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Toric pluripotential theory

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A tribute to Professor Józef Siciak

Abstract. We study finite energy classes of quasiplurisubharmonic functions in the setting of toric compact Kähler manifolds. We characterize toric quasiplurisubharmonic functions and give necessary and sufficient conditions for them to have finite (weighted) energy, both in terms of the associated convex function in $\mathbb{R}^n$, and through the integrability properties of its Legendre transform. We characterize log-Lipschitz convex functions on the Delzant polytope, showing that they correspond to toric quasiplurisubharmonic functions which satisfy a certain exponential integrability condition. In the particular case of dimension one, those log-Lipschitz convex functions of the polytope correspond to Hölder continuous toric quasisubharmonic functions.

Introduction. A *toric* compact Kähler manifold (X,ω,T) is an equivariant compactification of the torus $T=(\mathbb{C}^\star)^n$ equipped with an $(S^1)^n$-invariant Kähler metric ω. Then ω can be written as

$$\omega = dd^c F_0\circ L \quad \text{in } (\mathbb{C}^\star)^n,$$

where $F_0:\mathbb{R}^n\to\mathbb{R}$ is a smooth strictly convex function and

$$L:(\mathbb{C}^\star)^n\to\mathbb{R}^n,\qquad L(z_1,\dots,z_n)=(\log|z_1|,\dots,\log|z_n|). \tag{1}$$

The celebrated Atiyah–Guillemin–Sternberg theorem asserts that the moment map $\nabla F_0:\mathbb{R}^n\to\mathbb{R}^n$ sends $\mathbb{R}^n$ onto the interior of a compact convex polytope

$$P=\{\ell_i(s)\geq 0: 1\leq i\leq d\}\subset\mathbb{R}^n,$$

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where $d \geq n+1$ is the number of $(n-1)$-dimensional faces of P and

$$\ell_i(s) = \langle s, u_i \rangle - \lambda_i,$$

with $\lambda_i \in \mathbb{R}$ and u_i a primitive element of $\mathbb{Z}^n$.

Delzant [Del] observed that in this case P is what is now called "Delzant", i.e. there are exactly n faces of dimension $n-1$ meeting at each vertex, and the corresponding u_i's form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. He also showed conversely that there is exactly one (up to symplectomorphism) toric compact Kähler manifold $(X_P, \{\omega_P\}, T)$ associated to a Delzant polytope $P \subset \mathbb{R}^n$. Here $\{\omega_P\}$ denotes the cohomology class of the T-invariant Kähler form ω_P.

Let

$$G_0(s) := \sup_{x \in \mathbb{R}^n} (\langle x, s \rangle - F_0(x))$$

denote the Legendre transform of F_0. One has $G_0 = \infty$ in $\mathbb{R}^n \setminus P$ and, for $s \in \operatorname{int} P = \nabla F_0(\mathbb{R}^n)$,

$$G_0(s) = \langle x, s \rangle - F_0(x) \;\Leftrightarrow\; s \in \nabla F_0(x) \;\Leftrightarrow\; x \in \nabla G_0(s).$$

Guillemin [Gui] observed that a "natural" representative of the cohomology class $\{\omega_P\}$ is given by

$$G(s) = \frac{1}{2} \sum_{i=1}^{d} \ell_i(s) \log \ell_i(s).$$

We refer the reader to [CDG] for a neat proof of this beautiful formula of Guillemin. Observe that G is only log-Lipschitz regular on P, although the original Kähler potential is smooth.

The purpose of this note is to undertake a systematic study of toric pluripotential analysis. There are three ways to understand a toric quasi-plurisubharmonic function and its Monge–Ampère measure:

- by working directly on X and imposing toric symmetries,
- by looking at the corresponding object (convex function, real Monge–Ampère measure) in $\mathbb{R}^n$ after a logarithmic transformation, and understanding the asymptotic properties at infinity,
- by understanding the behavior near the boundary of the polytope of the Legendre transform of the corresponding convex function.

We refer to Section 3 for the definition of toric ω-plurisubharmonic (ω-psh) functions on X and the corresponding energy classes. If φ is ω-psh, we denote by F_φ the corresponding convex function on $\mathbb{R}^n$ and by G_φ its Legendre transform (see Sections 2 and 3).

Our main results are as follows. We first describe the class of toric ω-psh functions (see Proposition 3.2):

PROPOSITION A. *Let $F_P(x) = \max_{s\in P}\langle x, s\rangle$ denote the support function of the polytope P. The following are equivalent:*

(i) $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$;
(ii) $F_\varphi \le F_P + C$ *for some constant* C;
(iii) $G_\varphi = \infty$ *on* $\mathbb{R}^n \setminus P$;
(iv) $\nabla F_\varphi(\mathbb{R}^n) \subset P$.

We then characterize finite energy toric ω-psh functions and their weighted versions, showing in particular the following (see Theorem 3.6):

THEOREM B. *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. The following are equivalent:*

(i) $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$;
(ii) G_φ *is finite on* $\operatorname{int} P$;
(iii) F_φ *has full Monge–Ampère mass;*
(iv) *the Lelong numbers* $\nu(\varphi, p)$ *equal* 0 *for all* $p \in X$.

In Theorem 4.4 we study more regular toric ω-psh functions, characterizing the maximal log-Lipschitz regularity of Legendrian potentials:

THEOREM C. *Let $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$. The following properties are equivalent:*

(i) *there exists* $\varepsilon > 0$ *such that* $\exp(-\varepsilon\,\mathrm{PSH}_{\mathrm{tor}}(X,\omega)) \subset L^1(\mathrm{MA}(\varphi))$;
(ii) *the function* G_φ *is log-Lipschitz on* P.

It is tempting to think that these conditions are all equivalent to the fact that φ is Hölder continuous. This is easily seen to be the case when $n = 1$. We refer the interested reader to [DD$^+$] for more information, geometric motivations, and related questions connecting the Hölder continuity of Monge–Ampère potentials with the integrability properties of the associated complex Monge–Ampère measure.

The paper is organized as follows. In Section 1 we recall some basic facts about ω-psh functions on any compact Kähler manifold (X,ω), together with the definition and main properties of various energy classes following [GZ]. Section 2 deals with the relevant properties of convex functions and their Legendre transforms. In Section 3 we study energy classes of toric ω-psh functions on a toric compact Kähler manifold (X,ω), and in Section 4 we conclude by looking at questions about the higher regularity of such functions.

1. Finite energy classes. In this section we let (X,ω) be a compact Kähler manifold of dimension n, and we recall the definition of finite energy classes of quasiplurisubharmonic functions following [GZ].

1.1. Bedford–Taylor theory. A function on X is *quasiplurisubharmonic* if it is locally the sum of a psh function and a smooth one. In particular, quasiplurisubharmonic functions are upper semicontinuous and integrable.

DEFINITION 1.1. A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is *ω-psh* if it is quasiplurisubharmonic in X and if the current $\omega + dd^c\varphi$ is positive on X.

Let $\mathrm{PSH}(X,\omega)$ denote the set of all ω-psh functions on X. This is a closed subset of $L^1(X,\omega^n)$.

Bedford and Taylor [BT82] showed that one can define the *complex Monge–Ampère operator*

$$\mathrm{MA}(\varphi) := (\omega + dd^c\varphi)^n = (\omega + dd^c\varphi) \wedge \cdots \wedge (\omega + dd^c\varphi)$$

for all *bounded* ω-psh functions. They showed that whenever (φ_j) is a sequence of bounded ω-psh functions decreasing to φ, the sequence $\mathrm{MA}(\varphi_j)$ of measures converges weakly towards the measure $\mathrm{MA}(\varphi)$. Note also that

$$\int_X \mathrm{MA}(\varphi) = \int_X \omega^n =: V_\omega.$$

At the heart of Bedford–Taylor theory lies the following *maximum principle*: if u, v are bounded ω-psh functions, then

$$\text{(MP)} \qquad 1_{\{v<u\}} \mathrm{MA}(\max(u,v)) = 1_{\{v<u\}} \mathrm{MA}(u).$$

The maximum principle (MP) implies the so-called *comparison principle*: if u, v are bounded ω-psh functions then

$$\int_{\{v<u\}} \mathrm{MA}(u) \le \int_{\{v<u\}} \mathrm{MA}(v).$$

1.2. The class $\mathcal{E}(X,\omega)$. If $\varphi \in \mathrm{PSH}(X,\omega)$, we let

$$\varphi_j := \max(\varphi, -j) \in \mathrm{PSH}(X,\omega) \cap L^\infty(X).$$

It follows from Bedford–Taylor theory that the $\mathrm{MA}(\varphi_j)$ are well defined measures of total mass V_ω. The following monotonicity property holds:

$$\mu_j := 1_{\{\varphi > -j\}} \mathrm{MA}(\varphi_j) \text{ is an increasing sequence of Borel measures.}$$

The proof is an elementary consequence of (MP) (see [GZ, p. 445]). Since the μ_j have total mass bounded above by V_ω, we can define

$$\mu_\varphi := \lim_{j\to\infty} \mu_j,$$

which is a positive Borel measure on X of total mass $\le V_\omega$.

DEFINITION 1.2. We let

$$\mathcal{E}(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\omega) : \mu_\varphi(X) = V_\omega\}.$$

For $\varphi \in \mathcal{E}(X,\omega)$, we set $\mathrm{MA}(\varphi) := \mu_\varphi$.

The definition is justified by the following important fact proved in [GZ]: *The complex Monge–Ampère operator* $\varphi \mapsto \mathrm{MA}(\varphi)$ *is well defined on the class* $\mathcal{E}(X,\omega)$*, in the sense that if* $\varphi \in \mathcal{E}(X,\omega)$ *then for every decreasing sequence* $\varphi_j \searrow \varphi$ *of bounded* ω*-psh functions the measures* $\mathrm{MA}(\varphi_j)$ *converge weakly on* X *towards* μ_φ*.*

Every bounded ω-psh function clearly belongs to $\mathcal{E}(X,\omega)$. The class $\mathcal{E}(X,\omega)$ also contains many ω-psh functions which are unbounded. When X is a compact Riemann surface, $\mathcal{E}(X,\omega)$ is the set of ω-sh functions whose Laplacian does not charge polar sets.

REMARK 1.3. If $\varphi \in \mathrm{PSH}(X,\omega)$ is normalized so that $\varphi \le -1$, then $-(-\varphi)^\varepsilon$ belongs to $\mathcal{E}(X,\omega)$ whenever $0 \le \varepsilon < 1$ (see e.g. [CGZ]). The functions which belong to the class $\mathcal{E}(X,\omega)$, although usually unbounded, have relatively mild singularities. In particular they have zero Lelong number at every point.

It is shown in [GZ] that the maximum principle (MP) and the comparison principle continue to hold in the class $\mathcal{E}(X,\omega)$. The latter can be characterized as the largest class for which the complex Monge–Ampère operator is well defined and the maximum principle holds.

1.3. Weighted energy classes. Let $\mathcal{W}$ denote the set of all functions $\chi : \mathbb{R}^- \to \mathbb{R}^-$ such that χ is increasing and $\chi(-\infty) = -\infty$.

DEFINITION 1.4. We let $\mathcal{E}_\chi(X,\omega)$ be the set of ω-psh functions with finite χ-energy,

$$\mathcal{E}_\chi(X,\omega) := \{\varphi \in \mathcal{E}(X,\omega) : \chi(-|\varphi|) \in L^1(X, \mathrm{MA}(\varphi))\}.$$

When $\chi(t) = -(-t)^p$, $p > 0$, we set $\mathcal{E}^p(X,\omega) = \mathcal{E}_\chi(X,\omega)$.

