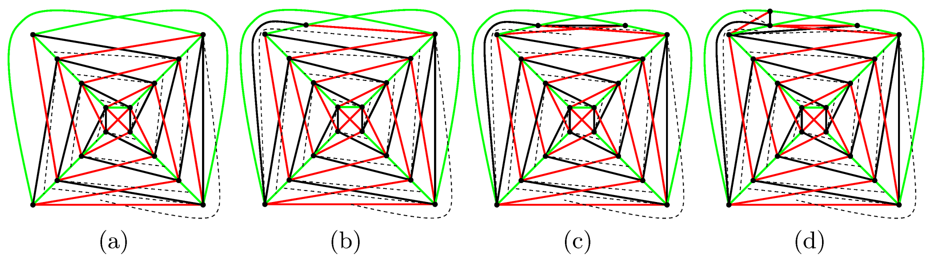


Fig. 3. 1-planar packings of three paths with n' vertices, with a cutting curve for each internal vertex of the black path. (a) $n' = 16$; (b) $n' = 17$; (c) $n' = 18$; (d) $n' = 19$.

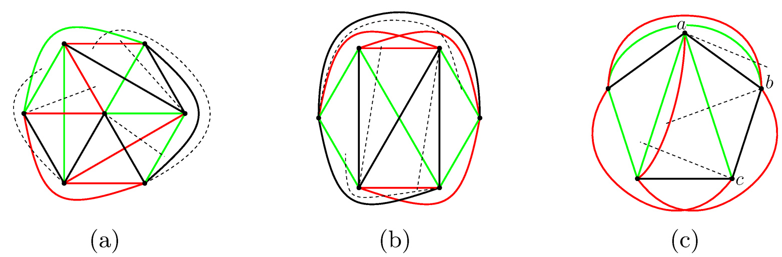


Fig. 4. 1-planar packings of three paths with n' vertices; a cutting curve for each internal vertex of the black path is also shown. (a) $n' = 7$. (b) $n' = 6$. (c) $n' = 5$.

parallel edges; in the second case, the cutting curves of the vertices labeled a and c will remove the parallel edges.

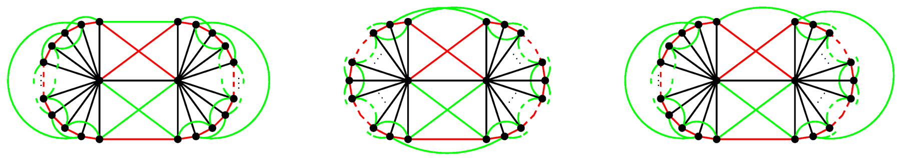
If only one vertex of P has leaves attached, we have only one cutting curve and thus it is not possible to intersect both pairs of parallel edges. To handle this case we use an ad-hoc technique which is described in Appendix B. □

The next theorem gives a complete characterization for the case when the backbone of T has length four.

Theorem 3. *Two paths and a caterpillar T whose backbone contains $n' = 4$ vertices admit a 1-planar packing if and only if $n \geq 6$ and $deg_T(v) \leq n - 3$ for every vertex v.*

Proof. Since the length of the backbone is four, we have exactly two non-leaf vertices v_1 and v_2. Denote by n_i the number of leaves adjacent to v_i $(i = 1, 2)$ and assume $n_1 \leq n_2$. We distinguish different cases depending on the values of n_1 and n_2. If $n_1 = 1$, then we have $deg_T(v_2) = n - 1$ and by Property 1 a 1-planar packing of P_1, P_2 and T does not exist. Assume then that $n_1 \geq 2$.

We start with the case when $n_1 \geq 5$. In this case we construct a 1-planar packing according to different techniques depending on the parity of n_1 and n_2. Figs. 5(a), 5(b), and 5(c) show the construction for the cases when n_1 and n_2 are both even, when they are both odd, and when they have different parity,



(a) even-even (b) odd-odd (c) even-odd

Fig. 5. Illustration for Theorem 3.

respectively. If $n_1 < 5$ we have different ad-hoc constructions that depend on the values of n_1 and n_2. All cases are shown in Fig. 9 in Appendix B. ◻

Lemmas 2 and 3, together with Theorem 3 imply the next theorem.

