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An incomplete equilibrium with a stochastic annuity

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Abstract We prove the global existence of an incomplete, continuous-time finite-agent Radner equilibrium in which exponential agents optimise their expected utility over both running consumption and terminal wealth. The market consists of a traded annuity, and along with unspanned income, the market is incomplete. Set in a Brownian framework, the income is driven by a multidimensional diffusion and in particular includes mean-reverting dynamics. The equilibrium is characterised by a system of fully coupled quadratic backward stochastic differential equations, a solution to which is proved to exist under Markovian assumptions. We also show that the equilibrium allocations lead to Pareto-optimal allocations only in exceptional situations.

Keywords Incomplete markets · Radner equilibrium · Annuity · BSDE · Systems of BSDEs · Unspanned income

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1 Introduction

We prove the existence and study Pareto optimality of a Radner equilibrium in an incomplete, continuous-time finite-agent market setting. The economic agents act as price takers in a fully competitive setting and maximise exponential utility from running consumption and terminal wealth. An annuity in one net supply is traded on a financial market, and it pays a constant running and terminal dividend to its shareholders. The agents choose between consuming their income and dividend streams or investing in the annuity.

Although our setting and the income dynamics are quite general, our financial market looks relatively simple at first glance. The only available asset is the annuity, and the agents' only choice at any given moment is how much to consume, keeping in mind that the only way to transfer wealth from one time to the next is through the annuity. This apparent simplicity is quite misleading, since the scarcity of the available traded assets leads to market incompleteness, a notorious difficulty in equilibrium analysis. Indeed, the fewer assets the agents have at their disposal, the less efficient the market becomes and the harder it becomes to use the standard tools such as the representative agent approach. In our case, this lack of assets is pushed to its limit and it turns out that our market is Pareto-efficient only in very special cases, which we completely characterise.

Admittedly, it would be more realistic to consider markets with several assets, both risky and riskless, where the incompleteness is derived from the constraints on each asset's ability to incorporate all the risk present in the environment. We believe that the exploration of such problems is one of the most interesting and important topics of future research in this area. Unfortunately, the formidable mathematical difficulties present in virtually all such problems leave them outside the scope of the techniques available to us today.

One of the advantages of our model is its ability to incorporate various income stream dynamics, including unspanned mean-reverting income streams (which have been studied extensively for their empirical relevance; see e.g. Wang [13, 14] and Cochrane [6]). To the best of our knowledge, our model is the first with exponential agents to incorporate unspanned mean-reverting income in equilibrium and prove the existence of such an equilibrium. The general income streams we study lead to stochastic annuity dynamics, which prevent a money market account from being replicated by trading in the annuity in equilibrium.

Telmer [11] studies a related equilibrium model with two power utility investors on an infinite horizon. Rather than a stochastic annuity, the agents in Telmer [11] are allowed to trade in a zero-net-supply riskless bond at every time period. The results are numerical and show an effect on asset prices (i.e., interest rates) when agents cannot hedge against their income streams. Our model proves equilibrium existence and quantifies the impact of unspanned random endowment on equilibrium annuity prices.

Our approach crucially relies on the presence of a traded annuity. We also need utility functions of exponential type and a Markovian assumption on the dynamics of the income streams in order to obtain conveniently structured individual agent problems, amenable to a BSDE analysis. Even so, the analysis involves a non-standard ansatz for the value function, as we need to formally treat the asset price A as a quantity that in standard models plays the role of a money market account. We are not the first to introduce a traded annuity into an equilibrium model (see e.g. Vayanos and Vila [12], Calvet [2], Christensen et al. [5, 4] and Weston [15]). Our contribution is to recognise the role of a traded annuity price in the individual agent value functions, even when general income streams render the annuity dynamics computationally intractable.

The backward stochastic differential equation (BSDE)/PDE-system approach to incomplete market equilibria dates back to Žitković [19], Zhao [18], Choi and Larsen [3], Kardaras et al. [8] and Xing and Žitković [16], with the early work relying on a smallness-type assumption on some ingredient of the model (the time horizon, size of the endowment, etc.) The mathematical analysis of the present paper is quite involved and relies heavily on some recent results of Xing and Žitković [16], which overcome smallness conditions and treat the existence and stability of solutions to quadratic systems of BSDEs. Moreover, the applicability of those results in our setting is not at all immediate and is contingent on a number of a priori estimates specific to our model.

