

**A PRIORI ESTIMATES FOR THE 3D COMPRESSIBLE
FREE-BOUNDARY EULER EQUATIONS WITH SURFACE
TENSION IN THE CASE OF A LIQUID**

MARCELO M. DISCONZI*

Department of Mathematics, Vanderbilt University
Nashville, TN 37240, USA

IGOR KUKAVICA

Department of Mathematics, University of Southern California
Los Angeles, CA 91107, USA

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ABSTRACT. We derive a priori estimates for the compressible free-boundary Euler equations with surface tension in three spatial dimensions in the case of a liquid. These are estimates for local existence in Lagrangian coordinates when the initial velocity and initial density belong to H^3 , with an extra regularity condition on the moving boundary, thus lowering the regularity of the initial data. Our methods are direct and involve two key elements: the boundary regularity provided by the mean curvature and a new compressible Cauchy invariance.

1. Introduction. In this paper we derive a priori estimates for the compressible free-boundary Euler equations with surface tension in three space dimensions (Theorem 1.1 below) in the case of a liquid. Our a priori estimates provide bounds for the Lagrangian velocity and Lagrangian density in H^3 , an improvement in regularity as compared to [27].

The compressible free-boundary Euler equations in a domain of \mathbb{R}^3 are given by

$$\frac{\partial u}{\partial t} + \nabla_u u + \frac{1}{\varrho} \nabla p = 0 \quad \text{in } \mathcal{D}, \quad (1a)$$

$$\frac{\partial \varrho}{\partial t} + \nabla_u \varrho + \varrho \operatorname{div}(u) = 0 \quad \text{in } \mathcal{D}, \quad (1b)$$

$$p = p(\varrho) \quad \text{in } \mathcal{D}, \quad (1c)$$

$$p = \sigma \mathcal{H} \quad \text{on } \partial \mathcal{D}, \quad (1d)$$

$$(\partial_t + u^\mu \partial_{x^\mu})|_{\partial \mathcal{D}} \in T \partial \mathcal{D}, \quad (1e)$$

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* Corresponding author: marcelo.disconzi@vanderbilt.edu.

$$u(0, \cdot) = u_0, \quad \varrho(0, \cdot) = \varrho_0, \quad \Omega(0) = \Omega_0, \quad (1f)$$

$$\text{where } \mathcal{D} = \bigcup_{0 \leq t < T} \{t\} \times \Omega(t). \quad (1g)$$

Above, the quantities $u = u(t, x)$, $p = p(t, x)$, $\varrho = \varrho(t, x)$ are the velocity, pressure, and density of the fluid; $\Omega(t) \subset \mathbb{R}^3$ is the moving (i.e., changing over time) domain, which may be written as $\Omega(t) = \eta(t)(\Omega_0)$, where η is the flow of u ; σ is a non-negative constant known as the coefficient of surface tension. Equation (1c) is the equation of state, indicating that the pressure is a given function of the density. In (1d), \mathcal{H} is the mean curvature of the moving (time-dependent) boundary $\partial\Omega(t)$; and $T\partial\mathcal{D}$ is the tangent bundle of $\partial\mathcal{D}$. The equation (1e) means that the boundary $\partial\Omega(t)$ moves at a speed equal to the normal component of u . The quantity u_0 is the velocity at time zero, ϱ_0 is the density at time zero, and Ω_0 is the domain at the initial time. The symbol ∇_u is the derivative in the direction of u , often written as $u \cdot \nabla$. The unknowns in (1) are u , ϱ , and $\Omega(t)$. Note that \mathcal{H} , $T\partial\mathcal{D}$, and p are functions of the unknowns and, therefore, are not known a priori, and have to be determined alongside a solution to the problem.

We focus on the case when $\sigma > 0$ and consider the model case when

$$\Omega_0 \equiv \Omega = \mathbb{T}^2 \times (0, 1).$$

Denoting coordinates on Ω by (x^1, x^2, x^3) , set

$$\Gamma_1 = \mathbb{T}^2 \times \{x^3 = 1\}$$

and

$$\Gamma_0 = \mathbb{T}^2 \times \{x^3 = 0\},$$

so that $\partial\Omega = \Gamma_0 \cup \Gamma_1$. The general domain can then be handled as in [68, Remark 4.2]. We assume that the lower boundary does not move, and thus $\eta(t)(\Gamma_0) = \Gamma_0$, where η is the flow of the vector field u . We introduce the Lagrangian velocity, pressure, and density, respectively, by $v(t, x) = u(t, \eta(t, x))$, $q(t, x) = p(t, \eta(t, x))$, and $R(t, x) = \varrho(t, \eta(t, x))$, or simply $v = u \circ \eta$, $q = p \circ \eta$, and $R = \varrho \circ \eta$. Therefore,

$$\partial_t \eta = v. \quad (2)$$

Denoting by ∇ the derivative with respect to the spatial variables x , introduce the matrix

$$a = (\nabla \eta)^{-1},$$

which is well defined for η near the identity. Equation (1c) gives $q = q(R)$, i.e., the equation of state written in Lagrangian variables. From a we obtain the cofactor matrix

$$A = Ja, \quad (3)$$

where

$$J = \det(\nabla \eta). \quad (4)$$

As a consequence of these definitions, we have the Piola identity

$$\partial_\beta A^{\beta\alpha} = \partial_\beta (Ja^{\beta\alpha}) = 0. \quad (5)$$

(The identity (5) can be verified by direct computation using the explicit form of a given in (22) below, or cf. [46, p. 462].) Above and throughout we adopt the following agreement.

Notation 1. We denote by ∂_α spatial derivatives, i.e., $\partial_\alpha = \partial/\partial x^\alpha$, for $\alpha = 1, 2, 3$. Greek indices (α, β , etc.) range from 1 to 3 and Latin indices (i, j , etc.), range from 1 to 2. Repeated indices are summed over their range. Indices shall be raised and lowered with the Euclidean metric. We write $\partial^\alpha = \delta^{\alpha\beta}\partial_\beta$.

In terms of v , q , R , and a , the system (1) becomes

$$R\partial_t v^\alpha + a^{\mu\alpha}\partial_\mu q = 0 \quad \text{in } [0, T) \times \Omega, \quad (6a)$$

$$\partial_t R + Ra^{\mu\alpha}\partial_\mu v_\alpha = 0 \quad \text{in } [0, T) \times \Omega, \quad (6b)$$

$$\partial_t a^{\alpha\beta} + a^{\alpha\gamma}\partial_\mu v_\gamma a^{\mu\beta} = 0 \quad \text{in } [0, T) \times \Omega, \quad (6c)$$

$$q = q(R) \quad \text{in } [0, T) \times \Omega, \quad (6d)$$

$$a^{\mu\alpha}N_\mu q + \sigma|a^T N| \Delta_g \eta^\alpha = 0 \quad \text{on } [0, T) \times \Gamma_1, \quad (6e)$$

$$v^\mu N_\mu = 0 \quad \text{on } [0, T) \times \Gamma_0, \quad (6f)$$

$$\eta(0, \cdot) = \text{id}, \quad R(0, \cdot) = \varrho_0, \quad v(0, \cdot) = v_0, \quad (6g)$$

where id is the identity diffeomorphism on Ω , N is the unit outer normal to $\partial\Omega$, a^T is the transpose of a , $|\cdot|$ is the Euclidean norm, and Δ_g is the Laplacian of the metric g_{ij} induced on $\partial\Omega(t)$ by the embedding η . Explicitly,

$$g_{ij} = \partial_i \eta \cdot \partial_j \eta = \partial_i \eta^\mu \partial_j \eta_\mu, \quad (7)$$

where \cdot is the Euclidean inner product, and

$$\Delta_g(\cdot) = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j (\cdot)), \quad (8)$$

with g the determinant of the matrix (g_{ij}) . In (6e), $\Delta_g \eta^\alpha$ simply means Δ_g acting on the scalar function η^α , for each $\alpha = 1, 2, 3$; see Lemma 2.2 below for some important identities used to obtain (6e).

Since $\eta(0, \cdot) = \text{id}$, the initial Lagrangian and Eulerian velocities agree, i.e., $v_0 = u_0$. Clearly, v_0 is orthogonal to Γ_0 in view of (6f). Note that

$$a(0, \cdot) = I, \quad (9)$$

where I is the identity matrix, in light of (6g). It also follows from the above definitions that J satisfies

$$\partial_t J - J a^{\alpha\beta} \partial_\alpha v_\beta = 0 \quad \text{in } [0, T) \times \Omega \quad (10)$$

and

$$RJ = R(0) = \varrho_0 \quad \text{in } [0, T) \times \Omega. \quad (11)$$

Physically, the equation of state has to satisfy $q'(R) > 0$ (pressure cannot decrease with an increase in density). Mathematically, this assumption guarantees the coercivity of the kinetic term for R in the energy. Here, we shall adopt a slightly more restrictive equation of state that allows us to simplify the estimates. We assume there exists a constant $A_q > 0$ such that for all R in a certain interval $[a, b]$, we have

$$q'(R) \geq A_q \quad \text{and} \quad \left(\frac{q(R)}{R} \right)' \geq A_q. \quad (12)$$

By Lemma 2.1(x) below, the first condition follows from the second if we allow A_q to be decreased if necessary. Importantly, the condition (12) is satisfied for equations of state of the form $q(R) = \alpha R^{1+\gamma}$, where $\alpha > 0$ and $\gamma > 0$ are constants (with

further assumptions on the constants and the range of R , (12) is also satisfied by $q(R) = \alpha R^{1+\gamma} + \beta$, where $\beta > 0$.

Notation 2. Sobolev spaces are denoted by $H^s(\Omega)$ (or simply by H^s when no confusion can arise), with the corresponding norm denoted by $\|\cdot\|_s$; note that $\|\cdot\|_0$ refers to the L^2 norm. We denote by $H^s(\partial\Omega)$ the Sobolev space of maps defined on $\partial\Omega$, with the corresponding norm $\|\cdot\|_{s,\partial}$, and similarly the space $H^s(\Gamma_1)$ with the norm $\|\cdot\|_{s,\Gamma_1}$. The L^p norms on Ω and Γ_1 are denoted by $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^p(\Gamma_1)}$ or $\|\cdot\|_{L^p}$ when no confusion can arise. We use \restriction to denote restriction, and Δ is the Euclidean Laplacian in Ω .

We now state our main result.

Theorem 1.1. *Let Ω be as described above and let $\sigma > 0$ in (6). Let v_0 be a smooth vector field on Ω , and ϱ_0 a smooth positive function on Ω bounded away from zero from below. Let $q: (0, \infty) \rightarrow (0, \infty)$ be a smooth function satisfying (12), in a neighborhood of ϱ_0 . Then, there exist a $T_* > 0$ and a constant C_* , depending only on*

$$\sigma, \|v_0\|_3, \|v_0\|_{3,\Gamma_1}, \|\varrho_0\|_3, \|\varrho_0\|_{3,\Gamma_1}, \|\operatorname{curl} v_0\|_{2.5+\delta}, \text{ and } \|(\Delta \operatorname{div} v_0)|\Gamma_1\|_{-1,\Gamma_1},$$

where $\delta \in (0, 0.5]$, such that any smooth solution (v, R) to (6) with initial condition (v_0, ϱ_0) and defined on the time interval $[0, T_*]$, satisfies

$$\|v\|_3 + \|\partial_t v\|_2 + \|\partial_t^2 v\|_1 + \|\partial_t^3 v\|_0 + \|R\|_3 + \|\partial_t R\|_2 + \|\partial_t^2 R\|_1 + \|\partial_t^3 R\|_0 \leq C_*.$$

The dependence of T_* and C_* on a higher norm on the boundary Γ_1 comes from the usual problems caused by the moving boundary in free-boundary problems. The technical difficulties leading to the necessity of including such higher norm are similar to those in [56] (see Section 3.3 and Remark 3 below). The assumption on $(\Delta \operatorname{div} v_0)|\Gamma_1$ is technical. It can be understood as a consequence of the fact that our techniques generalize methods previously applied to incompressible fluids in [42], where of course the condition is immediately satisfied as $\operatorname{div} v_0 = 0$ then. A regularity condition on the normal derivatives of the normal component of v_0 would suffice, but the assumption on $(\Delta \operatorname{div} v_0)|\Gamma_1$ is simpler to state. We remark that control of $\operatorname{curl} v$ in $H^{2.5+}$ follows from an argument similar to [68] combined with a simple estimate for the divergence which is omitted here.

Without attempting to be exhaustive, we now briefly review the literature on problem (6), and it is instructive to first recall some results for the incompressible free-boundary Euler equations.

The first existence result for incompressible free-boundary inviscid fluids is that of Nalimov [80], followed by [13, 34, 64, 81, 86, 87, 91, 95, 96, 99, 100]. Despite their importance, all these works consider simplifying assumptions, mostly irrotationality. It has not been until fairly recently, with the works of Lindblad [75] for $\sigma = 0$, Coutand and Shkoller [29] for $\sigma \geq 0$, and Shatah and Zeng [89, 90], also for $\sigma \geq 0$, and more recently by the first author and Ebin [40] for $\sigma > 0$, that existence and uniqueness for the incompressible free-boundary Euler equations have been addressed in full generality. Since the early 2000's, research on this topic has blossomed, as is illustrated by the sample list [1, 3, 4, 5, 6, 7, 10, 8, 2, 9, 12, 11, 15, 14, 16, 17, 18, 19, 20, 22, 23, 26, 30, 33, 35, 36, 37, 38, 39, 42, 47, 48, 49, 52, 55, 54, 53, 57, 60, 58, 59, 65, 67, 68, 69, 70, 71, 73, 77, 82, 83, 85, 88, 97, 98].

Although we are concerned here with $\sigma > 0$, it is worth mentioning that the free-boundary Euler equations behave differently for $\sigma = 0$ and $\sigma > 0$. In view

of a counter-example to well-posedness by Ebin [45], an extra condition (known as Taylor sign condition in the incompressible case), has to be imposed when $\sigma = 0$. However, it seems more difficult to obtain local existence in lower regularity spaces when $\sigma > 0$ compared to $\sigma = 0$ due to the presence of two space derivatives of η on the free boundary.

For the compressible free-boundary Euler equations (6), besides the difference between $\sigma > 0$ and $\sigma = 0$ referred above, a further distinction that needs to be made is between a liquid, when $\varrho_0 \geq \lambda > 0$, where λ is a constant, and a gas, when ϱ_0 can be zero, the former being the situation treated here. Existence and uniqueness of solutions for (6) have been proved by Lindblad [74] for the case of a liquid with $\sigma = 0$, by Coutand and Shkoller [32] for a gas with $\sigma = 0$ (see also [94]), and by Coutand, Hole, and Shkoller [27] for a liquid with $\sigma \geq 0$. Earlier and related works are [21, 24, 25, 28, 31, 62, 63, 72, 79, 92, 93]. Further, and more recent results, are [50, 61, 76, 78].

In this work we restricted ourselves to derive a priori estimates, hence a solution is assumed to be given. Therefore, there is no need to state compatibility conditions for the initial data. But we remind the reader that such conditions are necessary for construction of solutions. We also note that in our setting, compatibility conditions will be different on Γ_1 and on Γ_0 (see, e.g., [27], for the compatibility conditions on Γ_1 , and [41] for those on Γ_0).

Assumption 1. *For the rest of the paper, we work under the assumptions of Theorem 1.1 and denote by (v, q) a smooth solution to (6). We also assume that Ω , Γ_1 , and Γ_0 are as described above.*

1.1. Strategy and organization of the paper. The paper is organized as follows. Theorem 1.1 states the main result. Section 2 contains the preliminary estimates of the coefficients and the Lagrangian map. We also introduce the notation used in the rest of the paper. Section 3 contains the energy estimates. First, we start with the energy equality for the third time derivatives (cf. (36) below). Special care is required for the boundary integral, which is treated with complete details in Subsection 3.1.4. Two time derivative energy equality is written in (71) below, with the estimates given in Section 3.2. We emphasize that the obtained terms are not of lower order as they contain one more space derivative. We also point out that we can not use the H^3 energy equality with no time derivatives, since there is an interior term which can not be treated by the methods from the rest of the paper; instead, we need to rely on the div-curl estimates to obtain control of the H^3 norms of the velocity and the density. Section 4 contains estimates for the curl of the velocity; the main building block is a new Cauchy invariance formula, generalizing the incompressible version from [56, 68]. The conclusion of the proof, where all the bounds are suitably combined, is provided in the last section.

Several of the terms that appear in our energy identities, especially in the case of some boundary integrals, cannot be bounded directly. To control them, we explore the structure of the equations and make frequent use of several geometric identities. These lead to a cancellation of top-order terms, allowing us to close the estimates.

2. Auxiliary results. In this section we state some preliminary results that are employed in the proof of Theorem 1.1 below.

Lemma 2.1. *Assume that $\|v\|_3, \|R\|_3 \leq M$, where $M \geq 1$. Then, there exists a constant $C > 0$ such that if $T \in [0, 1/CM^2]$ and (v, q) is defined on $[0, T]$, the following inequalities hold for $t \in [0, T]$:*

- (i) $\|\eta\|_3 \leq C$.
- (ii) $\|a\|_2 \leq C$.
- (iii) $\|\partial_t a\|_{L^p} \leq C\|\nabla v\|_{L^p}$, $1 \leq p \leq \infty$.
- (iv) $\|\partial_\alpha \partial_t a\|_{L^p} \leq C\|\nabla v\|_{L^{p_1}}\|\partial_\alpha a\|_{L^{p_2}} + C\|\partial_\alpha \nabla v\|_{L^p}$, where $1/p = 1/p_1 + 1/p_2$, and $1 \leq p, p_1, p_2 \leq 6$.
- (v) $\|\partial_t a\|_s \leq C\|\nabla v\|_s$, $0 \leq s \leq 2$.
- (vi) $\|\partial_t^2 a\|_s \leq C\|\nabla v\|_s\|\nabla v\|_{L^\infty} + C\|\nabla \partial_t v\|_s$, $0 \leq s \leq 1$.
- (vi)' $\|\partial_t^2 a\|_1 \leq C\|\nabla v\|_{5/4}^2 + C\|\nabla \partial_t v\|_1$.
- (vii) $\|\partial_t^3 a\|_{L^p} \leq C\|\nabla v\|_{L^p}\|\nabla v\|_{L^\infty}^2 + C\|\nabla \partial_t v\|_{L^p}\|\nabla v\|_{L^\infty} + C\|\nabla \partial_t^2 v\|_{L^p}$, $1 \leq p < \infty$.
- (viii) $J \geq 1/2$.
- (ix) Furthermore, if ϵ is sufficiently small and $T \leq \epsilon/CM^2$ then, for $t \in [0, T]$, we have

$$\|a^{\alpha\beta} - \delta^{\alpha\beta}\|_2 \leq \epsilon$$

and

$$\|a^{\alpha\mu}a^\beta_\mu - \delta^{\alpha\beta}\|_2 \leq \epsilon.$$

In particular, the form $a^{\alpha\mu}a^\beta_\mu$ satisfies the ellipticity estimate

$$a^{\alpha\mu}a^\beta_\mu \xi_\alpha \xi_\beta \geq \frac{1}{C}|\xi|^2.$$

(x) $C^{-1} \leq R \leq C$.