Theorem 4. *Two paths and a 5-legged caterpillar T with n vertices admit a 1-planar packing if and only if $n \geq 6$ and $deg_T(v) \leq n-3$ for every vertex v.*

5 1-planar Packings with Constant Edge Crossings

The technique described in the previous section constructs 1-planar drawings that have a linear number of crossings. A natural question is whether it is possible to compute a 1-planar packing with a constant number of crossings. In this section we prove that seven (resp. fourteen) crossings suffice for three paths (resp. cycles). It is worth remarking that a 1-planar packing of three paths has at least three crossings (because it has $3n-3$ edges), while a 1-planar packing of three cycles has at least six crossings (because it has $3n$ edges).

Theorem 5. *Three paths with $n \geq 6$ vertices can be packed into a 1-plane graph with at most 7 edge crossings.*

Proof. We prove the statement by showing how to construct a 1-planar drawing with at most 7 crossings of a graph that is the union of three paths. Suppose first that $n = 7 + 3k$ for $k \in \mathbb{N}$. If $k = 0$, we draw the union of the three paths with 7 vertices as shown in Fig. 4(b). The drawing is 1-planar and has three crossings in total. Suppose now that $k > 0$. We consider three rays r_0, r_1, r_2 with a common origin pairwise forming a 120° angle and we place k vertices on each line. We denote by $u_{i,1}, u_{i,2}, \ldots, u_{i,k}$ the vertices of line r_i ($i = 0, 1, 2$) in the order they appear along r_i starting from the origin (see Fig. 6(a)). In the following, when working with the indices of the rays r_i, indices will be taken modulo 3. To draw path P_i ($i = 0, 1, 2$) we draw the edges $(u_{i,1}, u_{i+1,1})$, $(u_{i,j}, u_{i+1,j-1})$, and $(u_{i,j}, u_{i+1,j})$ (for $j = 2, \ldots, k$) as straight-line segments. Notice that, these edges form a zig-zagging path between the vertices of rays r_i and r_{i+1}, so P_i passes trough all vertices of r_i and r_{i+1} but not through the vertices of r_{i+2}. To include these missing vertices in P_i we draw as edges of P_i the straight-line

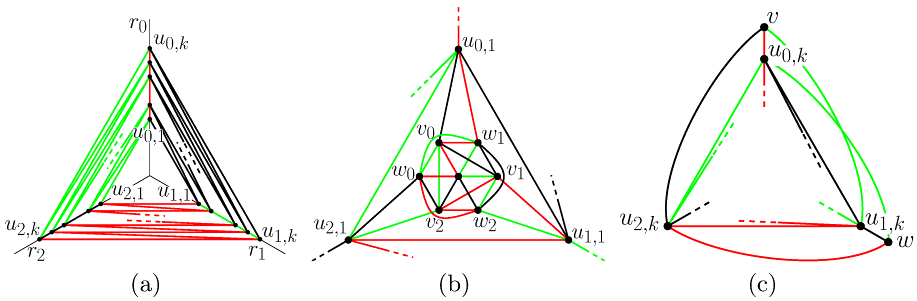


Fig. 6. Illustration for Theorem 5.

edges $(u_{i+2,j}, u_{i+2,j+1})$ (for $j = 1, 2, \ldots, k-1$). In this way we draw two disjoint sub-paths for each path P_i, namely a zig-zagging path between r_i and r_{i+1} and a straight-line path along r_{i+2}. Moreover, we only draw $3k$ edges and therefore there are still 7 missing vertices (and 8 missing edges) in each path. To add the missing vertices and edges and to connect the two sub-paths of each path, we construct a drawing Γ_0 of three paths P'_0, P'_1, P'_2 with seven vertices as in the case when $k = 0$. Denote with v_i and w_i the end-vertices of P'_i in Γ_0. We place Γ_0 inside the triangle $u_{0,1}, u_{1,1}, u_{2,1}$ and add the edges $(v_i, u_{i,1})$ and $(w_i, u_{i+2,1})$. It is easy to see (see also Fig. 6(b)) that these six edges can be added so that the drawing is still 1-planar and so that the total number of crossings is 6. This concludes the proof for $n = 7 + 3k$. If $n = 7 + 3k + 1$ we start with the same construction as in the previous case and then add an extra vertex v outside the triangle $u_{1,k}, u_{2,k}, u_{3,k}$. Notice that each of these three vertices is the end-vertex of two of the three paths with $7 + 3k$ vertices. Thus we can extend each path to include v by connecting it to each of the three vertices $u_{1,k}, u_{2,k}, u_{3,k}$ in a planar way (see Fig. 6(c) ignoring vertex w). If $n = 7 + 3k + 2$, then we add two extra vertices outside the triangle $u_{0,k}, u_{1,k}, u_{2,k}$ and connect both of them to the three vertices $u_{0,k}, u_{1,k}, u_{2,k}$ (recall that each of these three vertices is the end-vertex of two distinct paths with $7 + 3k$ vertices). In this case however the addition of the two extra vertices causes the creation of one crossing. Thus the final drawing is 1-planar and the total number of crossings is at most 7 (see Fig. 6(c)). This concludes the proof for $n \geq 7$. If $n = 6$ we construct a 1-planar packing of three paths with three crossings in total as shown in Fig. 4(a). □