Notation and conventions. For $J, d \in \mathbb{N}$, the set of $J \times d$-matrices is denoted by $\mathbb{R}^{J\times d}$. The Euclidean space $\mathbb{R}^J$ is identified with the set $\mathbb{R}^{J\times 1}$, i.e., vectors in $\mathbb{R}^J$ are columns by default. The ith row of a matrix $Z \in \mathbb{R}^{J\times d}$ is denoted by Z^i, and $|\cdot|$ denotes the Euclidean norm on either $\mathbb{R}^{J\times d}$ or $\mathbb{R}^J$.

We work on a finite interval $[0, T]$ with $T > 0$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the usual augmentation of the filtration generated by a d-dimensional Brownian motion B. Leb denotes Lebesgue measure on $[0, T]$.

Stochastic integrals with respect to B are taken for $\mathbb{R}^{1\times d}$-valued (row) processes as if dB were a column of its components, i.e., $\int \sigma(t)\, dB_t$ stands for the (vector) stochastic integral $\sum_{j=1}^{d} \int \sigma_j(t)\, dB^j_t$. Similarly, for a process Z with values in $\mathbb{R}^{J\times d}$, $\int Z_t\, dB_t$ is an $\mathbb{R}^J$-valued process whose components are the stochastic integrals of the rows Z^i of Z with respect to dB_t.

For a function defined on a domain in $\mathbb{R}^d$, the derivative Du is always assumed to take row-vector values, i.e., $Du(x) \in \mathbb{R}^{d\times 1}$. If u is $\mathbb{R}^J$-valued, the Jacobian Du is as usual interpreted as an element of $\mathbb{R}^{J\times d}$. The Hessian D^2u of a scalar-valued function takes values in $\mathbb{R}^{d\times d}$, and we have no need for Hessians of vector-valued maps in this paper.

To relieve the notation, we omit the time index from many expressions involving stochastic processes, but keep (and abuse) the notation dt for an integral with respect to Lebesgue measure.

The set of all $\mathbb{F}$-progressively measurable process is denoted by Prog. $L^r(\mathrm{Prog})$ denotes the set of all $c \in \mathrm{Prog}$ with $\int_0^T |c|^r\, dt < \infty$ a.s. The same notation is used for scalar, vector or matrix-valued processes – the distinction will always be clear from the context.

The set of all adapted, continuous and uniformly bounded processes is denoted by $\mathcal{S}^\infty$. A martingale M for which there exists a constant $C > 0$ such that

$$\mathbb{E}[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau] \leq C \text{ a.s.}$$

for each stopping time τ is said to be a *BMO-martingale*, and the set of such martingales is denoted by BMO (we refer the reader to Kazamaki [9] for all the necessary background). The family of all processes $\sigma \in L^2(\mathrm{Prog})$ such that $\int \sigma\, dB$ is a BMO-martingale is denoted by bmo.

2 The problem

2.1 Model primitives

The model primitives can be divided into three groups. In the first one, we describe the uncertain environment underlying the entire economy. In the second, we postulate the form of the dynamics of the traded asset and in the third, we describe the characteristics of individual agents. A single real consumption good is taken as the numeraire throughout.

For $d \in \mathbb{N}$, we start with an $\mathbb{R}^d$-valued *state process* ξ whose dynamics is given by

$$d\xi_t = \Lambda(t, \xi_t)\, dt + \Sigma(t, \xi_t)\, dB_t, \qquad \xi_0 = x_0 \in \mathbb{R}^d, \tag{2.1}$$

where the functions $\Lambda : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\Sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable and satisfy the following regularity assumption:

Assumption 2.1 (Regularity of the state process) There exists a constant $K > 0$ such that for all $t, t' \in [0, T]$, $x, x' \in \mathbb{R}^d$ and $z \in \mathbb{R}^{d\times 1}$, we have

1) $|\Lambda(t, x)| \le K$ and $|\Lambda(t, x) - \Lambda(t, x')| \le K|x - x'|$;
2) $|\Sigma(t, x)| \le K$ and $|\Sigma(t, x) - \Sigma(t, x')| \le K(\sqrt{|t' - t|} + |x - x'|)$;
3) $|\Sigma(t, x)z| \ge \frac{1}{K}|z|$.

Remark 2.2 Under Assumption 2.1, the SDE (2.1) admits a unique strong solution. The full significance of the assumptions above, however, will only become apparent in later sections and is related to the ability to use certain existence results for systems of backward stochastic differential equations.