We list here a few important properties of these classes and refer the reader to [GZ, BEGZ] for the proofs:

- $\mathcal{E}(X,\omega) = \bigcup_{\chi\in\mathcal{W}} \mathcal{E}_\chi(X,\omega)$;
- $\mathrm{PSH}(X,\omega) \cap L^\infty(X) = \bigcap_{\chi\in\mathcal{W}} \mathcal{E}_\chi(X,\omega)$;
- the classes $\mathcal{E}^p(X,\omega)$ are convex;
- $\varphi \in \mathcal{E}^p(X,\omega)$ if and only if for any (resp. for one) sequence of bounded ω-psh functions $\varphi_j \searrow \varphi$, $\sup_j \int_X |\varphi_j|^p \,\mathrm{MA}(\varphi_j) < \infty$;
- if φ_j is a sequence of ω-psh functions decreasing to $\varphi \in \mathcal{E}^p(X,\omega)$, then the measures $|\varphi_j|^p \,\mathrm{MA}(\varphi_j)$ converge weakly to $|\varphi|^p \,\mathrm{MA}(\varphi)$.

2. Facts on convex functions. We collect here a few properties of convex functions which will be used later. Some of these are well known, but proofs are included for the convenience of the reader (see also [BB, Section 2]).

2.1. Subgradients and Monge–Ampère measures. Let $F:\mathbb{R}^n\to\mathbb{R}$ be a convex function. The *subgradient* of F at x is the set

$$\nabla F(x)=\{s\in\mathbb{R}^n : F(y)\geq F(x)+\langle y-x,s\rangle,\ \forall y\in\mathbb{R}^n\}.$$

We let

$$\nabla F(\mathbb{R}^n):=\bigcup_{x\in\mathbb{R}^n}\nabla F(x).$$

The *Legendre transform* G of F is the lower semicontinuous convex function defined by

$$G:\mathbb{R}^n\to(-\infty,\infty],\qquad G(s)=\sup_{x\in\mathbb{R}^n}(\langle x,s\rangle-F(x)).$$

Then F is the Legendre transform of G,

$$F(x)=\sup_{s\in\mathbb{R}^n}(\langle x,s\rangle-G(s)),$$

and one has

$$G(s)=\langle x,s\rangle-F(x)\ \Leftrightarrow\ s\in\nabla F(x)\ \Leftrightarrow\ x\in\nabla G(s).$$

Lemma 2.1. *Let $F:\mathbb{R}^n\to\mathbb{R}$ be a convex function.*

(i) *If F is smooth and strictly convex then $\nabla F:\mathbb{R}^n\to\mathbb{R}^n$ is injective, and hence an open map.*
(ii) *If $s_0\in\nabla F(\mathbb{R}^n)$ then $G(s_0)<\infty$. Conversely, if $G(s)<\infty$ for all s in an open ball $B(s_0,r)$ then $s_0\in\nabla F(\mathbb{R}^n)$.*
(iii) *Let $F_j:\mathbb{R}^n\to\mathbb{R}$, $j\geq 1$, be convex functions. Then $F_j\searrow F$ pointwise on $\mathbb{R}^n$ if and only if the Legendre transforms satisfy $G_j\nearrow G$ pointwise on $\mathbb{R}^n$.*

Proof. (i) If $p\neq q$ and $f(t):=F((1-t)p+tq)$ then $f''(t)>0$, so $f'(0)=\langle\nabla F(p),q-p\rangle<f'(1)=\langle\nabla F(q),q-p\rangle$. Hence $\nabla F(p)\neq\nabla F(q)$.

(ii) By the definition of the subgradient, if $s_0\in\nabla F(x)$ then $\langle y,s_0\rangle-F(y)\leq\langle x,s_0\rangle-F(x)$ for all $y\in\mathbb{R}^n$, so $G(s_0)=\langle x,s_0\rangle-F(x)<\infty$. Conversely, by shrinking r we may assume that $G<M$ on $B(s_0,r)$ for some constant M, hence $\langle x,s\rangle-F(x)\leq M$ for all $x\in\mathbb{R}^n$ and $s\in B(s_0,r)$. Let $\widetilde F(x)=F(x)-\langle x,s_0\rangle+M$. It follows that $\widetilde F(x)\geq\langle x,s-s_0\rangle$ for all $s\in B(s_0,r)$, hence $\widetilde F(x)\geq r\|x\|$. Therefore $\widetilde F$ assumes a global minimum, i.e. there exists $x_0\in\mathbb{R}^n$ such that $\widetilde F(x)\geq\widetilde F(x_0)$. Thus $0\in\nabla\widetilde F(x_0)=\nabla F(x_0)-s_0$.

(iii) Assume that $F_j\searrow F$. Then $G_j\nearrow\widetilde G$, where $\widetilde G$ is lower semicontinuous, convex and $\widetilde G\leq G$. If $\widetilde F$ is the Legendre transform of $\widetilde G$ then we have $F_j\geq\widetilde F\geq F$. We conclude that $\widetilde F=F$ and so $\widetilde G=G$. The converse follows by a similar argument. ∎

REMARK 2.2. Note that if $F(x) = e^x$, $x \in \mathbb{R}$, then we have $G(0) = 0$ but $0 \notin F'(\mathbb{R})$, so the hypothesis that $G(s) < \infty$ in a neighborhood of s_0 is needed in the condition (ii) of Lemma 2.1.

LEMMA 2.3. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function. If χ is a continuous function with compact support on $\mathbb{R}^n$ then*

$$\int_{(\mathbb{C}^\star)^n} (\chi \circ L)(dd^c F \circ L)^n = \int_{\mathbb{R}^n} \chi \, \mathrm{MA}_{\mathbb{R}}(F),$$

where L is defined in (1), $d = \partial + \overline{\partial}$, $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$, *and* $\mathrm{MA}_{\mathbb{R}}(F)$ *is the real Monge–Ampère measure of F.*

Proof. Approximating F by a decreasing sequence of smooth convex functions it suffices to assume that F is smooth. Recall that in this case $\mathrm{MA}_{\mathbb{R}}(F)$ is the measure defined by

$$\mathrm{MA}_{\mathbb{R}}(F) = n! \det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right] dV,$$

where V denotes the Lebesgue measure on the corresponding Euclidean space. Note that the function $F \circ L$ is plurisubharmonic on $(\mathbb{C}^\star)^n$ and

$$\frac{\partial^2 (F \circ L)}{\partial z_i \partial \overline{z_j}} = \frac{1}{4 z_i \overline{z}_j}\left(\frac{\partial^2 F}{\partial x_i \partial x_j} \circ L\right),$$

hence

$$\det\left[\frac{\partial^2 (F \circ L)}{\partial z_i \partial \overline{z_j}}\right] = \frac{1}{4^n \prod_j |z_j|^2}\left(\det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right] \circ L\right).$$

It follows that

$$\begin{aligned}
(dd^c F \circ L)^n &= \left(\frac{i}{\pi}\right)^n (\partial\overline{\partial} F \circ L)^n \\
&= n!\left(\frac{i}{\pi}\right)^n \det\left[\frac{\partial^2 (F \circ L)}{\partial z_i \partial \overline{z_j}}\right] dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n \\
&= n!\left(\frac{2}{\pi}\right)^n \det\left[\frac{\partial^2 (F \circ L)}{\partial z_i \partial \overline{z_j}}\right] dV(z) \\
&= n!\left(\frac{2}{\pi}\right)^n \frac{1}{4^n \prod_j |z_j|^2}\left(\det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right] \circ L\right) dV(z) \\
&= \frac{n!}{(2\pi)^n} \det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}(\log r_1, \ldots, \log r_n)\right] \frac{dr_1 \ldots dr_n}{r_1 \ldots r_n}\, d\theta_1 \ldots d\theta_n,
\end{aligned}$$

where we use polar coordinates $z_j = r_j e^{i\theta_j}$. Changing variables $x_j := \log r_j$

we obtain

$$\begin{aligned}\int\limits_{(\mathbb{C}^\star)^n} (\chi \circ L)\,(dd^c F \circ L)^n \\ &= n! \int\limits_{(0,\infty)^n} \left(\chi \det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right]\right)(\log r_1, \ldots, \log r_n)\, \frac{dr_1 \ldots dr_n}{r_1 \ldots r_n} \\ &= n! \int\limits_{\mathbb{R}^n} \chi(x) \det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}(x)\right] dV(x) = \int\limits_{\mathbb{R}^n} \chi\, \mathrm{MA}_{\mathbb{R}}(F).\end{aligned}$$

For a non-smooth convex function F the positive measure $\mathrm{MA}_{\mathbb{R}}(F)$ is the real Monge–Ampère measure of F (in the sense of Alexandrov; see [Gut]). ∎

The following lemma is proved using an idea of Al Taylor [T].

Lemma 2.4. *If $F_1, F_2 : \mathbb{R}^n \to \mathbb{R}$ are convex functions with $F_2(x) \to \infty$ as $\|x\| \to \infty$ and $F_1(x) \le F_2(x)$ for all $x \in \mathbb{R}^n$, then*

$$\int\limits_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_1) \le \int\limits_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_2).$$

Proof. Fix a compact $K \subset (\mathbb{C}^\star)^n$, a number $\varepsilon > 0$, and consider on $(\mathbb{C}^\star)^n$ the plurisubharmonic function

$$u := \max\{F_1 \circ L, (1+\varepsilon)F_2 \circ L - C\},$$

where the constant $C > 0$ is chosen so that $u = F_1 \circ L$ in a neighborhood of K. Since $F_2(x) \to \infty$ as $\|x\| \to \infty$ it follows that $F_1 \le F_2 \le (1+\varepsilon)F_2 - C$ on $\mathbb{R}^n \setminus \mathcal{K}$ for some compact $\mathcal{K} \subset \mathbb{R}^n$. Then $L^{-1}(\mathcal{K}) \subset (\mathbb{C}^\star)^n$ is compact and $u = (1+\varepsilon)F_2 \circ L - C$ on $(\mathbb{C}^\star)^n \setminus L^{-1}(\mathcal{K})$. We infer that

$$(1+\varepsilon)^n \int\limits_{(\mathbb{C}^\star)^n} (dd^c F_2 \circ L)^n = \int\limits_{(\mathbb{C}^\star)^n} (dd^c u)^n \ge \int\limits_{K} (dd^c F_1 \circ L)^n.$$

We conclude by using Lemma 2.3 and by letting $K \nearrow (\mathbb{C}^\star)^n$ and $\varepsilon \searrow 0$. ∎

2.2. Growth properties. Let P be a (compact) convex body in $\mathbb{R}^n$. Its *support function*, which is also known as the *indicator function*, is the convex function

$$F_P(x) := \max_{s \in P} \langle x, s\rangle.$$

Its Legendre transform is the convex function

$$G_P(x) = \begin{cases} 0 & \text{if } x \in P, \\ \infty & \text{if } x \notin P. \end{cases}$$

If $P_\vartheta := \vartheta + P$ is the image of P under the translation by ϑ, and $P_\lambda := \lambda P$ is the image of P under the dilation by $\lambda > 0$, then

$$F_{P_\vartheta}(x) = F_P(x) + \langle \vartheta, x\rangle, \qquad G_{P_\vartheta}(s) = G_P(s-\vartheta),$$
$$F_{P_\lambda}(x) = F_P(\lambda x) = \lambda F_P(x), \quad G_{P_\lambda}(s) = G_P(s/\lambda).$$

LEMMA 2.5. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function with Legendre transform G. The following are equivalent:*

(i) $F \le F_P + C$ *for some constant* C;
(ii) $G = \infty$ *on* $\mathbb{R}^n \setminus P$;
(iii) $\nabla F(\mathbb{R}^n) \subset P$.