The construction of Theorem 5 can be extended to three cycles.

Theorem 6 (*). *Three cycles with $n \geq 20$ vertices can be packed into a 1-plane graph with at most 14 edge crossings.*

6 From Triples to Quadruples

In this section we extend the study of 1-planar packings from triples of graphs to quadruples of graphs. By Property 1 a 1-planar packing of four graphs does not exist if all graphs are connected, because the number of edges of the four graphs is higher than the number of edges allowed in a 1-planar graph. We consider therefore a quadruple consisting of three paths and a perfect matching. Notice that, in this case the number of vertices n has to be even.

Theorem 7. *Three paths and a perfect matching with $n \geq 24$ vertices admit a 1-planar packing. If $n \leq 10$, the quadruple does not admit a 1-planar packing.*

Proof. Three paths and a perfect matching have a total of $3(n-1)+\frac{n}{2}=\frac{7n}{2}-3$ edges. Since a 1-planar graph has at most $4n-8$ edges, a 1-planar packing of three paths and a perfect matching exists only if $\frac{7n}{2}-3 \leq 4n-8$, i.e., if $n \geq 10$. If $n = 10$, we have $\frac{7n}{2}-3 = 32$ and $4n-8 = 32$, which means that any 1-planar packing of three paths and a perfect matching with $n = 10$ vertices is an optimal 1-planar graph. It is known that every optimal 1-planar graph has at least eight vertices of degree exactly six [1]. On the other hand, in any 1-planar packing of three paths and a perfect matching all vertices, except the at most six end-vertices of the three paths, have degree seven, which implies that a 1-planar packing of three paths and a perfect matching does not exist.

We now prove that a 1-planar packing exists if $n \geq 24$. Based on the fact that in any 1-planar packing of three paths and a perfect matching all vertices have degree seven except at most six, we construct the desired 1-planar packing starting from a 1-planar graph G such that all its vertices have degree at least seven (except possibly at most six); we then partition the edges of G into five sets; three of these sets form a spanning path each, the fourth one forms a perfect matching, and the fifth one contains edges that will not be part of the 1-planar packing. For every $n = 8k$ and $k \geq 3$ it is possible to construct a 1-planar graph with n vertices each having degree at least seven as follows. We start with $k-1$ cycles $C_1, C_2, \ldots, C_{k-1}$. Each cycle C_i $(1 \leq i \leq k-1)$ has eight vertices $v_{i,j}$ with $0 \leq j \leq 7$. Cycle C_i, for $1 \leq i \leq k-2$, is embedded inside cycle C_{i+1} and is connected to it with edges $(v_{i,j}, v_{i+1,j})$ for each $0 \leq j \leq 7$. We have a cycle with four vertices u_0, u_1, u_2, u_3 embedded inside C_1 and connected to it with edges $(u_j, v_{1,2j})$ and $(u_j, v_{1,2j+1})$. Finally, we have a cycle with four vertices w_0, w_1, w_2, w_3 embedded outside C_{k-1} and connected to it with edges $(w_j, v_{k-1,2j})$ and $(w_j, v_{k-1,2j+1})$. The graph G' described so far has n vertices, is planar, all its vertices have degree four, and each vertex is incident to at most one face of size three (see Fig. 7(a)). By adding two crossing edges inside each face of size four, we obtain a 1-planar graph G with n vertices where each vertex has degree at least seven. The graph G and the partition of the edges of G in five sets defining three paths and a matching is shown in Fig. 7(b). If n is not a multiple of 8, then it will be $n = 8k + r$ with $0 < r < 8$ and r even (because n is even). In this case we construct G' as explained above and then we extend the paths $u_0, v_{1,1}, \ldots, v_{k-1,1}$ and $u_1, v_{1,2}, \ldots, v_{k-1,2}$ to the left with 1, 2 or 3 vertices each; we then suitably add some new edges and remove some of the edges of G'. The graph G is then obtained, as in the previous case, by adding a pair of crossing edges inside each face of size four. The resulting graph G and a partition of its edges in five sets defining three paths and a matching is shown in Figs. 7(c), 7(d), and 7(e), for the cases when $r = 2$, $r = 4$, and $r = 6$, respectively. $\square$