Our market consists of a single real asset A in one net supply, whose dynamics we postulate to be of the form

$$dA_t = (A_t\mu_t - 1)\, dt + A_t\sigma_t\, dB_t, \qquad A_T = 1, \tag{2.2}$$

with the processes μ and σ to be determined in equilibrium. This asset can be interpreted as an annuity which pays a dividend at the rate 1 during $[0, T]$, as well as a unit lump sum payment at time T.

Let Γ, the *coefficient space*, denote the set of all pairs $\gamma = (\mu, \sigma)$ of processes in bmo, with μ scalar-valued and σ $\mathbb{R}^{1\times d}$-valued. For simplicity, we often identify the market A^γ with its coefficient pair $\gamma = (\mu, \sigma)$ and talk simply about *the market* γ. The set of all markets given by (2.2) is not bijectively parametrised by Γ as not every $\gamma \in \Gamma$ defines a market. Indeed, the terminal condition $A_T = 1$ imposes a nontrivial relationship between μ and σ; for example, if μ is deterministic, σ either has to vanish or one of its components has to be truly stochastic. The set of those $\gamma \in \Gamma$ that do define a market is denoted by Γ_v and its elements are said to be *viable*. There are a finite number $I \in \mathbb{N}$ of economic agents, each of which is characterised by the following three elements:

1) The *risk-aversion coefficient* $\alpha^i > 0$: It fully characterises the agent's utility function U^i which is of exponential form,

$$U^i(c) = -\frac{1}{\alpha^i} e^{-\alpha^i c} \qquad \text{for } c \in \mathbb{R}.$$

2) The *random-endowment (stochastic income) rate*: Each agent receives an endowment of the consumption good at the rate $e^i_t = e^i(t, \xi_t)$, as well as a lump sum $e^i_T = e^i(T, \xi_T)$ at time T, for some function $e^i : [0, T] \times \mathbb{R}^d \to \mathbb{R}$.

3) The *initial holding* $\pi^i_0 \in \mathbb{R}$ is the initial number of shares of the annuity A held by the agent.

With the *cumulative endowment rate* defined by $e = \sum_{i=1}^I e^i$, we impose the following regularity conditions:

Assumption 2.3 (Regularity of the endowment rates)

1) Each e^i is bounded and continuous, and its terminal section $e^i(T, \cdot)$ is α-Hölder-continuous for some $\alpha \in (0, 1]$.

2) The cumulative endowment process $e_t = e(t, \xi_t)$, $t \in [0, T]$, is a semimartingale with the decomposition

$$e(t, \xi_t) = e(0, x_0) + \int_0^t \mu_e(s, \xi_s)\, ds + \int_0^t \sigma_e(s, \xi_s)\, dB_s,$$

where the drift function $\mu_e : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is bounded and continuous and $(\sigma_e(s, \xi_s))$ is a bmo process.

We often overload our notation and write e^i both for the deterministic function $e^i : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and the stochastic process $e^i_t = e^i(t, \xi_t)$, $t \in [0, T]$. The same applies to other functions applied to (t, ξ_t) – such as e or μ_e.

Remark 2.4 It is worth stopping here to give a few examples of state processes ξ and functions e^i which satisfy all the regularity conditions imposed so far. Once the coefficients Λ and Σ for ξ are picked so as to satisfy Assumption 2.1, Assumption 2.3 is easy to check for a sufficiently smooth e^i by a simple application of Itô's formula.

The more interesting observation is that there is room for improvement. It may seem that the boundedness imposed in Assumption 2.1 rules out some of the most important classes of state processes such as the classical mean-reverting (Ornstein–Uhlenbeck) processes. This is not the case, as we have the freedom to choose both the state process ξ and the deterministic function e^i applied to it, while only caring about the resulting composition. We illustrate what we mean by that with a simple example. The reader can easily add the required bells and whistles to it, and adapt it to other similar frameworks.

