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# Conditional Davis pricing

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**Abstract** We study the set of Davis (marginal utility-based) prices of a financial derivative in the case where the investor has a non-replicable random endowment. We give a new characterisation of the set of all such prices, and provide an example showing that even in the simplest of settings – such as Samuelson's geometric Brownian motion model –, the interval of Davis prices is often a non-degenerate subinterval of the set of all no-arbitrage prices. This is in stark contrast to the case with a constant or replicable endowment where non-uniqueness of Davis prices is exceptional. We provide formulas for the endpoints of these intervals and illustrate the theory with several examples.

Dedicated to the memory of Mark Davis.

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## 1 Introduction

We consider an investor trading in a frictionless but incomplete financial market with stock price dynamics modelled by a locally bounded semimartingale $S$. The investor receives a random endowment $B > 0$ at a future time $T > 0$, and we seek to price a contingent claim with payoff $\varphi$ at time $T$. In many cases of interest, the interval of arbitrage-free prices of $\varphi$ takes on an extreme form: it is open and its endpoints are given by the (essential) infimum and supremum of $\varphi$. To reduce the size of this interval – ideally to a single point –, additional input is needed. Classically, this comes via utility theory and leads to the notion of indifference (reservation) prices. Its proper home within microeconomic theory is in the context of Hicksian demand theory (see e.g. Hicks [17, Chap. 3.D] and Mas-Colell et al. [31, Sect. 3.E]). We refer the reader to Carmona [4, Chap. 2] and Föllmer and Schied [15, Chap. 8] for further general information and thorough historical overviews of various ways to price unspanned payoffs in incomplete models (in Appendix A, we discuss several related derivative pricing methods including reservation prices). For example, Cochrane and Saá Requejo [6] and the extension Björk and Slinko [3] use so-called good deal bounds based on the Hansen–Jagannathan bound for the Sharpe ratio to reduce the width of the interval of arbitrage-free prices.

Reservation prices often come with a high computational overhead which reduces their tractability and applicability (see e.g. Munk [32]). However, when $|\varphi| \ll |B|$, asymptotic analysis can be used to simplify the problem. A formal linearisation of the indifference pricing equation (A.3) in Appendix A around $q = 0$ produces the derivative pricing method introduced into mathematical finance by Mark Davis in [9]; it goes by the names *Davis pricing* and *marginal utility-based pricing*.

Let us briefly and informally describe the conditional version of Davis' construction, which is the focus of this paper. Its main ingredients are a utility function $U : (0, \infty) \to \mathbb{R}$, a random endowment $B$, and the claim with payoff $\varphi$ to be priced. Consider an agent who receives the endowment $B$ at a future time $T > 0$, but also has access to any quantity $q$ of the claim $\varphi$ for the unit price $p$, as well as to a financial market with a zero-interest bond and a risky asset with price process $S$. In an effort to invest optimally in the resulting market, the agent faces the optimisation problem

$$\sup_{q\in\mathbb{R}} \sup_{\pi\in\mathcal{A}} \mathbb{E}\bigg[U\bigg(q(\varphi - p) + B + \int_0^T \pi_t dS_t\bigg)\bigg] \tag{1.1}$$

with $\pi$ in a suitable set $\mathcal{A}$ of admissible trading strategies (defined in Sect. 2 below). A constant $p \in \mathbb{R}$ is then called a *conditional Davis price of* $\varphi$ (conditional on the presence of the random endowment $B$) if the (first) supremum in (1.1) is attained at $q = 0$. In other words, $p$ is a Davis price if the agent is indifferent between being able to buy or sell any quantity of $\varphi$ at unit price $p$, and not having access to trade $\varphi$ at all.

Conditional Davis prices as described above can also be seen as a variation of the classical case where the random endowment $B$ is replaced by a constant initial wealth $x > 0$, but where the utility function is no longer deterministic. We could consider the random endowment $B$ as a part of the preference structure of the agent, i.e., think of $x \mapsto U(x + B(\omega))$ as a stochastic utility function and view $\mathbb{E}[U(q\varphi + B)]$ as the expected utility of the position $q\varphi$.

As mentioned above, the defining equation for Davis prices can be seen as a linearised version of the defining equation for indifference prices. In Appendix A below, we show that the extremal solutions to the linearised equation provide good approximations to the solutions of the indifference-pricing equation (as well as other related derivative prices). This shows that conditional Davis prices can be seen as a limit of several standard utility-based pricing concepts. We expand more on this idea in Appendix A, but we note that this interpretation gains additional traction in the conditional case with an unspanned endowment $B$. Because the random endowment $B$ is the totality of the investor's endowed holdings, $B$ is typically orders of magnitude larger than the size of the claim's payoff $\varphi$.

Due to the typical strict concavity of utility functions, one expects in (1.1) at least some non-trivial demand for $\varphi$ ($q > 0$) at prices $p' < p$ and at least some non-trivial supply ($q < 0$) when $p' > p$; this heuristic observation leads to a generic expectation of uniqueness of Davis prices. While uniqueness of Davis prices holds for utilities defined on the entire real line,[1] it can fail for utilities defined only on $(0, \infty)$. Theorem 3.1(ii) in Hugonnier et al. [20] gives an example of an incomplete model, a constant endowment $B \equiv x > 0$ and a payoff $\varphi$ with a non-trivial interval of Davis prices. This is often treated as a rare pathological case as the construction requires sophisticated functional-theoretic machinery, and a hope remained that "most" relevant models do not exhibit such behaviour and that Davis pricing can still be used to assign a single price to any "reasonable" payoff $\varphi$. As we show in the present paper, even such a hope is unfounded: We show that in Samuelson's geometric Brownian motion model with constant coefficients, there exists a whole spectrum of explicit random endowments $B > 0$ and payoffs $\varphi$ with a non-trivial and explicitly computable interval of Davis prices (and both $B$ and $\varphi$ are bounded random variables).

Even though the concept of a Davis price has been around for several decades, previous studies do not cover the conditional case where the endowment $B > 0$ is unspanned beyond some special cases and under strong regularity conditions. For example, Hugonnier and Kramkov [19] define Davis prices in their Remark 1, but only investigate the underlying utility maximisation problem; Hugonnier et al. [20] study Davis prices when the random endowment $B \equiv x > 0$ is constant; and Kramkov and Sîrbu [24] study asymptotic expansions under a stringent decay assumption which forces Davis prices to be unique (the decay condition is from [20, Theorem 3.1(i)]).

[1] The theory of Davis pricing is simpler for utility functions, such as the exponential utility, defined on $\mathbb{R}$. For example, in any such model, Davis prices are unique because Bellini and Frittelli [2] have shown that the dual utility minimiser is a countably additive probability measure.

In Siorpaes [34], non-uniqueness of Davis prices is established for "extreme" random endowments $B$ (a class we rule out by assuming that $B \in \mathbb{L}^\infty_{++}$).

There exist natural conditions on the market model (see Kramkov and Weston [25]) such that every contingent claim with a bounded payoff $\varphi$ admits a unique Davis price. However, these conditions apply only in the case with spanned endowment and, as we shall see, no longer guarantee uniqueness in the presence of a general unspanned random endowment $B$.

The goal of the present paper is to develop the theory of Davis pricing in the conditional case with a bounded random endowment $B > 0$ described above under minimal assumptions, and to look into uniqueness issues such as the one raised in [20].

**Assuming throughout** that the utility function $U$ is defined only on the positive half-line, our main results fall into several categories:

1) For an arbitrary bounded endowment $B$, we give a new, simple and natural characterisation (Theorem 3.5) of Davis prices of the payoff $\varphi \in \mathbb{L}^\infty(\mathbb{P})$ as the set of all numbers of the form $\langle \varphi, \frac{\hat{\mathbb{Q}}}{\hat{\mathbb{Q}}[\Omega]} \rangle \in \mathbb{R}$, where $\hat{\mathbb{Q}} \in \mathrm{ba}(\mathbb{P})$ ranges through the set of finitely additive minimisers of the associated dual utility problem (as introduced in Cvitanić et al. [7]). Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $\mathbb{L}^\infty(\mathbb{P})$ and its topological dual $\mathrm{ba}(\mathbb{P}) := (\mathbb{L}^\infty(\mathbb{P}))^*$.

2) We give a whole class of non-pathological non-uniqueness examples (Example 6.3) of Davis prices in the conditional case. In fact, we show that the interval of Davis prices is non-trivial in the standard Samuelson–Black–Scholes model and with any utility function in the power (CRRA) family, as soon as both $B$ and $\varphi$ are non-constant, bounded and uniformly Lipschitz ($W^{1,\infty}$) functions of an independent Brownian motion.

3) As is well known, questions of uniqueness, existence and computability of Davis prices are intimately linked to differentiability properties of the value function $\mathfrak{U}$ given by

$$\mathfrak{U}(X) := \sup_{\pi \in \mathcal{A}} \mathbb{E}\bigg[U\bigg(X + \int_0^T \pi_t \, dS_t\bigg)\bigg]. \tag{1.2}$$

In this context, we give an example (Example 4.3) showing that this function need not be differentiable, even in "constant" $\varphi$-directions, and even when $U(\xi) := \log \xi$, $\xi > 0$. More precisely, the restriction $\mathfrak{U}(B + \cdot)$ to the set $\mathbb{R}$ (identified with the set of deterministic $\varphi$) can fail to be differentiable at 0, which is in the interior of its effective domain. This is in contrast to Kramkov and Schachermayer [23, Theorem 2.2] which ensures differentiability of such a restriction when there is no random endowment ($B \equiv 0$). The same example also serves as a counterexample to a statement in [7], which is further discussed in the Erratum by Cvitanić et al. [8].

4) On the constructive side, we show that under a mild growth condition on the utility function $U(\xi)$ around $\xi = 0$, one-sided directional derivatives of $\mathfrak{U}$ exist on $\mathbb{L}^\infty_{++}$ and can be characterised as values of a new linear stochastic control problem (Proposition 4.7). We solve this problem explicitly under the additional assumption of minimal (unique) superreplicability from Larsen et al. [28] placed on $B$ and $\varphi$. This gives us explicit formulas for the two endpoints of the interval of Davis prices. As an offshoot, we show additionally that $\mathfrak{U}$ is Gâteaux-differentiable at each minimally superreplicable $B$.

5) We show in Appendix A that the endpoints of the set of Davis prices of the payoff $\varphi$ correspond to the positive and negative small-quantity limits of indifference prices (as well as certainty equivalents and marginal utility-based prices). This universal limiting property holds even when Davis prices are not unique and thereby gives an additional economic justification for the use of Davis pricing in our conditional setting.

The paper is organised as follows. The model is described, the terminology set, standing assumptions are imposed and the preliminary analysis of our central utility maximisation problem is performed in Sect. 2. In Sect. 3, we define conditional Davis prices, characterise them from the dual point of view and lay out some of the first consequences of this characterisation. Directional derivatives of the primal utility maximisation problem are studied in Sect. 4, which also contains the explicit example of non-smoothness mentioned in 3) above. Section 4 also gives a characterisation of the directional derivative in terms of a linear stochastic control problem. Section 5 recalls the definition of unique superreplicability from [28] (now called *minimal* superreplicability) and provides a family of examples of minimally superreplicable claims. The main result of Sect. 5 gives an explicit expression for the directional derivative of the utility maximisation value function under the minimal superreplicability condition. This result is subsequently used in Sect. 6 to give explicit formulas for the interval of conditional Davis prices in a general setting. We use these formulas in two examples, one of which supports our claim that non-uniqueness of conditional Davis prices occurs even in the simplest of settings. Appendix A discusses some popular utility-based pricing methods and relates them to conditional Davis prices in the small-quantity limit.

## 2 The setup and assumptions

### 2.1 Notation

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with a finite time horizon $T \in (0, \infty)$; $\mathcal{L}^2$ denotes the family of all progressively measurable processes $\pi$ with $\int_0^T |\pi_u|^2\, du < \infty$ a.s. For a semimartingale $S$, $L(S)$ denotes the set of all predictable $S$-integrable processes, and the stochastic integral of $\pi \in L(S)$ with respect to $S$ is denoted by either $\pi \cdot S$ or $\int_0^\cdot \pi_u\, dS_u$. The stochastic exponential of a semimartingale $Y$ is denoted by $\mathcal{E}(Y)$.

The subspaces of various $\mathbb{L}^p$-spaces consisting of nonnegative random variables (or sets of set functions taking nonnegative values) get a subscript $+$, while $\mathbb{L}^\infty_{++}$ denotes the family of all nonnegative essentially bounded random variables which are essentially bounded away from 0, i.e., $\mathbb{L}^\infty_{++} := \bigcup_{x>0}(x + \mathbb{L}^\infty_+)$. The space of finite finitely additive set functions on $(\Omega, \mathcal{F})$, absolutely continuous with respect to $\mathbb{P}$, is denoted by ba$(\mathbb{P})$ (or simply ba if no confusion can arise). The space of finite $\sigma$-additive measures absolutely continuous with respect to $\mathbb{P}$ is often identified with $\mathbb{L}^1_+$, while the space ba is identified with the topological dual $(\mathbb{L}^\infty)^*$ of $\mathbb{L}^\infty$. For $\mu \in \mathrm{ba}_+$, $\mu^r$ denotes the regular part of $\mu$, i.e, the (set-wise) largest $\sigma$-additive measure dominated by $\mu$. The singular part $\mu - \mu^r$ is denoted by $\mu^s$, making $\mu = \mu^r + \mu^s$ the Yosida–Hewitt decomposition of $\mu$.

For a concave (convex) function $u$, $\partial u$ denotes its superdifferential (subdifferential), while $\partial_{x+}u$ and $\partial_{x-}u$ denote the right and left derivative, respectively, in the argument $x$.

### 2.2 The market model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space which satisfies the usual conditions and $(S_t)_{t\in[0,T]}$ a locally bounded semimartingale; $\mathcal{M}$ denotes the set of all $\mathbb{P}$-equivalent (countably additive) probability measures $\mathbb{Q}$ on $\mathcal{F}$ for which $S$ is a $\mathbb{Q}$-local martingale.