Proof. To show (i)⇒(ii), if $F \le F_P + C$ then $G \ge G_P - C$, so $G = G_P = \infty$ on $\mathbb{R}^n \setminus P$. For (ii)⇒(iii), if $s \in \nabla F(\mathbb{R}^n)$ then $G(s) < \infty$, hence $s \in P$ by (ii).

To prove (iii)⇒(i), let $x \in \mathbb{R}^n$ and note that if $s \in \nabla F(\mathbb{R}^n)$ then $s \in P$, so $\langle s, x\rangle \le F_P(x)$. Since F is locally Lipschitz along the line $\mathbb{R} \ni t \mapsto tx$, we have

$$F(x) - F(0) = \int_0^1 \frac{d}{dt} F(tx)\, dt = \int_0^1 \langle \nabla F(tx), x\rangle\, dt \le \int_0^1 F_P(x)\, dt = F_P(x). \ \blacksquare$$

LEMMA 2.6. *Let $F_0 : \mathbb{R}^n \to \mathbb{R}$ be a smooth strictly convex function such that $F_P - C \le F_0 \le F_P + C$ for some constant C. Then $\nabla F_0 : \mathbb{R}^n \to \operatorname{int} P$ is bijective and $\nabla G_0 : \operatorname{int} P \to \mathbb{R}^n$ is its inverse, where G_0 is the Legendre transform of F_0. Moreover, if χ is a continuous function with compact support on $\mathbb{R}^n$ then*

$$\int_{\mathbb{R}^n} \chi\, \mathrm{MA}_{\mathbb{R}}(F_0) = n! \int_{\operatorname{int} P} \chi \circ \nabla G_0\, dV \quad \text{and} \quad \int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_0) = n!\, \mathrm{vol}(P).$$

Proof. By Lemmas 2.5 and 2.1(i), $\nabla F_0 : \mathbb{R}^n \to P$ is injective. As $F_P - C \le F_0$ we see that $G_0 \le G_P + C$, so $G_0 \le C$ on P. Thus $\operatorname{int} P \subset \nabla F_0(\mathbb{R}^n)$ by Lemma 2.1(ii), and hence $\nabla F_0(\mathbb{R}^n) = \operatorname{int} P$ since ∇F_0 is open. If $x, x' \in \nabla G_0(s)$ then $s = \nabla F_0(x) = \nabla F_0(x')$, so $x = x'$. Hence G_0 is differentiable on $\operatorname{int} P$ and $\nabla G_0 = (\nabla F_0)^{-1}$. The remaining assertions follow by the change of variables $x = \nabla G_0(s)$, $s = \nabla F_0(x)$, so

$$dV(s) = \det\left[\frac{\partial^2 F_0}{\partial x_i \partial x_j}\right] dV(x) = \frac{1}{n!} \mathrm{MA}_{\mathbb{R}}(F_0)(x). \ \blacksquare$$

LEMMA 2.7. *If $0 \in \operatorname{int} P$ then there exist constants $a, b > 0$ such that*

$$b\|x\| \le F_P(x) \le a\|x\|, \qquad \forall x \in \mathbb{R}^n.$$

Proof. If $a, b > 0$ are such that the closed balls $\overline{B}(0,b)$, $\overline{B}(0,a)$ satisfy $\overline{B}(0,b) \subset P \subset \overline{B}(0,a)$, then

$$b\|x\| = F_{\overline{B}(0,b)} \le F_P(x) \le F_{\overline{B}(0,a)} = a\|x\|. \ \blacksquare$$

LEMMA 2.8. *Assume that $0 \in \operatorname{int} P$ and let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function with Legendre transform G, such that $F \le F_P + C$ for some constant C. The following are equivalent:*

(i) *$G(s) < \infty$ for all $s \in \operatorname{int} P$;*
(ii) *for every $\varepsilon \in (0,1)$ there is $M_\varepsilon > 0$ with $F \ge (1-\varepsilon)F_P - M_\varepsilon$ on $\mathbb{R}^n$.*

Moreover, these conditions imply that $\int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F) = n!\operatorname{vol}(P)$.

Proof. Note that

$$F_{(1-\varepsilon)P}(x) = \sup_{s \in P} \langle x, (1-\varepsilon)s\rangle = (1-\varepsilon)F_P(x).$$

Assume that $G(s) < \infty$ for all $s \in \operatorname{int} P$. Since $0 \in \operatorname{int} P$, $(1-\varepsilon)P \subset \operatorname{int} P$ for $\varepsilon \in (0,1)$, so there exists $M_\varepsilon > 0$ such that $G \le M_\varepsilon$ on $(1-\varepsilon)P$. It follows that

$$F(x) \ge \sup_{s \in (1-\varepsilon)P} (\langle x, s\rangle - G(s)) \ge F_{(1-\varepsilon)P}(x) - M_\varepsilon = (1-\varepsilon)F_P(x) - M_\varepsilon.$$

Conversely, if $F \ge (1-\varepsilon)F_P - M_\varepsilon = F_{(1-\varepsilon)P} - M_\varepsilon$, then $G \le G_{(1-\varepsilon)P} + M_\varepsilon$, so $G(s) \le M_\varepsilon$ for $s \in (1-\varepsilon)P$. As $\varepsilon \searrow 0$ this implies that $G(s) < \infty$ for all $s \in \operatorname{int} P$.

By Lemma 2.7 we have $F_P(x) \to \infty$ as $\|x\| \to \infty$. Since $F \le F_P + C$, Lemmas 2.4 and 2.6 imply that

$$\int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F) \le \int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_P) = \int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_0) = n!\operatorname{vol}(P),$$

where F_0 is a function as in Lemma 2.6. Note that by (ii), $F(x) \to \infty$ as $\|x\| \to \infty$, hence Lemma 2.4 again shows that

$$\int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F) \ge (1-\varepsilon)^n \int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_P), \qquad \forall \varepsilon \in (0,1).$$

Letting $\varepsilon \to 0$ finishes the proof. ∎

We conclude this section with the following lemma:

LEMMA 2.9. *Let P be a compact convex body in $\mathbb{R}^n$ with non-empty interior and $G : P \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous convex function. Then*

$$|\inf_P G| \le \frac{1}{(2^{1/(n+1)} - 1)\operatorname{vol}(P)} \int_P |G|\, dV.$$

Proof. If $G \ge 0$ on P then $\int_P G\, dV \ge \inf_P G \cdot \operatorname{vol}(P)$ and we are done. Otherwise, consider the convex set $S = \{G < 0\} \subset P$. It suffices to show that if $p \in \operatorname{int} S$ then

$$-G(p) \le \frac{1}{(2^{1/(n+1)} - 1)\operatorname{vol}(P)} \int_P |G|\, dV.$$

We assume without loss of generality that $p = 0$ and use spherical coordinates. For $\theta \in S^{n-1}$ let $0 < a(\theta) \le b(\theta)$ be defined by $a(\theta)\theta \in \partial S$ and $b(\theta)\theta \in \partial P$. If σ is the area measure on S^{n-1} then

$$\operatorname{vol}(P) = \int_{S^{n-1}} \frac{b(\theta)^n}{n}\, d\sigma(\theta).$$

By convexity it follows that

$$\begin{aligned} G(t\theta) &\le \frac{-G(0)}{a(\theta)}(t - a(\theta)) \le 0 \quad \text{if } 0 \le t \le a(\theta),\\ G(t\theta) &\ge \frac{-G(0)}{a(\theta)}(t - a(\theta)) \ge 0 \quad \text{if } a(\theta) < t \le b(\theta). \end{aligned}$$

Thus

$$\begin{aligned} \int_P |G|\, dV \ge{}& \int_{S^{n-1}} \int_0^{a(\theta)} \frac{-G(0)}{a(\theta)}(a(\theta) - t)t^{n-1}\, dt\, d\sigma(\theta)\\ &+ \int_{S^{n-1}} \int_{a(\theta)}^{b(\theta)} \frac{-G(0)}{a(\theta)}(t - a(\theta))t^{n-1}\, dt\, d\sigma(\theta)\\ ={}& \frac{-G(0)}{n(n+1)} \int_{S^{n-1}} \left(\frac{nb(\theta)^{n+1}}{a(\theta)} - (n+1)b(\theta)^n + 2a(\theta)^n \right) d\sigma(\theta). \end{aligned}$$

Note that

$$\begin{aligned} f(a) &:= \frac{nb^{n+1}}{a} - (n+1)b^n + 2a^n\\ &\ge f(b2^{-1/(n+1)}) = (n+1)(2^{1/(n+1)} - 1)b^n \end{aligned}$$

for $0 < a \le b$. Therefore

$$\begin{aligned} \int_P |G|\, dV &\ge \frac{-G(0)}{n}(2^{1/(n+1)} - 1) \int_{S^{n-1}} b(\theta)^n\, d\sigma(\theta)\\ &= -G(0)(2^{1/(n+1)} - 1)\operatorname{vol}(P), \end{aligned}$$

and we are done. ■

3. Toric energy classes. Let (X, ω) be a toric compact Kähler manifold of dimension n. Then X is a compactification of the complex torus $(\mathbb{C}^\star)^n$ such that the canonical action by multiplication of $(\mathbb{C}^\star)^n$ on itself extends to a holomorphic action of $(\mathbb{C}^\star)^n$ on X. Moreover, there exists a smooth strictly convex function $F_0 : \mathbb{R}^n \to \mathbb{R}$ such that $\omega|_{(\mathbb{C}^\star)^n} = dd^c F_0 \circ L$, where L is defined in (1). If P is the compact convex polytope determined by X then $\nabla F_0 : \mathbb{R}^n \to \operatorname{int} P$ is bijective and we may assume that $0 \in \operatorname{int} P$. Let G_0 denote the Legendre transform of F_0.

3.1. Toric quasiplurisubharmonic functions. A *toric ω-psh function* on X is an ω-psh function φ that is invariant under the $(S^1)^n$ action induced by the $(\mathbb{C}^\star)^n$ action on X. We denote by $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ the class of such functions. It follows that there exists a convex function $F_\varphi : \mathbb{R}^n \to \mathbb{R}$ such that

$$F_\varphi \circ L = F_0 \circ L + \varphi \quad \text{ on } (\mathbb{C}^\star)^n \subset X.$$

We denote by G_φ the Legendre transform on F_φ. Note that F_φ is continuous on $\mathbb{R}^n$, hence φ is continuous on $(\mathbb{C}^\star)^n$.

We define the energy classes of toric ω-psh functions by

$$\begin{aligned}\mathcal{E}_{\mathrm{tor}}(X,\omega) &= \mathrm{PSH}_{\mathrm{tor}}(X,\omega) \cap \mathcal{E}(X,\omega),\\ \mathcal{E}^p_{\mathrm{tor}}(X,\omega) &= \mathrm{PSH}_{\mathrm{tor}}(X,\omega) \cap \mathcal{E}^p(X,\omega),\\ \mathcal{E}_{\chi,\mathrm{tor}}(X,\omega) &= \mathrm{PSH}_{\mathrm{tor}}(X,\omega) \cap \mathcal{E}_\chi(X,\omega),\end{aligned}$$

where $p > 0$ and $\chi \in \mathcal{W}$ (see Sections 1.2, 1.3).