Proof. The proofs of (i)–(vii) and (ix) are very similar to [56, Lemma 3.1] and [66, Lemma 3.1], making the necessary adjustments for $\|v\|_3 \leq M$ (in [56], $\|v\|_{3.5} \leq M$ is used). The statement (x) follows from

$$\|R(t) - R(0)\|_{L^\infty} \leq C \left\| \int_0^t Ra^{\mu\alpha} \partial_\mu v_\alpha \right\|_{L^\infty} \leq C \int_0^t \|R\|_3 \|v\|_3 \leq CM^2T$$

by (6b). The inequality (viii) is proven analogously, using (10) instead of (6b). \square

Notation 3. In the rest of the paper, the symbol C denotes a positive sufficiently large constant. It can vary from expression to expression, but it is always independent of (v, R) . We also write $X \lesssim Y$ to mean $X \leq CY$. The a priori estimates require for T to be sufficiently small so that it satisfies $TM \leq 1/C$, where M is an upper bound on the norm of the solution (cf. Lemma 2.1 below). In several estimates it suffices to keep track of the number of derivatives so we write ∂^ℓ to denote any derivative of order ℓ and $\bar{\partial}^\ell$ to denote any derivative of order ℓ on the boundary, i.e., with respect to x^i . We use upper-case Latin indices to denote x^i or t , so $\bar{\partial}_A$ means ∂_t or ∂_i .

Remark 1. (Simple lower order estimates and symbolic notation) In the subsequent sections, we use the following consequence of Lemma 2.1. Let Q be a rational function of derivatives of η with respect to x^i ,

$$Q = Q(\partial_1 \eta^1, \partial_2 \eta^1, \partial_1 \eta^2, \partial_2 \eta^2, \partial_1 \eta^3, \partial_2 \eta^3).$$

More precisely, we are given a map $Q: \mathcal{D} \rightarrow \mathbb{R}$, where \mathcal{D} is a domain in \mathbb{R}^6 , and consider the composition of Q with $D(\eta|\Gamma_1)$, where D means the derivative. Assume that $0 \notin \overline{\mathcal{D}}$ and that $(1, 0, 0, 1, 0, 0) \in \mathcal{D}$. Assume that the derivatives of Q belong to $H^s(\mathcal{D}')$, where $1 < s \leq 1.5$ and \mathcal{D}' is some small neighborhood of $(1, 0, 0, 1, 0, 0)$. The application we have in mind is when Q is a combination of the terms \sqrt{g} and g^{ij} . It is not difficult to check that such terms satisfy the assumptions just stated on Q . In this regard, note that at time zero g is the Euclidean metric on Γ_1 , and that $(1, 0, 0, 1, 0, 0)$ corresponds to $D(\eta(0)|\Gamma_1)$.

In what follows it suffices to keep track of the generic form of some expressions so we write Q symbolically as

$$Q = Q(\bar{\partial}\eta).$$

Then

$$\bar{\partial}_A Q(\bar{\partial}\eta) = \tilde{Q}_\alpha^i(\bar{\partial}\eta) \bar{\partial}_A \partial_i \eta^\alpha,$$

where the terms $\tilde{Q}_\alpha^i(\bar{\partial}\eta)$ are also rational function of derivatives of η with respect to x^i . Note that $\tilde{Q}_\alpha^i(\bar{\partial}\eta)$ are simply the partial derivatives of Q evaluated at $\bar{\partial}\eta$. We write the last equality symbolically as

$$\bar{\partial}_A Q(\bar{\partial}\eta) = \tilde{Q}(\bar{\partial}\eta) \bar{\partial}_A \bar{\partial}\eta.$$

For $s > 1$, we have the estimate

$$\|\bar{\partial}_A Q(\bar{\partial}\eta)\|_{s, \Gamma_1} \leq C_1 \|\tilde{Q}(\bar{\partial}\eta)\|_{s, \Gamma_1} \|\bar{\partial}_A \bar{\partial}\eta\|_{s, \Gamma_1},$$

where C_1 depends only on s and on the domain Γ_1 . The term $\|\tilde{Q}(\bar{\partial}\eta)\|_{s, \Gamma_1}$ can be estimated in terms of the Sobolev norm of the map \tilde{Q} , i.e., $\|\tilde{Q}\|_{H^s(\mathcal{D})}$, and the Sobolev norm of $\bar{\partial}\eta$, i.e., $\|\bar{\partial}\eta\|_{s, \Gamma_1}$. Under the conditions of Lemma 2.1, we have

$$\|\bar{\partial}\eta - \bar{\partial}\eta(0)\|_{L^\infty(\Gamma_1)} \leq \int_0^t \|\partial_t \bar{\partial}\eta\|_{L^\infty(\Gamma_1)} \leq C_2 t \|v\|_3 \leq C_2 M t,$$

where C_2 depends only on the domain Γ_1 and we used that $H^{1.5}(\Gamma_1)$ embeds into $C^0(\Gamma_1)$. Therefore, if t is very small, we can guarantee that

$$\bar{\partial}\eta(\Gamma_1) \subset \mathcal{D}',$$

and thus, shrinking \mathcal{D} if necessary, we can assume that the derivatives of Q are in $H^s(\mathcal{D})$ for $1 < s \leq 1.5$, and, therefore, that $\|\tilde{Q}\|_{H^s(\mathcal{D})}$ is bounded for $s \leq 1.5$. Since Lemma 2.1 also provides a bound for $\|\bar{\partial}\eta\|_{s, \Gamma_1}$, $s \leq 1.5$, we conclude that

$$\|\bar{\partial}_A Q(\bar{\partial}\eta)\|_{s, \Gamma_1} \leq C \|\bar{\partial}_A \bar{\partial}\eta\|_{s, \Gamma_1}, \text{ for } 1 < s \leq 1.5,$$

where C depends only on M , s , and Γ_1 , and provided that t is small enough. The above also shows that

$$\|Q(\bar{\partial}\eta)\|_{s, \Gamma_1} \leq C \|\bar{\partial}\eta\|_{s, \Gamma_1}, \text{ for } 1 < s \leq 1.5.$$

We also need some geometric identities that may be known to specialists, but we state them below and provide some of the corresponding proofs for the reader's convenience.

Lemma 2.2. *Let n denote the unit outer normal to $\eta(\Gamma_1)$. Then*

$$n \circ \eta = \frac{a^T N}{|a^T N|}. \tag{13}$$

Denoting by τ the tangent bundle of $\overline{\eta(\Omega)}$ and by ν the normal bundle of $\eta(\Gamma_1)$, the canonical projection $\Pi: \tau \upharpoonright \eta(\Gamma_1) \rightarrow \nu$ is given by

$$\Pi_\beta^\alpha = \delta_\beta^\alpha - g^{kl} \partial_k \eta^\alpha \partial_l \eta_\beta. \quad (14)$$

Furthermore, the following identities hold:

$$\Pi_\lambda^\alpha \Pi_\beta^\lambda = \Pi_\beta^\alpha, \quad (15)$$

$$J|a^T N| = \sqrt{g}, \quad (16)$$

$$\sqrt{g} \Delta_g \eta^\alpha = \sqrt{g} g^{ij} \partial_{ij}^2 \eta^\alpha - \sqrt{g} g^{ij} g^{kl} \partial_k \eta^\alpha \partial_l \eta^\mu \partial_{ij}^2 \eta_\mu, \quad (17)$$

$$-\Delta_g(\eta^\alpha | \Gamma_1) = \mathcal{H} \circ \eta n^\alpha \circ \eta, \quad (18)$$

$$\partial_t(n_\mu \circ \eta) = -g^{kl} \partial_k v^\tau \hat{n}_\tau \partial_l \eta_\mu, \quad (19)$$

and

$$\partial_i(n_\mu \circ \eta) = -g^{kl} \partial_{ik} \eta^\tau \hat{n}_\tau \partial_l \eta_\mu. \quad (20)$$

Proof. Letting $r = \eta | \Gamma_1$, we know that $n \circ \eta$ is given by (see e.g. [51])

$$n \circ \eta = \frac{\partial_1 r \times \partial_2 r}{|\partial_1 r \times \partial_2 r|}. \quad (21)$$

By $\det(\nabla \eta) = J$, we have

$$a = \frac{1}{J} \begin{bmatrix} \partial_2 \eta^2 \partial_3 \eta^3 - \partial_3 \eta^2 \partial_2 \eta^3 & \partial_3 \eta^1 \partial_2 \eta^3 - \partial_2 \eta^1 \partial_3 \eta^3 & \partial_2 \eta^1 \partial_3 \eta^2 - \partial_3 \eta^1 \partial_2 \eta^2 \\ \partial_3 \eta^2 \partial_1 \eta^3 - \partial_1 \eta^2 \partial_3 \eta^3 & \partial_1 \eta^1 \partial_3 \eta^3 - \partial_3 \eta^1 \partial_1 \eta^3 & \partial_3 \eta^1 \partial_1 \eta^2 - \partial_1 \eta^1 \partial_3 \eta^2 \\ \partial_1 \eta^2 \partial_2 \eta^3 - \partial_2 \eta^2 \partial_1 \eta^3 & \partial_2 \eta^1 \partial_1 \eta^3 - \partial_1 \eta^1 \partial_2 \eta^3 & \partial_1 \eta^1 \partial_2 \eta^2 - \partial_2 \eta^1 \partial_1 \eta^2 \end{bmatrix}. \quad (22)$$

Using (22) to compute $Ja^T N$ and comparing with $\partial_1 r \times \partial_2 r$, one verifies that

$$Ja^T N = \partial_1 r \times \partial_2 r,$$

and then (13) follows from (21).

To prove (14), we use (13) to write

$$(\delta_\lambda^\alpha - g^{kl} \partial_k \eta^\alpha \partial_l \eta_\lambda) n^\lambda \circ \eta = \frac{a^{\mu\alpha} N_\mu}{|a^T N|} - \frac{g^{kl} \partial_k \eta^\alpha \partial_l \eta_\lambda a^{\mu\lambda} N_\mu}{|a^T N|}.$$

Contracting $g^{kl} \partial_l \eta_\lambda a^{\mu\lambda} N_\mu$ with g_{mk} gives

$$\begin{aligned} g_{mk} g^{kl} \partial_l \eta_\lambda a^{\mu\lambda} N_\mu &= \partial_m \eta_\lambda a^{3\lambda} \\ &= \partial_m \eta_1 (\partial_1 \eta^2 \partial_2 \eta^3 - \partial_2 \eta^2 \partial_1 \eta^3) + \partial_m \eta_2 (\partial_2 \eta^1 \partial_1 \eta^3 - \partial_1 \eta^1 \partial_2 \eta^3) \\ &\quad + \partial_m \eta_3 (\partial_1 \eta^1 \partial_2 \eta^2 - \partial_2 \eta^1 \partial_1 \eta^2) \\ &= 0. \end{aligned} \quad (23)$$

Above, the first equality follows because $N = (0, 0, 1)$ (and $g_{mk} g^{kl} = \delta_m^l$), the second equality uses (22), and the third equality follows upon setting $m = 1$ and then $m = 2$ and observing that in each case all the terms cancel out. Thus, contracting (23) with g^{mn} ,

$$g^{nl} \partial_l \eta_\lambda a^{\mu\lambda} N_\mu = 0,$$

and hence

$$(\delta_\lambda^\alpha - g^{kl} \partial_k \eta^\alpha \partial_l \eta_\lambda) n^\lambda \circ \eta = \frac{a^{\mu\alpha} N_\mu}{|a^T N|}.$$

To conclude the proof of (14), we need to verify that $\Pi(X) = 0$ if X is tangent to $\eta(\Gamma_1)$. Since the tangent space to $\eta(\Gamma_1)$ is spanned by $\partial_j \eta$, for $j = 1, 2$, it suffices to verify the identity for these vectors. We have

$$\Pi_\alpha^\mu \partial_j \eta_\mu = (\delta_\alpha^\mu - g^{kl} \partial_k \eta_\alpha \partial_l \eta^\mu) \partial_j \eta_\mu = \partial_j \eta_\alpha - g^{kl} \partial_k \eta_\alpha g_{lj} = 0,$$

where we used $g_{lj} = \partial_l \eta^\mu \partial_j \eta_\mu$ and $g^{kl} g_{lj} = \delta_j^k$. Thus, (14) is proven.

The identity (15) follows from the fact that Π is a projection operator or, alternatively, by direct computation using (14). The identity (16) follows from (13), (21), and the standard formula (see e.g. [51])

$$\frac{\partial_1 r \times \partial_2 r}{|\partial_1 r \times \partial_2 r|} = \frac{1}{\sqrt{g}} \partial_1 r \times \partial_2 r.$$

In order to prove (17), recall that (see e.g. [51])

$$\Delta_g \eta^\alpha = g^{ij} \partial_{ij}^2 \eta^\alpha - g^{ij} \Gamma_{ij}^k \partial_k \eta^\alpha, \quad (24)$$

where Γ_{ij}^k are the Christoffel symbols. Recalling (7), a direct computation using the definition of the Christoffel symbols gives

$$\Gamma_{ij}^k = g^{kl} \partial_l \eta^\mu \partial_{ij}^2 \eta_\mu, \quad (25)$$

and (17) follows from (24) and (25).

The identity (18) is a standard formula for the mean curvature of an embedding into \mathbb{R}^3 (see e.g. [51] or [84]).

The identities (19) and (20) are well-known, but we provide their proofs for the reader's convenience. Denote $\hat{n} = n \circ \eta$. Since $\{\partial_1 \eta, \partial_2 \eta, \hat{n}\}$ are linearly independent, we can write

$$\bar{\partial}_A \hat{n} = a^1 \partial_1 \eta + a^2 \partial_2 \eta + b \hat{n}. \quad (26)$$

Taking the dot product with \hat{n} we see that $b = 0$, since $\bar{\partial}_A \hat{n} \cdot \hat{n} = 0$ in view of $\hat{n} \cdot \hat{n} = 1$, and the fact that $\partial_i \eta$ is tangent to the embedding. Taking the dot product with $\partial_1 \eta$ and $\partial_2 \eta$, and using the definition (7), we obtain

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} \partial_1 \eta \cdot \bar{\partial}_A \hat{n} \\ \partial_2 \eta \cdot \bar{\partial}_A \hat{n} \end{pmatrix}.$$

Using $\partial_l \eta \cdot \bar{\partial}_A \hat{n} = -\bar{\partial}_A \partial_l \eta \cdot \hat{n}$ (which follows from $\partial_l \eta \cdot \hat{n} = 0$) to eliminate $\bar{\partial}_A \hat{n}$ on the right-hand side, solving for a^1 and a^2 , and using the result into (26), produces (19) when $\bar{\partial}_A = \partial_t$ and (20) when $\bar{\partial}_A = \partial_i$. \square

For future reference, we record the identity

$$\bar{\partial}_A (\sqrt{g} g^{ij}) = \sqrt{g} \left(\frac{1}{2} g^{ij} g^{kl} - g^{lj} g^{ik} \right) \bar{\partial}_A g_{kl}, \quad (27)$$

which follows from the well-known identities (see e.g. [84]),

$$\bar{\partial}_A g = g g^{kl} \bar{\partial}_A g_{kl},$$

and

$$\bar{\partial}_A g^{ij} = -g^{lj} g^{ik} \bar{\partial}_A g_{kl}.$$

We also need the following result about a gain or regularity of the moving boundary.

Notation 4. From here on, we use $P(\cdot)$, with indices attached when appropriate, to denote a general polynomial expression of its arguments.

Proposition 1. *Assume that that conditions of Lemma 2.1 are valid. We have the estimate*

$$\|\eta\|_{3.5,\Gamma_1} \leq P(\|R\|_{1.5,\Gamma_1}).$$

Proof. We would like to apply elliptic estimates to (6e). While we do not know a priori that the coefficients g_{ij} have enough regularity for an application of standard elliptic estimates, we can use improved estimates for coefficients with lower regularity as in [43]. For this, it suffices to check that g_{ij} has small oscillation, in the following sense.

Given $r > 0$ and $x \in \Gamma_1$, set

$$\text{osc}_x(g^{ij}) = \frac{1}{\text{vol}(B_r(x))} \int_{B_r(x)} \left| g^{ij}(y) - \frac{1}{\text{vol}(B_r(x))} \int_{B_r(x)} g^{ij}(z) dz \right| dy$$

and

$$g_R = \sup_{x \in \Gamma_1} \sup_{r \leq R} \text{osc}_x(g^{ij}).$$

We need to verify that there exists $\tilde{R} \leq 1$ such that

$$g_{\tilde{R}} \leq \rho, \quad (28)$$

where ρ is sufficiently small.

Since $g^{ij} \in H^{1.5}(\Gamma_1)$, we have $g^{ij} \in C^{0,\alpha}(\Gamma_1)$ with $0 < \alpha < 0.5$ fixed. Thus, for $y \in B_r(x)$,

$$\begin{aligned} \left| g^{ij}(y) - \frac{1}{\text{vol}(B_r(x))} \int_{B_r(x)} g^{ij}(z) dz \right| &= \left| \frac{1}{\text{vol}(B_r(x))} \int_{B_r(x)} (g^{ij}(y) - g^{ij}(z)) dz \right| \\ &\leq \sup_{z \in B_r(x)} |g^{ij}(y) - g^{ij}(z)| \leq C_\alpha r^\alpha. \end{aligned}$$

Hence,

$$g_{\tilde{R}} \leq C_\alpha R^\alpha,$$

and we can ensure (28). Therefore, the results of [43] imply that

$$\begin{aligned} \|\eta^\alpha\|_{3.5,\Gamma_1} &\leq C(\|a^{\mu\alpha} N_\mu q\|_{1.5,\Gamma_1} + \|\eta^\alpha\|_{1.5,\Gamma_1}) \\ &\leq C(\|a\|_{1.5,\Gamma_1} \|q\|_{1.5,\Gamma_1} + \|\eta\|_{1.5,\Gamma_1}), \end{aligned}$$

where C depends on $\|g_{ij}\|_{1.5,\Gamma_1}$. Or yet,

$$\|\eta^\alpha\|_{3.5,\Gamma_1} \leq C\|q\|_{1.5,\Gamma_1} + C\|\eta\|_3 \leq C\|q\|_{1.5,\Gamma_1} + C \leq P(\|R\|_{1.5,\Gamma_1}).$$

We remark that [43] deals only with Sobolev spaces of integer order, but since the estimates are linear on the norms we can extend them to fractional order Sobolev spaces as well. \square

Corollary 1. *Under the same assumptions of Proposition 1,*

$$\|\eta\|_{4.5,\Gamma_1} \leq P(\|R\|_{2.5,\Gamma_1}). \quad (29)$$

Proof. Since g_{ij} involves only tangential derivatives of η , by Proposition 1 we have an estimate for g^{ij} in $H^{2.5}(\Gamma_1)$. We can thus use elliptic regularity to bootstrap the estimate on η restricted to Γ_1 to $H^{4.5}(\Gamma_1)$. \square

We conclude this section with a compressible version of the Cauchy invariance (see, e.g., [68] for the incompressible case).

Proposition 2. *Let (v, R) be a smooth solution to (6) defined on $[0, T]$. Then*

$$\varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu\partial_\gamma\eta_\mu = \omega_0^\alpha \quad (30)$$

for $0 \leq t < T$. Here, $\varepsilon^{\alpha\beta\gamma}$ is the totally anti-symmetric symbol with $\varepsilon^{123} = 1$ and ω_0 is the vorticity at time zero.