**Standing Assumption 2.1** (NFLVR) $\mathcal{M} \neq \emptyset$.

*Remark 2.2* We assume that the asset price process $S$ is locally bounded and postulate the existence of a local martingale measure. While it is possible to relax our setting to the non-locally-bounded case (as used in e.g. [7]), it is not possible to relax Assumption 2.1 to the existence of a supermartingale deflator only. Indeed, the presence of a non-replicable endowment $B$ makes the admissibility class which produces only nonnegative wealth processes too small to host an optimiser. This delicate issue is discussed and illustrated in Larsen [27, Example 2.2]. To keep the focus of the current paper on the issues directly related to conditional Davis pricing, we have opted for a set of assumptions which is slightly stronger than absolutely necessary.

**Standing Assumption 2.3** $B \in \mathbb{L}^\infty_{++}$ *and* $\varphi \in \mathbb{L}^\infty$.

*Remark 2.4* In the spirit of Remark 2.2 above, Standing Assumption 2.3 may be replaced by only requiring $B$ and $\varphi$ to be bounded in absolute value by a constant plus the outcome of a maximal wealth process. We do not implement that generalisation, but simply point the reader to [19, Lemma 1] for details and terminology.

### 2.3 Gains and admissibility

The investor's gains process has the dynamics

$$(\pi \cdot S)_t := \int_0^t \pi_u \, dS_u, \qquad t \in [0, T],$$

for some $\pi \in L(S)$. We call $\pi \in L(S)$ *admissible* if the gains process is uniformly bounded below by a constant, in which case we write $\pi \in \mathcal{A}$. The set of terminal outcomes (gains) of admissibly strategies is denoted by $\mathcal{K}$, i.e., we define

$$\mathcal{K} := \{(\pi \cdot S)_T \, : \, \pi \in \mathcal{A}\}.$$

### 2.4 The primal problem

While all the necessary notation is defined below, we refer the reader to [23] and [7] for a wider context of and further references to the utility maximisation theory within mathematical finance. Let $U$ be a utility function on $(0,\infty)$ — a strictly concave, strictly increasing and continuously differentiable function with $U'(0+)=+\infty$ and $U'(+\infty)=0$. When necessary, we extend the domain of $U$ artificially to $\mathbb{R}$ by setting $U(x)=-\infty$ for $x<0$ and $U(0)=\inf_{x>0}U(x)$. Finally, $U$ is said to be *reasonably elastic* (as defined in [23]) if

$$\limsup_{x\to\infty}\frac{xU'(x)}{U(x)}<1.$$

Even though we need this for some of our results, we do not impose the condition of reasonable elasticity from the start.

For a random variable $X\in\mathbb{L}^\infty_{++}$, we set

$$\mathfrak{U}(X):=\sup_{G\in\mathcal{K}}\mathbb{E}[U(X+G)], \tag{2.1}$$

where we use the convention that $\mathbb{E}[U(X)]=-\infty$ if $\mathbb{E}[U(X)^-]=+\infty$. We note that this definition is a restatement of (1.2) in the introduction. For $X\in\mathbb{L}^\infty_{++}$, we have $\mathfrak{U}(X)\geq U(\operatorname{essinf}X)>-\infty$, which implies that $\mathfrak{U}$ is $(-\infty,\infty]$-valued on $\mathbb{L}^\infty_{++}$. In (2.5) below, we impose a dual properness assumption which among other things ensures that $\mathfrak{U}$ is finitely valued on $\mathbb{L}^\infty_{++}$.

### 2.5 The dual utility maximisation problem

The set $\mathcal{M}$ of equivalent local martingale measures can be identified – via Radon–Nikodým derivatives with respect to $\mathbb{P}$ – with a subset of $\mathbb{L}^1_+(\mathbb{P})$ and embedded naturally into $\mathrm{ba}(\mathbb{P})=\mathbb{L}^\infty(\mathbb{P})^*\supseteq\mathbb{L}^1(\mathbb{P})$. We define $\overline{\mathcal{M}}^*$ as the weak*-closure of $\mathcal{M}$ and $\mathcal{D}\subseteq\mathrm{ba}_+(\mathbb{P})$ as the family of all $y\mathbb{Q}$, where $y\in[0,\infty)$ are constants and $\mathbb{Q}\in\overline{\mathcal{M}}^*$. For $B\in\mathbb{L}^\infty_{++}$, the *dual utility functional* can now be defined by

$$\mathbb{V}_B(\mu):=\sup_{X\in\mathbb{L}^\infty}\big(\mathbb{E}[U(B+X)]-\langle\mu,X\rangle\big),\qquad \mu\in\mathrm{ba}(\mathbb{P}).$$

In particular, $\mathbb{V}_B$ is convex, lower weak*-semicontinuous on $\mathrm{ba}(\mathbb{P})$ and bounded from below by $\mathbb{E}[U(B)]\in\mathbb{R}$. For the remainder of the paper, we impose a *properness* assumption. While not the weakest possible in our setting, this allows us to deal swiftly, and yet with only a minor loss of generality, with several technical points that are not central to the message of the paper.

**Standing Assumption 2.5** (Properness) *There exist $y_0\in(0,\infty)$ and $\mathbb{Q}_0\in\mathcal{M}$ such that $\mu_0:=y_0\mathbb{Q}_0$ satisfies*

$$\mathbb{V}_B(\mu_0)<\infty. \tag{2.2}$$

Thanks to a minimal modification of Owen and Žitković [33, Lemma 2.1] and the discussion before it, $\mathbb{V}_B$ admits the representation

$$\mathbb{V}_B(\mu)=\mathbb{E}\left[V\left(\frac{d\mu^r}{d\mathbb{P}}\right)\right]+\langle\mu,B\rangle,\qquad \mu\in\mathcal{D}, \tag{2.3}$$

where $V$ is the (strictly convex) dual utility function defined by

$$V(y) := \sup_{x>0}\big(U(x) - xy\big), \qquad y > 0.$$

Consequently, Fenchel's inequality and (2.2) guarantee that the primal value function $\mathfrak{U}$ satisfies $\mathfrak{U}(X) < \infty$ for all $X \in \mathbb{L}^\infty_{++}$. Furthermore, (2.2) also ensures that the corresponding dual value function defined by

$$\mathfrak{V}(B) := \inf_{\mu\in\mathcal{D}} \mathbb{V}_B(\mu), \qquad B \in \mathbb{L}^\infty_{++}, \tag{2.4}$$

is finitely valued. For $B \in \mathbb{L}^\infty_{++}$, we let $\hat{\mathcal{D}}(B)$ denote the set of all minimisers, i.e.,

$$\hat{\mathcal{D}}(B) = \{\mu \in \mathcal{D} : \mathfrak{V}(B) = \mathbb{V}_B(\mu)\}.$$

The next two results collect some basic facts that we need in the sequel.

**Lemma 2.6** *Assume that $B \in \mathbb{L}^\infty_{++}$ and that $\mathbb{A} : \mathcal{D} \to [0,\infty)$ is a nonnegative weak*-lower semicontinuous functional. Then we have*:

(1) *Any minimising sequence for $\mathbb{V}_B + \mathbb{A}$ is bounded in total mass.*

(2) *The set of all minimisers of $\mathbb{V}_B + \mathbb{A}$ is nonempty and weak*-compact.*

*Proof* For (1), we let $(\mu_n)_{n\in\mathbb{N}}$ be a minimising sequence for $\mathbb{V}_B + \mathbb{A}$. By the definition of $\mathcal{D}$, it can be written in the form

$$\mu_n = y_n\mathbb{Q}_n, \qquad \text{where } \mathbb{Q}_n \in \overline{\mathcal{M}}^* \text{ and } (y_n)_{n\in\mathbb{N}} \subseteq [0,\infty).$$

Using the representation (2.3) and the Standing Assumption (2.2), we get the estimate

$$\begin{aligned}\mathbb{V}_B(\mu_0) + \mathbb{A}(\mu_0) &\ge \limsup_{n\to\infty}\left(\mathbb{E}\left[V\left(y_n\frac{d\mathbb{Q}_n^r}{d\mathbb{P}}\right)\right] + y_n\langle\mathbb{Q}_n, B\rangle\right)\\ &\ge \limsup_{n\to\infty}\big(V(y_n) + y_n \operatorname{essinf} B\big),\end{aligned}$$

where the first inequality follows from the positivity of $\mathbb{A}$ and the minimising property of the sequence $(\mu_n)_{n\in\mathbb{N}}$, and the second is produced by Jensen's inequality and the decreasing property of $V$. Since $\lim_{y\to\infty} V'(y) = 0$ and $\operatorname{essinf} B > 0$, we conclude that the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded from above, which establishes (1). Note that this bound is universal because it only depends on $\mu_0$ from Assumption 2.5.

For (2), we pick a minimising sequence $(\mu_n)_{n\in\mathbb{N}}$ and use (1) to establish the existence of a constant $\bar{y} > 0$ such that $\mu_n(\Omega) \le \bar{y}$ for all $n$. Therefore, the family $(\mu_n)_{n\in\mathbb{N}}$ lives in a bounded subset $\bar{y}\overline{\mathcal{M}}^*$ of ba, and we can use the Banach–Alaoglu theorem to conclude that there is a subnet $(\mu_\alpha)_\alpha$ of $(\mu_n)_{n\in\mathbb{N}}$ with $\mu_\alpha \to \hat{\mu} \in$ ba in the weak*-sense. It remains to use the weak*-closedness of $\mathcal{D}$ and the lower semicontinuity of the functional $\mathbb{V}_B + \mathbb{A}$ to conclude that $\hat{\mu}$ is a minimiser over $\mathcal{D}$.

We denote by $\hat{\mathcal{D}} \subseteq \mathcal{D}$ the non-empty and closed set of all minimisers for $\mathbb{V}_B + \mathbb{A}$. We use (1) to see that the set $\{\mu(\Omega) : \mu \in \hat{\mathcal{D}}\}$ is bounded. That in turn allows us to use the Banach–Alaoglu theorem to conclude that $\hat{\mathcal{D}}$ is weak*-compact. □

**Lemma 2.7** *For $B \in \mathbb{L}^\infty_{++}$, the set $\hat{\mathcal{D}}(B)$ is a nonempty weak*-compact subset of* $\mathrm{ba}(\mathbb{P})$, *and there exists a nonnegative random variable $\hat{Y} = \hat{Y}(B)$ such that $\mathbb{P}[\hat{Y} > 0] > 0$ and*

$$\hat{Y} = \frac{d\mu^r}{d\mathbb{P}} \qquad \text{for all } \mu \in \hat{\mathcal{D}}(B).$$

*Furthermore, the strong duality $\mathfrak{U}(B) = \mathfrak{V}(B)$ holds for all $B \in \mathbb{L}^\infty_{++}$.*

*Proof* The nonemptyness and compactness of $\hat{\mathcal{D}}(B)$ follow directly from Lemma 2.6 with $\mathbb{A} := 0$. From Cvitanić et al. [7, Theorem 3.1], the regular part $\hat{Y}$ of all dual optimisers is known to be unique. To see that $\hat{Y} \not\equiv 0$, we argue by contradiction and suppose that $\mathbb{P}[\hat{Y} = 0] = 1$. In that case, Standing Assumption 2.5 and the representation (2.3) imply that $V(0) < \infty$ and so, thanks to Jensen's inequality, we have $\mathbb{V}_B(\mu) < \infty$ for all $\mu \in \mathcal{D}$. In particular, we have for some $\mathbb{Q} \in \mathcal{M}$ and $\hat{\mu} \in \hat{\mathcal{D}}(B)$ that

$$\mathbb{V}_B(\mu^\varepsilon) < \infty, \qquad \text{where } \mu_\varepsilon := \varepsilon\mathbb{Q} + (1-\varepsilon)\hat{\mu}, \varepsilon \in [0, 1].$$

Because the regular-part functional is additive, we have

$$\mu^r_\varepsilon = \varepsilon\mathbb{Q} + (1-\varepsilon)\hat{\mu}^r = \varepsilon\mathbb{Q}.$$

Therefore, the minimality of $\hat{\mu}$ implies that

$$\mathbb{E}\left[V\left(\varepsilon\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] + \langle \mu_\varepsilon, B\rangle = \mathbb{V}_B(\mu_\varepsilon) \geq \mathbb{V}_B(\hat{\mu}) \geq V(0) + \langle \hat{\mu}, B\rangle.$$

Fatou's lemma then yields

$$\langle \mathbb{Q} - \hat{\mu}, B\rangle \geq \liminf_{\varepsilon \searrow 0} \frac{1}{\epsilon}\left(V(0) - \mathbb{E}\left[V\left(\varepsilon\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right) = -V'(0) = +\infty.$$

This is a contradiction because $B \in \mathbb{L}^\infty_{++}$ ensures that the left-hand side is finite.

Finally, to establish the strong duality property, we define the nested sequence of weak*-compact dual sets

$$\mathcal{D}_n := \{\mu \in \mathcal{D} : \mu(\Omega) \leq n\}, \qquad n \in \mathbb{N},$$

as well as the primal set

$$\mathcal{C} := (\mathcal{K} - \mathbb{L}^0_+) \cap \mathbb{L}^\infty = \{X \in \mathbb{L}^\infty : \langle \mathbb{Q}, X\rangle \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{M}\}.$$

For a proof of the last identity, see e.g. Larsen and Žitković [29, Corollary 3.4(1)]. As a consequence, we have for $X \in \mathbb{L}^\infty(\mathbb{P})$ the identity

$$\lim_{n\to\infty} \sup_{\mu \in \mathcal{D}_n} \langle \mu, X\rangle = \sup_{\mu \in \mathcal{D}} \langle \mu, X\rangle = \begin{cases} 0, & X \in \mathcal{C}, \\ +\infty, & X \notin \mathcal{C}. \end{cases} \tag{2.5}$$

The minimax theorem (see e.g. Zălinescu [35, Theorem 2.10.2]) can then be used to conclude that

$$\mathfrak{V}(B) = \lim_{n\to\infty} \inf_{\mu\in\mathcal{D}_n} \sup_{X\in\mathbb{L}^\infty(\mathbb{P})} \big(\mathbb{E}[U(X+B)] - \langle\mu, X\rangle\big) = \sup_{X\in\mathcal{C}} \mathbb{E}[U(X+B)],$$

with the last equality justified by the monotone convergence theorem. □

## 3 Conditional Davis prices and a dual characterisation

After the introduction of the necessary terminology and notation, we turn to the definition of the central concept of the paper. A wider economic context and a comparison to other arbitrage- and utility-based pricing concepts are provided in Appendix A below.