We begin with the following simple lemma:

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$-C \le F_0(x) - F_P(x) \le C, \quad \forall x \in \mathbb{R}^n.$$

Proof. Since $\nabla F_0(\mathbb{R}^n) \subset P$, by Lemma 2.5 we have $F_0 \le F_P + C_1$ for some constant C_1. Let $\omega' \in \{\omega\}$ be a Kähler form with associated convex function F such that its Legendre transform G is given by Guillemin's formula. Then $F \circ L = F_0 \circ L + \theta$ for some smooth ω-psh function θ. Hence $F \le F_0 + C_2$ and $G \ge G_0 - C_2$, for some constant C_2. Since G is bounded above on P it follows that $G_0 \le G_P + C_3$, and so $F_0 \ge F_P - C_3$, for some constant C_3. ∎

Our next result gives a characterization of toric ω-psh functions:

Proposition 3.2. *The following are equivalent:*

(i) $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$;
(ii) $F_\varphi \le F_P + C$ *for some constant* C;
(iii) $G_\varphi = \infty$ *on* $\mathbb{R}^n \setminus P$;
(iv) $\nabla F_\varphi(\mathbb{R}^n) \subset P$.

Proof. If $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ then φ is bounded above on X, hence $F_\varphi \le F_0 + C'$ for some constant C', and (ii) follows by Lemma 3.1.

Conversely, if (ii) holds then by Lemma 3.1, $F_\varphi \le F_0 + C'$ for some constant C', hence $\varphi \le C'$ on $(\mathbb{C}^\star)^n \subset X$. Since $X \setminus (\mathbb{C}^\star)^n$ is an analytic set invariant under the $(S^1)^n$ action, we conclude that φ extends to an ω-psh function on X which is $(S^1)^n$ invariant.

The remaining equivalences (ii)⇔(iii)⇔(iv) follow from Lemma 2.5. ∎

Proposition 3.3. *If* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *then*

$$\sup_X \varphi \le C_P + \frac{1}{(2^{1/(n+1)}-1)\,\mathrm{vol}(P)} \int_P |G_\varphi|\,dV,$$

where $C_P = \sup_P G_0 = \sup_{\mathbb{R}^n}(F_P - F_0)$.

Proof. Note that, for a constant C, one has $F_\varphi - F_0 \le C$ on $\mathbb{R}^n$ if and only if $G_0 - G_\varphi \le C$ on P. It follows that

$$\sup_X \varphi = \sup_{\mathbb{R}^n}(F_\varphi - F_0) = \sup_P (G_0 - G_\varphi) \le C_P - \inf_P G_\varphi,$$

and the proposition follows from Lemma 2.9. ∎

Example 3.4. Let $X = \mathbb{F}_1$ be the blow up of $\mathbb{P}^2$ at a toric point p. It is a geometrically ruled surface. We let F denote a generic fiber, E be the exceptional divisor, and $H = E + F$ the total transform of a line through p. The cohomology classes of F and H are both semipositive and generate $H^{1,1}(X,\mathbb{R})$. Any Kähler class $\{\omega\}$ is cohomologous to $aH + bF$, with $a, b > 0$. In coordinates $z \in (\mathbb{C}^\star)^2$ it can be represented as

$$\omega = a\omega_1 + b\omega_2, \quad \text{where} \quad \omega_1 = \tfrac{1}{2} dd^c \log(1 + \|z\|^2), \quad \omega_2 = dd^c \log \|z\|.$$

The convex function associated to ω is

$$F_0(x) = \tfrac{1}{2} a \log(1 + e^{2x_1} + e^{2x_2}) + \tfrac{1}{2} b \log(e^{2x_1} + e^{2x_2}),$$

and $P = \overline{\nabla F_0(\mathbb{R}^2)}$ is the polytope

$$P = \{s_1 \ge 0,\ s_2 \ge 0,\ b \le s_1 + s_2 \le a + b\}.$$

Thus $d = 4$, $\ell_1(s) = s_1$, $\ell_2(s) = s_2$, $\ell_3(s) = a + b - s_1 - s_2$, and $\ell_4(s) = s_1 + s_2 - b$. For $s \in P$, the Legendre transform of F_0 is given by

$$\begin{aligned} G_0(s) = \tfrac{1}{2}[&s_1 \log s_1 + s_2 \log s_2 + (a + b - s_1 - s_2) \log(a + b - s_1 - s_2) \\ &+ (s_1 + s_2 - b) \log(s_1 + s_2 - b) - (s_1 + s_2) \log(s_1 + s_2) - a \log a]. \end{aligned}$$

3.2. The class $\mathcal{E}_{\mathrm{tor}}(X,\omega)$

Definition 3.5. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $F \le F_P + C$. We say that F has *full Monge–Ampère mass* if

$$\int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F) = \int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_0) = n!\,\mathrm{vol}(P).$$

Recall that a *toric point* of X is a point fixed by the action of the complex torus $(\mathbb{C}^\star)^n$ on X.

Theorem 3.6. *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. *The following are equivalent:*

(i) $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$;
(ii) G_φ *is finite on* $\mathrm{int}\,P$;
(iii) F_φ *has full Monge–Ampère mass;*

(iv) *for every* $\varepsilon > 0$ *there exists a compact set* $K_\varepsilon \subset (\mathbb{C}^\star)^n$ *such that*

$$\varphi(z) \geq -\varepsilon \max\{\big|\log|z_1|\big|, \dots, \big|\log|z_n|\big|\} \text{ on } (\mathbb{C}^\star)^n \setminus K_\varepsilon;$$

(v) *the Lelong numbers* $\nu(\varphi, p)$ *equal* 0 *for all* $p \in X$*;*

(vi) *the Lelong numbers* $\nu(\varphi, p)$ *equal* 0 *at all toric points* $p \in X$*.*

Proof. To prove that (i)⇒(ii), if $\varphi \in \mathcal{E}_{\mathrm{tor}}(X, \omega)$ then $\varphi \in \mathcal{E}_{\chi,\mathrm{tor}}(X, \omega)$ for some function $\chi \in \mathcal{W}$. By Proposition 3.9, $G_\varphi \in L_\chi(P)$, so $G_\varphi < \infty$ a.e. on P. Since G_φ is convex, this implies that $G_\varphi(s) < \infty$ for all $s \in \operatorname{int} P$. The implication (ii)⇒(iii) follows from Lemma 2.8.

We next prove that (iii)⇒(i). Consider the measure $\langle \omega_\varphi^n \rangle$ defined as the non-pluripolar product of the positive closed currents $\omega_\varphi := \omega + dd^c\varphi$ [BEGZ, Definition 1.1]. As φ is locally bounded on $(\mathbb{C}^*)^n$, the Bedford–Taylor product $\omega_\varphi^n = \omega_\varphi \wedge \cdots \wedge \omega_\varphi$ is well defined on $(\mathbb{C}^*)^n$ [BT76, BT82]. Since $(\mathbb{C}^*)^n = X \setminus A$, where A is an analytic subset of X, it follows from [BEGZ, p. 204, Proposition 1.6] that $\langle \omega_\varphi^n \rangle$ is the trivial extension of ω_φ^n to X. Then

$$\begin{aligned} \int_X \langle \omega_\varphi^n \rangle &= \int_{(\mathbb{C}^*)^n} \omega_\varphi^n = \int_{(\mathbb{C}^*)^n} (dd^c F_\varphi \circ L)^n \\ &= \int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_\varphi) = \int_{\mathbb{R}^n} \mathrm{MA}_{\mathbb{R}}(F_0) = \int_X \omega^n. \end{aligned}$$

Therefore $\langle \omega_\varphi^n \rangle$ has full mass, so $\varphi \in \mathcal{E}_{\mathrm{tor}}(X, \omega)$ and $\mathrm{MA}(\varphi) = \langle \omega_\varphi^n \rangle$ [BEGZ, Section 2].

To show that (ii)⇒(iv), let $\varepsilon > 0$. Using Lemmas 2.8 and 3.1 we get

$$\varphi = (F_\varphi - F_0) \circ L \geq -\varepsilon F_P \circ L - C - M_\varepsilon$$

with some constants $C, M_\varepsilon > 0$. By Lemma 2.7 there exists a constant $a > 0$ such that $F_P(x) \leq a \max\{|x_1|, \dots, |x_n|\}$. These imply that

$$\varphi(z) \geq -2a\varepsilon \max\{\big|\log|z_1|\big|, \dots, \big|\log|z_n|\big|\}$$

for $z \in (\mathbb{C}^\star)^n \setminus K_\varepsilon$, where $K_\varepsilon = \{\varepsilon F_P \circ L \leq C + M_\varepsilon\}$.

Conversely, to prove that (iv)⇒(ii), we let $\varepsilon \in (0, 1)$ and by applying Lemma 2.8 we need to show that there exists $M_\varepsilon > 0$ such that $F_\varphi \geq (1 - \varepsilon)F_P - M_\varepsilon$. By Lemma 2.7 we have $F_P(x) \geq b \max\{|x_1|, \dots, |x_n|\}$ for some constant $b > 0$. Using (iv) and Lemma 3.1 we obtain

$$\begin{aligned} F_\varphi(L(z)) &= F_0(L(z)) + \varphi(z) \\ &\geq F_P(L(z)) - C - b\varepsilon \max\{\big|\log|z_1|\big|, \dots, \big|\log|z_n|\big|\} \\ &\geq (1 - \varepsilon)F_P(L(z)) - C \end{aligned}$$

for $z \in (\mathbb{C}^\star)^n \setminus K_\varepsilon$, where $K_\varepsilon \subset (\mathbb{C}^\star)^n$ is a compact set. Since F_φ, F_P are continuous this implies that $F_\varphi(L(z)) \geq (1 - \varepsilon)F_P(L(z)) - M_\varepsilon$ on $(\mathbb{C}^\star)^n$ for some constant $M_\varepsilon > C$.

Recall that functions in the class $\mathcal{E}(X,\omega)$ have zero Lelong number at each point. To complete the proof we assume that $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ has zero Lelong number at all the toric points of X and show that (iv) holds.

Let $\varepsilon > 0$ and let $p_1,\dots,p_N$ be the toric points of X. We denote as before by $z=(z_1,\dots,z_n)$ the coordinates on the complex torus $(\mathbb{C}^\star)^n \subset X$. For $1 \le j \le N$, there exists an open set V_j such that $p_j \in V_j \subset X$, and a biholomorphic map $\varPhi_j : V_j \to \mathbb{C}^n$ such that $\varPhi_j(p_j)=0$, $(\mathbb{C}^\star)^n \subset V_j$, $\varPhi_j((\mathbb{C}^\star)^n)=(\mathbb{C}^\star)^n$, and $X = V_1\cup\dots\cup V_N$ (see e.g. [ALZ, Proposition 4.4 and its proof]). Moreover, if $\varPhi_j(z)=\zeta=(\zeta_1,\dots,\zeta_n)$ then

$$(\log|\zeta_1|,\dots,\log|\zeta_n|) = A_j(\log|z_1|,\dots,\log|z_n|), \qquad \text{where } A_j \in \mathrm{GL}_n(\mathbb{Z}).$$

We denote by $\|A_j\|_\infty$ the operator norm of A_j with respect to the sup norm (i.e. $\|A_jx\|_\infty \le \|A_j\|_\infty\|x\|_\infty$) and let $\gamma := \max\{\|A_1\|_\infty,\dots,\|A_N\|_\infty\}$. Since X is compact, we can find $R>0$ such that $X=\bigcup_{j=1}^N \varPhi_j^{-1}(\varDelta^n(0,R))$, where $\varDelta^n(0,R)\subset\mathbb{C}^n$ is the open polydisc of radius R centered at 0.