Proof. Compute

$$\begin{aligned} \partial_t(\varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu\partial_\gamma\eta_\mu) &= \varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu\partial_\gamma v_\mu + \varepsilon^{\alpha\beta\gamma}\partial_\beta\partial_t v^\mu\partial_\gamma\eta_\mu = \varepsilon^{\alpha\beta\gamma}\partial_\beta\partial_t v^\mu\partial_\gamma\eta_\mu \\ &= -\frac{1}{R}\varepsilon^{\alpha\beta\gamma}\partial_\beta(a^{\lambda\mu}\partial_\lambda q)\partial_\gamma\eta_\mu + \frac{1}{R^2}\varepsilon^{\alpha\beta\gamma}a^{\lambda\mu}\partial_\lambda q\partial_\beta R\partial_\gamma\eta_\mu, \end{aligned}$$

where we used the anti-symmetry of $\varepsilon^{\alpha\beta\gamma}$ and (6a). From $a\nabla\eta = I$, we have

$$\partial_\beta(a^{\lambda\mu}\partial_\gamma\eta_\mu) = \partial_\beta a^{\lambda\mu}\partial_\gamma\eta_\mu + a^{\lambda\mu}\partial_\gamma\partial_\beta\eta_\mu = 0,$$

and thus

$$\begin{aligned} \partial_t(\varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu\partial_\gamma\eta_\mu) &= \frac{1}{R}\partial_\lambda q a^{\lambda\mu}\varepsilon^{\alpha\beta\gamma}\partial_\beta\partial_\gamma\eta_\mu - \frac{1}{R}\varepsilon^{\alpha\beta\gamma}a^{\lambda\mu}\partial_\gamma\eta_\mu\partial_\beta\partial_\lambda q \\ &\quad + \frac{1}{R^2}\varepsilon^{\alpha\beta\gamma}a^{\lambda\mu}\partial_\lambda q\partial_\beta R\partial_\gamma\eta_\mu \\ &= 0 - \frac{1}{R}\varepsilon^{\alpha\beta\lambda}\partial_\beta\partial_\lambda q + \frac{1}{R^2}\varepsilon^{\alpha\beta\gamma}a^{\lambda\mu}\partial_\lambda q\partial_\beta R\partial_\gamma\eta_\mu \\ &= \frac{1}{R^2}\varepsilon^{\alpha\beta\gamma}\partial_\gamma q\partial_\beta R = q'(R)\frac{1}{R^2}\varepsilon^{\alpha\beta\gamma}\partial_\gamma R\partial_\beta R = 0, \end{aligned}$$

where we used again the anti-symmetry of $\varepsilon^{\alpha\beta\gamma}$ and the identity $a^{\lambda\mu}\partial_\gamma\eta_\mu = \delta_\gamma^\lambda$. Integrating in time yields the result. \square

3. Energy estimates. In this section we derive estimates for v , R , $v \cdot N$, and their time derivatives.

Assumption 2. *Throughout this section, we suppose that the hypotheses of Lemma 2.1 hold; we make frequent use of its conclusions without mentioning it every time. The reader is also reminded of (2), which is often going to be used without mention as well. We assume further that T is as in part (ix) of that lemma, and that (v, q) are defined on $[0, T]$.*

Notation 5. We use $\tilde{\epsilon}$ to denote a small positive constant which may vary from expression to expression. Typically, $\tilde{\epsilon}$ comes from choosing the time sufficiently small, from Lemma 2.1, or from the Cauchy inequality with epsilon. The important point to keep in mind, which can be easily verified in the expressions containing $\tilde{\epsilon}$, is that once all estimates are obtained, we can fix $\tilde{\epsilon}$ to be sufficiently small in order to close the estimates.

Notation 6. Recalling Notation 4, we denote

$$\begin{aligned} \mathcal{P} &= P(\|v\|_3, \|\partial_t v\|_2, \|\partial_t^2 v\|_1, \|\partial_t^3 v\|_0, \|R\|_3, \|\partial_t R\|_2, \\ &\quad \|\partial_t^2 R\|_1, \|\partial_t^3 R\|_0, \|\Pi\bar{\partial}\partial_t^2 v\|_{0,\Gamma_1}, \|\Pi\bar{\partial}^2\partial_t v\|_{0,\Gamma_1}) \end{aligned}$$

and

$$\mathcal{P}_0 = P \left(\sigma, \frac{1}{\sigma}, \|v_0\|_3, \|v_0\|_{3,\Gamma_1}, \|\varrho_0\|_3, \|\varrho_0\|_{3,\Gamma_1}, \|(\Delta \operatorname{div} v_0)|\Gamma_1\|_{-1,\Gamma_1} \right),$$

where we abbreviate

$$\|\Pi \bar{\partial} \partial_t^2 v\|_{0,\Gamma_1}^2 = \int_{\Gamma_1} \delta^{ij} \delta^{\beta\gamma} \Pi_{\mu\gamma} \partial_j \partial_t^2 v^\mu \Pi_{\beta\alpha} \partial_i \partial_t^2 v^\alpha = \int_{\Gamma_1} \Pi_\mu^\beta \partial^i \partial_t^2 v^\mu \Pi_\beta^\alpha \partial_i \partial_t^2 v_\alpha.$$

Notation 7. We shall use the following abbreviated notation:

$$\begin{aligned} \mathcal{N}(t) \equiv \mathcal{N} = & \|v\|_3^2 + \|\partial_t v\|_2^2 + \|\partial_t^2 v\|_1^2 + \|\partial_t^3 v\|_0^2 + \|R\|_3^2 + \|\partial_t R\|_2^2 + \|\partial_t^2 R\|_1^2 \\ & + \|\partial_t^3 R\|_0^2 + \|\Pi \bar{\partial} \partial_t^2 v\|_{0,\Gamma_1}^2 + \|\Pi \bar{\partial}^2 \partial_t v\|_{0,\Gamma_1}^2. \end{aligned}$$

Before starting with a priori estimates, we record an additional regularity of η which is combined below with Corollary 1 and Proposition 2. As in [68], Proposition 2 implies

$$\|\operatorname{curl} \eta\|_{H^{2.5+\delta}} \leq \|\eta\|_{H^3} + \tilde{\epsilon} \|\eta\|_{H^{3.5+\delta}} + C \int_0^t \|v\|_{H^3} \|\eta\|_{H^{3.5+\delta}} + Ct \|\omega_0\|_{H^{2.5+\delta}}, \quad (31)$$

where $\delta \in (0, 0.5]$. In order to control the divergence of η , we start with $A^{\alpha\beta} \partial_\alpha \eta_\beta = 3J$, which leads to

$$\operatorname{div} \eta = 3J + (\delta^{\alpha\beta} - A^{\alpha\beta})(\partial_\alpha \eta - \delta_{\alpha\beta}) + (3 - \operatorname{Tr} A).$$

Now, let

$$\tilde{\eta} = \eta - x.$$

Using (22), we have

$$\begin{aligned} \operatorname{Tr} A = 3 = & \partial_2 \tilde{\eta}^2 \partial_3 \tilde{\eta}^3 - \partial_3 \tilde{\eta}^2 \partial_2 \tilde{\eta}^3 + \partial_1 \tilde{\eta}^1 \partial_3 \tilde{\eta}^3 - \partial_3 \tilde{\eta}^1 \partial_1 \tilde{\eta}^3 \\ & + \partial_1 \tilde{\eta}^1 \partial_2 \tilde{\eta}^2 - \partial_2 \tilde{\eta}^1 \partial_1 \tilde{\eta}^2 - 2 \operatorname{div} \eta + 3, \end{aligned}$$

which gives

$$\begin{aligned} \operatorname{div} \tilde{\eta} = J - 1 + & \frac{1}{3} (\delta^{\alpha\beta} - A^{\alpha\beta}) \partial_\alpha \tilde{\eta}_\beta \\ & - \frac{1}{3} \left(\partial_2 \tilde{\eta}^2 \partial_3 \tilde{\eta}^3 - \partial_3 \tilde{\eta}^2 \partial_2 \tilde{\eta}^3 + \partial_1 \tilde{\eta}^1 \partial_3 \tilde{\eta}^3 - \partial_3 \tilde{\eta}^1 \partial_1 \tilde{\eta}^3 + \partial_1 \tilde{\eta}^1 \partial_2 \tilde{\eta}^2 - \partial_2 \tilde{\eta}^1 \partial_1 \tilde{\eta}^2 \right). \end{aligned} \quad (32)$$

Now, by (22), the entries of $\delta^{\alpha\beta} - A^{\alpha\beta}$ are either of the form $\partial \tilde{\eta} \partial \tilde{\eta}$ or of the form $\partial \tilde{\eta} \partial \tilde{\eta} + \partial \tilde{\eta}$. Differentiating (32), we get

$$\|\nabla \operatorname{div} \tilde{\eta}\|_{H^{1.5+\delta}} \leq C + \|J\|_{H^{2.5+\delta}} + \|\nabla \operatorname{div} \tilde{\eta}\|_{H^{1.5+\delta}} \int_0^t \mathcal{P} \quad (33)$$

and thus

$$\|\operatorname{div} \eta\|_{H^{2.5+\delta}} \leq C + \|J\|_{H^{2.5+\delta}} + \|\nabla \operatorname{div} \eta\|_{H^{1.5+\delta}} \int_0^t \mathcal{P} + \int_0^t \mathcal{P}. \quad (34)$$

3.1. Three time derivatives. In this section we derive the estimate

$$\|\partial_t^3 v\|_0^2 + \|\partial_t^3 R\|_0^2 + \|\Pi \bar{\partial} \partial_t^2 v\|_{0,\Gamma_1}^2 \leq \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad (35)$$

where we recall that Π is given by (14).

3.1.1. *Energy identity.* We begin by establishing the identity

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} R(0) \partial_t^3 v^\beta \partial_t^3 v_\beta + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{R(0)}{R} \bar{q}'(R) (\partial_t^3 R)^2 + \int_{\Gamma_1} \partial_t^3 (J a^{\alpha\beta} q) \partial_t^3 v_\beta N_\alpha \\
&= - \int_{\Omega} \frac{R(0)}{R} \left(\partial_t^3 (R a^{\alpha\beta} \partial_\alpha v_\beta) - R a^{\alpha\beta} \partial_t^3 \partial_\alpha v_\beta \right) \partial_t^3 \left(\frac{q}{R} \right) \\
&\quad + \int_{\Omega} R(0) \left(\partial_t^3 \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^3 \left(\frac{q}{R} \right) \right) \partial_t^3 \partial_\alpha v_\beta \\
&\quad - 3 \int_{\Omega} R(0) \frac{\bar{q}''(R)}{R} \partial_t^4 R \partial_t^2 R \partial_t R - \int_{\Omega} R(0) \frac{\bar{q}'''(R)}{R} \partial_t^4 R (\partial_t R)^3 \\
&\quad + \frac{1}{2} \int_{\Omega} R(0) \partial_t \left(\frac{\bar{q}'(R)}{R} \right) (\partial_t^3 R)^2,
\end{aligned} \tag{36}$$

where

$$\bar{q}(R) = \frac{q(R)}{R}.$$

To obtain it, we first multiply (6a) by J (replacing α with β), differentiate three times in t , contract with $\partial_t^3 v_\beta$, and integrate. We obtain

$$\int_{\Omega} \partial_t^3 (J R \partial_t v^\beta) \partial_t^3 v_\beta + \int_{\Omega} \partial_t^3 (J a^{\alpha\beta} \partial_\alpha q) \partial_t^3 v_\beta = 0.$$

Using the Piola identity (5) and integrating by parts in ∂_α , we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} R(0) \partial_t^3 v^\beta \partial_t^3 v_\beta + \int_{\Gamma_1} \partial_t^3 (J a^{\alpha\beta} q) \partial_t^3 v_\beta N_\alpha = \int_{\Omega} \partial_t^3 (J a^{\alpha\beta} q) \partial_t^3 \partial_\alpha v_\beta,$$

where we also used (11), that $R(0) = \varrho_0$, and the fact that the boundary integral vanishes on Γ_0 .

Now we write

$$\begin{aligned}
\int_{\Omega} \partial_t^3 (J a^{\alpha\beta} q) \partial_t^3 \partial_\alpha v_\beta &= \int_{\Omega} R(0) \partial_t^3 \left(a^{\alpha\beta} \frac{q}{R} \right) \partial_t^3 \partial_\alpha v_\beta \\
&= \int_{\Omega} R(0) a^{\alpha\beta} \partial_t^3 \left(\frac{q}{R} \right) \partial_t^3 \partial_\alpha v_\beta + \int_{\Omega} R(0) \left(\partial_t^3 \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^3 \left(\frac{q}{R} \right) \right) \partial_t^3 \partial_\alpha v_\beta \\
&= \int_{\Omega} \frac{R(0)}{R} \partial_t^3 (R a^{\alpha\beta} \partial_\alpha v_\beta) \partial_t^3 \left(\frac{q}{R} \right) \\
&\quad - \int_{\Omega} \frac{R(0)}{R} \left(\partial_t^3 (R a^{\alpha\beta} \partial_\alpha v_\beta) - R a^{\alpha\beta} \partial_t^3 \partial_\alpha v_\beta \right) \partial_t^3 \left(\frac{q}{R} \right) \\
&\quad + \int_{\Omega} R(0) \left(\partial_t^3 \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^3 \left(\frac{q}{R} \right) \right) \partial_t^3 \partial_\alpha v_\beta \\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

The terms \mathcal{I}_2 and \mathcal{I}_3 correspond to the first and second terms on the right side of (36) respectively. To handle \mathcal{I}_1 , we use the density equation (6b) to eliminate the spatial derivative:

$$\mathcal{I}_1 = \int_{\Omega} \frac{R(0)}{R} \partial_t^3 (R a^{\alpha\beta} \partial_\alpha v_\beta) \partial_t^3 \left(\frac{q}{R} \right) = - \int_{\Omega} \frac{R(0)}{R} \partial_t^4 R \partial_t^3 \bar{q}.$$

Since

$$\partial_t^3 (\bar{q}(R)) = \bar{q}'(R) \partial_t^3 R + 3 \bar{q}''(R) \partial_t^2 R \partial_t R + \bar{q}'''(R) (\partial_t R)^3,$$

we have

$$\begin{aligned}\mathcal{I}_1 &= - \int_{\Omega} R(0) \frac{\bar{q}'(R)}{R} \partial_t^4 R \partial_t^3 R - 3 \int_{\Omega} R(0) \frac{\bar{q}''(R)}{R} \partial_t^4 R \partial_t^2 R \partial_t R \\ &\quad - \int_{\Omega} R(0) \frac{\bar{q}'''(R)}{R} \partial_t^4 R (\partial_t R)^3 \\ &= \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.\end{aligned}$$

The terms \mathcal{I}_{12} and \mathcal{I}_{13} give the third and the fourth terms on the right side of (36). For \mathcal{I}_{11} , we write

$$\mathcal{I}_{11} = -\frac{1}{2} \frac{d}{dt} \int_{\Omega} R(0) \frac{\bar{q}'(R)}{R} (\partial_t^3 R)^2 + \frac{1}{2} \int_{\Omega} R(0) \partial_t \left(\frac{\bar{q}'(R)}{R} \right) (\partial_t^3 R)^2. \quad (37)$$

The first term on the right side leads to the second term on the left side of (36), while the second term on the right side of (37) gives the last term in (36).

Denote the terms on the right side of (36) by \mathcal{J}_1 – \mathcal{J}_5 .

3.1.2. *Estimates of \mathcal{J}_1 , \mathcal{J}_3 , \mathcal{J}_4 , and \mathcal{J}_5 .* In this section we estimate \mathcal{J}_1 , \mathcal{J}_3 , \mathcal{J}_4 , and \mathcal{J}_5 . We begin with

$$\mathcal{J}_1 = - \int_{\Omega} \frac{R(0)}{R} \left(\partial_t^3 (Ra^{\alpha\beta} \partial_{\alpha} v_{\beta}) - Ra^{\alpha\beta} \partial_t^3 \partial_{\alpha} v_{\beta} \right) \partial_t^3 \left(\frac{q}{R} \right). \quad (38)$$

First observe that

$$\left\| \partial_t^3 \left(\frac{q}{R} \right) \right\|_{L^2(\Omega)} \leq P(\|\partial_t^3 R\|_{L^2(\Omega)}, \|\partial_t^2 R\|_{L^2(\Omega)}, \|\partial_t R\|_{L^2(\Omega)}, \|R\|_{L^2(\Omega)}) \leq \mathcal{P}.$$

When the expression in parentheses in (38) involving three time derivatives is expanded and one of them canceled, we obtain eight terms, which are all bounded in a similar way. For instance, we have

$$\|\partial_t^3 Ra^{\alpha\beta} \partial_{\alpha} v_{\beta}\|_{L^2(\Omega)} \leq C \|\partial_t^3 R\|_{L^2(\Omega)} \|a^{\alpha\beta}\|_{L^{\infty}(\Omega)} \|\partial_{\alpha} v_{\beta}\|_{L^{\infty}(\Omega)} \leq \mathcal{P}$$

and

$$\|R \partial_t^3 a^{\alpha\beta} \partial_{\alpha} v_{\beta}\|_{L^2(\Omega)} \leq C \|R\|_{L^{\infty}(\Omega)} \|\partial_t^3 a^{\alpha\beta}\|_{L^2(\Omega)} \|\partial_{\alpha} v_{\beta}\|_{L^{\infty}(\Omega)} \leq \mathcal{P},$$

as well as

$$\|\partial_t^3 Ra^{\alpha\beta} \partial_{\alpha} v_{\beta}\|_{L^2(\Omega)} \leq C \|\partial_t^2 R\|_{L^4(\Omega)} \|\partial_t a^{\alpha\beta}\|_{L^4(\Omega)} \|\partial_t^2 \partial_{\alpha} v_{\beta}\|_{L^2(\Omega)} \leq \mathcal{P}.$$

After estimating all the terms in this manner, we obtain

$$\mathcal{J}_1 \leq \mathcal{P}.$$

Next, we treat the term

$$\begin{aligned}\mathcal{J}_3 &= -3 \int_{\Omega} R(0) \frac{\bar{q}''(R)}{R} \partial_t^4 R \partial_t^2 R \partial_t R \\ &= \frac{d}{dt} \left(-3 \int_{\Omega} R(0) \frac{\bar{q}''(R)}{R} \partial_t^3 R \partial_t^2 R \partial_t R \right) \\ &\quad + 3 \int_{\Omega} R(0) \partial_t^3 R \partial_t \left(\frac{\bar{q}''(R)}{R} \partial_t^2 R \partial_t R \right) \\ &= \frac{d}{dt} \mathcal{J}_{31} + \mathcal{J}_{32}.\end{aligned} \quad (39)$$

For the first term in (39), we have

$$\begin{aligned}\mathcal{J}_{31}(t) &\lesssim \|R(0)\|_{L^\infty(\Omega)} \|R^{-1}\|_{L^\infty(\Omega)} \|\partial_t^3 R\|_{L^2(\Omega)} \|\partial_t^2 R\|_{L^2(\Omega)} \|\partial_t R\|_{L^\infty(\Omega)} \\ &\lesssim \|R(0)\|_{L^\infty(\Omega)} \|R^{-1}\|_{L^\infty(\Omega)} \|\partial_t^3 R\|_{L^2(\Omega)} \|\partial_t^2 R(0)\|_{L^2(\Omega)} \|\partial_t R\|_{L^\infty(\Omega)} \\ &\quad + \|R(0)\|_{L^\infty(\Omega)} \|R^{-1}\|_{L^\infty(\Omega)} \|\partial_t^3 R\|_{L^2(\Omega)} \|\partial_t R\|_{L^\infty(\Omega)} \int_0^t \|\partial_t^3 R\|_{L^2(\Omega)} dt.\end{aligned}$$

Using Lemma 2.1(x) as well as the Sobolev and Young's inequalities, we get

$$\mathcal{J}_{31}(t) \leq \tilde{\epsilon} \|\partial_t^3 R\|_0^2 + \tilde{\epsilon} \|\partial_t R\|_2^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}$$

where we also used

$$\|\partial_t R\|_1^2 \leq \left\| \partial_t R(0) + \int_0^t \partial_t R \right\|_1^2 \lesssim \|\partial_t R(0)\|_1^2 + \left\| \int_0^t \partial_t R \right\|_1^2 \leq \mathcal{P}_0 + \int_0^t \mathcal{P}$$

and Jensen's inequality. Also,

$$\mathcal{J}_{31}(0) \lesssim C \|\partial_t^3 R(0)\|_{L^2(\Omega)} \|\partial_t^2 R(0)\|_{L^2(\Omega)} \|\partial_t^2 R(0)\|_{L^\infty(\Omega)} \leq \mathcal{P}_0.$$