**Definition 3.1** Given a random endowment $B \in \mathbb{L}^\infty_{++}$, a number $p \in \mathbb{R}$ is said to be a $B$*-conditional Davis price* (or simply a *conditional Davis price* if $B$ is clear from the context) for a contingent claim $\varphi \in \mathbb{L}^\infty$ if

$$\mathfrak{U}\big(B + q(\varphi - p)\big) \le \mathfrak{U}(B), \qquad \forall q \in \mathbb{R}. \tag{3.1}$$

The set of all $B$-conditional Davis prices of $\varphi$ is denoted by $P(\varphi|B)$.

### 3.1 A dual characterisation

The following characterisation in Theorem 3.5 of the set of Davis prices in terms of the dual set $\hat{\mathcal{D}}(B)$ is going to play a central role throughout the paper. It rests on several lemmas given below.

**Definition 3.2** For $B \in \mathbb{L}^\infty_{++}$, a random variable $R \in \mathbb{L}^\infty$ is said to be $B$*-irrelevant*, denoted by $R \in \mathcal{I}(B)$, if

$$\mathfrak{U}(B + qR) \le \mathfrak{U}(B), \qquad \forall q \in \mathbb{R}. \tag{3.2}$$

**Lemma 3.3** *$\mathcal{I}(B)$ is a nonempty, weak*-closed linear subspace in $\mathbb{L}^\infty$, and $p$ is a $B$-conditional Davis price of $p$ if and only if $\varphi - p \in \mathcal{I}(B)$.*

*Proof* The function $\mathfrak{U}$ is concave at $B$, so $\mathcal{I}(B)$ is the set of those directions $R$ with the property that the directional derivative of $\mathfrak{U}$ in both directions $R$ and $-R$ are nonpositive. In other words, we have

$$\sup_{\mu\in\partial\mathfrak{U}(B)} \langle R, \mu\rangle \le 0 \quad \text{and} \quad \sup_{\mu\in\partial\mathfrak{U}(B)} -\langle R, \mu\rangle \le 0,$$

where $\partial\mathfrak{U}(B) \subseteq \mathrm{ba}(\mathbb{P})$ is the superdifferential of $\mathfrak{U}$. Therefore $\mathcal{I}(B)$ is the annihilator of $\partial\mathfrak{U}(B)$, i.e.,

$$\mathcal{I}(B) = \{R \in \mathbb{L}^\infty : \langle\mu, R\rangle = 0 \text{ for all } \mu \in \partial\mathfrak{U}(B)\},$$

which implies the first statement. The second statement is just restating the definition. □

**Lemma 3.4** *A random variable $R \in \mathbb{L}^\infty$ is $B$-irrelevant if and only if*

$$\inf_{\mu \in \mathcal{D}} \big(\mathbb{V}_B(\mu) + |\langle \mu, R\rangle|\big) = \inf_{\mu \in \mathcal{D}} \mathbb{V}_B(\mu). \tag{3.3}$$

*Proof* Because $\mathcal{I}(B)$ is a vector space, we can scale $R$ so that without loss of generality, we can assume that $B \pm R \in \mathbb{L}^\infty_{++}$. Then by the minimax theorem (see [35, Theorem 2.10.2]), we have

$$\begin{aligned}\inf_{\mu \in \mathcal{D}} \big(\mathbb{V}_B(\mu) + |\langle \mu, R\rangle|\big) &= \inf_{\mu \in \mathcal{D}} \sup_{|q|\le 1} \big(\mathbb{V}_B(\mu) + q\langle \mu, R\rangle\big) \\ &= \sup_{|q|\le 1} \inf_{\mu \in \mathcal{D}} \big(\mathbb{V}_B(\mu) + q\langle \mu, R\rangle\big) = \sup_{|q|\le 1} \mathfrak{U}(B + qR).\end{aligned}$$

The same equality with $R = 0$ implies that (3.3) is equivalent to

$$\mathfrak{U}(B) = \sup_{|q|\le 1} \mathfrak{U}(B + qR). \tag{3.4}$$

Since the function $\mathfrak{U}$ is finite-valued at $B$ as well as in an $\mathbb{L}^\infty$-open ball around $B$, both sides of (3.2) are finite-valued for small enough $q$. Thanks to the concavity of $\mathfrak{U}$, it is enough to check (3.2) only for $q$ in a neighbourhood of 0 to determine whether $R \in \mathcal{I}(B)$, and this is exactly what (3.4) provides. □

By Lemma 2.7, we have $\mu(\Omega) > 0$ for each $\mu \in \hat{\mathcal{D}}(B)$. Therefore the family

$$\hat{\mathcal{D}}_0(B) := \left\{ \frac{1}{\mu(\Omega)}\mu \,:\, \mu \in \hat{\mathcal{D}}(B) \right\} \tag{3.5}$$

is a well-defined nonempty family of finitely additive probabilities. We now have everything set up for our main characterisation of conditional Davis prices.

**Theorem 3.5** *For $\varphi \in \mathbb{L}^\infty(\mathbb{P})$, the following two statements are equivalent:*
(1) *$p \in P(\varphi|B)$, i.e., $p$ is a $B$-conditional Davis price of $\varphi$.*
(2) *$p = \langle \mathbb{Q}, \varphi\rangle$ for some $\mathbb{Q} \in \hat{\mathcal{D}}_0(B)$.*
*In particular, $P(\varphi|B)$ is a nonempty compact subinterval of $\mathbb{R}$.*

*Proof* (1) ⇒ (2) According to Lemma 2.6 with $\mathbb{A}(\mu) := |\langle \mu, \varphi - p\rangle|$, the functional $\mu \mapsto \mathbb{V}_B(\mu) + |\langle \mu, \varphi - p\rangle|$ admits a minimiser $\hat{\mu}$. By Lemma 3.4, the same $\hat{\mu}$ must minimise the functional $\mu \mapsto \mathbb{V}_B(\mu)$ as well, and so $\hat{\mu} \in \hat{\mathcal{D}}(B)$ and $\langle \hat{\mu}, \varphi - p\rangle = 0$.

(2) ⇒ (1) Suppose that $p$ is such that $\langle \mu^*, \varphi - p\rangle = 0$ for some $\mu^* \in \hat{\mathcal{D}}(B)$. Then for any $\mu$, we have

$$\mathbb{V}_B(\mu^*) + |\langle \mu^*, \varphi - p\rangle| = \mathbb{V}_B(\mu^*) \le \mathbb{V}_B(\mu) \le \mathbb{V}_B(\mu) + |\langle \mu, \varphi - p\rangle|,$$

and Lemma 3.4 can be used.

Finally, Lemma 2.6 and the continuity of the map $\mu \mapsto \langle \mu, \varphi\rangle$ ensure that the set $P(\varphi|B)$ is compact. □

### 3.2 First consequences

A reinterpretation in the setting of portfolios with convex constraints leads to the following dual characterisation which is used in proof of Proposition 3.7 below.

**Corollary 3.6** *Suppose that $U$ is reasonably elastic. Then for each constant $c \geq 0$ and each $R \in \mathbb{L}^\infty$, we have*

$$\inf_{\mu\in\mathcal{D}} \big(\mathbb{V}_B(\mu) + c|\langle \mu, R\rangle|\big) = \inf_{y\geq 0, \mathbb{Q}\in\mathcal{M}} \big(\mathbb{V}_B(y\mathbb{Q}) + c|\langle y\mathbb{Q}, R\rangle|\big).$$

*Proof* Let $\mathcal{C} := (\mathcal{K} - \mathbb{L}^0_+) \cap \mathbb{L}^\infty$ and let $\mathcal{C}'$ be the family of all random variables $X' \in \mathbb{L}^\infty$ of the form

$$X' = B + X + qR, \qquad \text{where } X \in \mathcal{C}, q \in [-c, c].$$

The support function $\alpha_{\mathcal{C}'}$ for the set $\mathcal{C}'$ is then given by

$$\begin{aligned} \alpha_{\mathcal{C}'}(\mu) &= \sup_{X'\in\mathcal{C}'} \langle \mu, X'\rangle \\ &= \langle \mu, B\rangle + \sup_{X\in\mathcal{C},\ q\in[-c,c]} \big(\langle \mu, X\rangle + q\langle \mu, R\rangle\big) \\ &= \langle \mu, B\rangle + c|\langle \mu, R\rangle| + \begin{cases} 0, & \mu \in \mathcal{D}, \\ +\infty, & \mu \notin \mathcal{D}, \end{cases} \end{aligned}$$

where the last equality follows from (2.5) above. Therefore,

$$\inf_{\mu\in\mathcal{D}} \big(\mathbb{V}_B(\mu) + c|\langle \mu, R\rangle|\big) = \inf_{\mu\in\mathrm{ba}(\mathbb{P})} \big(\mathbb{V}_0(\mu) + \alpha_{\mathcal{C}'}(\mu)\big). \tag{3.6}$$

Moreover, the set $\mathcal{C}$ is weak*-closed by Theorem 4.2 in Delbaen and Schachermayer [10]; hence so is $\mathcal{C}'$. Hence the assumptions of [29, Proposition 3.14] are satisfied (via [29, Corollary 3.4]), and so the infimum on the right-hand side of (3.6) can be replaced by an infimum over $\sigma$-additive measures. □

Our next two consequences of Theorem 3.5 provide a partial generalisation and an alternative method of proof for [20, Theorem 3.1].

**Proposition 3.7** *Suppose that $B \in \mathbb{L}^\infty_{++}$, $U$ is reasonably elastic and the dual problem (2.4) admits a non-$\sigma$-additive optimiser. Then there exists $A \in \mathcal{F}$ such that $\varphi = \mathbf{1}_A$ has multiple $B$-conditional Davis prices.*

*Proof* Let $(\mu_n)_{n\in\mathbb{N}}$ be a minimising sequence for $\inf_{\mu\in\mathcal{D}} \mathbb{V}_B(\mu)$. By Corollary 3.6, we can assume that each $\mu$ is countably additive. Moreover, Lemma 2.6, (1) guarantees that the total-mass sequence $(\mu_n(\Omega))_{n\in\mathbb{N}}$ is bounded. By extracting a subsequence, we can assume that $(\mu_n(\Omega))_{n\in\mathbb{N}}$ converges to a constant $y \geq 0$. By Lemma 2.7, this conclusion can be strengthened to $y > 0$.

We suppose first that $(\mu_n)_{n\in\mathbb{N}}$ is not weak*-convergent. Then two of its convergent subnets have different limits, and both of these are elements of $\hat{\mathcal{D}}(B)$ with the same total mass $y > 0$. Hence the set $\hat{\mathcal{D}}_0(B)$ of (3.5) is not a singleton and by Theorem 3.5, there exists $\varphi = \mathbf{1}_A$, with $A \in \mathcal{F}$, with two different conditional Davis prices.

On the other hand, if $(\mu_n)_{n\in\mathbb{N}}$ converges to $\hat{\mu}$ in the weak*-sense, then $\hat{\mu} \in \hat{\mathcal{D}}(B)$. Furthermore, by the Vitali–Hahn–Saks theorem (see Dunford and Schwartz [13, Corollary III.7.3]), the limit $\hat{\mu}$ is countably additive. Hence the set $\hat{\mathcal{D}}(B)$ has at least two different elements — one countably additive and one, by assumption, not. Then a random variable $\varphi = \mathbf{1}_A$ with two different conditional Davis prices can be constructed as above. □

The next consequence of Theorem 3.5 gives a sufficient condition (analogous to that of [20, Theorem 3.1]) for the uniqueness of conditional Davis prices. Before we state it, we recall that, under the condition of reasonable elasticity, Cvitanić et al. [7] show that there exists a process $\hat{\pi} \in \mathcal{A}$ such that $\hat{X} := (\hat{\pi} \cdot S)_T + B$ satisfies

$$\mathbb{E}[U(\hat{X})] = \mathfrak{U}(B) \qquad \text{and} \qquad U'(\hat{X}) = \frac{d\hat{\mu}^r}{d\mathbb{P}}, \tag{3.7}$$

where $\hat{\mu} \in \hat{D}(B)$. The random variable $\hat{X}$ with this property is $\mathbb{P}$-a.s. unique.

**Corollary 3.8** *Suppose $U$ is reasonably elastic and that $|\varphi| \le c\hat{X}$ for some constant $c \ge 0$, where $\hat{X}$ is as in (3.7). Then the set $P(\varphi|B)$ of $B$-conditional Davis prices for $\varphi \in \mathbb{L}^\infty$ is a singleton.*

*Proof* In view of Lemma 2.7 and Theorem 3.5, it is enough to show that $\langle \hat{\mu}^s, \hat{X} \rangle = 0$ for each $\hat{\mu} \in \hat{\mathcal{D}}(B)$. This in turn follows directly from the first part of Eq. (4.7) in [7]. □

## 4 Differentiability of the primal value function

The purpose of this section is to study differentiability properties of the *primal value function* $u : \mathbb{R}^2 \to [-\infty, \infty)$ defined by

$$u(x, q) := \mathfrak{U}(B + x + q\varphi), \tag{4.1}$$

where $\mathfrak{U}$ is defined in (2.1). We start by providing some motivation in Sect. 4.1. Then we present an example which shows that directional differentiability – even in the most "benign" directions – cannot be expected in general. Next we give a characterisation of the one-sided directional derivatives in terms of a linear control problem. We hope that both our counterexample and the later characterisation hold some independent interest outside of the context of Davis pricing. We use the obtained results in later sections to give a workable characterisation of the interval of conditional Davis prices.

### 4.1 Relevance of differentiability

The value function $u$ in (4.1) is a standard object of interest in the context of Davis pricing, and the relevance of its differentiability properties has been noted by several authors (including Davis in [9]). To explain why that is the case, we start by giving a characterisation of Davis prices in terms of superdifferentials in Proposition 4.1, and then provide a short discussion in Remark 4.2 that follows it. The case $B \equiv x > 0$ can already be found in Hugonnier and Kramkov [19, Remark 1] and Hugonnier et al. [20, Eq. (3.12)]. Consequently, Proposition 4.1 is relevant because it covers the case of an unspanned endowment $B \in \mathbb{L}^\infty_{++}$.

**Proposition 4.1** *For each* $(y, r) \in \partial u(0, 0)$, *we have* $y > 0$, *and*

$$P(\varphi|B) = \{r/y : (y, r) \in \partial u(0, 0)\}, \tag{4.2}$$

*where* $\partial u(0, 0)$ *denotes the supergradient of* $u$ *at* $(0, 0)$.