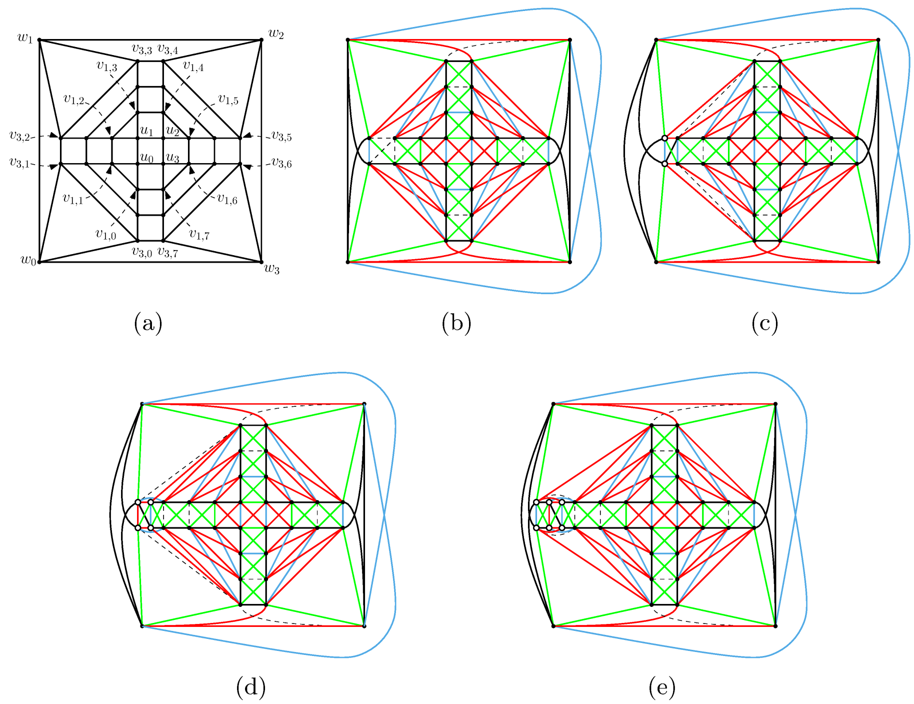


Fig. 7. Illustration for Theorem 7.

7 Open Problems

In this paper we initiated the study of the 1-planar packing problem. We find that this is a fertile and still largely unexplored research subject. We conclude the paper with a list of open problems.

- Theorem 2 holds only for $n = 7$. Do two caterpillars (or more general trees) and a path admit a 1-planar packing if they have more than 7 vertices?
- Can Theorem 4 be extended to general caterpillars? What about two paths and a tree more complex than a caterpillar, for example a binary tree?
- Is it possible to compute a 1-planar packing of three paths or cycles with the minimum number of crossings (three and six, respectively)? Can we compute 1-planar packings with few crossings for triples of other types of trees?
- Theorem 7 states that a quadruple consisting of three paths and a perfect matching has no 1-planar packing if $n \leq 10$, while it admits one when $n \geq 24$. What happens for $10 < n \leq 22$?

We conclude this section by pointing at the more general research direction of extending the packing problem to other families of beyond planar graphs [3].