The second term in (39), \mathcal{J}_{32} , is simpler, as we just apply Hölder's inequality and write

$$\begin{aligned}\mathcal{J}_{32} &\lesssim \|R(0)\|_{L^\infty(\Omega)} \|\partial_t^3 R\|_{L^2(\Omega)} \left(\left\| \frac{\bar{q}'''(R)}{R} \partial_t R \right\|_{L^\infty(\Omega)} \|\partial_t^2 R\|_{L^2(\Omega)} \|\partial_t R\|_{L^\infty(\Omega)} \right. \\ &\quad + \left\| \frac{\bar{q}''(R)}{R^2} \partial_t R \right\|_{L^\infty(\Omega)} \|\partial_t^2 R\|_{L^2(\Omega)} \|\partial_t R\|_{L^\infty(\Omega)} \\ &\quad + \left\| \frac{\bar{q}''(R)}{R} \right\|_{L^\infty(\Omega)} \|\partial_t^3 R\|_{L^2(\Omega)} \|\partial_t R\|_{L^\infty(\Omega)} \\ &\quad \left. + \left\| \frac{\bar{q}''(R)}{R} \right\|_{L^\infty(\Omega)} \|\partial_t^2 R\|_{L^4(\Omega)}^2 \right) \\ &\leq \mathcal{P}.\end{aligned}$$

The term \mathcal{J}_4 is treated similarly to \mathcal{J}_3 by differentiating by parts in time. Namely, we have

$$\begin{aligned}\mathcal{J}_4 &= - \int_\Omega R(0) \frac{\bar{q}'''(R)}{R} \partial_t^4 R (\partial_t R)^3 \\ &= \frac{d}{dt} \left(- \int_\Omega R(0) \frac{\bar{q}'''(R)}{R} \partial_t^3 R (\partial_t R)^3 \right) + \int_\Omega R(0) \partial_t^3 R \partial_t \left(\frac{\bar{q}'''(R)}{R} (\partial_t R)^3 \right) \quad (40) \\ &= \frac{d}{dt} \mathcal{J}_{41} + \mathcal{J}_{42}.\end{aligned}$$

The pointwise terms are estimated using Hölder and Sobolev inequalities as

$$\begin{aligned}\mathcal{J}_{41}(t) &\lesssim \|R(0)\|_{L^\infty(\Omega)} \|R^{-1}\|_{L^\infty(\Omega)} \|\partial_t^3 R\|_{L^2(\Omega)} \|\partial_t R\|_{L^6(\Omega)}^3 \\ &\lesssim \|\partial_t^3 R\|_0 \|\partial_t R\|_1^3 \\ &\lesssim \tilde{\epsilon} \|\partial_t^3 R\|_0^2 + \mathcal{P}_0 + \int_0^t \mathcal{P}\end{aligned}$$

and

$$\mathcal{J}_{41}(0) \lesssim \|\partial_t^3 R(0)\|_0 \|\partial_t R(0)\|_1^3 \leq \mathcal{P}_0.$$

For the second term \mathcal{J}_{42} in (40), we use Hölder's inequality, yielding

$$\begin{aligned} \mathcal{J}_{42} &\lesssim \|R(0)\|_{L^\infty(\Omega)} \|\partial_t^3 R\|_{L^2(\Omega)} \left(\left\| \frac{\bar{q}'''(R)}{R} \right\|_{L^\infty(\Omega)} \|\partial_t R\|_{L^6(\Omega)}^3 \right. \\ &\quad + \left\| \frac{\bar{q}'''(R)}{R^2} \right\|_{L^\infty(\Omega)} \|\partial_t R\|_{L^8(\Omega)}^4 \\ &\quad \left. + \left\| \frac{\bar{q}'''(R)}{R} \right\|_{L^\infty(\Omega)} \|\partial_t R\|_{L^\infty(\Omega)}^2 \|\partial_t^2 R\|_{L^2(\Omega)} \right) \\ &\leq \mathcal{P}. \end{aligned}$$

Finally, the last term \mathcal{J}_5 can be bounded using Hölder's inequality

$$\begin{aligned} \mathcal{J}_5 &= \frac{1}{2} \int_{\Omega} R(0) \partial_t \left(\frac{\bar{q}'(R)}{R} \right) (\partial_t^3 R)^2 \\ &\lesssim \|R(0)\|_{L^\infty(\Omega)} \left\| \partial_t \left(\frac{\bar{q}'(R)}{R} \right) \right\|_{L^\infty(\Omega)} \|\partial_t^3 R\|_{L^4(\Omega)}^2 \leq \mathcal{P}. \end{aligned}$$

Remark 2. (Recurrent estimates of lower order terms) Ideas similar to the above, relying on a combination of Sobolev embedding, Young and Jensen's inequalities, and interpolation, shall be used throughout the paper to estimate lower order terms, many times without explicit mention. Before proceeding further, we illustrate in detail how a typical lower order is bounded.

Consider $\|\partial_t^2 v\|_{0.5+\delta} \|\partial_t^3 v\|_0$, where $\delta > 0$ is small. Interpolating

$$\|\partial_t^2 v\|_{0.5+\delta} \lesssim \|\partial_t^2 v\|_1^{0.5+\delta} \|\partial_t^2 v\|_0^{0.5-\delta},$$

and using the Cauchy inequality with ϵ , we find

$$\|\partial_t^2 v\|_{0.5+\delta} \|\partial_t^3 v\|_0 \lesssim C(\tilde{\epsilon}) \|\partial_t^2 v\|_0^{1-2\delta} \|\partial_t^2 v\|_1^{1+2\delta} + \tilde{\epsilon} \|\partial_t^3 v\|_0^2.$$

Next, choosing $p = 2/(1+2\delta)$ and $q = 2/(1-2\delta)$, we apply Young's inequality with ϵ to get

$$\begin{aligned} \|\partial_t^2 v\|_{0.5+\delta} \|\partial_t^3 v\|_0 &\lesssim C(\tilde{\epsilon}) (C(\epsilon') \|\partial_t^2 v\|_0^2 + \epsilon' \|\partial_t^2 v\|_1^2) + \tilde{\epsilon} \|\partial_t^3 v\|_0^2 \\ &\lesssim C(\tilde{\epsilon}, \epsilon') \|\partial_t^2 v\|_0^2 + \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + \tilde{\epsilon} \|\partial_t^3 v\|_0^2, \end{aligned}$$

where in the second step we chose ϵ' so small that $C(\tilde{\epsilon})\epsilon' \leq \tilde{\epsilon}$. The fundamental theorem of calculus and Jensen's inequality provide

$$\|\partial_t^2 v\|_0^2 \lesssim \|\partial_t^2 v(0)\|_0^2 + \left(\int_0^t \|\partial_t^3 v\|_0 \right)^2 \lesssim \|\partial_t^2 v(0)\|_0^2 + t \int_0^t \|\partial_t^3 v\|_0^2.$$

We conclude that for t less than a certain fixed T , we have

$$\|\partial_t^2 v\|_{0.5+\delta} \|\partial_t^3 v\|_0 \lesssim \mathcal{P}_0 + \tilde{\epsilon} \mathcal{N} + \int_0^t \mathcal{P}.$$

3.1.3. *Estimate of \mathcal{J}_2 .* There is a part of the integral

$$\mathcal{J}_2 = \int_{\Omega} R(0) \left(\partial_t^3 \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^3 \left(\frac{q}{R} \right) \right) \partial_t^3 \partial_{\alpha} v_{\beta}, \quad (41)$$

which can not be estimated using integration by parts and Hölder estimates and involves a special cancellation, namely the “tricky” term

$$T = \int_0^t \int_{\Omega} \partial_t^3 A^{\mu\alpha} \partial_t^3 \partial_{\mu} v_{\alpha} q, \quad (42)$$

where, recall, $A = Ja$. It obeys the following estimate.

Lemma 3.1. *The term T given by (42) satisfies the estimate*

$$T \leq \tilde{\epsilon} \|\Pi \bar{\partial} \partial_t^2 v\|_{0,\Gamma_1}^2 + \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.$$

Proof of Lemma 3.1. From (22), we may write

$$\begin{aligned} A^{1\alpha} &= \varepsilon^{\alpha\lambda\tau} \partial_2 \eta_\lambda \partial_3 \eta_\tau, \\ A^{2\alpha} &= -\varepsilon^{\alpha\lambda\tau} \partial_1 \eta_\lambda \partial_3 \eta_\tau, \\ A^{3\alpha} &= \varepsilon^{\alpha\lambda\tau} \partial_1 \eta_\lambda \partial_2 \eta_\tau. \end{aligned}$$

Expanding the index μ in (42), we have

$$\begin{aligned} T &= \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_2 \partial_t^2 v_\lambda \partial_3 \eta_\tau \partial_1 \partial_t^3 v_\alpha + \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_2 \eta_\lambda \partial_3 \partial_t^2 v_\tau \partial_1 \partial_t^3 v_\alpha \\ &\quad - \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^2 v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t^3 v_\alpha - \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_1 \eta_\lambda \partial_3 \partial_t^2 v_\tau \partial_2 \partial_t^3 v_\alpha \\ &\quad + \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^2 v_\lambda \partial_2 \eta_\tau \partial_3 \partial_t^3 v_\alpha + \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_1 \eta_\lambda \partial_2 \partial_t^2 v_\tau \partial_3 \partial_t^3 v_\alpha \\ &\quad + L_1 \\ &= T_1 + \cdots + T_6 + L_1 \end{aligned} \tag{43}$$

where L_1 denotes lower order terms, which are all of the form

$$\begin{aligned} \int_0^t \int_\Omega q \partial \partial_t v \partial v \partial \partial_t^3 v &= \int_\Omega q \partial \partial_t v \partial v \partial \partial_t^2 v|_0^t - \int_0^t \int_\Omega \partial_t q \partial \partial_t v \partial v \partial \partial_t^2 v \\ &\quad - \int_0^t \int_\Omega q \partial \partial_t^2 v \partial v \partial \partial_t^2 v \\ &\leq \|q\|_{L^\infty} \|\nabla v\|_{L^\infty} \|\nabla \partial_t v\|_0 \|\nabla \partial_t^2 v\|_0 + \mathcal{P}_0 + \int_0^t \mathcal{P} \\ &\lesssim \|v\|_2^{1/2} \|v\|_3^{1/2} \|\partial_t v\|_1 \|\partial_t^2 v\|_1 + \mathcal{P}_0 + \int_0^t \mathcal{P} \\ &\lesssim \tilde{\epsilon} \|v\|_3^2 + \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + \mathcal{P}_0 + \int_0^t \mathcal{P}. \end{aligned}$$

We group the leading terms in (43) as $T_1 + T_3$, $T_4 + T_6$, and $T_2 + T_5$. Integrating by parts in time in T_3 , we find

$$\begin{aligned} T_1 + T_3 &= \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_2 \partial_t^2 v_\lambda \partial_3 \eta_\tau \partial_1 \partial_t^3 v_\alpha + \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^3 v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t^2 v_\alpha \\ &\quad - \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^2 v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t^2 v_\alpha + L_2 \\ &= \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_2 \partial_t^2 v_\lambda \partial_3 \eta_\tau \partial_1 \partial_t^3 v_\alpha + \int_0^t \int_\Omega q \epsilon^{\lambda\alpha\tau} \partial_1 \partial_t^3 v_\alpha \partial_3 \eta_\tau \partial_2 \partial_t^2 v_\lambda \\ &\quad - \int_0^t \int_\Omega q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^2 v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t^2 v_\alpha + L_2 \end{aligned} \tag{44}$$

$$\begin{aligned}
&= \int_0^t \int_{\Omega} q\epsilon^{\alpha\lambda\tau} \partial_2 \partial_t^2 v_{\lambda} \partial_3 \eta_{\tau} \partial_1 \partial_t^3 v_{\alpha} - \int_0^t \int_{\Omega} q\epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^3 v_{\alpha} \partial_3 \eta_{\tau} \partial_2 \partial_t^2 v_{\lambda} \\
&\quad - \int_{\Omega} q\epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^2 v_{\lambda} \partial_3 \eta_{\tau} \partial_2 \partial_t^2 v_{\alpha} + L_2 \\
&= - \int_{\Omega} q\epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^2 v_{\lambda} \partial_3 \eta_{\tau} \partial_2 \partial_t^2 v_{\alpha} + L_2,
\end{aligned}$$

where from the first to the second line we relabeled the indices $\alpha \leftrightarrow \lambda$ in the second integral, from the second to the third we used that $\epsilon^{\lambda\alpha\tau} = -\epsilon^{\alpha\lambda\tau}$, and from the third to the fourth we observed that the first two integrals cancel each other. The symbol L_2 denotes the lower order terms, which are treated below. We now analyze the term

$$T_{13} = - \int_{\Omega} q\epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^2 v_{\lambda} \partial_3 \eta_{\tau} \partial_2 \partial_t^2 v_{\alpha}.$$

We have

$$T_{13} = - \int_{\Omega} q\epsilon^{\alpha\lambda 3} \partial_1 \partial_t^2 v_{\lambda} \partial_3 \eta_3 \partial_2 \partial_t^2 v_{\alpha} - \int_{\Omega} q\epsilon^{\alpha\lambda i} \partial_1 \partial_t^2 v_{\lambda} \partial_3 \eta_i \partial_2 \partial_t^2 v_{\alpha}, \quad (45)$$

where the last integral may be bounded by

$$\tilde{\epsilon} \|\partial_t^2 v\|_1^2$$

because $\eta(0) = \text{id}$, so that $\partial_3 \eta_i = O(\tilde{\epsilon})$ for small time; we also used $q \leq C$ by Lemma 2.1(x). For the first integral in (45), again by the initial condition, we have that $\partial_3 \eta_3 = 1 + O(\tilde{\epsilon})$ and thus

$$\begin{aligned}
&- \int_{\Omega} q\epsilon^{\alpha\lambda 3} \partial_1 \partial_t^2 v_{\lambda} \partial_3 \eta_3 \partial_2 \partial_t^2 v_{\alpha} \\
&= - \int_{\Omega} q\epsilon^{\alpha\lambda 3} \partial_1 \partial_t^2 v_{\lambda} \partial_2 \partial_t^2 v_{\alpha} - \int_{\Omega} q\epsilon^{\alpha\lambda 3} \partial_1 \partial_t^2 v_{\lambda} O(\tilde{\epsilon}) \partial_2 \partial_t^2 v_{\alpha}
\end{aligned}$$

where the last integral is also bounded by $\tilde{\epsilon} \|\partial_t^2 v\|_1^2$. For the remaining integral, we expand $\epsilon^{\alpha\lambda 3}$:

$$\begin{aligned}
- \int_{\Omega} q\epsilon^{\alpha\lambda 3} \partial_1 \partial_t^2 v_{\lambda} \partial_2 \partial_t^2 v_{\alpha} &= - \int_{\Omega} (q\epsilon^{123} \partial_1 \partial_t^2 v_2 \partial_2 \partial_t^2 v_1 + q\epsilon^{213} \partial_1 \partial_t^2 v_1 \partial_2 \partial_t^2 v_2) \\
&= - \int_{\Omega} (q\partial_1 \partial_t^2 v_2 \partial_2 \partial_t^2 v_1 - q\partial_1 \partial_t^2 v_1 \partial_2 \partial_t^2 v_2),
\end{aligned}$$

after using $\epsilon^{123} = 1 = -\epsilon^{213}$. We integrate by parts the ∂_2 in the first term and the ∂_1 in the second term to find

$$\begin{aligned}
- \int_{\Omega} q\epsilon^{\alpha\lambda 3} \partial_1 \partial_t^2 v_{\lambda} \partial_2 \partial_t^2 v_{\alpha} &= \int_{\Omega} (q\partial_2 \partial_1 \partial_t^2 v_2 \partial_t^2 v_1 - q\partial_t^2 v_1 \partial_1 \partial_2 \partial_t^2 v_2) \\
&\quad + \int_{\Omega} (\partial_1 \partial_t^2 v_2 \partial_t^2 v_1 \partial_2 q - \partial_t^2 v_1 \partial_2 \partial_t^2 v_1 \partial_1 q) \\
&= 0 + \int_{\Omega} (\partial_1 \partial_t^2 v_2 \partial_t^2 v_1 \partial_2 q - \partial_t^2 v_1 \partial_2 \partial_t^2 v_1 \partial_1 q),
\end{aligned}$$

where the last integral obeys

$$\int_{\Omega} (\partial_1 \partial_t^2 v_2 \partial_t^2 v_1 \partial_2 q - \partial_t^2 v_1 \partial_2 \partial_t^2 v_1 \partial_1 q) \leq C \|\partial_t^2 v\|_1 \|\partial_t^2 v\|_0 \|\nabla q\|_{L^\infty(\Omega)}$$

$$\begin{aligned} &\leq \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + C \|\partial_t^2 v\|_0^2 \|\nabla q\|_{L^\infty(\Omega)}^2 \leq \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + C \|\partial_t^2 v\|_0^2 \|R\|_2^{1/2} \|R\|_3^{1/2} \\ &\leq \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + \tilde{\epsilon} \|R\|_3^2 + \mathcal{P}_0 + \int_0^t \mathcal{P}. \end{aligned}$$

The symbol L_2 in (44), denotes the sum of

$$\int_{\Omega} q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t^2 v_{\lambda} \partial_3 \eta_{\tau} \partial_2 \partial_t^2 v_{\alpha} |_{t=0} \leq \mathcal{P}_0$$

and

$$\int_0^t \int_{\Omega} \epsilon^{\alpha\lambda\tau} \partial_t \left(q \partial_1 \partial_t^2 v_{\lambda} \partial_3 \eta_{\tau} \right) \partial_2 \partial_t^2 v_{\alpha} \leq \int_0^t \mathcal{P}.$$

For the sum of T_1 and T_3 , we conclude

$$T_1 + T_3 \leq \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + \tilde{\epsilon} \|R\|_3^2 + \mathcal{P}_0 + \int_0^t \mathcal{P}.$$

The terms $T_4 + T_6$ and $T_2 + T_5$ are handled in the same way, with one extra step. In the last step above, we integrated ∂_1 and ∂_2 by parts. For $T_4 + T_6$ we integrate by parts the derivatives ∂_2 and ∂_3 ; this last one produces the boundary term

$$\int_{\Gamma_1} q \partial_t^2 v_2 \partial_2 \partial_t^2 v_3.$$

(Note that the same integral over Γ_0 vanishes by (6f).) To bound this term, we recall (14), which allows us to relate $\Pi \bar{\partial} \partial_t^2 v$ and $\bar{\partial} \partial_t^2 v_3$ and write

$$\begin{aligned} \left| \int_{\Gamma_1} q \partial_t^2 v_2 \partial_2 \partial_t^2 v_3 \right| &= \left| \int_{\Gamma_1} q \partial_t^2 v_2 (\Pi_{\lambda}^3 \partial_2 \partial_t^2 v^{\lambda} + g^{kl} \partial_k \eta_3 \partial_l \eta_{\lambda} \partial_2 \partial_t^2 v^{\lambda}) \right| \\ &\lesssim \tilde{\epsilon} \|\Pi \bar{\partial} \partial_t^2 v\|_{0,\Gamma_1}^2 + \|q\|_{1.5,\Gamma_1}^2 \|\partial_t^2 v\|_{0,\Gamma_1}^2 \\ &\quad + \|q \partial_t^2 v_2 g^{kl} \partial_k \eta_3 \partial_l \eta_{\lambda}\|_{0.5,\Gamma_1} \|\partial_2 \partial_t^2 v^{\lambda}\|_{-0.5,\Gamma_1} \\ &\lesssim \tilde{\epsilon} \|\Pi \bar{\partial} \partial_t^2 v\|_{0,\Gamma_1}^2 + \|q\|_{1.5,\Gamma_1}^2 \|\partial_t^2 v\|_{0,\Gamma_1}^2 \\ &\quad + \|q\|_{1.5,\Gamma_1} \|g^{-1}\|_{1.5,\Gamma_1} \|\bar{\partial} \eta_3\|_{1.5,\Gamma_1} \|\bar{\partial} \eta_3\|_{1.5,\Gamma_1} \|\partial_t^2 v\|_{0.5,\Gamma_1}^2. \end{aligned} \tag{46}$$