*Proof* Since $B \in \mathbb{L}^\infty_{++}$, $u$ is concave and finite-valued around $(0, 0)$, and the first statement follows from the fact that $x \mapsto u(x, 0)$ is strictly increasing on its effective domain which contains 0 in its interior.

Definition 3.1 translates into the statement that

$$p \in P(\varphi|B) \quad \text{if and only if} \quad u(0, 0) \geq u(-q' p, q') \text{ for all } q'.$$

By concavity, this is equivalent to the nonpositivity of the directional derivative of $u$ at $(0, 0)$ in the directions $(p, -1)$ and $(-p, 1)$, i.e.,

$$\inf_{(y,r)\in\partial u(0,0)} (-r + py) \leq 0 \quad \text{and} \quad \inf_{(y,r)\in\partial u(0,0)} (r - py) \leq 0.$$

By the convexity of the supergradient, this is equivalent to the existence of a pair $(y, r)$ in $\partial u(0, 0)$ such that $py = r$. □

*Remark 4.2* Equation (4.2) is often used to explain the relationship between differentiability of the value function $u$ and the uniqueness of the Davis price in the unconditional setting. Indeed, when $B$ is replicable, the function $u(x, 0)$ is differentiable in the variable $x$ (by Kramkov and Schachermayer [23, Theorem 2.1]) so that all elements $(y, r)$ of $\partial u(0, 0)$ have the same $y$, given by $y = \frac{\partial}{\partial x} u(x, 0)|_{x=0}$. Therefore, we have multiple Davis prices if and only if we have multiple values of the $r$-components in elements of the supergradient $\partial u(0, 0)$. That occurs if and only if the left and the right derivatives of $q \mapsto u(0, q)$ at $q = 0$ do not match, which is in turn equivalent to the lack of differentiability at 0.

The case where $B$ is unspanned is more subtle. As we shall see in Example 4.3 below, when a non-replicable random endowment is present, $u$ is no longer necessarily differentiable in $x$. That means that both $y$ and $r$ are potentially allowed to vary across different elements $(y, r)$ of the supergradient. To complicate the situation even more, it may happen, however, that each such pair has the same quotient, making the Davis price unique. A simple example is when (as in Example 4.3 below) $u$ is not differentiable in $x$ at $x = 0$ and $\varphi \equiv 1$. Indeed, in that case, $(x, q) \mapsto u(x, q) = u(x + q, 0)$ is clearly not differentiable at $(0, 0)$, but the unique conditional Davis price of $\varphi \equiv 1$ is 1 (the latter claim follows from Corollary 5.9 below and can also be found in [20, Remark 3.2]).

### 4.2 An example of non-differentiability

Our next example shows that the value function $u$ may fail to be differentiable in $x$, i.e., that $\mathfrak{U}$ may fail to be differentiable even in "constant directions". The same example exhibits another, equivalent property: The set $\hat{\mathcal{D}}(B)$ of dual minimisers may contain measures with different total masses. In other words, $\hat{\mu}(\Omega)$ need not be constant over $\hat{\mu} \in \hat{\mathcal{D}}(B)$. As a supplement to the rigorous argument in Example 4.3 below, we note that the total mass of a dual optimiser carries the interpretation of a Lagrange multiplier corresponding to the initial wealth. Multiple Lagrange multipliers translate directly to multiple elements in the $x$-superdifferential of the primal value function, and hence imply the failure of the value function's differentiability at $x$. Once we introduce the concept of *minimal superreplicability* in the next section, we shall see how we can regain differentiability in certain cases of interest.

For simplicity and concreteness, we base the example on Kramkov and Schachermayer [23, Example 5.1'] and use the following notation and conventions. All random variables $X$ are defined on the sample space $\Omega := \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, measures are identified with sequences in $\ell^1_+$, and for $\mathbb{Q} = (q_n)_{n\in\mathbb{N}_0} \in \ell^1_+$, we write $\langle \mathbb{Q}, X\rangle$ for $\sum_{n=0}^{\infty} q_n X_n$ whenever $X = (X_n)_{n\in\mathbb{N}_0} \in \ell^\infty$.

*Example 4.3* We start by recalling the elements of (a special case of) the one-period model in [23, Example 5.1'], where $\Omega := \mathbb{N}_0$ and $\mathbb{P} = (p_n)_{n\in\mathbb{N}_0}$ with

$$p_0 := \frac{3}{4}, \qquad p_n := \frac{2^{-n}}{4} \quad \text{for } n \in \mathbb{N}.$$

The one-period stock price increment $\Delta S = (\Delta S_n)$ is defined as

$$\Delta S_0 := 1 \qquad \text{and} \qquad \Delta S_n := \frac{1-n}{n} \quad \text{for } n \in \mathbb{N}.$$

With $U := \log$, we consider the utility maximisation problem

$$\sup_{\pi\in[-x,x]} \mathbb{E}[U(x + \pi \Delta S)], \qquad x > 0. \tag{4.3}$$

Let $\mathcal{Q}$ denote the set of all finite martingale measures, i.e.,

$$\mathcal{Q} := \{\mathbb{Q} \in \ell^1_+ : \langle \mathbb{Q}, \Delta S\rangle = 0\},$$

and let $\mathcal{M} := \{\mathbb{Q} \in \mathcal{Q} : \langle \mathbb{Q}, 1\rangle = 1\}$. Because the conjugate utility function $V$ is given by $V(y) = -1 - \log y$, the dual problem to (4.3) is given by

$$v(y) := \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = V(y) + v^*, \quad \text{where } v^* := \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[-\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right].$$

Let $(\mathbb{Q}^N)_{N\in\mathbb{N}} \subseteq \mathcal{M}$ be a minimising sequence for $v^*$ (equivalently, for $v(y)$). We claim that there exists a function $f \in \ell^\infty$ such that $\langle \mathbb{Q}^N, f\rangle$ does not converge in $\mathbb{R}$. Indeed, if not, the sequence $(\mathbb{Q}^N)_{N\in\mathbb{N}}$ – interpreted as a sequence in $\ell^1_+$ – would be weakly Cauchy in $\ell^1$. By a theorem of Steinhaus (see Wojatszcyk [36, Corollary 14]) which states that $\ell^1$ is weakly sequentially complete, the sequence $(\mathbb{Q}^N)_{N\in\mathbb{N}}$ would admit a weak limit $\mathbb{Q}^*$ in $\ell^1_+$. Since any weak limit of any minimising sequence must also belong to $\mathcal{M}$, $\mathbb{Q}^*$ would necessarily be a minimiser for $v^*$. However, as shown in [23, Example 5.1'], this contradicts the strict supermartingale property of the (unique) dual minimiser.

As a consequence of the above, for a given minimising sequence $(\mathbb{Q}^N)_{N\in\mathbb{N}}$, there exists a random variable $H \in \ell^\infty$ such that

$$\langle \mathbb{Q}^N, H\rangle \text{ does not converge in } \mathbb{R} \text{ as } N \to \infty.$$

Because $\langle \mathbb{Q}^N, 1\rangle = 1$ for each $N$, we can assume that $H \geq 1$. Moreover, there exist two subsequences $(\mathbb{Q}^{1,N})_{N\in\mathbb{N}}$ and $(\mathbb{Q}^{2,N})_{N\in\mathbb{N}}$ of $(\mathbb{Q}^N)_{N\in\mathbb{N}}$ such that the limits

$$y_1 = \lim_{N\to\infty} \langle \mathbb{Q}^{1,N}, H\rangle \;\text{ and }\; y_2 = \lim_{N\to\infty} \langle \mathbb{Q}^{2,N}, H\rangle \text{ exist with } y_1 \neq y_2. \tag{4.4}$$

With $H$ as above, we define $B := 1/H$ and a new stock price process with the increment

$$\Delta\tilde{S} := B\,\Delta S.$$

Then we consider the log-utility maximisation problem with the random endowment $B$, the stock price increment $\Delta\tilde{S}$ and the value function

$$\tilde{u}(x) := \sup_{\pi\in\mathbb{R}} \mathbb{E}[U(B + x + \pi\Delta\tilde{S})], \qquad x \in \mathbb{R}, \tag{4.5}$$

with the convention $\mathbb{E}[U(x + \pi\Delta\tilde{S} + B)] = -\infty$ if $\mathbb{E}[U(x + \pi\Delta\tilde{S} + B)^-] = +\infty$. We note that $\tilde{u}$ is the section at $q = 0$ of the function $u(x, q)$ defined in (4.1) in this particular setup.

The associated dual problem[2] is given by

$$\begin{aligned}
\tilde{v}(y) &:= \inf_{\tilde{\mathbb{Q}}\in\tilde{\mathcal{M}}} \mathbb{E}\bigg[V\bigg(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\bigg)\bigg] + y\langle\tilde{\mathbb{Q}}, B\rangle \\
&= -1 + \inf_{\tilde{\mathbb{Q}}\in\tilde{\mathcal{M}}} \bigg(\mathbb{E}\bigg[-\log\bigg(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\bigg)\bigg] + y\langle\tilde{\mathbb{Q}}, B\rangle\bigg) \\
&= -1 + \mathbb{E}[\log B] + \inf_{\tilde{\mathbb{Q}}\in\tilde{\mathcal{M}}} \bigg(\mathbb{E}\bigg[-\log\bigg(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}B\bigg)\bigg] + y\langle\tilde{\mathbb{Q}}B, 1\rangle\bigg) \\
&= \mathbb{E}[\log B] + \inf_{\tilde{\mathbb{Q}}\in\tilde{\mathcal{M}}} \bigg(\mathbb{E}\bigg[V\bigg(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}B\bigg)\bigg] + y\langle\tilde{\mathbb{Q}}B, 1\rangle\bigg), \qquad y > 0,
\end{aligned}$$

where the sets of measures $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{M}}$ are defined by

$$\tilde{\mathcal{Q}} := \{\tilde{\mathbb{Q}} \in \ell^1_+ : \langle\tilde{\mathbb{Q}}, \Delta\tilde{S}\rangle = 0\} \qquad \text{and} \qquad \tilde{\mathcal{M}} := \{\tilde{\mathbb{Q}} \in \tilde{\mathcal{Q}} : \langle\tilde{\mathbb{Q}}, 1\rangle = 1\},$$

[2] It has been shown in Larsen and Žitković [29, Lemma 3.12] that under the reasonable elasticity condition, infimisation over the set of countably additive martingale measures – as opposed to its finitely additive enlargement as in Cvitanić et al. [7] – leads to the same value function.

and the measure $\tilde{\mathbb{Q}}B \in \ell^1_+$ is defined by

$$\tilde{\mathbb{Q}}B(\Gamma) := \mathbb{E}^{\tilde{\mathbb{Q}}}[B 1_\Gamma], \qquad \Gamma \subseteq \Omega.$$

Because $\tilde{\mathbb{Q}} \in \tilde{\mathcal{Q}}$ if and only if $\mathbb{Q} = \tilde{\mathbb{Q}}B \in \mathcal{Q}$, we have

$$\begin{aligned}
\inf_{y>0} \tilde{v}(y) &= \mathbb{E}[\log B] + \inf_{y>0} \inf_{\mathbb{Q}\in\mathcal{M}} \left( \mathbb{E}\left[ V\left( y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] + y \right) \\
&= \mathbb{E}[\log B] + \inf_{y>0} \big( v(y) + y \big) \\
&= \mathbb{E}[\log B] + \inf_{y>0} \big( V(y) + y + v^* \big) \\
&= \mathbb{E}[\log B] + v^*.
\end{aligned}$$

By using the minimising sequences $(\mathbb{Q}^{1,N})_{N\in\mathbb{N}}$ and $(\mathbb{Q}^{2,N})_{N\in\mathbb{N}}$ constructed above, we define the sequence of probability measures

$$\tilde{\mathbb{Q}}^{i,N} := \frac{\mathbb{Q}^{i,N} H}{\langle \mathbb{Q}^{i,N} H, 1\rangle} \in \tilde{\mathcal{M}} \qquad \text{for } i = 1, 2.$$

In other words, we define the probability measures

$$\tilde{\mathbb{Q}}^{i,N}(\Gamma) := \frac{\mathbb{E}^{\mathbb{Q}^{i,N}}[H 1_\Gamma]}{\mathbb{E}^{\mathbb{Q}^{i,N}}[H]} \qquad \text{for } \Gamma \subseteq \Omega,\ i = 1, 2,\ N \in \mathbb{N}.$$

We can use (4.4) and the fact that $(\mathbb{Q}^{i,N})_{N\in\mathbb{N}}$, $i = 1, 2$, are minimising sequences for $v^*$ to see that

$$\begin{aligned}
\mathbb{E}\left[ V\left( y_i \frac{d\tilde{\mathbb{Q}}^{i,N}}{d\mathbb{P}} \right) \right] + y_i \langle \tilde{\mathbb{Q}}^{i,N}, B\rangle &= \mathbb{E}\left[ V\left( \frac{y_i H}{\langle \mathbb{Q}^{i,N} H, 1\rangle} \frac{d\mathbb{Q}^{i,N}}{d\mathbb{P}} \right) \right] + \frac{y_i}{\langle \mathbb{Q}^{i,N} H, 1\rangle} \\
&= \mathbb{E}\left[ V\left( \frac{y_i H}{\langle \mathbb{Q}^{i,N} H, 1\rangle} \right) \right] + \frac{y_i}{\langle q^{i,N} H, 1\rangle} \\
&\quad - \mathbb{E}\left[ \log \frac{d\mathbb{Q}^{i,N}}{d\mathbb{P}} \right] \\
&\longrightarrow \mathbb{E}[\log B] + v^* \\
&= \inf_{y>0} \tilde{v}(y).
\end{aligned}$$

Clearly, $\tilde{v}(y_i) \geq \inf_{y>0} \tilde{v}(y)$ for $i = 1, 2$, which implies that $(\tilde{\mathbb{Q}}^{i,N})_{N\in\mathbb{N}}$ is a minimising sequence for $\tilde{v}(y_i)$. Therefore, $\tilde{v}(y_1) = \tilde{v}(y_2) = \inf_{y>0} \tilde{v}(y)$ which implies that $\tilde{v}$ is constant on $[y_1, y_2]$. This in turn implies that the conjugate function to $\tilde{v}$ (and by duality, this conjugate function equals the primal value function) fails to be differentiable at 0. Indeed, the entire segment $[y_1, y_2]$ belongs to the primal value function's supergradient at zero. □

*Remark 4.4* 1) The construction of the random endowment $B$ in Example 4.3 above rests on the weak sequential completeness property of $\ell^1$, which in fact holds for any $\mathbb{L}^1$-space. Example 4.3 above is therefore generic in the sense that it can be applied to any model which produces non-trivial singular components in the dual optimiser for the log-investor (with constant endowment). This implies that there also exist random endowments in the Brownian setting of [23, Example 5.1] which produce a non-differentiable primal utility function.