We observe that $(\varPhi_j^{-1})^\star(\omega|_{(\mathbb{C}^\star)^n}) = dd^cF_0^j\circ L$, where $L(\zeta_1,\dots,\zeta_n) = (\log|\zeta_1|,\dots,\log|\zeta_n|)$ and $F_0^j:\mathbb{R}^n\to\mathbb{R}$ is a smooth strictly convex function such that $F_0^j\circ L$ extends to a smooth plurisubharmonic function on $\mathbb{C}^n$. It follows that $\nabla F_0^j(\mathbb{R}^n)\subset(0,\infty)^n$. Moreover, there exists a convex function $F_\varphi^j:\mathbb{R}^n\to\mathbb{R}$ such that $\nabla F_\varphi^j(\mathbb{R}^n)\subset[0,\infty)^n$ and $F_\varphi^j\circ L = F_0^j\circ L+\varphi\circ\varPhi_j^{-1}$ on $(\mathbb{C}^\star)^n$. The function $F_\varphi^j\circ L$ extends to a plurisubharmonic function on $\mathbb{C}^n$ and has Lelong number $\nu(F_\varphi^j\circ L,0)=0$, since Lelong numbers are invariant under biholomorphic maps. Hence there exists $r_j=r_j(\varepsilon)>0$ such that $F_\varphi^j(\log r,\dots,\log r)\ge\frac12\varepsilon\log r$ for $0<r<r_j$. Since F_φ^j is increasing in each variable, this implies

$$F_\varphi^j(\log|\zeta_1|,\dots,\log|\zeta_n|)\ge\tfrac12\varepsilon\min\{\log|\zeta_1|,\dots,\log|\zeta_n|\}$$

if $\min\{|\zeta_1|,\dots,|\zeta_n|\}<r_j$. So there exists a constant $M_\varepsilon>0$ such that

$$\varphi\circ\varPhi_j^{-1}(\zeta)=F_\varphi^j\circ L(\zeta)-F_0^j\circ L(\zeta)\ge\tfrac12\varepsilon\min\{\log|\zeta_1|,\dots,\log|\zeta_n|\}-M_\varepsilon$$

for $\zeta\in\varDelta^n(0,R)$ with $\min\{|\zeta_1|,\dots,|\zeta_n|\}<r_j$. By shrinking r_j we obtain

$$\varphi\circ\varPhi_j^{-1}(\zeta)\ge\varepsilon\min\{\log|\zeta_1|,\dots,\log|\zeta_n|\}=-\varepsilon\max\bigl\{\bigl|\log|\zeta_1|\bigr|,\dots,\bigl|\log|\zeta_n|\bigr|\bigr\}$$

for $\zeta\in\varDelta^n(0,R)$ with $\min\{|\zeta_1|,\dots,|\zeta_n|\}<r_j$. It follows that

$$\varphi(z)\ge-\gamma\varepsilon\max\bigl\{\bigl|\log|z_1|\bigr|,\dots,\bigl|\log|z_n|\bigr|\bigr\}$$

if $z\in U_j:=(\mathbb{C}^\star)^n\cap\varPhi_j^{-1}(\varDelta^n(0,R)\cap\{\zeta\in\mathbb{C}^n:\min\{|\zeta_1|,\dots,|\zeta_n|\}<r_j\})$. We note that $U_\varepsilon:=\bigcup_{j=1}^N U_j\subset(\mathbb{C}^\star)^n$ is open and $K_\varepsilon:=(\mathbb{C}^\star)^n\setminus U_\varepsilon$ is compact. Moreover the above lower estimate on φ holds on $(\mathbb{C}^\star)^n\setminus K_\varepsilon$. ■

COROLLARY 3.7. *If $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$ and χ is a non-negative continuous function on $\mathbb{R}^n$ then*

$$\int_{(\mathbb{C}^\star)^n} (\chi\circ L)(dd^c F_\varphi \circ L)^n = \int_{\mathbb{R}^n} \chi\,\mathrm{MA}_{\mathbb{R}}(F_\varphi) = n! \int_{\operatorname{int} P} \chi(\nabla G_\varphi(s))\,dV(s).$$

Proof. The first equality follows from Lemma 2.3. The second one follows from [BB, Lemma 2.7], since F_φ has full Monge–Ampère mass by Theorem 3.6. ∎

EXAMPLES 3.8. If $X = \mathbb{P}^n$ is the complex projective space and ω is the Fubini–Study Kähler form, then $F_0(x) = \frac{1}{2}\log(1+\sum_{i=1}^n e^{2x_i})$ and $P = \overline{\nabla F_0(\mathbb{R}^n)}$ is the simplex

$$P = \Big\{s : s_i \geq 0,\ 1\leq i\leq n,\ \sum_{i=1}^n s_i \leq 1\Big\}.$$

Thus $d = n+1$, $\ell_i(s) = s_i$ for $1 \leq i \leq n$, and $\ell_{n+1}(s) = 1-\sum_{i=1}^n s_i$. The Legendre transform of F_0 is

$$G_0(s) = \frac{1}{2}\Big[\sum_{i=1}^n s_i \log s_i + \Big(1-\sum_{j=1}^n s_j\Big)\log\Big(1-\sum_{j=1}^n s_j\Big)\Big].$$

This coincides with the function given by Guillemin's formula.

1° Let $[z] = [z_0 : z_1 : \ldots : z_n]$ denote the homogeneous coordinates on $\mathbb{P}^n$. The function

$$\varphi_1[z] = \log|z_1| - \log\|z\|$$

is ω-psh and toric. It does not belong to the class $\mathcal{E}_{\mathrm{tor}}(X,\omega)$ since it has positive Lelong numbers along the toric hyperplane $(z_1 = 0)$. The associated convex function $F_1 : \mathbb{R}^n \to \mathbb{R}$ is given by $F_1(x) = x_1$ and its Legendre transform is

$$G_1(s) = 0 \quad \text{if } s = (1,0,\ldots,0),$$

and $G_1(s) = \infty$ otherwise.

2° The function

$$\varphi_2[z] = \max_{1\leq i\leq n} \log|z_i| - \log\|z\|$$

is ω-psh and toric. It does not belong to the class $\mathcal{E}_{\mathrm{tor}}(X,\omega)$ since it has one positive Lelong number at the point $[1 : 0 : \cdots : 0]$. The corresponding convex function is $F_2(x) = \max_{1\leq i\leq n} x_i$ and its Legendre transform is

$$G_2(s) = 0 \quad \text{if } s \in \Big\{s_i \geq 0,\ 1\leq i\leq n,\ \sum_{i=1}^n s_i = 1\Big\},$$

and $G_2(s) = \infty$ otherwise.

3.3. The classes $\mathcal{E}_{\chi,\mathrm{tor}}(X,\omega)$. If $\chi \in \mathcal{W}$, we define $L_\chi(P)$ to be the set of lower semicontinuous functions $G : P \to \mathbb{R} \cup \{\infty\}$ such that

$$\int_P -\chi\Big(\min_P G - G(s)\Big)\, dV(s) < \infty.$$

PROPOSITION 3.9. *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. If $\chi \in \mathcal{W}$ and $\varphi \in \mathcal{E}_{\chi,\mathrm{tor}}(X,\omega)$ then $G_\varphi \in L_\chi(P)$. Conversely, if $p \geq 1$ and $G_\varphi \in L^p(P)$ then $\varphi \in \mathcal{E}^p_{\mathrm{tor}}(X,\omega)$.*

Proof. For the first claim, let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega) \cap L^\infty(X)$ be such that F_φ is smooth and strictly convex, and $F_\varphi \leq F_P \leq F_0$. We prove the following a priori estimate:

$$\int_{\mathrm{int}\,P} -\chi(-G_\varphi(s))\, dV(s) \leq \frac{1}{n!} \int_X -\chi(\varphi)\,\mathrm{MA}(\varphi).$$

Note that $\varphi \leq 0$ on X and $G_\varphi \geq G_P = 0$ on P. Moreover, by Proposition 4.1 below and Lemma 2.6, $\nabla F_\varphi : \mathbb{R}^n \to \mathrm{int}\,P$ is bijective. Since $F_0 \geq F_P$ and $F_\varphi(x) = \langle x, s\rangle - G_\varphi(s)$ for $x = \nabla G_\varphi(s)$, we obtain

$$\begin{aligned}(F_0 - F_\varphi) \circ \nabla G_\varphi(s) &\geq (F_P - F_\varphi) \circ \nabla G_\varphi(s) \\ &= F_P(\nabla G_\varphi(s)) - \langle \nabla G_\varphi(s), s\rangle + G_\varphi(s) \geq G_\varphi(s) \geq 0,\end{aligned}$$

where $s \in \mathrm{int}\,P$ and the last estimate follows from the definition of F_P. Applying Lemmas 2.6 and 2.3 we get

$$\begin{aligned}\int_{\mathrm{int}\,P} -\chi(-G_\varphi(s))\, dV(s) &\leq \int_{\mathrm{int}\,P} -\chi((F_\varphi - F_0) \circ \nabla G_\varphi(s))\, dV(s) \\ &= \frac{1}{n!} \int_{\mathbb{R}^n} -\chi(F_\varphi - F_0)\,\mathrm{MA}_{\mathbb{R}}(F_\varphi) \\ &= \frac{1}{n!} \int_{(\mathbb{C}^\star)^n} -\chi(\varphi)\,\mathrm{MA}(\varphi).\end{aligned}$$

Let now $\varphi \in \mathcal{E}_{\chi,\mathrm{tor}}(X,\omega)$ be such that $F_\varphi \leq F_P - 1$. There exists a sequence $\varphi_j \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega) \cap L^\infty(X)$ such that $\varphi_j \searrow \varphi$, the associated functions F_{φ_j} are smooth and strictly convex, and $F_{\varphi_j} \leq F_P$. Then

$$\int_X -\chi(\varphi_j)\,\mathrm{MA}(\varphi_j) \to \int_X -\chi(\varphi)\,\mathrm{MA}(\varphi)$$

as $j \to \infty$. Since $G_{\varphi_j} \nearrow G_\varphi$ it follows by the a priori estimate applied to φ_j and by the monotone convergence theorem that

$$\int_P -\chi(-G_\varphi(s))\, dV(s) \leq \frac{1}{n!} \int_X -\chi(\varphi)\ \mathrm{MA}(\varphi) < \infty,$$

so $G_\varphi \in L_\chi(P)$. This concludes the proof of the first claim.