Using that $\bar{\partial} \eta_3 = 0$ at $t = 0$, we may write $\bar{\partial} \eta_3 = \int_0^t \bar{\partial} v_3$ to estimate

$$\int_{\Gamma_1} q \partial_t^2 v_2 \partial_2 \partial_t^2 v_3 \lesssim \tilde{\epsilon} \|\Pi \bar{\partial} \partial_t^2 v\|_{0,\Gamma_1}^2 + \tilde{\epsilon} \mathcal{N} + \mathcal{P} \int_0^t \mathcal{P},$$

and the proof is concluded. \square

Now, we complete the treatment of \mathcal{J}_2 by estimating the rest of the terms appearing in (41), i.e., by bounding the expression

$$\begin{aligned} \mathcal{J}_2 - T &= \int_{\Omega} R(0) \left(\partial_t^3 \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^3 \left(\frac{q}{R} \right) \right) \partial_t^3 \partial_{\alpha} v_{\beta} \\ &\quad - \int_{\Omega} R(0) \partial_t^3 \left(\frac{a^{\alpha\beta}}{R} \right) \partial_t^3 \partial_{\alpha} v_{\beta} q \end{aligned}$$

which we may rewrite as

$$\begin{aligned} \mathcal{J}_2 - T &= \int_{\Omega} R(0) \left(\partial_t^3 (a^{\alpha\beta} \bar{q}) - a^{\alpha\beta} \partial_t^3 \bar{q} - \partial_t^3 a^{\alpha\beta} \bar{q} \right) \partial_t^3 \partial_{\alpha} v_{\beta} \\ &\quad - \int_{\Omega} R(0) \left(\partial_t^3 (a^{\alpha\beta} R^{-1}) - \partial_t^3 a^{\alpha\beta} R^{-1} \right) \partial_t^3 \partial_{\alpha} v_{\beta} q. \end{aligned} \tag{47}$$

After time integration, the first integral in (47) equals

$$\begin{aligned} & 3 \int_0^t \int_{\Omega} R(0) (\partial_t^2 a^{\alpha\beta} \partial_t \bar{q} + \partial_t a^{\alpha\beta} \partial_t^2 \bar{q}) \partial_t^3 \partial_{\alpha} v_{\beta} \\ &= 3 \int_{\Omega} R(0) (\partial_t^2 a^{\alpha\beta} \partial_t \bar{q} + \partial_t a^{\alpha\beta} \partial_t^2 \bar{q}) \partial_t^2 \partial_{\alpha} v_{\beta} |_0^t \\ & \quad - 3 \int_0^t \int_{\Omega} R(0) \partial_t (\partial_t^2 a^{\alpha\beta} \partial_t \bar{q} + \partial_t a^{\alpha\beta} \partial_t^2 \bar{q}) \partial_t^2 \partial_{\alpha} v_{\beta}. \end{aligned}$$

The second term is bounded by $\int_0^t \mathcal{P}$, while the pointwise term at $t = 0$ by \mathcal{P}_0 . It is easy to check that the pointwise term at t is bounded by

$$\begin{aligned} & \|\partial_t^2 v\|_1 (\|\partial_t v\|_1^{1/2} \|\partial_t v\|_2^{1/2} + \|v\|_2^2) \|\partial_t R\|_1 \\ & \quad + \|\partial_t^2 v\|_1 \|v\|_1^{1/2} \|v\|_2^{1/2} (\|\partial_t R\|_1^2 + \|\partial_t^2 R\|_0^{1/2} \|\partial_t^2 R\|_1^{1/2}) \\ & \lesssim \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + \tilde{\epsilon} \|\partial_t^2 R\|_1^2 + \mathcal{P}_0 + \int_0^t \mathcal{P}. \end{aligned} \quad (48)$$

The second integral in (47) is treated the same way, resulting in the bound as in (48) but with an additional term

$$\tilde{\epsilon} \|\partial_t^3 R\|_0^2.$$

3.1.4. Estimate of the boundary integral. We now estimate the boundary integral on the left-hand side of (36) or, rather, its time integral, which in view of (6e) and (16) can be written as

$$\int_0^t \int_{\Gamma_1} \partial_t^3 (J a^{\alpha\beta} q) \partial_t^3 v_{\beta} N_{\alpha} = -\sigma I_1, \quad (49)$$

where

$$I_1 = \int_0^t \int_{\Gamma_1} \partial_t^3 (\sqrt{g} \Delta_g \eta^{\alpha}) \partial_t^3 v_{\alpha}. \quad (50)$$

We shall repeatedly use the identity

$$\sqrt{g} \Delta_g \eta^{\alpha} = \sqrt{g} g^{ij} \Pi_{\mu}^{\alpha} \partial_{ij}^2 \eta^{\mu}. \quad (51)$$

The identity (51) follows from (24) and (25) since

$$\begin{aligned} \sqrt{g} g^{ij} \Delta_g \eta^{\alpha} &= \sqrt{g} g^{ij} \partial_{ij}^2 \eta^{\alpha} - \sqrt{g} g^{ij} g^{kl} \partial_l \eta_{\mu} \partial_{ij}^2 \eta^{\mu} \partial_k \eta^{\alpha} \\ &= \sqrt{g} g^{ij} \partial_{ij}^2 \eta^{\mu} (\delta_{\mu}^{\alpha} - g^{kl} \partial_k \eta^{\alpha} \partial_l \eta_{\mu}) \end{aligned}$$

and the term inside the parentheses equals Π_{μ}^{α} by (14). Using (51) and applying the Leibniz rule, we may split

$$\begin{aligned} I_1 &= \int_0^t \int_{\Gamma_1} \partial_t^3 (\sqrt{g} \Delta_g \eta^{\alpha}) \partial_t^3 v_{\alpha} = \int_0^t \int_{\Gamma_1} \partial_t^3 (\sqrt{g} g^{ij} \Pi_{\mu}^{\alpha} \partial_{ij}^2 \eta^{\mu}) \partial_t^3 v_{\alpha} \\ &= \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \Pi_{\mu}^{\alpha} \partial_{ij}^2 \partial_t^2 v^{\mu} \partial_t^3 v_{\alpha} + 3 \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij} \Pi_{\mu}^{\alpha}) \partial_{ij}^2 \partial_t v^{\mu} \partial_t^3 v_{\alpha} \\ & \quad + 3 \int_0^t \int_{\Gamma_1} \partial_t^2 (\sqrt{g} g^{ij} \Pi_{\mu}^{\alpha}) \partial_{ij}^2 v^{\mu} \partial_t^3 v_{\alpha} + \int_0^t \int_{\Gamma_1} \partial_t^3 (\sqrt{g} g^{ij} \Pi_{\mu}^{\alpha}) \partial_{ij}^2 \eta^{\mu} \partial_t^3 v_{\alpha} \\ &= I_{11} + 3I_{12} + 3I_{13} + I_{14}. \end{aligned}$$

Estimate of I_{11} . In order to bound I_{11} , we integrate by parts in ∂_i and then in t to obtain

$$\begin{aligned} I_{11} &= - \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \Pi_\mu^\alpha \partial_j \partial_t^2 v^\mu \partial_i \partial_t^3 v_\alpha - \int_0^t \int_{\Gamma_1} \partial_i (\sqrt{g} g^{ij} \Pi_\mu^\alpha) \partial_j \partial_t^2 v^\mu \partial_t^3 v_\alpha \\ &= -\frac{1}{2} \int_{\Gamma_1} \sqrt{g} g^{ij} \Pi_\mu^\alpha \partial_j \partial_t^2 v^\mu \partial_i \partial_t^2 v_\alpha + \frac{1}{2} \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij} \Pi_\mu^\alpha) \partial_j \partial_t^2 v^\mu \partial_i \partial_t^2 v_\alpha \\ &\quad - \int_0^t \int_{\Gamma_1} \partial_i (\sqrt{g} g^{ij} \Pi_\mu^\alpha) \partial_j \partial_t^2 v^\mu \partial_t^3 v_\alpha + \frac{1}{2} \int_{\Gamma_1} \sqrt{g} g^{ij} \Pi_\mu^\alpha \partial_j \partial_t^2 v^\mu \partial_i \partial_t^2 v_\alpha|_0 \\ &= I_{111} + I_{112} + I_{113} + I_{114}. \end{aligned}$$

The first term on the right produces a coercive term, as we may write

$$\begin{aligned} I_{111} &= -\frac{1}{2} \int_{\Gamma_1} \sqrt{g} g^{ij} \Pi_\mu^\beta \partial_j \partial_t^2 v^\mu \Pi_\beta^\alpha \partial_i \partial_t^2 v_\alpha \\ &= -\frac{1}{2} \int_{\Gamma_1} \delta^{ij} \Pi_\mu^\beta \partial_j \partial_t^2 v^\mu \Pi_\beta^\alpha \partial_i \partial_t^2 v_\alpha - \frac{1}{2} \int_{\Gamma_1} (\sqrt{g} g^{ij} - \delta^{ij}) \Pi_\mu^\beta \partial_j \partial_t^2 v^\mu \Pi_\beta^\alpha \partial_i \partial_t^2 v_\alpha \\ &= I_{1111} + I_{1112}. \end{aligned}$$

Since

$$\|\sqrt{g} g^{ij} - \delta^{ij}\|_{1.5, \Gamma_1} \leq \|\sqrt{g} g^{ij} - \delta^{ij}\|_{1.5, \Gamma_1} \leq C t \|\partial_t \bar{\partial} \eta\|_{1.5, \Gamma_1} \leq C t \|v\|_3 \leq \tilde{\epsilon},$$

the second term is absorbed in the first provided $T \leq 1/CM$ for a sufficiently large C . Thus

$$I_{111} \leq -\frac{1}{4} \|\Pi \bar{\partial} \partial_t^2 v\|_{0, \Gamma_1},$$

so that (recall (49))

$$-\sigma I_{111} \geq \frac{\sigma}{4} \|\Pi \bar{\partial} \partial_t^2 v\|_{0, \Gamma_1}.$$

The term I_{112} is rewritten as

$$\begin{aligned} I_{112} &= \frac{1}{2} \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_j \partial_t^2 v^\mu \partial_i \partial_t^2 v_\alpha + \frac{1}{2} \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t \Pi_\mu^\alpha \partial_j \partial_t^2 v^\mu \partial_i \partial_t^2 v_\alpha \\ &= \frac{1}{2} \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij}) \Pi_\mu^\sigma \partial_j \partial_t^2 v^\mu \Pi_\sigma^\alpha \partial_i \partial_t^2 v_\alpha + \frac{1}{2} \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t \Pi_\mu^\alpha \partial_j \partial_t^2 v^\mu \partial_i \partial_t^2 v_\alpha \\ &= I_{1121} + I_{1122}, \end{aligned}$$

where we used $\Pi_\mu^\alpha = \Pi_\mu^\sigma \Pi_\sigma^\alpha$. We have

$$I_{1121} \lesssim \int_0^t \|\partial_t (\sqrt{g} g^{ij})\|_{L^\infty(\Gamma_1)} \|\Pi \bar{\partial} \partial_t^2 v\|_{0, \Gamma_1}^2,$$

and since by (27)

$$\|\partial_t (\sqrt{g} g^{ij})\|_{L^\infty(\Gamma_1)} = \|Q(\bar{\partial} \eta) \bar{\partial} \partial_t \eta\|_{L^\infty(\Gamma_1)} = \|Q(\bar{\partial} \eta) \bar{\partial} v\|_{L^\infty(\Gamma_1)} \lesssim \|Q(\bar{\partial} \eta)\|_2 \|v\|_3,$$

we have

$$I_{1121} \leq \int_0^t \mathcal{P} \|\Pi \bar{\partial} \partial_t^2 v\|_{0, \Gamma_1}^2. \quad (52)$$

The term I_{1122} is more delicate. First, by $\Pi_\mu^\alpha = \Pi_\mu^\sigma \Pi_\sigma^\alpha$, we have

$$\begin{aligned} I_{1122} &= \frac{1}{2} \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t \Pi_\mu^\sigma \partial_j \partial_t^2 v^\mu \Pi_\sigma^\alpha \partial_i \partial_t^2 v_\alpha \\ &\quad + \frac{1}{2} \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \Pi_\mu^\sigma \partial_j \partial_t^2 v^\mu \partial_t \Pi_\sigma^\alpha \partial_i \partial_t^2 v_\alpha \\ &= \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t \Pi_\mu^\sigma \partial_j \partial_t^2 v^\mu \Pi_\sigma^\alpha \partial_i \partial_t^2 v_\alpha. \end{aligned}$$

Since

$$\Pi_\mu^\alpha = \hat{n}^\alpha \hat{n}_\mu, \quad (53)$$

where $\hat{n} = n \circ \eta$ (cf. (13) and (21)), we have

$$\partial_t \Pi_\mu^\alpha = \partial_t \hat{n}^\alpha \hat{n}_\mu + \hat{n}^\alpha \partial_t \hat{n}_\mu. \quad (54)$$

Therefore, I_{1122} may be rewritten as

$$\begin{aligned} I_{1122} &= \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \hat{n}^\sigma \partial_t \hat{n}_\mu \partial_j \partial_t^2 v^\mu \Pi_\sigma^\alpha \partial_i \partial_t^2 v_\alpha \\ &\quad + \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t \hat{n}^\sigma \hat{n}_\mu \partial_j \partial_t^2 v^\mu \Pi_\sigma^\alpha \partial_i \partial_t^2 v_\alpha \\ &= I_{11221} + I_{11222}. \end{aligned}$$

For the second term, we use

$$\hat{n}_\mu \partial_j \partial_t^2 v^\mu = \hat{n}_\tau \Pi_\mu^\tau \partial_j \partial_t^2 v^\mu$$

and thus I_{11222} is controlled by the right side of (52). For I_{11221} , we use (recall (19)),

$$\partial_t \hat{n}_\mu = -g^{kl} \partial_k v^\tau \hat{n}_\tau \partial_l \eta_\mu \quad (55)$$

which gives

$$I_{11221} = - \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \hat{n}^\sigma g^{kl} \partial_k v^\tau \hat{n}_\tau \partial_l \eta_\mu \partial_j \partial_t^2 v^\mu \Pi_\sigma^\alpha \partial_i \partial_t^2 v_\alpha. \quad (56)$$

From the equation (6a) for the velocity, we have $\partial_t v^\alpha = -(J/\rho_0) a^{\mu\alpha} \partial_\mu q$, and by the definition of a ,

$$\partial_t v^\mu \partial_l \eta_\mu = -\frac{J}{\rho_0} \partial_l q, \quad (57)$$

from where, by applying $\partial_j \partial_t$ to both sides,

$$\begin{aligned} \partial_j \partial_t^2 v^\mu \partial_l \eta_\mu &= -\frac{J}{\rho_0} \partial_{jl}^2 \partial_t q - \left(\partial_j \partial_t \left(\frac{J}{\rho_0} \partial_l q \right) - \frac{J}{\rho_0} \partial_{jl}^2 \partial_t q \right) \\ &\quad - \left(\partial_j \partial_t (\partial_t v^\mu \partial_l \eta_\mu) - \partial_j \partial_t^2 v^\mu \partial_l \eta_\mu \right), \end{aligned} \quad (58)$$

which we replace in (56). The commutators are easily controlled, so we only need to consider the main term

$$I_{11221} \stackrel{L}{=} \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \hat{n}^\sigma g^{kl} \partial_k v^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_{jl}^2 \partial_t q \Pi_\sigma^\alpha \partial_i \partial_t^2 v_\alpha \quad (59)$$

where we henceforth adopt:

Notation 8. We use $\stackrel{L}{=}$ to denote equality modulo lower order terms that can be controlled. Thus, $\stackrel{L}{=}$ in (59) indicates the leading term of I_{11221} .

Now, we integrate by parts in x_j , leading to

$$\begin{aligned} I_{11221} &\stackrel{L}{=} - \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \hat{n}^\sigma g^{kl} \partial_k v^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_l \partial_t q \Pi_\sigma^\alpha \partial_{ij}^2 \partial_t^2 v_\alpha \\ &\stackrel{L}{=} - \int_{\Gamma_1} \sqrt{g} g^{ij} \hat{n}^\sigma g^{kl} \partial_k v^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_l \partial_t q \Pi_\sigma^\alpha \partial_{ij}^2 \partial_t v_\alpha \Big|_0^t \\ &\quad + \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \hat{n}^\sigma g^{kl} \partial_k v^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_l \partial_t^2 q \Pi_\sigma^\alpha \partial_{ij}^2 \partial_t v_\alpha = I_{112211} + I_{112212}. \end{aligned}$$

At this point, we use the identity

$$g^{ij} \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu = -\frac{1}{\sigma} \frac{J}{\sqrt{g}} a^{\mu\alpha} N_\mu q,$$

which follows from (6e), (16), and (51), which after applying ∂_t^2 gives

$$g^{ij} \Pi_\mu^\alpha \partial_t \partial_{ij}^2 v^\mu = -\partial_t^2 \left(\frac{1}{\sigma} \frac{J}{\sqrt{g}} a^{\mu\alpha} N_\mu q \right) - \left(\partial_t^2 (g^{ij} \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu) - g^{ij} \Pi_\mu^\alpha \partial_t^2 \partial_{ij}^2 \eta^\mu \right).$$

After replacing the first term in I_{112211} and I_{112212} , the resulting terms may be controlled using $H^{1/2}(\Gamma_1)$ - $H^{-1/2}(\Gamma_1)$ duality. We illustrate this on the term where both time derivatives hit q , i.e., $-(1/\sigma)(J/\sqrt{g})a^{\mu\alpha}N_\mu\partial_t^2q$. After replacing this in I_{112212} , we get the term of the form

$$\int_0^t \int_{\Gamma_1} B^{jl} \partial_l \partial_t^2 q \partial_t^2 q,$$

which is estimated by

$$\begin{aligned} &\int_0^t \|\partial_l \partial_t^2 q\|_{H^{-1/2}(\Gamma_1)} \|B^{jl} \partial_t^2 q\|_{H^{1/2}(\Gamma_1)} \\ &\lesssim \int_0^t \|\partial_t^2 q\|_{H^{1/2}(\Gamma_1)} \|B\|_{H^{1/2+\delta}} \|\partial_t^2 q\|_{H^{1/2}(\Gamma_1)} \leq \int_0^t \mathcal{P} \|\partial_t^2 q\|_1 \leq \int_0^t \mathcal{P} \end{aligned}$$

where $\delta > 0$ is a small parameter.