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Appendix

A Additional Material for Section 2

Property 1 (*). A 1-planar packing of k connected n-vertex graphs $G_1, \ldots, G_k$ exists only if $k \leq 3$ and $n \geq 2k$. Moreover, $deg_{G_i}(v) \leq n - k$ for each vertex v.

Proof. If each G_i is connected, then it has at least $n-1$ edges and therefore any packing of $G_1, G_2, \ldots, G_k$ has at least $k(n-1)$ edges; since the complete graph with n vertices has $\frac{n(n-1)}{2}$ edges it must be $k(n-1) \leq \frac{n(n-1)}{2}$, that is $n \geq 2k$. On the other hand, a 1-planar graph has at most $4n-8$ edges, and therefore it must be $k(n-1) \leq 4n-8$, which implies $k \leq 3$. Moreover, if each G_i is connected $deg_{G_i}(v) \geq 1$ for each v and since $\sum_{i=1}^{k} deg_{G_i}(v) \leq n-1$, it must be $deg_{G_i}(v) \leq n-k$. □

B Additional Material for Section 4

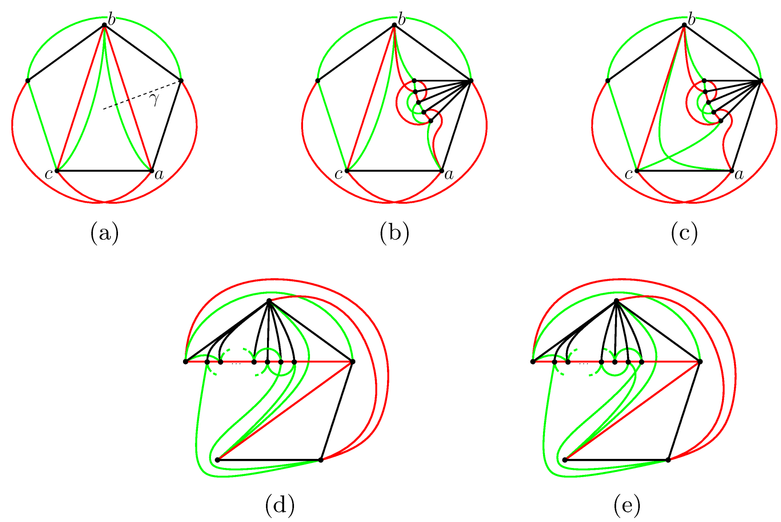


Fig. 8. Illustration for Lemma 3.

Lemma 3 (*). *Two paths and a 5-legged caterpillar T whose backbone contains $n' = 5$ vertices admit a 1-planar packing, unless T is a path.*

Proof. If T is a path, then P_1, P_2 and T are all paths of length five, and by Property 1 a 1-planar packing of P_1, P_2 and T does not exist. Suppose therefore

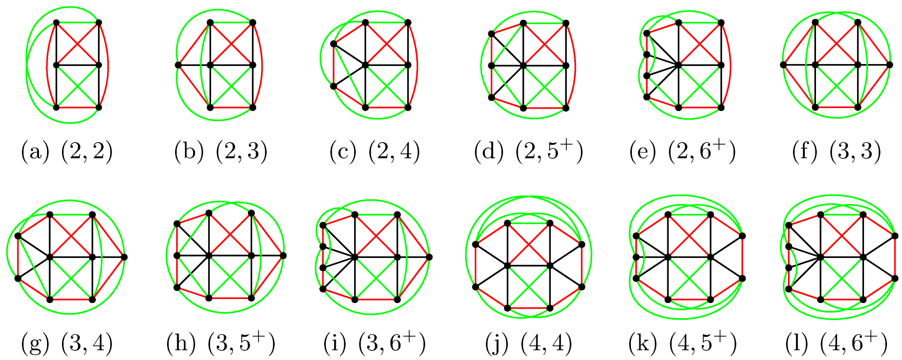


Fig. 9. Illustration for Theorem 3. Constructions for the cases when $n_1 < 5$. For each case the values (n_1, n_2) are indicated; 5^+ means $n_2 \geq 5$ with n_2 odd, while 6^+ means $n_2 \geq 6$ with n_2 even.