We assume $d=1$ and that we are interested in the random endowment rate $e^i(t,\eta_t)$, where e^i is a bounded and appropriately smooth function and (η_t) is an Ornstein–Uhlenbeck process with the dynamics

$$d\eta_t = \theta(\bar{\eta}-\eta_t)\,dt + \sigma_\eta\,dB_t$$

and parameters $\theta, \sigma_\eta > 0$ and $\eta_0, \bar{\eta} \in \mathbb{R}$. Since the drift function $x \mapsto \theta(\bar{\eta}-x)$ is not bounded, the process η does not satisfy the conditions of Assumption 2.1. The process η admits, however, an explicit expression in terms of a stochastic integral of a deterministic process with respect to the underlying Brownian motion, namely

$$\eta_t = \bar{\eta} + (\eta_0-\bar{\eta})e^{-\theta t} + \sigma_\eta e^{-\theta t}\int_0^t e^{\theta s}\,dB_s. \tag{2.3}$$

If we define the state process ξ by

$$d\xi_t = e^{-\theta t}\,dB_t, \qquad \xi_0 = 0,$$

i.e., if we set $\Lambda(t,x)=0$ and $\Sigma(t,x)=e^{-\theta t}$, the boundedness of the interval $[0,T]$ allows us to conclude that Λ and Σ satisfy Assumption 2.1. Moreover, by (2.3), the choice $f^i(t,x) = e^i(t, \bar{\eta} + (\eta_0-\bar{\eta})e^{-\theta t} + \sigma_\eta x)$ yields

$$f^i(t,\xi_t) = e^i(t,\eta_t).$$

In this way, we can represent a function of an interesting, but not entirely compliant state process η as a (modified) function of a regular state process ξ. The function f^i inherits the boundedness (and other regularity properties) of e^i, thanks to the boundedness from above and away from zero of the function $t \mapsto e^{-\theta t}$.

2.2 Admissibility and equilibrium

Definition 2.5 Given a viable set of coefficients $\gamma = (\mu,\sigma) \in \Gamma_v$, a pair (π, c) of scalar processes is said to be a *γ-admissible strategy* for agent i if

1) $|c| + |\pi(A^\gamma \mu - 1)| \in L^1(\text{Prog})$ and $\pi A^\gamma \sigma \in \text{bmo}$;

2) the *gains process* $X = X^{\pi,\gamma} = \pi A^\gamma$ is a semimartingale which satisfies the *self-financing condition*

$$dX = \pi\,dA^\gamma + (e^i - c + \pi)\,dt.$$

The set of all γ-admissible strategies for agent i is denoted by $\mathcal{A}^i_\gamma$, and the subset of $\mathcal{A}^i_\gamma$ consisting of the strategies with $\pi(0)=\pi_0$ is called $\mathcal{A}^i_\gamma(\pi_0)$.

For ease of later computation, we note that the dynamics of X/A are

$$d\left(\frac{X}{A}\right) = \frac{e^i - c + \pi}{A}dt.$$

Definition 2.6 We call $\gamma^* \in \Gamma_v$ a set of *equilibrium market coefficients* (and A^{γ^*} an *equilibrium market*) if there exist γ^*-admissible strategies $(\hat{\pi}^i, \hat{c}^i) \in \mathcal{A}^i_{\gamma^*}(\pi^i_0)$, $i = 1, \dots, I$, such that the following two conditions hold:

1) *Single-agent optimality:* For each i and all $(\pi, c) \in \mathcal{A}^i_{\gamma^*}(\pi^i_0)$, we have

$$\mathbb{E}\left[\int_0^T U^i(\hat{c}^i_t)\,dt\right] + \mathbb{E}[U^i(X_T^{\hat{\pi}^i,\hat{c}^i} + e^i_T)] \geq \mathbb{E}\left[\int_0^T U^i(c_t)\,dt\right] + \mathbb{E}[U^i(X_T^{\pi,c} + e^i_T)].$$

2) *Market clearing:*

$$\sum_{i=1}^I \hat{\pi}^i = 1 \text{ and } \sum_{i=1}^I \hat{c}^i = e + 1 \text{ on } [0,T) \qquad \text{and} \qquad \sum_{i=1}^I X_T^{\hat{\pi}^i,\hat{c}^i} = 1, \qquad \text{a.s.}$$

3 Results

3.1 A BSDE characterisation

Our first result is a characterisation of equilibria in terms of a system of backward stochastic differential equations (BSDEs). These systems consist of $1 + I$ equations, with the first component generally playing a different role from the other I. For that reason, it pays to depart slightly from the classical notation (Y, Z), where Y has as many components as there are equations and the integrand Z is a matrix process whose additional dimension reflects the number of driving Brownian motions. Instead, we use the notation $((a, Y), (\sigma, Z))$, where a is a scalar and Y is $\mathbb{R}^{I\times 1}$-valued. Similarly, σ and Z are $\mathbb{R}^{1\times d}$- and $\mathbb{R}^{I\times d}$-valued processes, respectively. As usual, we say that $((a, Y), (\sigma, Z))$ is an $(\mathcal{S}^\infty \times \text{bmo})$-solution if all the components of a and Y are in $\mathcal{S}^\infty$ and all components of (σ, Z) are in bmo. To simplify the notation, we also introduce the derived quantities

$$\frac{1}{\bar{\alpha}} := \sum_{i=1}^I \frac{1}{\alpha^i} \qquad \text{and} \qquad \kappa^i := \frac{\bar{\alpha}}{\alpha_i} > 0 \quad \text{so that} \sum_{i=1}^I \kappa^i = 1.$$