Before continuing, it is worthwhile to formalize the (55), (57), and (58) into the identity

$$\begin{aligned} \partial_t \hat{n}_\mu \partial_j \partial_t^2 v^\mu &= g^{kl} \partial_k v^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_{jl}^2 \partial_t q + g^{kl} \partial_k v^\tau \hat{n}_\tau \left(\partial_j \partial_t \left(\frac{J}{\rho_0} \partial_l q \right) - \frac{J}{\rho_0} \partial_{jl}^2 \partial_t q \right) \\ &\quad + g^{kl} \partial_k v^\tau \hat{n}_\tau \left(\partial_j \partial_t (\partial_t v^\mu \partial_l \eta_\mu) - \partial_j \partial_t^2 v^\mu \partial_l \eta_\mu \right). \end{aligned} \quad (60)$$

Also, similarly to (55), we have (recall (20))

$$\partial_i \hat{n}_\mu = -g^{kl} \partial_{ik} \eta^\tau \hat{n}_\tau \partial_l \eta_\mu,$$

whence, as for (60), we have

$$\begin{aligned} \partial_i \hat{n}_\mu \partial_j \partial_t^2 v^\mu &= g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_{jl}^2 \partial_t q + g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \left(\partial_j \partial_t \left(\frac{J}{\rho_0} \partial_l q \right) - \frac{J}{\rho_0} \partial_{jl}^2 \partial_t q \right) \\ &\quad + g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \left(\partial_j \partial_t (\partial_t v^\mu \partial_l \eta_\mu) - \partial_j \partial_t^2 v^\mu \partial_l \eta_\mu \right). \end{aligned} \quad (61)$$

Next, we consider

$$I_{113} = - \int_0^t \int_{\Gamma_1} \partial_i (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_j \partial_t^2 v^\mu \partial_t^3 v_\alpha - \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_i \Pi_\mu^\alpha \partial_j \partial_t^2 v^\mu \partial_t^3 v_\alpha$$

$$\begin{aligned}
&= - \int_0^t \int_{\Gamma_1} \partial_i(\sqrt{g}g^{ij})\hat{n}^\alpha \hat{n}_\mu \partial_j \partial_t^2 v^\mu \partial_t^3 v_\alpha \\
&\quad - \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} \partial_i \hat{n}_\mu \partial_j \partial_t^2 v^\mu \hat{n}^\alpha \partial_t^3 v_\alpha - \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} \hat{n}_\mu \partial_j \partial_t^2 v^\mu \partial_i \hat{n}^\alpha \partial_t^3 v_\alpha \\
&= I_{1131} + I_{1132} + I_{1133},
\end{aligned}$$

where we used $\Pi_\mu^\alpha = \hat{n}^\alpha \hat{n}_\mu$. The first term I_{1131} is of high order and can not be treated directly. It cancels with a term resulting from I_{14} further below; cf. (68). Using (61), we have

$$\begin{aligned}
I_{1132} &\stackrel{L}{=} - \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_{jl}^2 \partial_t q \hat{n}^\alpha \partial_t^3 v_\alpha \\
&\stackrel{L}{=} - \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_{jl}^2 \partial_t q \hat{n}^\alpha \partial_t^2 v_\alpha \Big|_0^t \\
&\quad + \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_{jl}^2 \partial_t^2 q \hat{n}^\alpha \partial_t^2 v_\alpha.
\end{aligned} \tag{62}$$

The first term is easily controlled since

$$\hat{n}^\alpha \partial_t^2 v_\alpha = \hat{n}^\alpha \hat{n}^\tau \hat{n}_\tau \partial_t^2 v_\alpha = \hat{n}^\tau \Pi_\tau^\alpha \partial_t^2 v_\alpha.$$

For the second term in (62), we use

$$q = -\sigma \Delta_g \eta^\alpha \hat{n}_\alpha,$$

which follows from $\hat{n}^\alpha q = -\sigma \Delta_g \eta^\alpha$ and consequently (recalling (51) and using $\Pi_\mu^\alpha = \hat{n}^\alpha \hat{n}_\mu$)

$$q = -\sigma g^{ij} \hat{n}_\mu \partial_{ij}^2 \eta^\mu, \tag{63}$$

and we obtain

$$I_{1132} \stackrel{L}{=} -\sigma \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_{jl}^2 (g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu) \hat{n}^\alpha \partial_t^2 v_\alpha.$$

Integrating by parts in x_l and then in x_i , we get

$$\begin{aligned}
I_{1132} &\stackrel{L}{=} \sigma \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{ikl}^3 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_j (g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu) \hat{n}^\alpha \partial_t^2 v_\alpha \\
&\quad + \sigma \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_j (g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu) \hat{n}^\alpha \partial_l \partial_t^2 v_\alpha \\
&= -\sigma \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{kl}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_{ij}^2 (g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu) \hat{n}^\alpha \partial_t^2 v_\alpha \\
&\quad - \sigma \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{kl}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_j (g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu) \hat{n}^\alpha \partial_i \partial_t^2 v_\alpha \\
&\quad + \sigma \int_0^t \int_{\Gamma_1} \sqrt{g}g^{ij} g^{kl} \partial_{ik}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_j (g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu) \hat{n}^\alpha \partial_l \partial_t^2 v_\alpha.
\end{aligned} \tag{64}$$

The last two integrals cancel by the symmetry property

$$\sum_{i,k,l=1}^2 (g^{ji} g^{kl} - g^{ik} g^{lj}) = 0 \tag{65}$$

(which is true for any symmetric matrix); this identity is proven by using $g^{lj} = g^{jl}$ in the last sum and then relabeling it. Thus we only need to treat the first term in (64). Integrating by parts in x_i , x_j , and then in t , we get

$$\begin{aligned} I_{11322} &\stackrel{L}{=} -\sigma \int_0^t \int_{\Gamma_1} \sqrt{g} g^{kl} \partial_{kl}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} (g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu) (g^{ij} \hat{n}^\alpha \partial_{ij}^2 \partial_t^2 v_\alpha) \\ &= -\frac{1}{2} \sigma \int_0^t \int_{\Gamma_1} \sqrt{g} g^{kl} \partial_{kl}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \partial_t (g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu g^{ij} \hat{n}^\alpha \partial_{ij}^2 \partial_t v_\alpha) \\ &\stackrel{L}{=} -\frac{1}{2} \sigma \int_{\Gamma_1} \sqrt{g} g^{kl} \partial_{kl}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu g^{ij} \hat{n}^\alpha \partial_{ij}^2 \partial_t v_\alpha \Big|_0^t \\ &\quad + \frac{1}{2} \sigma \int_0^t \int_{\Gamma_1} \partial_t \left(\sqrt{g} g^{kl} \partial_{kl}^2 \eta^\tau \hat{n}_\tau \frac{J}{\rho_0} \right) g^{mn} \hat{n}_\mu \partial_{mn}^2 \partial_t v^\mu g^{ij} \hat{n}^\alpha \partial_{ij}^2 \partial_t v_\alpha. \end{aligned}$$

It is easy to check that both terms can be controlled. For the first term on the far right, we use that $\partial_{kl}^2 \eta$ vanishes at $t = 0$. This completes the treatment of the term I_{11} .

Estimates of I_{12} and I_{13} . The term I_{12} is split as

$$\begin{aligned} I_{12} &= \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_{ij}^2 \partial_t v^\mu \partial_t^3 v_\alpha + \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t \Pi_\mu^\alpha \partial_{ij}^2 \partial_t v^\mu \partial_t^3 v_\alpha \\ &= \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_{ij}^2 \partial_t v^\mu \partial_t^2 v_\alpha \Big|_0^t - \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_{ij}^2 \partial_t^2 v^\mu \partial_t^2 v_\alpha \\ &\quad - \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij}) \partial_t \Pi_\mu^\alpha \partial_{ij}^2 \partial_t v^\mu \partial_t^2 v_\alpha - \int_0^t \int_{\Gamma_1} \partial_t^2 (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_{ij}^2 \partial_t v^\mu \partial_t^2 v_\alpha \\ &\quad + \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t \Pi_\mu^\alpha \partial_{ij}^2 \partial_t v^\mu \partial_t^3 v_\alpha \\ &= I_{121} + I_{122} + I_{123} + I_{124} + I_{125}. \end{aligned}$$

All the terms except I_{123} are estimated as above. For I_{123} , we use (54) and obtain

$$\begin{aligned} I_{123} &= - \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij}) \hat{n}^\mu \partial_{ij}^2 \partial_t v^\mu \partial_t \hat{n}^\alpha \partial_t^2 v_\alpha \\ &\quad - \int_0^t \int_{\Gamma_1} \partial_t (\sqrt{g} g^{ij}) \partial_t \hat{n}^\mu \partial_{ij}^2 \partial_t v^\mu \hat{n}^\alpha \partial_t^2 v_\alpha. \end{aligned}$$

The terms are treated as I_{11222} and I_{11221} respectively. This concludes the treatment of I_{12} .

The term I_{13} is handled analogously to I_{12} to I_{14} , so we omit the details.

Estimate of I_{14} . For I_{14} , we have

$$\begin{aligned} I_{14} &= \int_0^t \int_{\Gamma_1} \partial_t^3 (\sqrt{g} g^{ij} \Pi_\mu^\alpha) \partial_{ij}^2 \eta^\mu \partial_t^3 v_\alpha \\ &\stackrel{L}{=} \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t^3 \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu \hat{n}^\alpha \partial_t^3 v_\alpha + \int_0^t \int_{\Gamma_1} \partial_t^3 (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu \partial_t^3 v_\alpha \\ &\stackrel{L}{=} \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \partial_t^3 \hat{n}_\mu \partial_{ij}^2 \eta^\mu \hat{n}^\alpha \partial_t^3 v_\alpha + \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} \hat{n}_\mu \partial_{ij}^2 \eta^\mu \partial_t^3 \hat{n}^\alpha \partial_t^3 v_\alpha \\ &\quad + \int_0^t \int_{\Gamma_1} \partial_t^3 (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu \partial_t^3 v_\alpha = I_{141} + I_{142} + I_{143}, \end{aligned}$$

where we used (53) in the last step. The terms I_{142} and I_{143} are treated with similar methods (see below); here we focus on the high order term I_{141} . Since, by (55), we have

$$\partial_t^3 \hat{n}_\mu = -g^{kl} \partial_k \partial_t^2 v^\tau \hat{n}_\tau \partial_l \eta_\mu - (\partial_t^2 (g^{kl} \partial_k v^\tau \hat{n}_\tau \partial_l \eta_\mu) - g^{kl} \partial_k \partial_t^2 v^\tau \hat{n}_\tau \partial_l \eta_\mu)$$

we get

$$I_{141} \stackrel{L}{=} - \int_0^t \int_{\Gamma_1} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_t^2 v^\tau \hat{n}_\tau \partial_l \eta_\mu \partial_{ij}^2 \eta^\mu \hat{n}^\alpha \partial_t^3 v_\alpha. \quad (66)$$

At this point we need the identity

$$\partial_i (\sqrt{g} g^{ij}) = -\sqrt{g} g^{ij} g^{kl} \partial_{ij}^2 \eta^\mu \partial_l \eta_\mu, \quad (67)$$

which we prove next. First, by (27), we have

$$\begin{aligned} \partial_i (\sqrt{g} g^{ij}) &= \sqrt{g} \left(\frac{1}{2} g^{ij} g^{mn} - g^{im} g^{jn} \right) \partial_i g_{mn} \\ &= \sqrt{g} \left(\frac{1}{2} g^{ij} g^{mn} - g^{im} g^{jn} \right) \partial_i (\partial_m \eta^\mu \partial_n \eta_\mu) \\ &= \sqrt{g} \left(\frac{1}{2} g^{ij} g^{mn} - g^{im} g^{jn} \right) \partial_{im}^2 \eta^\mu \partial_n \eta_\mu \\ &\quad + \sqrt{g} \left(\frac{1}{2} g^{ij} g^{mn} - g^{im} g^{jn} \right) \partial_m \eta^\mu \partial_{in}^2 \eta_\mu. \end{aligned}$$

In the second term on the far right side, we relabel m and n and then factor out $\partial_{im}^2 \eta^\mu \partial_n \eta_\mu$. We get

$$\begin{aligned} \partial_i (\sqrt{g} g^{ij}) &= \sqrt{g} \left(\frac{1}{2} g^{ij} g^{mn} - g^{im} g^{jn} \right) \partial_{im}^2 \eta^\mu \partial_n \eta_\mu \\ &\quad + \sqrt{g} \left(\frac{1}{2} g^{ij} g^{mn} - g^{in} g^{jm} \right) \partial_n \eta^\mu \partial_{im}^2 \eta_\mu \\ &= \sqrt{g} (g^{ij} g^{mn} - g^{im} g^{jn} - g^{in} g^{jm}) \partial_{im}^2 \eta^\mu \partial_n \eta_\mu \\ &= -\sqrt{g} g^{im} g^{jn} \partial_{im}^2 \eta^\mu \partial_n \eta_\mu + \sqrt{g} \partial_{im}^2 \eta^\mu \partial_n \eta_\mu (g^{ij} g^{mn} - g^{in} g^{jm}). \end{aligned}$$

Since $\partial_{im}^2 \eta^\mu (g^{ij} g^{mn} - g^{in} g^{jm}) = 0$ due to anti-symmetry in i and m in the term in parenthesis, the identity (67) follows. Using (67) in (66), we get

$$I_{141} \stackrel{L}{=} \int_0^t \int_{\Gamma_1} \partial_i (\sqrt{g} g^{ij}) \partial_k \partial_t^2 v^\tau \hat{n}_\tau \hat{n}^\alpha \partial_t^3 v_\alpha. \quad (68)$$

As pointed out earlier, this term cancels with I_{1131} above.

As said, the terms I_{142} and I_{143} are treated with similar ideas as above. We illustrate this by estimating I_{143} . Integrating by parts in time

$$\begin{aligned} I_{143} &= I_{143,0} + \int_{\Gamma_1} \partial_t^3 (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu \partial_t^2 v_\alpha - \int_0^t \int_{\Gamma_1} \partial_t^4 (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu \partial_t^2 v_\alpha \\ &\quad - \int_0^t \int_{\Gamma_1} \partial_t^3 (\sqrt{g} g^{ij}) \partial_t (\Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu) \partial_t^2 v_\alpha \\ &= I_{143,0} + I_{1431} + I_{1432} + I_{1433}, \end{aligned}$$

where $I_{143,0}$ is controlled by \mathcal{P}_0 . Let us handle I_{1431} . Using (27) to write

$$\partial_t (\sqrt{g} g^{ij}) = \sqrt{g} (g^{ij} g^{kl} - 2g^{lj} g^{ik}) \partial_k v^\lambda \partial_l \eta_\lambda,$$

we have

$$\begin{aligned} \partial_t^3(\sqrt{g}g^{ij}) &\stackrel{L}{=} \partial_t^2(\sqrt{g}(g^{ij}g^{kl} - 2g^{jl}g^{ik}))\partial_k v^\lambda \partial_l \eta_\lambda \\ &\quad + \sqrt{g}(g^{ij}g^{kl} - 2g^{jl}g^{ik})\partial_k \partial_t^2 v^\lambda \partial_l \eta_\lambda. \end{aligned} \quad (69)$$

We split I_{1431} accordingly,

$$I_{1431} \stackrel{L}{=} I_{14311} + I_{14312},$$

and note I_{14311} that can be directly estimated producing

$$I_{14311} \leq \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + \mathcal{P} \int_0^t \mathcal{P}.$$

For I_{14312} , we time differentiate (57) and integrate by parts with respect to x^k to obtain

$$I_{14311} \leq \tilde{\epsilon} (\|\partial_t q\|_2^2 + \|\Pi \bar{\partial} \partial_t^2 v\|_{0,\Gamma_1}^2) + \mathcal{P} \int_0^t \mathcal{P}.$$

This produces an estimate for I_{1431} and I_{1433} is handled along the same lines.

Let us now investigate I_{1432} . Taking one further time derivative of (69) and using the resulting expression into I_{1432} , we see that the top term is

$$I_{1432,\text{top}} = \int_0^t \int_{\Gamma_1} \sqrt{g}(g^{ij}g^{kl} - 2g^{jl}g^{ik})\partial_k \partial_t^3 v^\lambda \partial_l \eta_\lambda \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu \partial_t^2 v_\alpha.$$

With the help of (57), we have

$$I_{1432,\text{top}} \stackrel{L}{=} \int_0^t \int_{\Gamma_1} \frac{J}{\varrho_0} \sqrt{g}(g^{ij}g^{kl} - 2g^{jl}g^{ik})\partial_k \partial_l \partial_t^2 q \Pi_\mu^\alpha \partial_{ij}^2 \eta^\mu \partial_t^2 v_\alpha.$$

Writing

$$\begin{aligned} &(g^{ij}g^{kl} - 2g^{jl}g^{ik})\partial_k \partial_l \partial_t^2 q \partial_{ij}^2 \eta^\mu \\ &= (g^{ij}g^{kl} - g^{jl}g^{ik})\partial_k \partial_l \partial_t^2 q \partial_{ij}^2 \eta^\mu - g^{jl}g^{ik}\partial_k \partial_l \partial_t^2 q \partial_{ij}^2 \eta^\mu, \end{aligned}$$

we observe that the first term cancels by (65). Writing now $\Pi_\mu^\alpha = \hat{n}^\alpha \hat{n}_\mu$ and invoking (63), we see that the resulting integral is estimated as the integral I_{1132} (see what follows (65)).

3.1.5. Finalizing the three time derivatives estimate. Combining the energy identity (36) with the estimates for \mathcal{J}_i , $i = 1, \dots, 5$ from Sections 3.1.2 and 3.1.3, and with the boundary estimates of Section 3.1.4 produces (35). In doing so, we use assumption (12) to bound the integral $\int_\Omega (R(0)/R) \bar{q}'(R) (\partial_t^3 R)^2$ from below.