that at least one internal vertex of the backbone P of T has some leaves attached. In this case we use an approach similar to the one described in the proof of Lemma 2. However, as we have just explained, a 1-planar packing of P_1', P_2' and P cannot exist in this case. We start with a 1-planar packing with two pairs of parallel edges. For each pair, one of the two parallel edges belongs to P_1' and the other one to P_2'. We will remove the parallel edges by performing the leaf addition operations. To this aim we must guarantee that there is a cutting curve for each pair of parallel edges. The 1-planar packing P_1', P_2' and P and the cutting curves for the internal vertices of P are shown in Fig. 4(c), for the case when we have at least two vertices with leaves attached. Indeed, if only two vertices have leaves attached, they are either consecutive along the backbone or not. In the first case, the cutting curves of the vertices labeled a and b will remove the parallel edges; in the second case, the cutting curves of the vertices labeled a and c will remove the parallel edges.

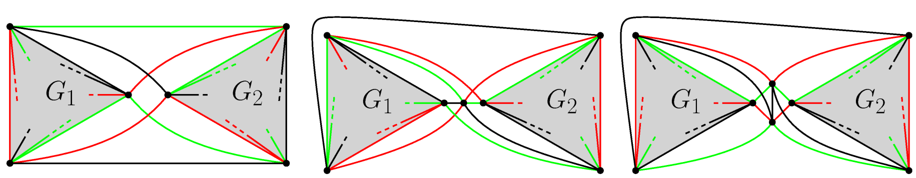
If only one vertex of P has leaves attached, we have only one cutting curve and thus it is not possible to intersect both pairs of parallel edges. To handle this case we distinguish two cases. If the only vertex with leaves attached is the middle vertex of the backbone, then we can adapt the technique used above as follows. Consider the 1-planar packing of P_1', P_2' and P shown in Fig. 8(a), where we have two parallel edges (a, b) and two parallel edges (b, c). Consider now the cutting curve γ shown in Fig. 8(a). This curve intersects the two parallel edges (a, b), thus, performing a leaf addition operation using that curve, we obtain a 1-planar packing of P_1, P_2 and T with the two parallel edges (b, c) (see Fig. 8(b)). These two parallel edges can be removed by modifying the drawing as follows (see also Fig. 8(c) for an illustration). Since the two edges crossed by γ are parallel edges, the leaf addition operation used must be one of those shown in Fig. 2. No matter which of the cases of Fig. 2 applies, one of the two edges incident to vertex a is non-crossed and can be disconnected from a and connected to c

without introducing any crossing. Call this edge e. The parallel edge (c, b) with the same color as e can be disconnected from c and connected to a only crossing e. With this modification we obtain the desired 1-planar packing. If the only vertex with leaves attached is the second (or fourth) vertex of the backbone, we compute a 1-planar packing of P_1, P_2 and T with an ad-hoc technique shown in Figs. 8(d) and 8(e) for an even or an odd number of leaves, respectively. □

C Additional Material for Section 5

Theorem 6 (*). *Three cycles with* $n \geq 20$ *vertices can be packed into a 1-plane graph with at most* 14 *edge crossings.*

Proof. Suppose first that $n \equiv 2 \pmod 3$. In this case, we partition the set V of the n vertices in two groups V_1 and V_2 of size $7 + 3k$ and $7 + 3h$, with $h, k \geq 1$. For each group V_i we compute a 1-planar packing G_i $(i = 1, 2)$ as described in the proof of Theorem 5. Each G_i has 6 crossings and it is embedded so that each path has both end-vertices on the external face (each of the three vertices of the external face is the end-vertex of two distinct paths). We create a 1-planar packing of three cycles with n vertices by connecting the two end-vertices of each path in G_1 with the two end-vertices of a path in G_2. This requires the addition of six edges that can be embedded so to form two crossings (see Fig. 10(a)). Thus, the total number of crossings in the final 1-planar packing is 14. If $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$, we proceed in a way similar to the previous case. We create two 1-planar packings G_1 and G_2 with $7 + 3k$ and $7 + 3h$ vertices $(h, k \geq 1)$ leaving out one or two vertices. When G_1 and G_2 are connected to create the 1-planar packing of three cycles we also add the missing one or two vertices as shown in Figs. 10(b) and 10(c). Also in this case when connecting G_1 and G_2 we have two additional crossings and a total of 14 crossings in the final 1-planar packing. □



(a) (b) (c)

Fig. 10. Illustration for Theorem 6.