2) Example 4.3 contradicts the claimed continuous differentiability of the primal value function stated in Cvitanić et al. [7, Theorem 3.1(i)] (we also refer the reader to the Erratum [8] for further discussion): the function $\tilde{u}$ defined in (4.5) within Example 4.3 is not differentiable at $x = 0$ which is an interior point of its domain.

### 4.3 A characterisation via a linear stochastic control problem

Even though the supergradient of $\mathfrak{U}$ at $B$ consists of finitely additive measures related to the solution of the dual problem, it is possible to give a characterisation of directional derivatives without any recourse to finite additivity. This is the most attractive feature of our linear characterisation in Proposition 4.7 below; moreover, as we shall see later, it also leads to explicit computations in many cases. The price we pay is the increased complexity of the linearised problem's domain.

**Throughout the remainder of the paper**, we impose the following assumption, where $\hat{X}$ is the primal optimiser characterised by (3.7), and whose existence is guaranteed by the assumption of reasonable elasticity.

**Assumption 4.5** $U$ is reasonably elastic, and there exists a constant $b > 0$ such that

$$\hat{X}\, U'\big((1-b)\hat{X}\big) \in \mathbb{L}^1(\mathbb{P}). \tag{4.6}$$

*Remark 4.6* Assumption 4.5 holds automatically if $U$ belongs to the class of CRRA (power) utilities

$$U(x) := \frac{x^p}{p} \text{ for } p \in (-\infty, 1) \setminus \{0\} \quad \text{or} \quad U(x) := \log x, \qquad x > 0.$$

More generally, suppose that $U$ admits an upper bound on the relative risk-aversion, i.e., $U \in C^2(0,\infty)$ and there exists a constant $c \in (0,\infty)$ such that

$$\frac{-xU''(x)}{U'(x)} \le c \qquad \text{for all } x > 0.$$

It follows directly that $x^c U''(x) + cx^{c-1}U'(x) \ge 0$ for all $x > 0$ which implies that the function $x^c U'(x)$ is nondecreasing on $(0,\infty)$. Therefore, $(\frac{1}{2}x)^c U'(\frac{1}{2}x) \le x^c U'(x)$, and so $U'(\frac{1}{2}x) \le 2^c U'(x)$ for all $x > 0$. By combining this inequality with the first-order condition (3.7), we see that

$$\hat{X} U'\left(\frac{1}{2}\hat{X}\right) \le 2^c \hat{X} U'(\hat{X}) = 2^c \hat{X} \frac{d\hat{\mu}^r}{d\mathbb{P}}.$$

Consequently, we have

$$\mathbb{E}\left[\hat{X}U'\left(\frac{1}{2}\hat{X}\right)\right] \le 2^c\mathbb{E}\left[\hat{X}\frac{d\hat{\mu}^r}{d\mathbb{P}}\right] \le 2^c\langle\hat{X},\hat{\mu}\rangle < \infty.$$

Hence (4.6) holds with $b := \frac{1}{2}$.

Given the optimiser $\hat{\pi} \in \mathcal{A}$ and the random variable $\hat{X}$, set $\Delta(\varphi) := \bigcup_{\varepsilon>0}\Delta^\varepsilon(\varphi)$, where $\Delta^\varepsilon(\varphi)$ denotes the class of all $\delta \in L(S)$ such that

$$\hat{\pi} + \varepsilon\delta \in \mathcal{A} \quad \text{and} \quad \hat{X} + \varepsilon\big(\varphi + (\delta\cdot S)_T\big) \ge 0. \tag{4.7}$$

Because $\mathcal{A}$ is a convex cone and $\hat{X} \ge 0$, the family $\Delta^\varepsilon(\varphi)$ is nonincreasing in $\varepsilon \ge 0$ in the sense that

$$\varepsilon_1 \le \varepsilon_2 \quad \text{implies} \quad \Delta^{\varepsilon_2}(\varphi) \subseteq \Delta^{\varepsilon_1}(\varphi).$$

Similarly, the family $\Delta^\varepsilon(\varphi)$ is nondecreasing in $\varphi \in \mathbb{L}^\infty$ in the sense that

$$\varphi_1 \le \varphi_2 \quad \text{implies} \quad \Delta^\varepsilon(\varphi_1) \subseteq \Delta^\varepsilon(\varphi_2).$$

**Proposition 4.7** *Under Assumption 4.5, we have for $\varphi \in \mathbb{L}^\infty(\mathbb{P})$ that*

$$\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\big(u(0,\varepsilon) - u(0,0)\big) = \sup_{\delta\in\Delta(\varphi)}\mathbb{E}\big[\hat{Y}\big((\delta\cdot S)_T + \varphi\big)\big],$$

*where $\hat{Y} := \frac{d\hat{\mu}^r}{d\mathbb{P}}$.*

*Proof* For small enough $\varepsilon > 0$, we can find $\pi^\varepsilon \in \mathcal{A}$ such that $X^\varepsilon = (\pi^\varepsilon\cdot S)_T + B + \varepsilon\varphi$ has the property that

$$\mathbb{E}[U(X_T^\varepsilon)] \ge \mathfrak{U}(B + \varepsilon\varphi) - \varepsilon^2.$$

For such an $\varepsilon > 0$, we define

$$\delta^\varepsilon = \frac{1}{\varepsilon}(\pi^\varepsilon - \hat{\pi}).$$

Since $\hat{\pi} + \varepsilon\delta^\varepsilon = \pi^\varepsilon \in \mathcal{A}$, the first part of (4.7) above holds. To see that the second part of (4.7) holds, we note that $\hat{X} + \varepsilon((\delta^\varepsilon\cdot S)_T + \varphi) = X^\varepsilon$ and $\mathbb{E}[U(X^\varepsilon)] > -\infty$ which implies $\hat{X} + \varepsilon((\delta^\varepsilon\cdot S)_T + \varphi) \ge 0$. Therefore, we have $\delta^\varepsilon \in \Delta^\varepsilon(\varphi)$. The concavity of $\mathfrak{U}$ then implies that

$$\begin{aligned}
\mathfrak{U}(B + \varepsilon\varphi) &\le \mathbb{E}[U(X^\varepsilon)] + \varepsilon^2 \\
&\le \mathbb{E}[U(\hat{X})] + \varepsilon\mathbb{E}\big[U'(\hat{X})\big(\varphi + (\delta^\varepsilon\cdot S)_T\big)\big] + \varepsilon^2 \\
&\le \mathfrak{U}(B) + \varepsilon\sup_{\delta\in\Delta^\varepsilon(\varphi)}\mathbb{E}\big[\hat{Y}\big((\delta\cdot S)_T + \varphi\big)\big] + \varepsilon^2 \\
&\le \mathfrak{U}(B) + \varepsilon\sup_{\delta\in\Delta(\varphi)}\mathbb{E}\big[\hat{Y}\big((\delta\cdot S)_T + \varphi\big)\big] + \varepsilon^2.
\end{aligned}$$

This produces the upper bound

$$\limsup_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\big(\mathfrak{U}(B+\varepsilon\varphi)-\mathfrak{U}(B)\big)\le \sup_{\delta\in\Delta(\varphi)}\mathbb{E}\big[\hat{Y}\big((\delta\cdot S)_T+\varphi\big)\big].$$

To prove the opposite inequality, we pick $\varepsilon_0>0$ and $\delta\in\Delta^{\varepsilon_0}(\varphi)$ so that $\hat{\pi}+\varepsilon_0\delta\in\mathcal{A}$ and $\hat{X}+\varepsilon_0 D\ge 0$, where

$$D=(\delta\cdot S)_T+\varphi.$$

Because $b>0$, we also have

$$\hat{X}+b\varepsilon_0 D\ge(1-b)\hat{X}.$$

Therefore, for $\varepsilon\in(0,\varepsilon_1)$ with $\varepsilon_1:=b\varepsilon_0$, we have

$$\hat{X}+\varepsilon D\ge(1-b)\hat{X}>0. \tag{4.8}$$

The concavity of $U$ implies that for $\varepsilon\in(0,\varepsilon_1)$, we have

$$U(\hat{X}+\varepsilon D)\ge U(\hat{X})+\varepsilon Y^\varepsilon D,\qquad \text{where } Y^\varepsilon=U'(\hat{X}+\varepsilon D).$$

Therefore, for $\varepsilon\in(0,\varepsilon_1)$, we obtain

$$\mathfrak{U}(B+\varepsilon\varphi)\ge\mathbb{E}[U(\hat{X}+\varepsilon D)]\ge\mathfrak{U}(B)+\varepsilon\mathbb{E}[Y^\varepsilon D].$$

In order to pass $\varepsilon$ to zero, we note that (4.8) gives

$$(Y^\varepsilon D)^-\le U'\big((1-b)\hat{X}\big)D^-\le U'\big((1-b)\hat{X}\big)\frac{1}{\varepsilon_0}\hat{X}, \tag{4.9}$$

which is integrable by assumption. The uniform bound in (4.9) allows us to use Fatou's lemma together with $Y^\varepsilon\to U'(\hat{X})=:\hat{Y}$ $\mathbb{P}$-a.s. to conclude that

$$\liminf_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\big(\mathfrak{U}(B+\varepsilon\varphi)-\mathfrak{U}(\varphi)\big)\ge\liminf_{\varepsilon\searrow 0}\mathbb{E}[Y^\varepsilon D]\ge\mathbb{E}[\hat{Y}D].\qquad\square$$

The goal of the following example is to show just how non-standard the linear optimisation problem of Proposition 4.7 is: Even under non-pathological conditions such as log-utility and a constant random endowment, its maximisers can be strict local martingales.

*Example 4.8* Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space supporting two independent Brownian motions $(Z,W)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ their augmented filtration up to some maturity $T>0$. We define the stock price dynamics to be

$$dS_t:=S_t(\lambda_t dt+dZ_t),\qquad S_0>0,$$

where the process $\lambda\in\mathcal{L}^2$ is such that the *minimal martingale density* (see Föllmer and Schweizer [16] for the definition and further discussion)

$$\mathcal{E}(-\lambda\cdot Z)_t:=e^{-\int_0^t\lambda_u dZ_u-\frac{1}{2}\int_0^t\lambda_u^2du},\qquad t\in[0,T],$$

fails the martingale property even though the set $\mathcal{M}$ of equivalent local martingale measures is nonempty. (An example of such a process $\lambda$ can be found in Delbaen and Schachermayer [11, Theorem 2.1], but its exact form it not important for this example.) As a consequence, the log-investor's dual optimiser $\hat{Y} := \hat{Y}_0 \mathcal{E}(-\lambda \cdot Z)$, where $\hat{Y}_0 > 0$ is a Lagrange multiplier, is a strict local martingale (see Kramkov and Schachermayer [23, Example 5.1 and Proposition 5.1] for details). We consider the constant endowment case where $B \equiv 1$ and the claim $\varphi$ is a positive constant. The fact that we are working with the log-utility function implies that $\hat{Y}_0 = 1$, which in our notation translates to

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\big(u(0,\varepsilon) - u(0,0)\big) = \varphi.$$

The log-utility function satisfies Assumption 4.5, which allows us to use Proposition 4.7 to conclude that

$$\sup_{\delta \in \Delta(\varphi)} \mathbb{E}\big[\hat{Y}_T\big((\delta \cdot S)_T + \varphi\big)\big] = \varphi. \tag{4.10}$$

For any $\delta \in \Delta(\varphi)$ for which the local martingale $\hat{Y}(\delta \cdot S)$ is a martingale, we have $\mathbb{E}[\hat{Y}_T\big((\delta \cdot S)_T + \varphi\big)] = \mathbb{E}[\hat{Y}_T \varphi]$. Since $\hat{Y}$ is a strict local martingale, this value $\mathbb{E}[\hat{Y}_T \varphi] = \varphi \mathbb{E}[\hat{Y}_T]$ is strictly smaller than the value $\varphi$ of the supremum in (4.10). On the other hand, the supremum in (4.10) is attained at any $\delta \in \Delta(\varphi)$ which satisfies the requirement $\mathbb{E}[\hat{Y}_T((\delta \cdot S)_T + \varphi)] = \varphi$. This requirement is in turn satisfied by any $\delta \in \Delta(\varphi)$ such that $(\delta \cdot S)_T + \varphi = \varphi \hat{X}_T$. □

## 5 Minimally superreplicable random variables

While the linear control problem of Proposition 4.7 provides a useful characterisation of $\mathfrak{U}$'s directional derivatives, it seems difficult to solve explicitly in full generality. The present section outlines a relevant class of payoffs $\varphi$ for which such a tractable solution is available. It involves the concept of minimal superreplicability, which is similar to the notion of unique superreplicability from Larsen et al. [28, Condition (B1)].

**Definition 5.1** A random variable $\psi \in \mathbb{L}^\infty(\mathbb{P})$ is said to be *boundedly replicable* if there exist a constant $\psi_0 \in \mathbb{R}$ and $\pi_\psi \in \mathcal{A} \cap (-\mathcal{A})$ such that

$$\psi = \psi_0 + (\pi_\psi \cdot S)_T.$$

It is called *minimally superreplicable (by* $\Psi$*)* if $\Psi \in \mathbb{L}^\infty(\mathbb{P})$ is boundedly replicable, $\Psi \geq \psi$ and

$$x + (\pi \cdot S)_T \geq \psi \quad \text{implies} \quad x + (\pi \cdot S)_T \geq \Psi$$

for all $x \in \mathbb{R}$ and $\pi \in \mathcal{A}$.