3.2. Two time derivatives. In this section we derive the estimate

$$\|\bar{\partial} \partial_t^2 v\|_0^2 + \|\bar{\partial} \partial_t^2 R\|_0^2 + \|\Pi \bar{\partial}^2 \partial_t v\|_{0,\Gamma_1}^2 \leq \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad (70)$$

where \mathcal{P} inside the integral now also depends on $\|\eta\|_{H^{3.5+\delta}}$. The energy equality for two time derivatives of (v, R) reads

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_\Omega R(0) \partial_t^2 \partial^i v^\beta \partial_t^2 \partial_i v_\beta + \frac{1}{2} \frac{d}{dt} \int_\Omega \frac{R(0)}{R} \bar{q}'(R) \partial_t^2 \partial^i R \partial_t^2 \partial_i R \\ &+ \int_{\Gamma_1} \partial_t^2 \partial^i (J a^{\alpha\beta} q) \partial_t^2 \partial_i v_\beta N_\alpha \end{aligned}$$

$$\begin{aligned}
&= - \int_{\Omega} \frac{R(0)}{R} \left(\partial_t^2 \partial^i (Ra^{\alpha\beta} \partial_\alpha v_\beta) - Ra^{\alpha\beta} \partial_t^2 \partial^i \partial_\alpha v_\beta \right) \partial_t^2 \partial^i \left(\frac{q}{R} \right) \\
&\quad + \int_{\Omega} R(0) \left(\partial_t^2 \partial^i \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^2 \partial^i \left(\frac{q}{R} \right) \right) \partial_t^2 \partial_i \partial_\alpha v_\beta \\
&\quad - 2 \int_{\Omega} R(0) \frac{\bar{q}''(R)}{R} \partial_t^3 \partial^i R \partial_t \partial_i R \partial_t R - \int_{\Omega} R(0) \frac{\bar{q}''(R)}{R} \partial_t^3 \partial^i R \partial_i R \partial_t^2 R \\
&\quad - \int_{\Omega} R(0) \frac{\bar{q}'''(R)}{R} \partial_t^3 \partial^i R \partial_i R (\partial_t R)^2 \\
&\quad + \frac{1}{2} \int_{\Omega} R(0) \partial_t \left(\frac{\bar{q}'(R)}{R} \right) \partial_t^2 \partial^i R \partial_t^2 \partial_i R - \int \partial_i R(0) \partial_t^3 v^\beta \partial_t^2 \partial_i v_\beta. \tag{71}
\end{aligned}$$

In order to derive (71), we multiply (6a) (with α replaced by β) by J , then differentiate in t twice, differentiate in x_i once, and contract with $\partial_i \partial_t^2 v_\beta$ obtaining

$$\int \partial_t^2 \partial^i (JR \partial_t v^\beta) \partial_t^2 \partial_i v_\beta + \int_{\Omega} \partial_t^2 \partial^i (Ja^{\alpha\beta} \partial_\alpha q) \partial_t^2 \partial_i v_\beta = 0$$

from where, using (11),

$$\int R(0) \partial_t^2 \partial^i (\partial_t v^\beta) \partial_t^2 \partial_i v_\beta + \int_{\Omega} \partial_t^2 \partial^i (Ja^{\alpha\beta} \partial_\alpha q) \partial_t^2 \partial_i v_\beta = - \int \partial_i R(0) \partial_t^3 v^\beta \partial_t^2 \partial_i v_\beta.$$

Integrating by parts in x_α , we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} R(0) \partial_t^2 \partial^i v^\beta \partial_t^2 \partial_i v_\beta + \int_{\Gamma_1} \partial_t^2 \partial^i (Ja^{\alpha\beta} q) \partial_t^2 \partial_i v_\beta N_\alpha \\
&= \int_{\Omega} \partial_t^2 \partial^i (Ja^{\alpha\beta} q) \partial_t^2 \partial_i \partial_\alpha v_\beta - \int \partial_i R(0) \partial_t^3 \partial_t v^\beta \partial_t^2 \partial_i v_\beta
\end{aligned}$$

due to the boundary integral vanishing on Γ_0 . For the term on the right side, we have

$$\begin{aligned}
\int_{\Omega} \partial_t^2 \partial^i (Ja^{\alpha\beta} q) \partial_t^2 \partial_\alpha \partial_i v_\beta &= \int_{\Omega} R(0) \partial_t^2 \partial^i \left(a^{\alpha\beta} \frac{q}{R} \right) \partial_t^2 \partial_i \partial_\alpha v_\beta \\
&= \int_{\Omega} R(0) a^{\alpha\beta} \partial_t^2 \partial^i \left(\frac{q}{R} \right) \partial_t^2 \partial_i \partial_\alpha v_\beta \\
&\quad + \int_{\Omega} R(0) \left(\partial_t^2 \partial^i \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^2 \partial^i \left(\frac{q}{R} \right) \right) \partial_t^2 \partial_i \partial_\alpha v_\beta \\
&= \int_{\Omega} \frac{R(0)}{R} \partial_t^2 \partial^i (Ra^{\alpha\beta} \partial_\alpha v_\beta) \partial_t^2 \partial_i \left(\frac{q}{R} \right) \\
&\quad - \int_{\Omega} \frac{R(0)}{R} \left(\partial_t^2 \partial^i (Ra^{\alpha\beta} \partial_\alpha v_\beta) - Ra^{\alpha\beta} \partial_t^2 \partial^i \partial_\alpha v_\beta \right) \partial_t^2 \partial_i \left(\frac{q}{R} \right) \\
&\quad + \int_{\Omega} R(0) \left(\partial_t^2 \partial^i \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^2 \partial^i \left(\frac{q}{R} \right) \right) \partial_t^2 \partial_i \partial_\alpha v_\beta,
\end{aligned}$$

from where, using (6b),

$$\begin{aligned}
&\int_{\Omega} \partial_t^2 \partial^i (Ja^{\alpha\beta} q) \partial_t^2 \partial_\alpha \partial_i v_\beta \\
&= - \int_{\Omega} \frac{R(0)}{R} \partial_t^3 \partial^i R \partial_t^2 \partial_i \left(\frac{q}{R} \right)
\end{aligned}$$

$$\begin{aligned}
& - \int_{\Omega} \frac{R(0)}{R} \left(\partial_t^2 \partial^i (Ra^{\alpha\beta} \partial_{\alpha} v_{\beta}) - Ra^{\alpha\beta} \partial_t^2 \partial^i \partial_{\alpha} v_{\beta} \right) \partial_t^2 \partial_i \left(\frac{q}{R} \right) \\
& + \int_{\Omega} R(0) \left(\partial_t^2 \partial^i \left(a^{\alpha\beta} \frac{q}{R} \right) - a^{\alpha\beta} \partial_t^2 \partial^i \left(\frac{q}{R} \right) \right) \partial_t^2 \partial_i \partial_{\alpha} v_{\beta} \\
& = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

The terms \mathcal{I}_2 and \mathcal{I}_3 give the first and second terms on the right side of (71) respectively. In order to treat

$$\mathcal{I}_1 = - \int_{\Omega} \frac{R(0)}{R} \partial_t^3 \partial^i R \partial_t^2 \partial_i \bar{q},$$

we write

$$\partial_t^2 \partial_i (\bar{q}(R)) = \bar{q}'(R) \partial_t^2 \partial_i R + 2\bar{q}''(R) \partial_t \partial_i R \partial_t R + \bar{q}''(R) \partial_i R \partial_t^2 R + \bar{q}'''(R) (\partial_t R)^2 \partial_i R$$

and thus

$$\begin{aligned}
\mathcal{I}_1 & = - \int_{\Omega} R(0) \frac{\bar{q}'(R)}{R} \partial_t^3 \partial^i R \partial_t^2 \partial_i R - 2 \int_{\Omega} R(0) \frac{\bar{q}''(R)}{R} \partial_t^3 \partial^i R \partial_t \partial_i R \partial_t R \\
& - \int_{\Omega} R(0) \frac{\bar{q}''(R)}{R} \partial_t^3 \partial^i R \partial_i R \partial_t^2 R \\
& - \int_{\Omega} R(0) \frac{\bar{q}'''(R)}{R} \partial_t^3 \partial^i R \partial_i R (\partial_t R)^2 \\
& = \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13} + \mathcal{I}_{14}.
\end{aligned}$$

The terms \mathcal{I}_{12} , \mathcal{I}_{13} , and \mathcal{I}_{14} give the third, fourth, and fifth terms on the right side of (71) respectively. For \mathcal{I}_{11} , we write

$$\mathcal{I}_{11} = - \frac{1}{2} \frac{d}{dt} \int_{\Omega} R(0) \frac{\bar{q}'(R)}{R} \partial_t^2 \partial^i R \partial_t^2 \partial_i R + \frac{1}{2} \int_{\Omega} R(0) \partial_t \left(\frac{\bar{q}'(R)}{R} \right) \partial_t^2 \partial^i R \partial_t^2 \partial_i R. \quad (72)$$

The first term on the right side leads to the second term on the left side of (71), while the second term on the right side of (72) gives the sixth term in (71).

3.2.1. Treatment of the terms involving two time derivatives. The estimates for the right side of (71) is the same as the estimates of the corresponding terms in (36) and we thus do not provide full details. However, we still show how to treat the most involved term

$$S = \int_0^t \int_{\Omega} \partial_t^2 \partial^i A^{\mu\alpha} \partial_t^2 \partial_i \partial_{\mu} v_{\alpha} q.$$

As in (43), we have

$$\begin{aligned}
S & = \int_0^t \int_{\Omega} q \epsilon^{\alpha\lambda\tau} \partial_2 \partial_t \partial^i v_{\lambda} \partial_3 \eta_{\tau} \partial_1 \partial_1 \partial_t^2 \partial_i v_{\alpha} + \int_0^t \int_{\Omega} q \epsilon^{\alpha\lambda\tau} \partial_2 \eta_{\lambda} \partial_3 \partial_t \partial^i v_{\tau} \partial_1 \partial_t^2 \partial_i v_{\alpha} \\
& - \int_0^t \int_{\Omega} q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t \partial^i v_{\lambda} \partial_3 \eta_{\tau} \partial_2 \partial_t^2 \partial_i v_{\alpha} - \int_0^t \int_{\Omega} q \epsilon^{\alpha\lambda\tau} \partial_1 \eta_{\lambda} \partial_3 \partial_t \partial^i v_{\tau} \partial_2 \partial_t^2 \partial_i v_{\alpha} \\
& + \int_0^t \int_{\Omega} q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t \partial^i v_{\lambda} \partial_2 \eta_{\tau} \partial_3 \partial_t^2 \partial_i v_{\alpha} + \int_0^t \int_{\Omega} q \epsilon^{\alpha\lambda\tau} \partial_1 \eta_{\lambda} \partial_2 \partial_t v_{\tau} \partial_3 \partial_t^2 \partial_i v_{\alpha} + L_3 \\
& = S_1 + \cdots + S_6 + L_3,
\end{aligned}$$

where L_3 equals

$$\begin{aligned} \int_0^t \int_{\Omega} q \partial \bar{\partial} v \partial v \partial \partial_t^2 \bar{\partial} v &= \int_{\Omega} q \partial \bar{\partial} v \partial v \partial \bar{\partial} \partial_t v|_0^t - \int_0^t \int_{\Omega} \partial_t q \partial \bar{\partial} v \partial v \partial \bar{\partial} \partial_t v \\ &- \int_0^t \int_{\Omega} q \partial \bar{\partial} \partial_t v \partial v \partial \bar{\partial} \partial_t v \\ &\leq \|q\|_{L^\infty} \|\nabla v\|_{L^\infty} \|\nabla \bar{\partial} v\|_{L^2} \|\nabla \bar{\partial} v_t\|_{L^2} + \mathcal{P}_0 + \mathcal{P}. \end{aligned}$$

We group the leading terms as before; the analog for (44) is

$$\begin{aligned} S_1 + S_3 &= \int_0^t \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_2 \partial^i \partial_t v_\lambda \partial_3 \eta_\tau \partial_1 \partial_t^2 \partial_i v_\alpha + \int_0^t \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_1 \partial_t^2 \partial^i v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t \partial_i v_\alpha \\ &\quad - \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_1 \partial_t \partial^i v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t \partial_i v_\alpha + L_4 \\ &= \int_0^t \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_2 \partial_t \partial^i v_\lambda \partial_3 \eta_\tau \partial_1 \partial_t^2 \partial^i v_\alpha + \int_0^t \int_{\Omega} q \epsilon^{\lambda \alpha \tau} \partial_1 \partial_t^2 \partial^i v_\alpha \partial_3 \eta_\tau \partial_2 \partial_t \partial_i v_\lambda \\ &\quad - \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_1 \partial_t \partial^i v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t \partial_i v_\alpha + L_4 \\ &= \int_0^t \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_2 \partial_t \partial^i v_\lambda \partial_3 \eta_\tau \partial_1 \partial_t^2 \partial_i v_\alpha - \int_0^t \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_1 \partial_t^2 \partial^i v_\alpha \partial_3 \eta_\tau \partial_2 \partial_t \partial_i v_\lambda \\ &\quad - \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_1 \partial_t \partial^i v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t \partial_i v_\alpha + L_4 \\ &= 0 - \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_1 \partial_t \partial^i v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t \partial_i v_\alpha + \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_1 \partial_t \partial^i v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t \partial_i v_\alpha|_{t=0} + L_4. \end{aligned}$$

The symbol L_4 denotes the lower order terms, which are bounded below. The first term on the far right side is treated as

$$\begin{aligned} S_{13,\partial} &= - \int_{\Omega} q \epsilon^{\alpha \lambda \tau} \partial_1 \partial_t \partial^i v_\lambda \partial_3 \eta_\tau \partial_2 \partial_t \partial^i v_\alpha \\ &= - \int_{\Omega} q \epsilon^{\alpha \lambda 3} \partial_1 \partial_t \partial^i v_\lambda \partial_3 \eta_3 \partial_2 \partial_t v_\alpha - \int_{\Omega} q \epsilon^{\alpha \lambda i} \partial_1 \partial_t \partial^j v_\lambda \partial_3 \eta_i \partial_2 \partial_t \partial_j v_\alpha. \end{aligned}$$

The last integral is bounded by

$$\tilde{\epsilon} \|\partial_t \bar{\partial} v\|_1^2 \|q\|_{L^\infty(\Omega)} \leq C \tilde{\epsilon} \|\partial_t \bar{\partial} v\|_1^2$$

using $\eta(0) = \text{id}$ and thus $\partial_3 \eta_i = O(\tilde{\epsilon})$ for small time. Since $\partial_3 \eta_3 = 1 + O(\tilde{\epsilon})$, we have

$$\begin{aligned} - \int_{\Omega} q \epsilon^{\alpha \lambda 3} \partial_1 \partial_t^2 v_\lambda \partial_3 \eta_3 \partial_2 \partial_t^2 v_\alpha &= - \int_{\Omega} q \epsilon^{\alpha \lambda 3} \partial_1 \partial_t^2 v_\lambda \partial_2 \partial_t^2 v_\alpha \\ &\quad - \int_{\Omega} q \epsilon^{\alpha \lambda 3} \partial_1 \partial_t^2 v_\lambda O(\tilde{\epsilon}) \partial_2 \partial_t^2 v_\alpha. \end{aligned}$$

The last integral is bounded by $\tilde{\epsilon} \|\partial_t \bar{\partial} v\|_1^2 \|q\|_{L^\infty(\Omega)}$. For the remaining integral, we write

$$\begin{aligned} - \int_{\Omega} q \epsilon^{\alpha \lambda 3} \partial_1 \partial_t \partial^i v_\lambda \partial_2 \partial_t \partial_i v_\alpha &= - \int_{\Omega} (q \epsilon^{123} \partial_1 \partial_t \partial^i v_2 \partial_2 \partial_t \partial_i v_1 \\ &\quad + q \epsilon^{213} \partial_1 \partial_t \partial^i v_1 \partial_2 \partial_t \partial_i v_2) \\ &= - \int_{\Omega} (q \partial_1 \partial_t \partial^i v_2 \partial_2 \partial_t \partial_i v_1 - q \partial_1 \partial_t \partial^i v_1 \partial_2 \partial_t \partial_i v_2). \end{aligned}$$

We integrate by parts in both terms obtaining

$$\begin{aligned} - \int_{\Omega} q \epsilon^{\alpha\lambda 3} \partial_1 \partial_t \partial^i v_{\lambda} \partial_2 \partial_t \partial_i v_{\alpha} &= \int_{\Omega} (q \partial_2 \partial_1 \partial_t \partial^i v_2 \partial_t \partial_i v_1 - q \partial_t \partial^i v_1 \partial_1 \partial_2 \partial_t \partial_i v_2) \\ &\quad + \int_{\Omega} (\partial_1 \partial_t \partial^i v_2 \partial_t \partial_i v_1 \partial_2 q - \partial_t \partial^i v_1 \partial_2 \partial_t \partial_i v_1 \partial_1 q) \\ &= 0 + \int_{\Omega} (\partial_1 \partial_t \partial^i v_2 \partial_t \partial_i v_1 \partial_2 q - \partial_t \partial^i v_1 \partial_2 \partial_t \partial_i v_1 \partial_1 q), \end{aligned}$$

where the last integral obeys

$$\begin{aligned} \int_{\Omega} (\partial_1 \partial_t \partial^i v_2 \partial_t \partial_i v_1 \partial_2 q - \partial_t \partial^i v_1 \partial_2 \partial_t \partial_i v_1 \partial_1 q) &\leq C \|\partial_t \partial^i v\|_1 \|\partial_t \partial_i v\|_0 \|\partial q\|_{L^{\infty}(\Omega)} \\ &\leq \tilde{\epsilon} \|\partial_t \bar{\partial} v\|_1^2 + C \|\partial_t \bar{\partial} v\|_0^2 \|\partial q\|_{L^{\infty}(\Omega)}^2. \end{aligned}$$

The symbol L_4 above consists of the sum of the terms

$$\int_{\Omega} q \epsilon^{\alpha\lambda\tau} \partial_1 \partial_t \partial^i v_{\lambda} \partial_3 \eta_{\tau} \partial_2 \partial_t \partial_i v_{\alpha} |_{t=0} \leq \mathcal{P}_0$$

and

$$\int_0^t \int_{\Omega} \epsilon^{\alpha\lambda\tau} \partial_t (q \partial_1 \partial_t \partial^i v_{\lambda} \partial_3 \eta_{\tau}) \partial_2 \partial_t \partial_i v_{\alpha} \leq \int_0^t \mathcal{P}.$$

We thus conclude

$$S_1 + S_3 \leq \tilde{\epsilon} \|\partial_t \bar{\partial} v\|_1^2 \|q\|_{L^{\infty}(\Omega)} + C \|\partial_t \bar{\partial} v\|_0^2 \|\partial q\|_{L^{\infty}(\Omega)}^2 + \mathcal{P}_0 + \int_0^t \mathcal{P}.$$

As above, when treating $S_4 + S_6$ and $S_2 + S_5$ we obtain an extra boundary term of the type

$$\int_{\Gamma_1} q \partial_t \partial^i v_2 \partial_2 \partial_t \partial_i v_3,$$

which is bounded analogously to (46). In summary, we obtain

$$\begin{aligned} S &\leq \tilde{\epsilon} \|\Pi \bar{\partial}^2 \partial_t v\|_{0,\Gamma_1}^2 + \tilde{\epsilon} \|\partial_t \bar{\partial} v\|_1^2 \|q\|_{L^{\infty}(\Omega)} + C \|\partial_t \bar{\partial} v\|_0^2 \|\partial q\|_{L^{\infty}(\Omega)}^2 + \tilde{\epsilon} \|\partial_t \bar{\partial} v\|_{1,\Gamma_1}^2 \\ &\quad + C \|q\|_{L^{\infty}} \|\nabla v\|_{L^{\infty}} \|\nabla \partial_t v\|_0 \|\nabla \partial_t^2 v\|_0 + \tilde{\epsilon} \|\partial_t \bar{\partial} v\|_1^2 \|q\|_{L^{\infty}(\Gamma_1)}^2 \\ &\quad + C \|\partial_t \bar{\partial} v\|_0^2 \|q\|_{L^{\infty}(\Gamma_1)}^2 + \mathcal{P}_0 + \int_0^t \mathcal{P}. \end{aligned}$$

3.3. Estimates at $t = 0$. In the above estimates we had several expressions involving time derivatives of v and R evaluated at zero. Here, we show that these quantities may all be estimated in terms of \mathcal{P}_0 . More precisely, we show that

$$\begin{aligned} \|\partial_t v(0)\|_2 + \|\partial_t R(0)\|_2 + \|\partial_t^2 v(0)\|_1 \\ + \|\partial_t^2 R(0)\|_1 + \|\partial_t^3 v(0)\|_0 + \|\partial_t^3 R(0)\|_0 \leq \mathcal{P}_0, \end{aligned} \tag{73}$$

and

$$\|\partial_t v(0)\|_{2,\Gamma_1} + \|\partial_t^2 v(0)\|_{1,\Gamma_1} \leq \mathcal{P}_0. \tag{74}$$

In light of (11), the equation (6a) may be written as

$$\varrho_0 \partial_t v^{\alpha} + J a^{\mu\alpha} q'(R) \partial_{\mu} R = 0. \tag{75}$$

From (75) and (6b) we get $\|\partial_t v(0)\|_2 \leq \mathcal{P}_0$ and $\|\partial_t R(0)\|_2 \leq \mathcal{P}_0$. Differentiating (75) and (6b) in time and evaluating at zero gives $\|\partial_t^2 v(0)\|_1 \leq \mathcal{P}_0$ and $\|\partial_t^2 R(0)\|_1 \leq \mathcal{P}_0$.

\mathcal{P}_0 . Taking another time derivative of (75) and (6b) and evaluating at zero produces (73).

To obtain (74), we use (6a) to estimate terms in $v^i(0)$ and (6e) to estimate terms in $v^3(0)$. Evaluating (6a) at $t = 0$ with $\alpha = i$ and recalling (9) gives

$$\partial_t v^i(0) = -\frac{1}{R(0)} \delta^{ji} \partial_j R(0), \quad (76)$$

which implies $\|\partial_t v^i(0)\|_{2,\Gamma_1} \leq \mathcal{P}_0$ since $R(0) \in H^3(\Gamma_1)$. Note that the conclusion would not be true if we had a $\partial_3 R$ term, that is why $\alpha = 3$ has to be treated differently.