Conversely, let $p \geq 1$ and consider the space of Kähler potentials $\mathcal{H} = \{\varphi \in \mathcal{C}^\infty(X) : \omega + dd^c\varphi > 0\}$ endowed with the metric

$$d_p(\varphi_1,\varphi_2)=\inf\Big\{\int_0^1\Big(\int_X |\dot\varphi_t|^p\,\mathrm{MA}(\varphi_t)\Big)^{1/p}dt\Big\},\qquad (\varphi_1,\varphi_2)\in\mathcal{H}^2,$$

where the infimum is taken over all smooth paths $[0,1]\ni t\mapsto\varphi_t\in\mathcal{H}$ joining φ_1 to φ_2. It is shown in [Dar, Theorem 3] that if $\varphi_1,\varphi_2\in\mathcal{H}$ then

$$\int_X |\varphi_1-\varphi_2|^p\,\mathrm{MA}(\varphi_1)\le C_p d_p(\varphi_1,\varphi_2)^p \tag{2}$$

for some constant $C_p>1$ depending on p. On the other hand if $\varphi_1,\varphi_2\in\mathcal{H}\cap\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ are determined by convex functions F_1,F_2 with Legendre transforms G_1, G_2 then, by [G, Proposition 4.3],

$$d_p(\varphi_1,\varphi_2)^p=\int_P |G_1-G_2|^p\,dV. \tag{3}$$

If $\varphi\in\mathcal{H}\cap\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, we apply (2) and (3) with $\varphi_1=\varphi$ and $\varphi_2=0$, and obtain

$$\int_X |\varphi|^p\,\mathrm{MA}(\varphi)\le C_p\int_P |G_\varphi-G_0|^p\,dV\le 2^{p-1}C_p(\|G_\varphi\|^p_{L^p(P)}+\|G_0\|^p_{L^p(P)}).$$

Let now $\varphi\in\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ be such that $G_\varphi\in L^p(P)$, and take a sequence $\varphi_j\in\mathcal{H}\cap\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ such that $\varphi_j\searrow\varphi$. Then $G_{\varphi_j}\nearrow G_\varphi$, so by the above estimate applied to φ_j and by dominated convergence we conclude that $\sup_j\int_X|\varphi_j|^p\,\mathrm{MA}(\varphi_j)<\infty$. Hence $\varphi\in\mathcal{E}^p_{\mathrm{tor}}(X,\omega)$. ∎

Example 3.10. Let $X=\mathbb{P}^1$ and ω be the Fubini–Study Kähler form. By Examples 3.8 the corresponding convex function is $F_0(x)=\frac12\log(1+e^{2x})$ and $P=\overline{F_0'(\mathbb{R})}=[0,1]$. Let φ be the toric ω-sh function associated to the convex function $F(x):=F_P(x)=\max(x,0)$. Note that the Legendre transform of F is $G=0$ on $[0,1]$, and $dd^cF(\log|z|)$ is the (normalized) Lebesgue measure on the unit circle $S^1\subset\mathbb{P}^1$. We consider the sequence $\{\varphi_j\}$ of toric ω-sh functions defined by the convex functions

$$F_j(x)=(1-\varepsilon_j)F(x)+\varepsilon_j\max(x,-C_j),$$

where ε_j decreases to 0, while C_j increases to ∞. A straightforward computation shows that the corresponding Legendre transforms are

$$G_j(s)=\max(C_j(\varepsilon_j-s),0),\qquad 0\le s\le 1.$$

Note that

$$\begin{aligned}\varphi_j(z)-\varphi(z)&=-\varepsilon_j\log^+|z|+\varepsilon_j\max(\log|z|,-C_j)\\&=\begin{cases}-\varepsilon_jC_j, & |z|<e^{-C_j},\\ \varepsilon_j\log|z|, & e^{-C_j}\le|z|<1,\\ 0, & |z|\ge 1.\end{cases}\end{aligned}$$

Thus we obtain the following:

- $\varphi_j \to \varphi$ in L^1 if and only if $\varepsilon_j \to 0$;
- $\varphi_j \to \varphi$ in L^∞ if and only if $\varepsilon_j C_j \to 0$;
- $\varphi_j \to \varphi$ in $W^{1,2}$ (that is, the natural topology on $\mathcal{E}^1(X,\omega)$) if and only if $\varepsilon_j^2 C_j \to 0$.

3.4. Finite moments. It is tempting to think that one can characterize the condition $\varphi \in \mathcal{E}^q_{\mathrm{tor}}(X,\omega)$ by a finite moment condition, as follows. Let $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$, $\mu_{\mathbb{R}}(\varphi) := \mathrm{MA}_{\mathbb{R}}(F_\varphi)$, $0 < q < n$, and $q^* = nq/(n-q)$ denote the Sobolev conjugate exponent of q. Does one have

$$\varphi \in \mathcal{E}^{q^*}_{\mathrm{tor}}(X,\omega) \;\Leftrightarrow\; \int_{\mathbb{R}^n} \|x\|^q \, d\mu_{\mathbb{R}}(\varphi) < \infty?$$

This question was raised by E. Di Nezza, who showed in [DiN, Proposition 2.5] that, if $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$, $n \geq 2$ and $1 \leq q < n$, then

$$\int_{\mathbb{R}^n} \|x\|^q \, d\mu_{\mathbb{R}}(\varphi) < \infty \;\Rightarrow\; \varphi \in \mathcal{E}^{q^*}_{\mathrm{tor}}(X,\omega).$$

We have the following partial answer to this question in dimension $n = 1$, i.e. when $X = \mathbb{P}^1$ and $\omega = \omega_{\mathrm{FS}}$:

PROPOSITION 3.11. *Let $(X,\omega) = (\mathbb{P}^1, \omega_{\mathrm{FS}})$, $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$, and $0 < q < 1$.*

(i) *If $q \geq 1/2$ and $\int_{\mathbb{R}} |x|^q \, d\mu_{\mathbb{R}}(\varphi) < \infty$ then $\varphi \in \mathcal{E}^{q^*}_{\mathrm{tor}}(X,\omega)$.*
(ii) *If $\varphi \in \mathcal{E}^1_{\mathrm{tor}}(X,\omega)$ then $\int_{\mathbb{R}} |x|^q \, d\mu_{\mathbb{R}}(\varphi) < \infty$ for all $q < 1/2$.*
(iii) *There exists a function $\varphi \in \mathcal{E}^1_{\mathrm{tor}}(X,\omega)$ with $\int_{\mathbb{R}} |x|^{1/2} \, d\mu_{\mathbb{R}}(\varphi) = \infty$.*

Proof. Recall that in this case $P = [0,1]$ (see Example 3.10).

(i) Replacing φ by $\varphi + C$ we may assume that $\min_{[0,1]} G_\varphi = G_\varphi(a) = 0$ for some $a \in [0,1]$. Note that G_φ is convex and finite, so it is differentiable a.e. on $(0,1)$. Let $s,t \in (0,1)$ be such that $G'_\varphi(s), G'_\varphi(t)$ exist and t is between a and s. Since G_φ is convex and assumes its minimum at a, we have $|G'_\varphi(t)| \leq |G'_\varphi(s)|$. It follows that

$$\begin{aligned} 0 \leq G_\varphi(s) &= \Big|\int_a^s G'_\varphi(t)\,dt\Big| \leq \Big|\int_a^s |G'_\varphi(t)|^{1-q} |G'_\varphi(t)|^q \,dt\Big| \\ &\leq |G'_\varphi(s)|^{1-q} \Big|\int_a^s |G'_\varphi(t)|^q \,dt\Big| \leq |G'_\varphi(s)|^{1-q} \int_0^1 |G'_\varphi(t)|^q \,dt. \end{aligned}$$

Using Corollary 3.7 we obtain

$$\begin{aligned} \int_0^1 G_\varphi(s)^{q/(1-q)}\,ds &\leq \Big(\int_0^1 |G'_\varphi(t)|^q\,dt\Big)^{q/(1-q)} \int_0^1 |G'_\varphi(s)|^q\,ds \\ &= \Big(\int_0^1 |G'_\varphi(s)|^q\,ds\Big)^{1/(1-q)} = \Big(\int_{\mathbb{R}} |x|^q\,d\mu_{\mathbb{R}}(\varphi)\Big)^{1/(1-q)} < \infty. \end{aligned}$$

Since $q/(1-q) \geq 1$, Proposition 3.9 yields $\varphi \in \mathcal{E}^{q/(1-q)}_{\mathrm{tor}}(X,\omega)$.

(ii) Let $\varphi \in \mathcal{E}^1_{\rm tor}(X,\omega)$ and $q < 1/2$. Then $G_\varphi \in L^1(P)$ by Proposition 3.9. The conclusion follows by showing that if $F_\varphi \le F_P$ on $\mathbb{R}$ then

$$\int_{\mathbb{R}} |x|^q \, d\mu_{\mathbb{R}}(\varphi) \le \frac{2(1-q)}{1-2q} \|G_\varphi\|^q_{L^1}.$$

Note that it suffices to prove this in the case when φ is bounded. Indeed, if $\varphi \in \mathcal{E}^1_{\rm tor}(X,\omega)$ is such that $F_\varphi \le F_P$ on $\mathbb{R}$, then there exists a sequence of bounded toric ω-psh functions $\varphi_j \searrow \varphi$ such that $F_{\varphi_j} \le F_P$ on $\mathbb{R}$. Hence $0 \le G_{\varphi_j} \nearrow G_\varphi$. Since $\mu_{\mathbb{R}}(\varphi_j) \to \mu_{\mathbb{R}}(\varphi)$ weakly on $\mathbb{R}$ it follows by the monotone convergence theorem that

$$\int_{\mathbb{R}} |x|^q \, d\mu_{\mathbb{R}}(\varphi) \le \liminf_{j\to\infty} \int_{\mathbb{R}} |x|^q \, d\mu_{\mathbb{R}}(\varphi_j) \le \frac{2(1-q)}{1-2q} \|G_\varphi\|^q_{L^1}.$$

Assume that φ is a bounded toric ω-psh function such that $F_\varphi \le F_P$. Then $G_\varphi \ge 0$ is a continuous convex function on $[0,1]$, and we fix $a \in [0,1]$ such that $\min_{[0,1]} G_\varphi = G_\varphi(a) \ge 0$. Applying Hölder's inequality with $p = 1/(1-q)$ we deduce, since $1/(1-pq) = (1-q)/(1-2q) > 1$, that

$$\begin{aligned}
\int_0^a |G'_\varphi(s)|^q \, ds &= \int_0^a s^{-q}(-sG'_\varphi(s))^q \, ds \\
&\le \Big(\int_0^a (-sG'_\varphi(s)) \, ds\Big)^q \Big(\int_0^a s^{-pq} \, ds\Big)^{1/p} \\
&\le \Big(-aG_\varphi(a) + \int_0^a G_\varphi(s) \, ds\Big)^q \Big(\int_0^1 s^{-pq} \, ds\Big)^{1/p} \\
&\le \frac{1-q}{1-2q} \, \|G_\varphi\|^q_{L^1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_a^1 |G'_\varphi(s)|^q \, ds &= \int_a^1 (1-s)^{-q}((1-s)G'_\varphi(s))^q \, ds \\
&\le \Big(\int_a^1 (1-s)G'_\varphi(s) \, ds\Big)^q \Big(\int_a^1 (1-s)^{-pq} \, ds\Big)^{1/p} \\
&\le \Big(-(1-a)G_\varphi(a) + \int_a^1 G_\varphi(s) \, ds\Big)^q \Big(\int_0^1 (1-s)^{-pq} \, ds\Big)^{1/p} \\
&\le \frac{1-q}{1-2q} \|G_\varphi\|^q_{L^1}.
\end{aligned}$$

Using the last two estimates and Corollary 3.7 we obtain

$$\int_{\mathbb{R}} |x|^q \, d\mu_{\mathbb{R}}(\varphi) = \int_0^1 |G'_\varphi(s)|^q \, ds \le \frac{2(1-q)}{1-2q} \|G_\varphi\|^q_{L^1}.$$

(iii) Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ be determined by a convex function F_φ defined as follows on the prescribed intervals and smooth on $\mathbb{R}$:

$$F_\varphi(x) = \begin{cases} x - 2\sqrt{x}/\ln x & \text{if } x \geq e^3, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Note that $F_\varphi(x) \leq x \leq F_0(x) = \frac{1}{2}\log(1+e^{2x})$ for $x \geq 0$, and $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$ since F_φ has full Monge–Ampère mass. Moreover,

$$\frac{1}{18x^{3/2}\ln x} \leq F_\varphi''(x) = \frac{1-8(\ln x)^{-2}}{2x^{3/2}\ln x} \leq \frac{1}{2x^{3/2}\ln x} \qquad \text{for } x \geq e^3.$$

Therefore

$$\int_{\mathbb{R}} |x|^{1/2} F_\varphi''(x)\,dx \geq \frac{1}{18}\int_{e^3}^{\infty} \frac{1}{x\ln x}\,dx = \infty.$$

Since $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$, the measure $\mathrm{MA}(\varphi)$ does not charge polar sets. Hence

$$\begin{aligned}\int_X (-\varphi)\,\mathrm{MA}(\varphi) = \int_{\mathbb{R}} (F_0 - F_\varphi)F_\varphi'' &\leq C + \int_{e^3}^{\infty} \frac{2\sqrt{x}}{\ln x} F_\varphi''(x)\,dx \\ &\leq C + \int_{e^3}^{\infty} \frac{1}{x(\ln x)^2}\,dx < \infty,\end{aligned}$$

for some constant C, which implies that $\varphi \in \mathcal{E}^1_{\mathrm{tor}}(X,\omega)$. ∎

4. Higher regularity

4.1. Continuous toric functions. These can be characterized as follows:

Proposition 4.1. *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. The following are equivalent:*

(i) *φ is continuous on X;*
(ii) *$\varphi \in L^\infty(X)$;*
(iii) *$F_P - C \leq F_\varphi \leq F_P + C$ for some constant $C \geq 0$;*
(iv) *$G_P - C \leq G_\varphi \leq G_P + C$ for some constant $C \geq 0$.*

Moreover, we have in this case $\|F_\varphi - F_P\|_{L^\infty(\mathbb{R}^n)} = \|G_\varphi\|_{L^\infty(P)}$.