*Remark 5.2* 1) The need to use uniformly bounded gains processes for replication purposes such as in Definition 5.1 has long been recognised; see e.g. Shiryaev and Cherny [5, Definition 1.15] and the first part of Remark 3.2 in Hugonnier et al. [20]. To highlight the need to have $\pi_\psi \in \mathcal{A} \cap (-\mathcal{A})$ instead of just $\pi_\psi \in \mathcal{A}$, we offer the following example. Consider a claim with the constant (deterministic) payoff $\varphi \in (0,\infty)$ in a Samuelson model with the stock price process $dS_t := S_t dB_t$. According to Duffie [12, Chap. 6C], there exists a portfolio $\pi$ such that $-\pi \in \mathcal{A}$ and $0 + \int_0^T \pi_t dS_t = \varphi$, which produces the representation

$$2\varphi - \int_0^T \pi_t dS_t = \varphi.$$

This implies that both $\varphi$ and $2\varphi$ qualify as initial claim prices, and so this weaker notion of replicability fails the law of one price.

2) The representation in Definition 5.1 of a boundedly replicable claim $\psi$ in terms of $(\psi_0, \pi_\psi)$ is unique. Moreover, the process $\pi_\psi \cdot S$ is a bounded $\mathbb{Q}$-martingale for each $\mathbb{Q} \in \mathcal{M}$. Consequently, because each $\mu \in \mathcal{D}$ is the weak*-limit of a net $(y_\alpha \mathbb{Q}_\alpha)$ with $y_\alpha \in [0,\infty)$ and $\mathbb{Q}_\alpha \in \mathcal{M}$, we have

$$\langle \mu, (\pi_\psi \cdot S)_t \rangle = \lim_\alpha y_\alpha \langle \mathbb{Q}_\alpha, (\pi_\psi \cdot S)_t \rangle = 0.$$

3) Provided it exists, the random variable $\Psi$ in Definition 5.1 is unique. If the random variable $\Psi = \Psi_0 + (\pi_\Psi \cdot S)_T$ minimally superreplicates $\psi$, we have the representation

$$\Psi_0 = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\psi].$$

4) Minimal superreplicability is scale invariant: If $\psi$ is minimally superreplicable by $\Psi$, then $\alpha\psi$ is minimally superreplicable by $\alpha\Psi$ for $\alpha \geq 0$. It is also invariant under translation by boundedly replicable random variables. In particular, boundedly replicable random variables are minimally superreplicable.

*Example 5.3* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting two independent Brownian motions $(B, W)$ and $(\mathcal{F}_t)_{t \in [0,T]}$ their augmented filtration up to some maturity $T > 0$. We let $S$ be the Itô process

$$dS_t := S_t \sigma_t (\lambda_t dt + dB_t), \qquad S_0 > 0,$$

where $\sigma, \lambda \in \mathcal{L}^2$ are such that NFLVR holds. We focus on payoffs of the form $\varphi(W_T)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded uniformly Lipschitz function. To show that such contingent claims $\varphi(W_T)$ are minimally superreplicable by the constant $\sup_a \varphi(a)$, we start by assuming that

$$x + (\pi \cdot S)_T \geq \varphi(W_T) \qquad \text{a.s.}$$

for some $x \in \mathbb{R}$ and $\pi \in \mathcal{A}$. Then for each $t \in [0,T)$, we have

$$x + (\pi \cdot S)_t \geq \operatorname*{esssup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[x + (\pi \cdot S)_T | \mathcal{F}_t] \geq \operatorname*{esssup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\varphi(W_T) | \mathcal{F}_t]. \tag{5.1}$$

Lemma 5.4 below gives conditions under which the limit as $t \uparrow T$ of the right-hand side of (5.1) equals $\sup_a \varphi(a)$. When these conditions are met, the continuity of the paths of the stochastic integral with respect to $S$ implies that $x + (\pi \cdot S)_T \geq \sup_a \varphi(a)$. This in turn confirms that $\varphi(W_T)$ is minimally superreplicable by the constant $\sup_a \varphi(a)$.

**Lemma 5.4** *In the setting of Example 5.3 above with $\varphi : \mathbb{R} \to \mathbb{R}$ bounded and uniformly Lipschitz, assume that there exist a nonnegative function $f \in \mathbb{L}^1([0,T])$ and a predictable process $\nu^{(0)} \in \mathcal{L}^2$ such that*

1) $|\nu_u^{(0)}| \leq f(u)$ *for Lebesgue-almost all $u \in [0,T]$, $\mathbb{P}$-a.s.;*

2) *the stochastic exponential $Z_T^{(0)} := \mathcal{E}(-\lambda \cdot B - \nu^{(0)} \cdot W)_T$ is the Radon–Nikodým density of some $\mathbb{Q}^{(0)} \in \mathcal{M}$ with respect to $\mathbb{P}$.*

*Then*

$$\lim_{t \uparrow T} \operatorname*{esssup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\varphi(W_T)|\mathcal{F}_t] = \sup_a \varphi(a). \tag{5.2}$$

*Proof* For a bounded and predictable process $\delta$, we define the process $Z^{(\delta)}$ by

$$dZ_t^{(\delta)} = -Z_t^{(\delta)}\big(\lambda_t \, dB_t + (\nu_t^{(0)} + \delta_t)\, dW_t\big), \qquad Z_0^{(\delta)} = 1.$$

A simple calculation yields the expression

$$Z_T^{(\delta)} = Z_T^{(0)} \mathcal{E}(-\delta \cdot W^{(0)})_T,$$

where $W_t^{(0)} := W_t + \int_0^t \nu_u^{(0)}\, du$, $0 \leq t \leq T$, is a $\mathbb{Q}^{(0)}$-Brownian motion. With $\mathbb{E}^{(0)}$ denoting the expectation with respect to $\mathbb{Q}^{(0)}$, we have

$$\mathbb{E}[Z_T^{(\delta)}] = \mathbb{E}^{(0)}[\mathcal{E}(-\delta \cdot W^{(0)})_T] = 1,$$

where the last equality follows from the boundedness of $\delta$. Hence $Z^{(\delta)}$ is a (true) martingale and can be used as the density of a probability measure $\mathbb{Q}^{(\delta)} \in \mathcal{M}$.

To proceed, we fix $t_0 \in (0,T)$ and $a \in \mathbb{R}$ and define

$$\delta_t^{(a)} := \frac{1}{T - t_0}(W_{t_0} \mathbf{1}_{\{|W_{t_0}| \leq 1/(T-t_0)\}} - a)\mathbf{1}_{\{t \geq t_0\}}, \qquad t \in [t_0, T],$$

$$W_t^{(a)} := W_t + \int_0^t (\nu_u^{(0)} + \delta_u^{(a)})\, du, \qquad t \in [0,T].$$

Then we have

$$W_T - a = W_T^{(a)} - W_{t_0}^{(a)} - \int_{t_0}^T \nu_u^{(0)}\, du + W_{t_0} \mathbf{1}_{\{|W_{t_0}| > 1/(T-t_0)\}}.$$

The process $W^{(a)}$ is a $\mathbb{Q}^{(a)}$-Brownian motion, where $\mathbb{Q}^{(a)}$ is a short for $\mathbb{Q}^{(\delta^{(a)})}$. Therefore the bound $|\nu^{(0)}| \leq f$ implies that

$$\mathbb{E}^{\mathbb{Q}^{(a)}}\big[|W_T - a| \big| \mathcal{F}_{t_0}\big] \leq C(t_0),$$

where

$$C(t_0) := \sqrt{\frac{2(T-t_0)}{\pi}} + \int_{t_0}^{T} f_u\,du + |W_{t_0}|\mathbf{1}_{\{|W_{t_0}|>1/(T-t_0)\}}.$$

With $L_\varphi$ denoting the uniform Lipschitz constant of $\varphi$, we have

$$\left|\mathbb{E}^{\mathbb{Q}^{(a)}}[\varphi(W_T)|\mathcal{F}_{t_0}] - \varphi(a)\right| \le L_\varphi\,\mathbb{E}^{\mathbb{Q}^{(a)}}\left[|W_T - a|\middle|\mathcal{F}_{t_0}\right] \le L_\varphi C(t_0).$$

Therefore,

$$\begin{aligned}\limsup_{t_0\nearrow T}\operatorname*{esssup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\varphi(W_T)|\mathcal{F}_{t_0}] &\ge \limsup_{t_0\nearrow T} \mathbb{E}^{\mathbb{Q}^{(a)}}[\varphi(W_T)|\mathcal{F}_{t_0}]\\ &\ge \limsup_{t_0\nearrow T}\big(\varphi(a) - L_\varphi C(t_0)\big) = \varphi(a).\end{aligned}$$

It remains to note that the left-hand side above does not depend on $a$ and that $\sup_a \varphi(a)$ is a trivial upper bound in (5.2). □

*Example 5.5* We continue Example 5.3 by examining two cases in which Lemma 5.4 applies. In the first one, we simply take $f \equiv 0$. That can be done if and only if the minimal martingale density $\mathcal{E}(-\lambda \cdot B)$ defines a martingale, which is the case in many popular models including the incomplete models developed in Kraft [22] and Kim and Omberg [26].

In the second case, $\mathcal{E}(-\lambda \cdot B)$ is a strict local martingale, but NFLVR nevertheless still holds. A famous example of a model where this occurs is given in Delbaen and Schachermayer [11]. We present here a time-changed version (using the standard logarithmic time transform $t \mapsto -\log(1-t)$), as the original version in [11, Theorem 2.1] is defined on an infinite horizon. Using the notation of Example 5.3 with $T := 1$, we define the local martingales $(B'_t)_{t\in[0,1)}$ and $(W'_t)_{t\in[0,1)}$ by

$$B'_t := \int_0^t \frac{1}{\sqrt{1-u}}\,dB_u \quad \text{and} \quad W'_t := \int_0^t \frac{1}{\sqrt{1-u}}\,dW_u, \qquad t\in[0,1),$$

as well as the stopping times

$$\tau := \inf\left\{t>0 : \mathcal{E}(B') = \frac{1}{2}\right\} \quad \text{and} \quad \sigma := \inf\{t>0 : \mathcal{E}(W') = 2\}.$$

With the processes $(\lambda_t)_{t\in[0,1]}$ and $(\nu_t^{(0)})_{t\in[0,1]}$ defined by

$$\lambda_t := -\frac{\mathbf{1}_{[0,\sigma\wedge\tau]}(t)}{\sqrt{1-t}}, \qquad \nu_t^{(0)} := -\frac{\mathbf{1}_{[0,\sigma\wedge\tau]}(t)}{\sqrt{1-t}},$$

it remains to apply [11, Theorem 2.1] to conclude that the NFLVR condition is satisfied, but that the minimal martingale density $\mathcal{E}(-\lambda\cdot B)$ is a strict local martingale. Our Lemma 5.4 applies because $|\nu_t^{(0)}| \le \frac{1}{\sqrt{1-t}} \in \mathbb{L}^1([0,1])$.

We mention that Examples 5.3 and 5.5 as well as Lemma 5.4 will be used again in the examples in Sect. 6.

The next example shows that it is quite easy to construct bounded payoffs $\psi$ which fail to be minimally superreplicable.

*Example 5.6* We consider the one-period model with three states given by

$$\Delta S := (1, 0, -1)', \qquad \psi := (-1, 0, -1)',$$

where $a'$ denotes the transpose of the vector $a$. The set of pairs $(x, \pi)$ for which $x + \pi \Delta S \geq \psi$ is given by $x \geq 0$ and $\pi \in [-1-x, 1+x]$. However, the corresponding set of gain outcomes $(x+\pi, x, x-\pi)'$ with $x \geq 0$ and $\pi \in [-1-x, 1+x]$ does not contain a smallest element. Indeed, if $(a, b, c)$ is a smallest element, we have $a \leq -1, b \leq 0$ and $c \leq -1$, but such an element $(a, b, c)$ is not the outcome of any gains process $x + \pi \Delta S$ with $x \geq 0$ and $\pi \in [-1-x, 1+x]$.

The main technical result of this section is the following proposition.

**Proposition 5.7** *Under Assumption 4.5, suppose that $-B$ and $-(B+\varepsilon\varphi)$ are minimally superreplicable by $-\underline{B}$ and $-(\underline{B+\varepsilon\varphi})$, respectively, for all $\varepsilon > 0$ in some neighbourhood of* 0. *Then for each $\hat{\mu} \in \hat{\mathcal{D}}(B)$, we have*

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\big(u(0,\varepsilon) - u(0,0)\big) = \mathbb{E}[\hat{Y}\varphi] + \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\langle \hat{\mu}^s, \underline{B+\varepsilon\varphi} - \underline{B}\rangle, \tag{5.3}$$

*where* $\hat{Y} = \frac{d\hat{\mu}^r}{d\mathbb{P}}$.