Theorem 3.1 *Suppose that* $\sum_{i=1}^I \pi^i_0 = 1$ *and that each* e^i *is a bounded process and the cumulative endowment process* e *is a semimartingale with decomposition of the form* $\mu_e dt + \sigma_e dB$, *where* μ_e *and* σ_e *are bmo processes.*

1) *If* $((a, Y), (\sigma, Z))$ *is an* $(\mathcal{S}^\infty \times \text{bmo})$*-solution to*

$$\begin{aligned} da &= \sigma\,dB + \left(\bar{\alpha}\mu_e - \frac{1}{2}\sum_{\ell=1}^I \kappa^\ell |Z^\ell|^2 - e^{-a}\right) dt, & a_T &= 0, \\ dY^i &= Z^i\,dB + \left(\frac{1}{2}|Z^i|^2 + e^{-a}(1 + a + Y^i - \alpha^i e^i)\right) dt, & Y^i_T &= \alpha^i e^i_T, \end{aligned} \tag{3.1}$$

then $A = \exp(a)$ *is an equilibrium annuity price with market coefficients* $(\mu, \sigma) \in \Gamma_v$, *where* μ *is given by*

$$\mu = \bar{\alpha}\mu_e + \frac{1}{2}|\sigma|^2 - \frac{1}{2}\sum_{i=1}^{I}\kappa^i |Z^i|^2. \tag{3.2}$$

2) *Conversely, suppose that* $\gamma = (\mu, \sigma)$ *is an equilibrium. Then there exist unique processes* $(Y, Z) \in (\mathcal{S}^\infty \times \mathrm{bmo})^I$ *such that* $((\log A^\gamma, Y), (\sigma, Z))$ *solves* (3.1). *Moreover, the strategies* $\hat{c}^i$ *and* $\hat{\pi}^i$ *of Definition* 2.6 *are unique and given by*

$$\hat{c}^i = \frac{1}{\alpha^i}(\log A^\gamma + Y^i) + \frac{\hat{X}^i}{A^\gamma}, \qquad \hat{\pi}^i = \frac{\hat{X}^i}{A^\gamma} \qquad (\mathrm{Leb} \otimes \mathbb{P})\text{-}a.e., \tag{3.3}$$

where $\hat{X}^i$ *is the unique solution of the linear SDE*

$$d\hat{X}^i = \left(\mu \hat{X}^i + \left(e^i - \frac{1}{\alpha^i}(\log A^\gamma + Y^i) - \frac{\hat{X}^i}{A^\gamma}\right)\right) dt + \hat{X}^i \sigma \, dB \tag{3.4}$$

with the initial condition $\hat{X}^i_0 = \pi^i_0 A^\gamma_0$, *for* $1 \le i \le I$.

Remark 3.2 1) We note that the assumptions in Theorem 3.1, which provides a characterisation of an equilibrium, are implied by Assumption 2.3. The full force of Assumptions 2.1 and 2.3 will be needed to establish existence in Theorems 3.4 and 3.5 below.

2) Part 2) of Theorem 3.1 states that once an equilibrium γ is fixed, all other important objects featuring in the description of the financial market and agent strategies are uniquely determined. This is the reason we can (and do) talk about the processes A, Y^i, Z^i, $\hat{X}^i$, $\hat{c}^i$ and $\hat{\pi}^i$ *associated* to γ.