Proof. Assume that $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ is bounded. Using the notation from the proof of Theorem 3.6 we let $p_1,\ldots,p_N$ be the toric points of X and $p_j \in V_j \subset X$ be open sets with biholomorphic maps $\Phi_j : V_j \to \mathbb{C}^n$ such that $\Phi_j(p_j) = 0$, $(\mathbb{C}^\star)^n \subset V_j$, $\Phi_j((\mathbb{C}^\star)^n) = (\mathbb{C}^\star)^n$, $X = V_1 \cup \cdots \cup V_N$. If $L(\zeta) = (\log|\zeta_1|,\ldots,\log|\zeta_n|)$, $\zeta \in \mathbb{C}^n = \Phi_j(V_j)$, there exist convex functions $F^j_\varphi, F^j_0 : \mathbb{R}^n \to \mathbb{R}$ such that $F^j_\varphi \circ L, F^j_0 \circ L$ extend to a plurisubharmonic function, respectively to a smooth plurisubharmonic function, on $\mathbb{C}^n$, and

$F_\varphi^j \circ L = F_0^j \circ L + \varphi \circ \Phi_j^{-1}$ on $(\mathbb{C}^\star)^n$. Since φ is bounded and since polyradial plurisubharmonic functions on $\mathbb{C}^n$ are continuous, this shows that φ is continuous on each V_j, hence on X.

Using Lemma 3.1 we see immediately that (ii)⇔(iii), while (iii)⇔(iv) follows from the definition of the Legendre transform. Moreover, if (iii) holds with a constant C then (iv) holds with the same constant, and vice versa. This implies the last claim. ∎

We note that assertion (iv) in Proposition 4.1 is equivalent to the condition that $G_\varphi = \infty$ on $\mathbb{R}^n \setminus P$ and that G_φ is bounded above on P.

4.2. Log-Lipschitz Legendre transforms. Recall that a continuous function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *log-Lipschitz* if its modulus of continuity $\omega_u(x,r)$ is locally bounded from above by $Cr\log r$.

In order to prove Theorem C we need the following preliminary results.

LEMMA 4.2. *Let $n \geq 1$ and*

$$I(\lambda) = \int_0^1 (t^{n-1} + \lambda^{-1}) \log(1 + \lambda^{-1} t^{-n+1})\, dt, \qquad \lambda > 0.$$

If $0 < x \leq 1/e$ and $\lambda_x = (n+3)x\log(1/x)$ then $xI(\lambda_x) < 1$.

Proof. We have

$$\begin{aligned} I(\lambda) &\leq \int_0^1 (t^{n-1} + \lambda^{-1}) \log \frac{1+\lambda}{\lambda t^{n-1}}\, dt \\ &= \left(\frac{1}{\lambda} + \frac{1}{n}\right) \log\frac{1+\lambda}{\lambda} + \frac{n-1}{\lambda} + \frac{n-1}{n^2}. \end{aligned}$$

Since $x \leq 1/e$, we see that $1 + \lambda_x \leq 1 + (n+3)/e$ and

$$\log\frac{1+\lambda_x}{\lambda_x} \leq \log\frac{1}{x} + \log\left(\frac{1}{n+3} + \frac{1}{e}\right) - \log\log\frac{1}{x} < \log\frac{1}{x}.$$

Therefore

$$\begin{aligned} I(\lambda_x) &< \left(\frac{1}{\lambda_x} + \frac{1}{n}\right)\log\frac{1}{x} + \frac{n-1}{\lambda_x} + \frac{1}{n} \\ &= \frac{1}{(n+3)x} + \frac{(n-1)}{(n+3)x\log\frac{1}{x}} + \frac{1}{n}\log\frac{1}{x} + \frac{1}{n} \\ &\leq \frac{1}{x}\left(\frac{n}{n+3} + \frac{x}{n}\log\frac{1}{x} + \frac{x}{n}\right) \leq \frac{1}{x}\left(\frac{n}{n+3} + \frac{2}{ne}\right) < \frac{1}{x}. \end{aligned}$$ ∎

PROPOSITION 4.3. *Let $P \subset \mathbb{R}^n$ be a compact convex polytope and let $f : \operatorname{int} P \to \mathbb{R}$ be a locally Lipschitz function. If $e^{\varepsilon\|\nabla f\|} \in L^1(P)$ for some $\varepsilon > 0$ then f extends to a log-Lipschitz function on P.*

Proof. If $n = 1$ then $P = [a, b]$, and for $a < s_1 < s_2 < b$ it follows by Jensen's inequality that

$$|f(s_2) - f(s_1)| \le \frac{1}{\varepsilon}\int_{s_1}^{s_2} \varepsilon|f'(t)|\,dt \le \frac{s_2 - s_1}{\varepsilon}\log\biggl(\frac{1}{s_2 - s_1}\int_{s_1}^{s_2} e^{\varepsilon|f'(t|}\,dt\biggr)$$
$$\le \frac{s_2 - s_1}{\varepsilon}\log\frac{\|e^{\varepsilon|f'|}\|_{L^1[a,b]}}{s_2 - s_1}.$$

We next consider the case $n > 1$. Since P is convex there exists a constant $c \in (0, 1)$ with the following property: for every $s_1, s_2 \in P$ there exists a compact subset A of the hyperplane perpendicular to the segment $[s_1, s_2]$ at its midpoint such that $A \subset \operatorname{int} P$ and

$$(4)\quad c\|s_1 - s_2\|^{n-1} \le V_{n-1}(A) \le 1, \qquad \|s_1 - \sigma\| \le \frac{\|s_1 - s_2\|}{2c} \quad \text{for all } \sigma \in A,$$

where $V_{n-1}(A)$ is the $(n-1)$-dimensional Hausdorff measure of A. Note that we then have $\|s_2 - \sigma\| \le \|s_1 - s_2\|/2c$ for all $\sigma \in A$.

We will show that if $s_1, s_2 \in \operatorname{int} P$ are such that $\|s_1 - s_2\| \le 2/e$ then

$$(5)\qquad |f(s_1) - f(s_2)| \le 2C\|s_1 - s_2\|\log\frac{2}{c\|s_1 - s_2\|^n},$$

where

$$C = \frac{n+3}{\varepsilon c}\max(1, \|e^{\varepsilon\|\nabla f\|}\|_{L^1(P)}).$$

This clearly implies that f extends to a log-Lipschitz function on P.

Fix $s_1, s_2 \in \operatorname{int} P$ with $\|s_1 - s_2\| \le 2/e$ and let A be a set as in (4). Note that (5) follows if we prove that

$$(6)\qquad \biggl|f(s_1) - \frac{1}{V_{n-1}(A)}\int_A f\,dV_{n-1}\biggr| \le C\|s_1 - s_2\|\log\frac{2}{c\|s_1 - s_2\|^n},$$

since the same holds with s_2 in place of s_1.

We may assume that $s_1 = (0, a) \in \mathbb{R}^{n-1} \times \mathbb{R}$, with $a > 0$, and that $A = B \times \{0\} \subset \mathbb{R}^{n-1} \times \{0\}$. Then $\|s_1 - s_2\| = 2a$. We set $\sigma = (\sigma', 0) \in A$ for $\sigma' \in B$. Since f is locally Lipschitz on $\operatorname{int} P$ and, by (4), $\|s_1 - \sigma\| \le a/c$ for $\sigma \in A$, we obtain

$$(7)\quad \biggl|V_{n-1}(B)f(s_1) - \int_B f(\sigma)\,dV_{n-1}(\sigma')\biggr|$$
$$= \biggl|\int_B\int_0^1 \langle \nabla f((1-t)s_1 + t\sigma), s_1 - \sigma\rangle\,dt\,dV_{n-1}(\sigma')\biggr|$$
$$\le \int_B\int_0^1 \|\nabla f((1-t)s_1 + t\sigma)\|\,\|s_1 - \sigma\|\,dt\,dV_{n-1}(\sigma')$$

$$\leq \frac{a}{c}\int_B\int_0^1 \|\nabla f((1-t)s_1+t\sigma)\|\,dt\,dV_{n-1}(\sigma')$$
$$= \frac{1}{\varepsilon c}\int_B\int_0^1 \varepsilon\|\nabla f((1-t)s_1+t\sigma)\|\,t^{-n+1}\,d\mu,$$

where μ is the measure on $B\times[0,1]$ given by $d\mu = at^{n-1}dt dV_{n-1}$.

Consider the weight $\chi(x) = (x+1)\log(x+1) - x$, $x \geq 0$, with conjugate weight (Legendre transform) $\chi^\star(y) = e^y - y - 1$, $y \geq 0$, and the Orlicz spaces $L^\chi(B\times[0,1],\mu)$ and $L^{\chi^\star}(B\times[0,1],\mu)$. Recall that the norm of the space $L^\chi(B\times[0,1],\mu)$ is given by

$$\|g\|_\chi := \inf\Big\{\lambda > 0 : \int_{B\times[0,1]} \chi(|g|/\lambda)\,d\mu \leq 1\Big\},$$

and one has $\|g\|_\chi \leq \max(1, \int_{B\times[0,1]}\chi(|g|)\,d\mu)$.