*Proof* For $\varepsilon > 0$, we let $x_\varepsilon, x_0 \in \mathbb{R}$ and $\pi_\varepsilon, \pi_0 \in \mathcal{A} \cap (-\mathcal{A})$ be such that

$$\underline{B+\varepsilon\varphi} = \varepsilon x_\varepsilon + \varepsilon(\pi_\varepsilon \cdot S)_T \quad \text{and} \quad \underline{B} = x_0 + (\pi_0 \cdot S)_T\,.$$

Because $B$ is bounded away from zero and $\varphi \in \mathbb{L}^\infty(\mathbb{P})$, we can consider $\varepsilon > 0$ so small that $x_\varepsilon, x_0 > 0$. For $\delta \in \Delta^\varepsilon(\varphi)$, we have $\varepsilon\delta + \hat{\pi} \in \mathcal{A}$ and

$$\varepsilon(\delta \cdot S)_T + (\hat{\pi} \cdot S)_T \geq -\varepsilon\varphi - B.$$

Therefore, by the minimal superreplicability of $\underline{B+\varepsilon\varphi}$, we have

$$0 \leq x_\varepsilon + (\delta \cdot S)_T + \frac{1}{\varepsilon}(\hat{\pi} \cdot S)_T + (\pi_\varepsilon \cdot S)_T\,. \tag{5.4}$$

Since $\underline{B}$ minimally superreplicates $B$, we have $\underline{B} + (\hat{\pi} \cdot S)_T \geq 0$. Therefore, for any $\hat{\mu} \in \hat{\mathcal{D}}(B)$, the first part of Cvitanić et al. [7, Eq. (4.7)] produces

$$0 \leq \langle \hat{\mu}^s, \underline{B} + (\hat{\pi} \cdot S)_T\rangle \leq \langle \hat{\mu}^s, B + (\hat{\pi} \cdot S)_T\rangle = 0. \tag{5.5}$$

The second part of [7, Eq. (4.7)] ensures that $\langle \hat{\mu}, (\hat{\pi} \cdot S)_T\rangle = 0$, and combining this with $\langle \hat{\mu}^s, \underline{B} + (\hat{\pi} \cdot S)_T\rangle = 0$, we see

$$\langle \hat{\mu}^r, \underline{B} + (\hat{\pi} \cdot S)_T\rangle = \langle \hat{\mu}, \underline{B} + (\hat{\pi} \cdot S)_T\rangle = \langle \hat{\mu}, \underline{B}\rangle.$$

Because $\hat{Y} = \frac{d\hat{\mu}^r}{d\mathbb{P}}$, we obtain the representation

$$\mathbb{E}[\hat{Y}(\hat{\pi}\cdot S)_T] = \langle \hat{\mu}^s, \underline{B}\rangle.$$

The property $\varepsilon\delta + \hat{\pi} \in \mathcal{A}$ gives $\langle \hat{\mu}, \varepsilon(\delta\cdot S)_T + (\hat{\pi}\cdot S)_T\rangle \leq 0$ and $\langle \hat{\mu}, (\pi_\varepsilon \cdot S)_T\rangle = 0$ for each $\varepsilon > 0$. Therefore, by (5.4), we find

$$\begin{aligned}
&\mathbb{E}\Big[\hat{Y}\Big(x_\varepsilon + (\pi_\varepsilon\cdot S)_T + (\delta\cdot S)_T + \frac{1}{\varepsilon}(\hat{\pi}\cdot S)_T\Big)\Big] \\
&\leq \Big\langle \hat{\mu}, x_\varepsilon + (\pi_\varepsilon\cdot S)_T + (\delta\cdot S)_T + \frac{1}{\varepsilon}(\hat{\pi}\cdot S)_T\Big\rangle \\
&\leq \langle \hat{\mu}, x_\varepsilon\rangle.
\end{aligned} \tag{5.6}$$

To show that the upper bound in (5.6) above is attained, we pick

$$\delta_\varepsilon = \Big(\frac{x_\varepsilon}{x_0} - \frac{1}{\varepsilon}\Big)\hat{\pi} + \frac{x_\varepsilon}{x_0}\pi_0 - \pi_\varepsilon.$$

Because $x_\varepsilon > 0$ and $x_0 > 0$, one can check that $\delta_\varepsilon \in \Delta^\varepsilon(\varphi)$. Then we have

$$\begin{aligned}
&\mathbb{E}\Big[\hat{Y}\Big(x_\varepsilon + (\pi_\varepsilon\cdot S)_T + (\delta_\varepsilon\cdot S)_T + \frac{1}{\varepsilon}(\hat{\pi}\cdot S)_T\Big)\Big] \\
&= \mathbb{E}\Big[\hat{Y}\Big(x_\varepsilon + \frac{x_\varepsilon}{x_0}\big((\hat{\pi}+\pi_0)\cdot S\big)_T\Big)\Big] \\
&= \frac{x_\varepsilon}{x_0}\mathbb{E}\big[\hat{Y}\big(\underline{B} + (\hat{\pi}\cdot S)_T\big)\big] \\
&= \frac{x_\varepsilon}{x_0}\langle \hat{\mu}, \underline{B} + (\hat{\pi}\cdot S)_T\rangle \\
&= \langle \hat{\mu}, x_\varepsilon\rangle,
\end{aligned}$$

where the last equality follows from $\langle \hat{\mu}, \underline{B}\rangle = \langle \hat{\mu}, x_0\rangle$ and $\langle \hat{\mu}, (\hat{\pi}\cdot S)_T\rangle = 0$. Therefore, $\delta_\varepsilon$ indeed attains the upper bound of (5.6), and so we get

$$\begin{aligned}
&\sup_{\delta\in\Delta^\varepsilon(\varphi)} \mathbb{E}\big[\hat{Y}\big((\delta\cdot S)_T + \varphi\big)\big] \\
&= \langle \hat{\mu}, x_\varepsilon\rangle - \frac{1}{\varepsilon}\mathbb{E}[\hat{Y}(\underline{B+\varepsilon\varphi})] - \frac{1}{\varepsilon}\mathbb{E}[\hat{Y}(\hat{\pi}\cdot S)_T] + \mathbb{E}[\hat{Y}\varphi] \\
&= \langle \hat{\mu}^r, \varphi\rangle + \frac{1}{\varepsilon}\langle \hat{\mu}^s, \underline{B+\varepsilon\varphi} - \underline{B}\rangle.
\end{aligned}$$

The sets $\Delta^\varepsilon(\varphi)$ monotonically increase to $\Delta(\varphi)$ as $\epsilon \searrow 0$. This implies

$$\sup_{\delta\in\Delta^\varepsilon(\varphi)} \mathbb{E}\big[\hat{Y}\big((\delta\cdot S)_T + \varphi\big)\big] \nearrow \sup_{\delta\in\Delta(\varphi)} \mathbb{E}\big[\hat{Y}\big((\delta\cdot S)_T + \varphi\big)\big] \tag{5.7}$$

as $\varepsilon \searrow 0$. Because the left-hand side of (5.7) equals $\langle \hat{\mu}^r, \varphi\rangle + \frac{1}{\varepsilon}\langle \hat{\mu}^s, \underline{B+\varepsilon\varphi} - \underline{B}\rangle$, we see that (5.3) holds by Proposition 4.7. □

A first consequence of Proposition 5.7 is that the situation encountered in Example 4.3 cannot happen if $-B$ if minimally superreplicable. Indeed, the primal value function $\mathfrak{U}$ is then differentiable in all boundedly replicable directions.

**Corollary 5.8** *Suppose that Assumption 4.5 holds and that $-B$ is minimally superreplicable. Then there exists a constant $y_B > 0$ such that*

$$y_B = \hat{\mu}(\Omega) \qquad \textit{for each } \hat{\mu} \in \hat{\mathcal{D}}(B).$$

*Moreover, for each boundedly replicable claim $\varphi$, the two-sided limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(u(0,\varepsilon) - u(0,0)\big) = \langle \hat{\mu}, \varphi \rangle \qquad \textit{for each } \hat{\mu} \in \hat{\mathcal{D}}(B) \tag{5.8}$$

*exists.*

*Proof* First observe that for boundedly replicable $\varphi$, we have $\underline{B + \varepsilon\varphi} = \underline{B} + \varepsilon\varphi$. Then we can apply Proposition 5.7 to both $\varphi$ and $-\varphi$ to conclude that (5.8) holds. The first claim follows by setting $\varphi \equiv 1$. □

Now that we have identified the circumstances under which all dual minimisers $\hat{\mu} \in \hat{\mathcal{D}}(B)$ have the same total mass, the following result follows directly from Theorem 3.5 and Corollary 5.8 above.

**Corollary 5.9** *Suppose that Assumption 4.5 holds and that $-B$ is minimally superreplicable. Then each boundedly replicable $\varphi \in \mathbb{L}^\infty(\mathbb{P})$ has the unique $B$-conditional Davis price $\langle \hat{\mu}, \varphi \rangle / \hat{\mu}(\Omega)$.*

When $B$ is a constant (and more generally, when $B$ is boundedly replicable), it is known that the product of the primal and dual optimisers is a martingale (see e.g. [23, Theorem 2.2]). When the dual optimiser is only a finitely additive measure, the following corollary may serve as a surrogate. The result relies on Karatzas and Žitković [21] where a positive supermartingale deflator $(\hat{Y}_t)_{t\in[0,T]}$ is constructed from $\hat{\mu} \in \hat{\mathcal{D}}(B)$ (see [21, Eq. (2.5)]).

**Corollary 5.10** *Suppose that Assumption 4.5 holds, that $-B$ is minimally superreplicable by $-\underline{B}$, and write*

$$\underline{B} = x_0 + (\pi_0 \cdot S)_T.$$

*Let $\hat{\pi} \in \mathcal{A}$ denote the optimiser for the problem $\sup_{\pi\in\mathcal{A}} \mathbb{E}[U(B + (\pi \cdot S)_T)]$. Then the process*

$$\hat{Y}_t\Big(x_0 + \big((\pi_0 + \hat{\pi}) \cdot S\big)_t\Big), \qquad t \in [0,T],$$

*is a nonnegative martingale, where $(\hat{Y}_t)_{t\in[0,T]}$ is the supermartingale deflator corresponding to $\hat{\mu} \in \hat{\mathcal{D}}(B)$.*

*Proof* From [21, Theorem 2.10], we know that the process in question is a nonnegative supermartingale. Furthermore, also from [21], we have $\hat{Y}_T = \frac{d\mu^r}{d\mathbb{P}}$ and $\hat{Y}_0 \le \hat{\mu}(\Omega)$. To obtain the constant expectation property, we use (5.5) to get

$$\begin{aligned}\langle \hat{\mu}, x_0\rangle &= \langle \hat{\mu}, (\hat{\pi}\cdot S)_T + \underline{B}\rangle = \langle \hat{\mu}^r, (\hat{\pi}\cdot S)_T + \underline{B}\rangle \\ &= \mathbb{E}\big[\hat{Y}_T\big((\hat{\pi}\cdot S)_T + \underline{B}\big)\big] \le \hat{Y}_0 x_0 \le \hat{\mu}(\Omega)x_0,\end{aligned}$$

and the claimed martingale property follows. □

## 6 The interval of conditional Davis prices

This section closes the loop and gives an explicit expression for the interval of conditional Davis prices under the assumption of minimal superreplicability. The supergradient characterisation in Proposition 4.1, together with the properties of minimally superreplicable claims, allows us to compute the interval of conditional Davis prices quite explicitly in many cases of interest.

**Theorem 6.1** *Suppose that Assumption 4.5 holds and that $-B$ and $-(B+\varepsilon\varphi)$ are minimally superreplicable by $-\underline{B}$ and $-(\underline{B+\varepsilon\varphi})$, respectively, for all $\varepsilon$ in some neighbourhood of 0. Then the interval of $B$-conditional Davis prices of $\varphi$ is given by*

$$\frac{1}{y_B}\mathbb{E}[\hat{Y}\varphi] + \frac{1}{y_B}\left[\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\langle \hat{\mu}^s, \underline{B+\varepsilon\varphi} - \underline{B}\rangle, \lim_{\varepsilon\nearrow 0}\frac{1}{\varepsilon}\langle \hat{\mu}^s, \underline{B+\varepsilon\varphi} - \underline{B}\rangle\right], \tag{6.1}$$

*where $y_B$ is the common value of $\hat{\mu}(\Omega)$ for all $\hat{\mu} \in \hat{\mathcal{D}}(B)$.*

*Proof* By Corollary 5.8, the function $u(x,q)$ defined in (4.1) is differentiable in $x$ at $(x,q) = (0,0)$ with derivative $y_B > 0$. The interval of $B$-conditional Davis prices, according to Proposition 4.1, is given by

$$\frac{1}{y_B}[\partial_{q+}u(0,0), \partial_{q-}u(0,0)].$$

This in turn coincides with the expression in (6.1) thanks to Proposition 5.7. □

### 6.1 Two illustrative examples

We conclude by giving two illustrative examples, both in an incomplete Brownian setting, of situations where our results can be applied directly and lead to explicit formulas for the non-trivial interval of conditional Davis prices.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting two independent Brownian motions $(Z, W)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ their augmented filtration up to some maturity $T > 0$. In both examples, the stock price dynamics are given by a one-dimensional Itô process

$$dS_t = S_t\sigma_t(\lambda_t dt + dZ_t), \qquad S_0 > 0, \tag{6.2}$$

with processes $\sigma, \lambda \in \mathcal{L}^2$. With more driving Brownian motions than assets, this leads to an incomplete financial model. Both examples will feature an (unspanned) contingent claim paying out $\varphi(W_T)$ at time $T$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-constant, bounded and uniformly Lipschitz function. The major difference between the examples is that in the first, the random endowment degenerates ($B \equiv x$ for a constant $x > 0$), while in the second, the random endowment $B$ is not replicable.

The first example illustrates that even when $B \equiv x > 0$ is constant, our setting differs from that of Kramkov and Sîrbu [24] because the corresponding Davis prices are non-unique, whereas the growth condition placed on the claim's payoff in [24, Assumption 4] always produces unique Davis prices (the growth condition used in [24] originates from Hugonnier et al. [20, Theorem 3.1(i)]). In other words, the payoffs considered in the first example are not included in [24].

The second example backs up the claim we made in both the abstract and in the introduction: When the endowment $B$ is not replicable, the generic case is that Davis prices are non-unique.

*Example 6.2* We adopt the setting used in Example 5.3 above which is based on Delbaen and Schachermayer [11]. The endowment is taken to be $B \equiv x > 0$. It follows from Example 5.3 that the interval of arbitrage-free prices for $\varphi(W_T)$ is given by $(\underline{\varphi}, \overline{\varphi})$, where

$$\underline{\varphi} := \inf_{a\in\mathbb{R}} \varphi(a), \qquad \overline{\varphi} := \sup_{a\in\mathbb{R}} \varphi(a).$$

Theorem 6.1 with $B \equiv x > 0$ shows that the interval of the log-investor's Davis prices for $\varphi(W_T)$ is given by $[\underline{p}, \overline{p}]$, where

$$\underline{p} := \frac{1}{\hat{Y}_0}\mathbb{E}[\hat{Y}_T(\varphi - \underline{\varphi})] + \underline{\varphi}, \qquad \overline{p} := \overline{\varphi} - \frac{1}{\hat{Y}_0}\mathbb{E}[\hat{Y}_T(\overline{\varphi} - \varphi)].$$

Therefore, since the function $\varphi$ is not constant, we have

$$\overline{p} - \underline{p} = (\overline{\varphi} - \underline{\varphi})(1 - \mathbb{E}[\hat{Y}_T]/\hat{Y}_0) > 0.$$

*Example 6.3* In this example, we consider the Samuelson model used in Larsen et al. [28, Sect. 2] where the stock price dynamics are given by (6.2) with both $\sigma_t \equiv \sigma > 0$ and $\lambda_t \equiv \lambda > 0$ being constants. Let $U(\xi) := \frac{\xi^\gamma}{\gamma}$, $\xi > 0$, $\gamma < 1$, be an arbitrary utility function in the "power" family, with constant relative risk-aversion parameter (as usual $\gamma := 0$ is interpreted as the log-investor).