Remark 3. The estimate (76) shows why we require higher regularity for the initial data on the boundary. We want $\partial_t v \in H^2(\Gamma_1)$ in order to apply the div-curl estimates, as explained in Section 1.1. But this would not hold even at time zero without the regularity assumption on the boundary.

Differentiating (6e) with $\alpha = 3$ in time twice gives

$$\begin{aligned} \partial_t^2(\Delta_g \eta^3) &= -\frac{1}{\sigma} \frac{a^{\mu 3} N_\mu}{|a^T N|} q'(R) \partial_t^2 R - \frac{1}{\sigma} \partial_t \left(\frac{a^{\mu 3} N_\mu}{|a^T N|} \right) q'(R) \partial_t R \\ &\quad - \frac{1}{\sigma} \frac{a^{\mu 3} N_\mu}{|a^T N|} q''(R) (\partial_t R)^2 - \frac{1}{\sigma} \partial_t^2 \left(\frac{a^{\mu 3} N_\mu}{|a^T N|} \right) q(R). \end{aligned} \quad (77)$$

But from (8),

$$\begin{aligned} \partial_t^2(\Delta_g \eta^3)|_{t=0} &= \delta^{ij} \partial_{ij}^2 \partial_t v^3(0) + \partial_t g^{ij}(0) \partial_{ij}^2 v^3(0) - \delta^{ij} \partial^k v^3(0) \partial_{ij}^2 v_k(0) \\ &= \delta^{ij} \partial_{ij}^2 \partial_t v^3(0) + F_0, \end{aligned} \quad (78)$$

where in light of our assumptions $\|F_0\|_{1,\Gamma_1} \leq \mathcal{P}_0$. From (6b) we obtain $\|\partial_t R(0)\|_{1.5,\Gamma_1} \leq \mathcal{P}_0$ and $\|\partial_t^2 R(0)\|_{0.5,\Gamma_1} \leq \mathcal{P}_0$.

Using (6c) we find

$$\partial_t \left(\frac{a^{\mu 3} N_\mu}{|a^T N|} \right) = -\frac{1}{|a^T N|^3} \left(|a^T N|^2 a^{3\gamma} \partial_\mu v_\gamma a^{\mu 3} + a^{\sigma 3} N_\sigma \sum_{\beta=1}^3 a^{3\beta} a^{3\gamma} \partial_\mu v_\gamma a^{\mu\beta} \right).$$

We now differentiate this expression in time again, use (6c) once more, and evaluate it at zero. Combined with the previous estimates and (77) and (78), we conclude that, on Γ_1 ,

$$\delta^{ij} \partial_{ij}^2 v^3(0) = F_1,$$

where F_1 satisfies the estimate $\|F_1\|_{0.5,\Gamma_1} \leq \mathcal{P}_0$. By the elliptic theory, we then obtain $\|F_1\|_{2.5,\Gamma_1} \leq \mathcal{P}_0$, which combined with the previous estimate for $\partial_t v^i(0)$ gives $\|\partial_t v(0)\|_{2,\Gamma_1} \leq \mathcal{P}_0$.

The estimate for $\partial_t^2 v$ is obtained in a similar way, upon differentiating one more time in time and proceeding as above. We omit the details, but explain where the assumption on $(\Delta \operatorname{div} v_0)|\Gamma_1$ is used. Proceeding as just explained, we find (writing \sim to mean “up to lower order”) $\delta^{ij} \partial_{ij}^2 \partial_t^2 v^3(0) \sim \partial_t^3 R(0)$. But from (6a) and (6b) we obtain $\partial_t^3 R(0) \sim \Delta \operatorname{div} v(0)$, which requires $(\Delta \operatorname{div} v(0))|\Gamma_1$ in $H^{-1}(\Gamma_1)$ in order to produce $\partial_t^2 v^3(0)$ in $H^1(\Gamma_1)$ from the elliptic estimates.

4. Estimates for the curl. In this section, we obtain estimates for the curl of v and its time derivatives. First, we write (30) as

$$\varepsilon^{\alpha\beta\gamma}\partial_\beta v_\gamma = \varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu(\delta_{\gamma\mu} - \partial_\gamma\eta_\mu) + \omega_0^\alpha, \quad (79)$$

from which we obtain

$$\varepsilon^{\alpha\beta\gamma}\partial_\beta\partial_t v_\gamma = \varepsilon^{\alpha\beta\gamma}\partial_\beta\partial_t v^\mu(\delta_{\gamma\mu} - \partial_\gamma\eta_\mu), \quad (80)$$

where we used $\varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu\partial_\gamma v_\mu = 0$ and

$$\varepsilon^{\alpha\beta\gamma}\partial_\beta\partial_t^2 v_\gamma = \varepsilon^{\alpha\beta\gamma}\partial_\beta\partial_t^2 v^\mu(\delta_{\gamma\mu} - \partial_\gamma\eta_\mu) - \epsilon^{\alpha\beta\gamma}\partial_\beta\partial_t v^\mu\partial_\gamma v_\mu. \quad (81)$$

Since

$$\delta_{\gamma\mu} - \partial_\gamma\eta_\mu = - \int_0^t \partial_\gamma v_\mu,$$

the term $\delta_{\gamma\mu} - \partial_\gamma\eta_\mu$ can be made arbitrarily small for small time. Hence, the relevant norm of the terms proportional to $\delta_{\gamma\mu} - \partial_\gamma\eta_\mu$ on the right-hand side of (79), (80), and (81) can be absorbed into the left-hand side. We then have to estimate the remaining terms on the right-hand side.

From (79) we immediately get

$$\|\operatorname{curl} v\|_2^2 \lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}, \quad (82)$$

where we used Jensen's inequality.

Note that (80) gives

$$\|\operatorname{curl} \partial_t v\|_1^2 \lesssim \tilde{\epsilon} \|R\|_3^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad (83)$$

while (81) implies

$$\|\operatorname{curl} \partial_t^2 v\|_0^2 \lesssim \tilde{\epsilon} (\|v\|_3^2 + \|R\|_3^2) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (84)$$

5. Conclusion. The main goal of this section is to provide the necessary ingredients for the div-curl estimate and to show the concluding Gronwall argument.

5.1. Comparison between $\Pi\partial_t^a v$ and $\partial_t^a v^3$. In order to use div-curl estimates, we first show that our estimates for $\Pi\partial_t^a v$ are equivalent, modulo lower order terms, to estimates for $\partial_t^a v^3$. Recalling (14), for any vector field X we have

$$(\Pi\bar{\partial} X)^3 = \Pi_\lambda^3 \bar{\partial} X^\lambda = \bar{\partial} X^3 - g^{kl} \partial_k \eta^3 \partial_l \eta_\lambda \bar{\partial} X^\lambda. \quad (85)$$

Using $X = \partial_t^2 v$ and estimating (85) in the $H^{-0.5}(\Gamma_1)$ norm yields

$$\|\bar{\partial} \partial_t^2 v^3\|_{-0.5, \Gamma_1}^2 \lesssim \|\Pi\bar{\partial} \partial_t^2 v\|_{0, \Gamma_1}^2 + \|g^{kl} \partial_k \eta^3 \partial_l \eta_\lambda\|_{1.5, \Gamma_1}^2 \|\partial_t^2 v^\lambda\|_{0.5, \Gamma_1}^2.$$

We add $\|\partial_t^2 v^3\|_{-0.5, \Gamma_1}^2$ to both sides, use the fact that $\|\partial_t^2 v^3\|_{-0.5, \Gamma_1}^2 + \|\bar{\partial} \partial_t^2 v^3\|_{-0.5, \Gamma_1}^2$ is equivalent to $\|\partial_t^2 v^3\|_{0.5, \Gamma_1}^2$, invoke $\partial_k \eta^3 = \int_0^3 \partial_k v^3$, which holds since $\eta^3(0) = 1$, to conclude

$$\|\partial_t^2 v^3\|_{0.5, \Gamma_1}^2 \lesssim \tilde{\epsilon} \|\partial_t^2 v\|_1^2 + \|\Pi\bar{\partial} \partial_t^2 v\|_{0, \Gamma_1}^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad (86)$$

where the term $\|\partial_t^2 v^3\|_{-0.5, \Gamma_1}^2$ that appeared on the right-hand side was estimated using interpolation inequality, Young's inequality, and the fundamental theorem of calculus.

Similarly, using (85) with $X = \bar{\partial} \partial_t v$, estimating in the $H^{-0.5}(\Gamma_1)$ norm and adding $\|\partial_t v^3\|_{-0.5}^2 + \|\bar{\partial} \partial_t v^3\|_{-0.5, \Gamma_1}^2$ to both sides gives

$$\|\partial_t v^3\|_{1.5, \Gamma_1}^2 \lesssim \tilde{\epsilon} \|\partial_t v\|_2^2 + \|\Pi \bar{\partial}^2 \partial_t v\|_{0, \Gamma_1}^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (87)$$

We also need an estimate for $\|v^3\|_{2.5, \Gamma_1}$. This follows directly from the boundary condition, as we now show. Differentiating (6e) in time and setting $\alpha = 3$ yields

$$\begin{aligned} \sqrt{g} g^{ij} \partial_{ij}^2 v^3 - \sqrt{g} g^{ij} \Gamma_{ij}^k \partial_k v^3 &= -\partial_t (\sqrt{g} g^{ij}) \partial_{ij}^2 \eta^3 - \partial_t (\sqrt{g} g^{ij} \Gamma_{ij}^k) \partial_k \eta^3 \\ &\quad - \frac{1}{\sigma} \partial_t a^{\mu 3} N_\mu q - \frac{1}{\sigma} a^{\mu \sigma} N_\mu \partial_t q \text{ on } \Gamma_1 \end{aligned}$$

where we also used (24).

In light of Proposition 1, we have

$$\|g_{ij}\|_{2.5, \Gamma_1} \leq C$$

and

$$\|\Gamma_{ij}^k\|_{1.5, \Gamma_1} \leq C.$$

Thus, by the elliptic estimates for operators with coefficients bounded in Sobolev norms (see [29, 44]) we have

$$\begin{aligned} \|v^3\|_{2.5, \Gamma_1} &\leq C \|\partial_t (\sqrt{g} g^{ij}) \partial_{ij}^2 \eta^3\|_{0.5, \Gamma_1} + C \|\partial_t (\sqrt{g} g^{ij} \Gamma_{ij}^k) \partial_k \eta^3\|_{0.5, \Gamma_1} \\ &\quad + C \|\partial_t a^{\mu 3} N_\mu q\|_{0.5, \Gamma_1} + C \|a^{\mu \sigma} N_\mu \partial_t q\|_{0.5, \Gamma_1}, \end{aligned}$$

where C depends on the bounds for $\|g_{ij}\|_{2.5, \Gamma_1}$ and $\|\Gamma_{ij}^k\|_{1.5, \Gamma_1}$ stated above. The right-hand side is now estimated in a routine fashion, and we conclude

$$\|v^3\|_{2.5, \Gamma_1}^2 \lesssim \tilde{\epsilon} (\|v\|_3^2 + \|R\|_3^2) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (88)$$

5.2. Gronwall-type argument via barriers. We shall show that our estimates imply

$$\mathcal{N}(t) \leq C_0 P(\mathcal{N}(0)) + P(\mathcal{N}(t)) \int_0^t P(\mathcal{N}(s)) ds \quad (89)$$

where P is now a fixed polynomial and C_0 is a fixed positive constant. The inequality (89) implies, via a routine continuity argument that we now sketch for the reader's convenience, the boundedness of $\mathcal{N}(t)$ on a positive interval of time (cf. [67, Section 8] where a similar inequality was treated). Assume, without loss of generality, that P is strictly positive and non-decreasing, and denote $M = \mathcal{N}(0)$. Let

$$T_0 = \inf \left\{ t \geq 0 : \mathcal{N}(t) \geq 2C_0 P(M) = M_1 \right\} \in (0, \infty].$$

If $T_0 = \infty$, then $\mathcal{N}(t) \leq M_1$ for all $t \geq 0$. Otherwise, $T_0 \in (0, \infty)$, and thus

$$2C_0 P(M) = \mathcal{N}(T_0) \leq C_0 P(M) + P(M_1) \int_0^{T_0} P(M_1) ds = C_0 P(M) + T_0 P(M_1)^2,$$

from where $T_0 \geq C_0 P(M)/P(M_1)^2$. We thus conclude that

$$\mathcal{N}(t) \leq M_1, \quad t \in \left[0, \frac{C_0 P(M)}{P(M_1)^2} \right],$$

and the local boundedness is established.

5.3. Closing the estimates. It remains to establish (89). Recall the standard div-curl estimate

$$\|X\|_s \lesssim \|\operatorname{div} X\|_{s-1} + \|\operatorname{curl} X\|_{s-1} + \|X \cdot N\|_{\partial, s-0.5} + \|X\|_0. \quad (90)$$

From Sections 3, 4, and 5.1, we have estimates for the curl and normal component of v and their time derivatives, as well as estimates for $\|\partial_t^3 v\|_0$ and $\|\partial_t^3 R\|_0$. In order to apply (90), we need to estimate the divergence of v and its time derivatives.

Taking two time derivatives of the density equation (6b) leads to

$$\partial^\alpha \partial_t^2 v_\alpha = (\delta^{\mu\alpha} - a^{\mu\alpha}) \partial_\mu \partial_t^2 v_\alpha - \frac{1}{R} \left(\partial_t^2 (Ra^{\mu\alpha} \partial_\mu v_\alpha) - Ra^{\mu\alpha} \partial_\mu \partial_t^2 v_\alpha \right) - \frac{1}{R} \partial_t^3 R.$$

Taking the L^2 norm of both sides,

$$\begin{aligned} & \|\partial^\alpha \partial_t^2 v_\alpha\|_0 \\ & \lesssim \|(\delta^{\mu\alpha} - a^{\mu\alpha}) \partial_\mu \partial_t^2 v_\alpha\|_0 + \left\| \partial_t^2 (Ra^{\mu\alpha} \partial_\mu v_\alpha) - Ra^{\mu\alpha} \partial_\mu \partial_t^2 v_\alpha \right\|_0 + \|\partial_t^3 R\|_0, \end{aligned}$$

where we used Lemma 2.1(x). By expanding the derivatives in the second term and using Lemma 2.1(ix) we get

$$\|\operatorname{div} \partial_t^2 v\|_0 \leq \tilde{\epsilon} \|\partial_t^2 v\|_1 + C \|\partial_t^2 (Ra^{\mu\alpha}) \partial_\mu v_\alpha\|_0 + C \|\partial_t (Ra^{\mu\alpha}) \partial_t \partial_\mu v_\alpha\|_0 + C \|\partial_t^3 R\|_0.$$

Squaring and using (35) gives

$$\|\operatorname{div} \partial_t^2 v\|_0^2 \leq \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (91)$$

Now, in (90), taking $X = \partial_t^2 v$, $s = 1$, and squaring, recalling that $v \cdot N = 0$ on Γ_0 and $v \cdot N = v^3$ on Γ_1 , invoking (84), (91), (86), and (35), produces

$$\|\partial_t^2 v\|_1^2 \lesssim \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad (92)$$

where the lower order term $\|\partial_t^2 v\|_0$ was estimated in a standard fashion.

We now move to estimate $\partial_t^2 R$. First, write (6a) as

$$R \partial_t v^\alpha + q'(R) a^{\mu\alpha} \partial_\mu R = 0. \quad (93)$$

Taking ∂_t^2 of (93) gives

$$\partial^\alpha \partial_t^2 R \stackrel{L}{=} (\delta^{\mu\alpha} - a^{\mu\alpha}) \partial_\mu \partial_t^2 R - \frac{R}{q'(R)} \partial_t^3 v^\alpha,$$

where we recall Notation 8. Taking $\alpha = 1, 2, 3$ and invoking (35) produces

$$\|\partial_t^2 R\|_1^2 \lesssim \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad (94)$$

where we also used (12).

Next we estimate $\|\operatorname{div} \partial_t v\|_1$. From (6b) we have

$$\partial^\alpha \partial_t v_\alpha = (\delta^{\mu\alpha} - a^{\mu\alpha}) \partial_\mu \partial_t v_\alpha - \frac{1}{R} \left(\partial_t (Ra^{\mu\alpha} \partial_\mu v_\alpha) - Ra^{\mu\alpha} \partial_\mu \partial_t v_\alpha \right) - \frac{1}{R} \partial_t^2 R,$$

from where

$$\begin{aligned} \|\partial_t \operatorname{div} v\|_1 & \leq \|\delta^{\mu\alpha} - a^{\mu\alpha}\|_2 \|\partial_\mu \partial_t v_\alpha\|_1 \\ & + \left\| \frac{1}{R} \left(\partial_t (Ra^{\mu\alpha} \partial_\mu v_\alpha) - Ra^{\mu\alpha} \partial_\mu \partial_t v_\alpha \right) \right\|_1 + \left\| \frac{1}{R} \partial_t^2 R \right\|_1, \end{aligned}$$

leading to

$$\|\operatorname{div} \partial_t v\|_1^2 \leq \tilde{\epsilon} \|\partial_t v\|_2^2 + C \|\partial_t^2 R\|_1^2 + \mathcal{P}_0 + \int_0^t \mathcal{P}. \quad (95)$$

Setting $X = \partial_t v$, $s = 2$ and squaring (90), invoking (83), (95), (87), (70), and (94) produces

$$\|\partial_t v\|_2^2 \lesssim \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (96)$$

From (93) we may now estimate $\partial_t R$ in terms of $\partial_t^2 v$, so (96) gives

$$\|\partial_t R\|_2^2 \lesssim \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (97)$$

Finally, to bound $\|\operatorname{div} v\|_2$, note that

$$\partial^\alpha v_\alpha = (\delta^{\mu\alpha} - a^{\mu\alpha}) \partial_\mu v_\alpha - \frac{1}{R} \partial_t R$$

whence

$$\|\operatorname{div} v\|_2 \leq \|\delta^{\mu\alpha} - a^{\mu\alpha}\|_2 \|\partial_\mu v_\alpha\|_2 + \left\| \frac{1}{R} \partial_t R \right\|_2,$$

so that

$$\|\operatorname{div} v\|_2 \leq \tilde{\epsilon} \|v\|_3 + C \|\partial_t R\|_2 + \mathcal{P}_0 + \int_0^t \mathcal{P}. \quad (98)$$

In the same spirit as above, choosing now $X = v$, $s = 2$ and squaring (90), invoking (82), (98), (88), and (97) leads to

$$\|v\|_3^2 \lesssim \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (99)$$

Similarly to the foregoing, (93) gives an estimate for R in light of the estimate (96) for $\partial_t v$, so

$$\|R\|_3^2 \lesssim \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (100)$$

Estimates (92), (96), (99), (94), (97), (100), (35), and (70) now imply

$$\mathcal{N} \lesssim \tilde{\epsilon} \mathcal{N} + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \quad (101)$$

Note that \mathcal{P} inside the integral also depends on $\|\eta\|_{H^{3.5+\delta}}$. Using successive applications of Young's inequality, we can trade the polynomial expressions \mathcal{P} by polynomials in \mathcal{N} ; choosing $\tilde{\epsilon}$ small enough produces (89). Now, combining (29), (31), and (34) with the div-curl inequality (90) provides a Gronwall inequality for $\|\eta\|_{H^{3.5+\delta}}$. Coupling it with (101) then concludes the proof of Theorem 1.1.

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E-mail address: marcelo.disconzi@vanderbilt.edu

E-mail address: kukavica@usc.edu