Estimating the last integral in (7) by the multiplicative Hölder–Young inequality (see [BB$^+$, Proposition 2.15] or [RR]) we get

$$\text{(8)}\qquad \Big|V_{n-1}(B)f(s_1) - \int_B f(\sigma)\,dV_{n-1}(\sigma')\Big|$$
$$\leq \frac{2}{\varepsilon c}\big\|\varepsilon\|\nabla f((1-t)s_1+t\sigma)\|\big\|_{\chi^\star}\|t^{-n+1}\|_\chi.$$

If Γ is the cone in $\mathbb{R}^n$ with vertex s_1 and base A then

$$\int_\Gamma e^{\varepsilon\|\nabla f\|}\,dV_n = \int_{B\times[0,1]} e^{\varepsilon\|\nabla f((1-t)s_1+t\sigma)\|}\,d\mu.$$

Since $\chi^\star(y) < e^y$ it follows that

$$\big\|\varepsilon\|\nabla f((1-t)s_1+t\sigma)\|\big\|_{\chi^\star} \leq \max\Big(1, \int_{B\times[0,1]} e^{\varepsilon\|\nabla f((1-t)s_1+t\sigma)\|}\,d\mu\Big)$$
$$\leq \max\Big(1, \int_P e^{\varepsilon\|\nabla f\|}\,dV_n\Big).$$

It remains to estimate the second Orlicz norm in (8). We have

$$\int_{B\times[0,1]} \chi(t^{-n+1}/\lambda)\,d\mu$$
$$= \int_B\int_0^1\Big[\Big(\frac{t^{-n+1}}{\lambda}+1\Big)\log\Big(\frac{t^{-n+1}}{\lambda}+1\Big) - \frac{t^{-n+1}}{\lambda}\Big]at^{n-1}\,dt\,dV_{n-1},$$
$$\leq aV_{n-1}(B)\,I(\lambda),$$

where $I(\lambda)$ is the function from Lemma 4.2. Note that

$$aV_{n-1}(B) \leq \|s_1 - s_2\|/2 \leq 1/e,$$

since $c\|s_1 - s_2\|^{n-1} \le V_{n-1}(B) = V_{n-1}(A) \le 1$ by (4). Lemma 4.2 implies

$$aV_{n-1}(B)I(\lambda_0) \le 1 \quad \text{if } \lambda_0 = (n+3)aV_{n-1}(B)\log\frac{1}{aV_{n-1}(B)},$$

hence

$$\|t^{-n+1}\|_\chi = \inf\Big\{\lambda > 0 : \int_{B\times[0,1]} \chi(t^{-n+1}/\lambda)\,d\mu \le 1\Big\}$$
$$\le \lambda_0 \le (n+3)aV_{n-1}(B)\log\frac{1}{c\|s_1 - s_2\|^{n-1}a}.$$

By (8) we conclude that

$$\Big|f(s_1) - \frac{1}{V_{n-1}(B)}\int_B f(\sigma)\,dV_{n-1}(\sigma')\Big|$$
$$\le \frac{2(n+3)}{\varepsilon c}\max(1, \|e^{\varepsilon\|\nabla f\|}\|_{L^1(P)})a\log\frac{1}{c\|s_1 - s_2\|^{n-1}a}.$$

This yields (6), since $a = \|s_1 - s_2\|/2$. ∎

We now prove Theorem C stated in the Introduction.

THEOREM 4.4. *Let $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$. The following properties are equivalent:*

- (i) *there exists $\varepsilon > 0$ such that* $\exp(-\varepsilon\,\mathrm{PSH}_{\mathrm{tor}}(X,\omega)) \subset L^1(\mathrm{MA}(\varphi))$*;*
- (ii) *there exists $\varepsilon > 0$ such that $e^{\varepsilon\|\nabla G_\varphi\|} \in L^1(P)$;*
- (iii) *the function G_φ is log-Lipschitz on P;*
- (iv) *there exists a constant $C > 0$ such that $\|\nabla G_\varphi(s)\| \le C\log\frac{C}{\mathrm{dist}(s,\partial P)}$ for almost all $s \in \mathrm{int}\,P$.*

Recall that Guillemin's potentials are only log-Lipschitz continuous on the Delzant polytope P, although they correspond to smooth toric ω-psh functions on X. The observation we make here is that this regularity actually corresponds to a class of toric ω-psh functions which seem to be merely Hölder continuous on X (see Remark 4.5).

Proof of Theorem 4.4. We set $\mu_{\mathbb{R}}(\varphi) := \mathrm{MA}_{\mathbb{R}}(F_\varphi)$. Since $\varphi \in \mathcal{E}_{\mathrm{tor}}(X,\omega)$, the measure $\mathrm{MA}(\varphi)$ does not charge pluripolar sets, so by Lemma 2.3,

$$\int_X e^{-\varepsilon\psi}\,\mathrm{MA}(\varphi) = \int_{(\mathbb{C}^\star)^n} e^{-\varepsilon(F_\psi - F_0)\circ L}(dd^c F_\varphi \circ L)^n = \int_{\mathbb{R}^n} e^{-\varepsilon(F_\psi - F_0)}\,d\mu_{\mathbb{R}}(\varphi)$$

for every $\psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. From Lemma 3.1 and Proposition 3.2, it follows that (i) is equivalent to

- (i′) $\int_{\mathbb{R}^n} e^{-\varepsilon(F - F_P)}\,d\mu_{\mathbb{R}}(\varphi) < \infty$ for any convex function $F : \mathbb{R}^n \to \mathbb{R}$ with $F \le F_P + O(1)$ on $\mathbb{R}^n$.

To show (i′)⇔(ii), we may assume that $0 \in \operatorname{int} P$ and we fix constants $a, b > 0$ such that $\overline{B}(0,b) \subset P \subset \overline{B}(0,a)$. Then by Lemma 2.7, $b\|x\| \le F_P(x) \le a\|x\|$. If (i′) holds, we apply it with $F = 0$ to conclude by Corollary 3.7 that

$$\int\limits_{\operatorname{int} P} e^{\varepsilon b\|\nabla G_\varphi(s)\|}\, dV(s) = \int\limits_{\mathbb{R}^n} e^{\varepsilon b\|x\|}\, d\mu_{\mathbb{R}}(\varphi) \le \int\limits_{\mathbb{R}^n} e^{\varepsilon F_P(x)}\, d\mu_{\mathbb{R}}(\varphi) < \infty,$$

which gives (ii). Conversely, assume (ii) holds and let F be a function as in (i′). Then by Proposition 3.2, $\nabla F(\mathbb{R}^n) \subset P \subset \overline{B}(0,a)$, so

$$F(x) = F(0) + \int_0^1 \langle \nabla F(tx), x\rangle\, dt \ge -a\|x\| + F(0).$$

Therefore $F_P(x) - F(x) \le 2a\|x\| - F(0)$ and

$$\begin{aligned}\int\limits_{\mathbb{R}^n} e^{-\frac{\varepsilon}{2a}(F-F_P)}\, d\mu_{\mathbb{R}}(\varphi) &\le e^{-\frac{\varepsilon}{2a}F(0)} \int\limits_{\mathbb{R}^n} e^{\varepsilon\|x\|}\, d\mu_{\mathbb{R}}(\varphi)\\ &= e^{-\frac{\varepsilon}{2a}F(0)} \int\limits_{\operatorname{int} P} e^{\varepsilon\|\nabla G_\varphi(s)\|}\, dV(s) < \infty,\end{aligned}$$

so (i′) holds.

Proposition 4.3 shows that (ii) implies (iii). We next prove that (iii) implies (iv). Since G_φ is log-Lipschitz on the compact polytope P it follows that there exists a constant $C > 0$ such that $\|s - s'\| \le C/2$ and

$$|G_\varphi(s) - G_\varphi(s')| \le C\|s - s'\| \log \frac{C}{\|s - s'\|}$$

for all $s, s' \in P$. Let $s \in \operatorname{int} P$ be such that G_φ is differentiable at s, and $\nabla G_\varphi(s) \ne 0$, and let ν be the unit vector in the direction of $\nabla G_\varphi(s)$. We consider the convex function

$$g(t) = G_\varphi(s + t\nu), \qquad 0 \le t \le t^\star,$$

where $t^\star > 0$ is defined such that $s^\star := s + t^\star\nu \in \partial P$. Then $t^\star = \|s^\star - s\| \ge \operatorname{dist}(s, \partial P)$ and

$$\begin{aligned}\|\nabla G_\varphi(s)\| = g'(0) &\le \frac{g(t^\star) - g(0)}{t^\star} = \frac{G_\varphi(s^\star) - G_\varphi(s)}{\|s^\star - s\|}\\ &\le C \log \frac{C}{\|s^\star - s\|} \le C \log \frac{C}{\operatorname{dist}(s, \partial P)}.\end{aligned}$$

Finally, we note that (iv) clearly implies that (ii) holds with $\varepsilon > 0$ small enough. ∎

Remark 4.5. It is tempting to think that these conditions are all equivalent to the fact that φ is Hölder continuous. This is easily seen to be the case when $n = 1$. We refer the interested reader to [DD+] for more information, geometric motivations, and related questions connecting the Hölder continuity of Monge–Ampère potentials with the integrability properties of the associated complex Monge–Ampère measure.

Example 4.6. Fix $0 < \alpha < 1$ and consider the convex function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = e^{\alpha x}$ when $x \le 0$ and $F(x) = x + 1$ when $x \ge 0$. It determines a Hölder continuous toric ω_{FS}-psh function φ on $\mathbb{P}^1$ which is defined in $\mathbb{C}$ by

$$\varphi(z) = \begin{cases} |z|^\alpha - \log\sqrt{1+|z|^2} & \text{if } |z| \le 1, \\ \log|z| + 1 - \log\sqrt{1+|z|^2} & \text{if } |z| \ge 1. \end{cases}$$

We let the reader check that the Legendre transform of F is given by

$$G(s) = \begin{cases} \frac{s}{\alpha}\log\frac{s}{\alpha} - \frac{s}{\alpha} & \text{if } 0 \le s \le \alpha, \\ -1 & \text{if } \alpha \le s \le 1. \end{cases}$$

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References

[ALZ] C. Arezzo, A. Loi and F. Zuddas, *Some remarks on the symplectic and Kähler geometry of toric varieties*, Ann. Mat. Pura Appl. (4) 195 (2016), 1287–1304.

[BT76] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge–Ampère equation*, Invent. Math. 37 (1976), 1–44.

[BT82] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[BB] R. Berman and B. Berndtsson, *Real Monge–Ampère equations and Kähler–Ricci solitons on toric log Fano varieties*, Ann. Fac. Sci. Toulouse Math. 22 (2013), 649–711.

[BB$^+$] R. Berman, S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, *Kähler–Ricci flow and Ricci iteration on log-Fano varieties*, J. Reine Angew. Math., to appear.

[BEGZ] S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, *Monge–Ampère equations in big cohomology classes*, Acta Math. 205 (2010), 199–262.

[CDG] D. Calderbank, L. David and P. Gauduchon, *The Guillemin formula and Kähler metrics on toric symplectic manifolds*, J. Symplectic Geom. 1 (2002), 767–784.

[CGZ] D. Coman, V. Guedj and A. Zeriahi, *Domains of definition of complex Monge–Ampère operators on compact Kähler manifolds*, Math. Z. 259 (2008), 393–418.

[Dar] T. Darvas, *The Mabuchi geometry of finite energy classes*, Adv. Math. 285 (2015), 182–219.

[Del] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France 116 (1988), 315–339.

[DD+] J.-P. Demailly, S. Dinew, V. Guedj, H. H. Pham, S. Kołodziej and A. Zeriahi, *Hölder continuous solutions to Monge–Ampère equations*, J. Eur. Math. Soc. 16 (2014), 619–647.

[DiN] E. Di Nezza, *Finite pluricomplex energy measures*, Potential Anal. 44 (2016), 155–167.

[G] V. Guedj, *The metric completion of the Riemannian space of Kähler metrics*, arXiv:1401.7857 (2014).

[GZ] V. Guedj and A. Zeriahi, *The weighted Monge–Ampère energy of quasiplurisubharmonic functions*, J. Funct. Anal. 250 (2007), 442–482.

[Gui] V. Guillemin, *Kähler structures on toric varieties*, J. Differential Geom. 40 (1994), 285–309.

[Gut] C. Gutiérrez, *The Monge–Ampère Equation*, Progr. Nonlinear Differential Equations Appl. 44, Birkhäuser Boston, 2001.

[RR] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monogr. Textbooks Pure Appl. Math. 146, Dekker, 1991.

[T] B. A. Taylor, *An estimate for an extremal plurisubharmonic function on* $\mathbb{C}^n$, in: Séminaire d'Analyse, P. Lelong–P. Dolbeault–H. Skoda, Années 1981/1983, Lecture Notes in Math. 1028, Springer, 1983, 318–328.

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