The investor receives a random endowment of the form $B(W_T)$ at time $T > 0$, where $B$ is a non-constant, bounded and uniformly Lipschitz function. The payoff $\varphi$ whose $B$-conditional Davis prices we are computing as well as the quantities $\underline{\varphi}$ and $\overline{\varphi}$ are defined exactly as in Example 6.2 above. We also define the quantities

$$\underline{B}(\varepsilon) := \inf_{a\in\mathbb{R}} \big(B(a) + \varepsilon\varphi(a)\big), \quad \overline{B}(\varepsilon) := \sup_{a\in\mathbb{R}} \big(\varepsilon\varphi(a) - B(a)\big), \qquad \varepsilon \geq 0.$$

Larsen et al. [28, Proposition 2.4] states that the dual optimiser $\hat{\mathbb{Q}} \in \mathrm{ba}(\mathbb{P})$ for the utility maximisation problem with the random endowment of the form $B(W_T)$ has a non-trivial singular part in the Yosida–Hewitt decomposition $\hat{\mathbb{Q}} = \hat{\mathbb{Q}}^r + \hat{\mathbb{Q}}^s$ after a possible shift of the function $B$ by a constant. Moreover, such a shift can always be arranged so as to keep the values of $B$ positive and bounded away from 0. Therefore we assume without loss of generality that such a shift has already been performed, so that in particular we have $\underline{B}(0) > 0$. This loss-of-mass property for $\hat{\mathbb{Q}}^r$ can be partially quantified as follows. Gu et al. [14, Theorem 3.7] and Larsen and Žitković [30, Proposition 3.2] allow us to write $\frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}} = \hat{Y}_T$, where

$$d\hat{Y}_t = -\hat{Y}_t\left(\frac{\mu}{\sigma}dZ_t + \hat{\nu}_t dW_t\right), \qquad \hat{Y}_0 > 0,$$

for some process $\hat{\nu} \in \mathcal{L}^2$. The presence of the non-trivial singular part $\hat{\mathbb{Q}}^s$ implies that $\hat{Y}$ is a strict local martingale, i.e., $\mathbb{E}[\hat{Y}_T] < \hat{Y}_0$.

Example 5.3 takes care of the conditions of Theorem 6.1 dealing with minimal superreplicability. Indeed, both $-B$ and $-(B+\varepsilon\varphi)$ are of the form treated there and are therefore minimally superreplicable by $-\underline{B}$ and $-\underline{B}(\varepsilon)$, respectively, for $\varepsilon \geq 0$.

The last step before Theorem 6.1 is applied is to simplify the two $\varepsilon$-limits appearing in (6.1). That is an easy task thanks to the fact that the random variable $\frac{1}{\varepsilon}(\underline{B+\varepsilon\varphi} - \underline{B})$ is constant and equal to $\frac{1}{\varepsilon}(\underline{B}(\varepsilon) - \underline{B}(0))$. Theorem 6.1 guarantees that this quotient admits a left and a right limit at $\varepsilon = 0$, and we introduce the notation

$$\underline{B}'(0+) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\big(\underline{B}(\varepsilon) - \underline{B}(0)\big) \quad \text{and} \quad \underline{B}'(0-) := \lim_{\varepsilon \nearrow 0} \frac{1}{\varepsilon}\big(\underline{B}(\varepsilon) - \underline{B}(0)\big).$$

The total mass in $\hat{\mathbb{Q}}^s$ is given by $\hat{Y}_0 - \mathbb{E}[\hat{Y}_T]$, and so the interval of $B(W_T)$-conditional Davis prices for the payoff $\varphi(W_T)$ is given by $[\underline{p}, \overline{p}]$, where

$$\underline{p} := \frac{1}{\hat{Y}_0}\mathbb{E}\big[\hat{Y}_T\big(\varphi(W_T) - \underline{B}'(0+)\big)\big] + \underline{B}'(0+),$$

$$\overline{p} := \overline{B}'(0+) - \frac{1}{\hat{Y}_0}\mathbb{E}\big[\hat{Y}_T\big(\overline{B}'(0+) - \varphi(W_T)\big)\big].$$

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## Appendix A: Davis prices and related derivative prices

The purpose of this appendix is to place the notion of Davis pricing within the wider framework of arbitrage- and utility-based pricing concepts. We start by noting that conditional Davis prices are always arbitrage-free in the sense of the following definition from Siorpaes [34, Sect. 2].

**Definition A.1** A constant $p = p(\varphi) \in \mathbb{R}$ is called an *arbitrage-free price* of $\varphi$ if for all $\pi \in \mathcal{A}$ and $q' \in \mathbb{R}$, we have that

$$q'(\varphi - p) + \int_0^T \pi_u dS_u \geq 0 \quad \text{implies} \quad q'(\varphi - p) + \int_0^T \pi_u dS_u = 0, \qquad \mathbb{P}\text{-a.s.} \tag{A.1}$$

**Proposition A.2** *Given $B \in \mathbb{L}^\infty_{++}$, each $B$-conditional Davis price $p$ for a contingent claim $\varphi \in \mathbb{L}^\infty$ is also an arbitrage-free price of $\varphi$.*

*Proof* Let $p$ be a conditional Davis price and suppose to the contrary of (A.1) that we can find $q' \in \mathbb{R}$ and $\pi \in \mathcal{A}$ such that the nonnegative random variable

$$A := q'(\varphi - p) + \int_0^T \pi_u dS_u$$

is strictly positive with strictly positive probability, i.e., that we have $\mathbb{P}[A \geq 0] = 1$ and $\mathbb{P}[A > 0] > 0$. Then for $n \in \mathbb{N}$, the inequality (3.1) implies that

$$\begin{aligned} \mathfrak{U}(B) &\geq \mathfrak{U}\big(B + nq'(\varphi - p)\big) \\ &\geq \mathbb{E}\left[U\left(B + nq'(\varphi - p) + n\int_0^T \pi_u \, dS_u\right)\right] \\ &= \mathbb{E}[U(B + nA)]. \end{aligned}$$

It remains to let $n \to \infty$ and use the monotone convergence theorem to reach a contradiction with the fact that $\mathfrak{U}(B) < \infty$. □

Next, we relate several popular utility-based pricing methods to Davis prices (see e.g. Becherer [1]). As always, $B \in \mathbb{L}^\infty_{++}$ is the random endowment and $\varphi \in \mathbb{L}^\infty$ is the claim's payoff. However, we also allow a dependence on the quantity $q$ of claims held.

**Definition A.3** Let $q \neq 0$. A constant $m$ is called a *utility-based derivative price* for the payoff $\varphi$ at quantity $q$ if

$$q \in \operatorname*{argmax}_{q' \in \mathbb{R}} \mathfrak{U}\big(B + q'(\varphi - m)\big). \tag{A.2}$$

A constant $h = h(q) = h(q; \varphi | B)$ is called a *utility indifference (Hicks, reservation) price* for $\varphi$ at quantity $q$ if

$$\mathfrak{U}\Big(B + q\big(\varphi - h(q)\big)\Big) = \mathfrak{U}(B). \tag{A.3}$$

Up to a sign convention, these prices are further divided into utility indifference *buy* and *sell* prices (see e.g. [4, Chap. 2]). Finally, a constant $c = c(q) = c(q; \varphi | B)$ is called a *certainty equivalent* for $\varphi$ at quantity $q$ if

$$\mathfrak{U}(B + q\varphi) = \mathfrak{U}\big(B + qc(q)\big).$$

While the three concepts introduced above differ from each other in general, we show that under appropriate conditions, their limits as $q \to 0\pm$ coincide with the endpoints of the interval $P(\varphi|B)$ of conditional Davis prices.[3] To streamline the presentation, we define $p_{\min}$ and $p_{\max}$ by

$$[p_{\min}, p_{\max}] = P(\varphi|B).$$

Similarly, let $m_{\min}(q) \leq m_{\max}(q)$ be the endpoints of the interval of utility-based derivative prices at quantity $q$ defined in (A.2) above.

In Proposition A.4 below, we assume that both $\varphi$ and $B$ are positive, bounded and bounded away from zero. This entails virtually no loss of generality, but makes the value function $u(x,q)$ defined in (4.1) strictly increasing in each argument. On the other hand, the assumption that $-B$ is minimally superreplicable is a major one. We leave the question of the validity of Proposition A.4 without this for future research.

**Proposition A.4** *Suppose that Assumption 4.5 holds, $B, \varphi \in \mathbb{L}^\infty_{++}$ and $-B$ is minimally superreplicable. Then*

$$\lim_{q \searrow 0} c(q) = \lim_{q \searrow 0} h(q) = \lim_{q \searrow 0} m_{\max}(q) = \lim_{q \searrow 0} m_{\min}(q) = p_{\max} \tag{A.4}$$

*and*

$$\lim_{q \nearrow 0} c(q) = \lim_{q \nearrow 0} h(q) = \lim_{q \nearrow 0} m_{\max}(q) = \lim_{q \nearrow 0} m_{\min}(q) = p_{\min}. \tag{A.5}$$

*Proof* We prove only (A.4) as the proof of (A.5) is completely analogous. Since the value function $u$ from (4.1) is strictly increasing in each argument, $h(q)$ and $c(q)$ are well defined and unique for $q \neq 0$. Moreover, we have for all $q > 0$ the bounds

$$\begin{aligned} 0 < \operatorname{essinf} \varphi \leq c(q) \leq \operatorname{esssup} \varphi < \infty, \\ 0 < \operatorname{essinf} \varphi \leq h(q) \leq \operatorname{esssup} \varphi < \infty. \end{aligned} \tag{A.6}$$

Hence $\lim_{q \searrow 0} qc(q) = 0$ and we can let $q \searrow 0$ in

$$\begin{aligned} \frac{u(0,q) - u(0,0)}{q} &= \frac{\mathfrak{U}(B + q\varphi) - \mathfrak{U}(B)}{q} \\ &= \frac{\mathfrak{U}(B + qc(q)) - \mathfrak{U}(B)}{q} \\ &= \frac{\mathfrak{U}(B + qc(q)) - \mathfrak{U}(B)}{qc(q)} c(q), \end{aligned}$$

and use Corollary 5.8 (where the constant $y_B > 0$ is defined) to obtain

$$\partial_{q+} u(0,0) = \lim_{q \searrow 0} y_B c(q).$$

[3] We thank our AE and a referee for this suggestion.

It remains to use Proposition 4.1 to conclude that $\lim_{q\searrow 0} c(q)$ is indeed the right endpoint of the interval of $B$-conditional Davis prices.

To deal with the indifference price $h$, we use its definition together with the concavity of the value function $u$ to conclude that for $q>0$ and $\lambda\in(0,1)$, we have

$$\begin{aligned} u\big(-\lambda q\, h(\lambda q), \lambda q\big) &= u(0,0) \\ &= \lambda u\big(-q\, h(q), q\big) + (1-\lambda) u(0,0) \\ &\le u\big(-\lambda q\, h(q), \lambda q\big). \end{aligned}$$

Since $u$ is strictly increasing in its second argument, we have $h(\lambda q)\ge h(q)$. Therefore $q\mapsto h(q)$ is a nonincreasing function and by (A.6), its limit $h(0+) := \lim_{q\searrow 0} h(q)$ exists in $(0,\infty)$. We use that fact to pass to the limit $q\searrow 0$ in both sides of the equality

$$0 = \frac{u(-q\, h(q), q) - u(0,0)}{q}.$$

So the directional derivative of $u$ at $(0,0)$ in the direction $(-h(0+),1)$ is 0, and

$$\inf_{(y,r)\in\partial u(0,0)} \big(r - h(0+) y\big) = 0.$$

Corollary 5.8 implies that we have $(y,r)\in\partial u(0,0)$ only if $y=y_B$, which then yields $h(0+) = \frac{1}{y_B}\partial_{q+} u(0,0)$. Proposition 4.1 completes the argument.

Finally, we treat utility-based derivative prices. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers decreasing to 0 and define the sequence $(m_n)_{n\in\mathbb{N}}$ by $m_n := m_{\max}(q_n)$. Because the lower endpoints $m_{\min}(q_n)$ can be treated similarly, we omit the details for that case. By (A.2), the concave function

$$q' \mapsto u(-m_n q', q')$$

admits a maximum at $q'=q_n$. Therefore the partial derivatives of $u$ at $(-m_n q_n, q_n)$ are nonpositive in the directions $(-m_n, 1)$ and $(m_n, -1)$. The proof of Proposition 4.1 produces a pair $(y_n, r_n)\in\partial u(-m_n q_n, q_n)$ such that $r_n = m_n y_n$.

Because of (A.6), we have $0 < m_n \le \operatorname{esssup}\varphi$ so that $(-m_n q_n, q_n)\to(0,0)$. The graph of the supergradient correspondence of a concave function is closed (see e.g. Hiriart-Urruty and Lemaréchal [18, Proposition 6.2.1]) and locally bounded (see e.g. [18, Proposition 6.2.2]). Therefore, by passing to a subsequence if necessary, we may conclude that $(y_n, r_n)\to(y^*, r^*)$ for some $(y^*, r^*)\in\partial u(0,0)$. By Corollary 5.8, the value $y^*$ must equal $y_B>0$, and so $y_n>0$ for large enough $n$. This implies that $m_n\to m$, where $m = r^*/y_B$, and Proposition 4.1 guarantees that $m$ is a Davis price for the payoff $\varphi$.

On the other hand, for any conditional Davis price $p\in P(\varphi|B)$ and for any $(y, r_B)\in\partial u(0,0)$ such that $p = r/y_B$, the monotonicity of the supergradient correspondence, i.e.,

$$\langle(-m_n q_n, q_n) - (0,0), (y_n, r_n) - (y, r_B)\rangle \ge 0,$$

see e.g. [18, Proposition 6.1.1], implies that

$$(r_n - r) - m_n(y_n - y_B) \geq 0 \qquad \text{for all } n \in \mathbb{N}.$$

Since $y_n \to y_B$, $r_n \to r^*$ and $(m_n)_{n\in\mathbb{N}}$ is a bounded sequence, we conclude that $p = r/y_B \leq r^*/y_B = m$. Consequently, $m = p_{\max}$.

It remains to show that $m_{\max}(q) \to p_{\max}$ as $q \to 0$. If it did not, there would exist a sequence $(q_n)_{n\in\mathbb{N}}$ with $q_n \searrow 0$ such that $m_{\max}(q_n) \to p_0 \neq p_{\max}$. The same would be true for any subsequence of $(q_n)_{n\in\mathbb{N}}$, which contradicts the conclusion of the previous paragraph. □

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