Proof of Theorem 3.1 1) Having fixed an $(\mathcal{S}^\infty, \mathrm{bmo})$-solution $((a, Y), (\sigma, Z))$, we set $A = \exp(a)$ and define μ as in (3.2) so that A satisfies (2.2). With the market coefficients $\gamma = (\mu, \sigma)$ fixed, we pick an agent $i \in \{1, \dots, I\}$ and a pair $(\pi, c) \in \mathcal{A}^i_\gamma(\pi^i_0)$ and define processes X^i, $\tilde{V}^i$ and V^i by

$$X^i = \pi A, \qquad \tilde{V}^i = -\exp\left(-\alpha^i \frac{X^i}{A} - Y^i\right), \qquad V^i = \tilde{V}^i + \int_0^\cdot -\exp(-\alpha^i c_t)\, dt.$$

The self-financing property of (π, c) implies that the semimartingale decomposition of V^i is given by $dV^i = \mu_V \, dt + \sigma_V \, dB$, where

$$\mu_V = -\exp(-\alpha^i c) + \frac{-\tilde{V}^i}{A}\left(1 - \log\frac{-\tilde{V}^i}{A}\right) - \alpha^i c\left(\frac{-\tilde{V}^i}{A}\right), \qquad \sigma_V = -\tilde{V}^i Z^i.$$

Young's inequality implies that $\mu_V \le 0$ and that the coefficients μ_V and σ_V are regular enough to conclude that V^i is a supermartingale for all admissible (π, c). Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T U^i(c_s)\,ds\right]+\mathbb{E}[U^i(X_T^i+e_T^i)]\\
&=\frac{1}{\alpha^i}\mathbb{E}\left[\int_0^T -\exp(-\alpha^i c_s)\,ds\right]-\frac{1}{\alpha^i}\mathbb{E}\big[\exp\big(-\alpha^i(X_T^i+e_T^i)\big)\big]\\
&=\frac{1}{\alpha^i}\left(\mathbb{E}\left[\int_0^T -\exp(-\alpha^i c_s)\,ds\right]-\mathbb{E}[\exp(-\alpha^i X_T^i/A_T-Y_T^i)]\right)\\
&=\frac{1}{\alpha^i}\mathbb{E}[V_T^i]\le\frac{1}{\alpha^i}V_0^i=-\frac{1}{\alpha^i}\exp(-\alpha^i\pi_0^i-Y_0^i).
\end{aligned}$$

Next, in order to characterise the optimiser, we construct a consumption process for which $\mu_V=0$. More precisely, we let the process $\hat{X}^i$ be the unique solution of (3.4) and define $\hat{c}^i$ and $\hat{\pi}^i$ by (3.3). It follows immediately that $(\hat{\pi}^i,\hat{c}^i)\in\mathcal{A}_\gamma^i(\pi_0^i)$ and that the process $\hat{X}^i$ is the associated gains process. The choice of $\hat{c}^i$, through $\hat{X}^i$, makes the process V^i a martingale and the pair $(\hat{\pi}^i,\hat{c}^i)$ optimal for agent i.

Turning to market clearing, we define the process $F=a+\sum_{i=1}^I\kappa^i Y^i-\bar{\alpha}e$, whose dynamics are given by

$$dF=\left(\sigma+\sum_{i=1}^I\kappa^i Z^i-\bar{\alpha}\sigma_e\right)dB+\exp(-a)F\,dt,\qquad F_T=0. \tag{3.5}$$

In other words, the pair $(Y,\zeta)=(F,\sigma+\sum_{i=1}^I\kappa^i Z^i-\bar{\alpha}\sigma_e)$ is an $(\mathcal{S}^\infty\times\text{bmo})$-solution to the linear BSDE

$$dY=\zeta\,dB+\exp(-a)Y\,dt,\qquad Y_T=0.$$

Since a is bounded, the coefficients of this BSDE are globally Lipschitz and therefore, by the uniqueness theorem (see Zhang [17, Theorem 4.3.1]), we can conclude that $F\equiv 0$. This implies that

$$a+\sum_{i=1}^I\kappa^i Y^i=\bar{\alpha}e\qquad\text{on }[0,T],$$

and so

$$\sum_{i=1}^I\hat{c}^i=e+\frac{1}{A}\sum_{i=1}^I\hat{X}^i.$$

The form of the dynamics (3.4) of each $\hat{X}^i$ leads for $\hat{X}=\sum_{i=1}^I\hat{X}^i$ to the dynamics

$$d\hat{X}=\left(\mu\hat{X}-\frac{1}{A}\hat{X}\right)dt+\hat{X}\sigma\,dt. \tag{3.6}$$

The assumption that $\sum \pi^i_0 = 1$ implies that $\hat{X}_0 = A_0$, which in turn implies that the process A is also a solution to (3.6). By uniqueness, we must have $\hat{X} = A$ and conclude that the clearing conditions are satisfied.

2) We define $a = \log A^\gamma$ and note that it satisfies the equation

$$da = \sigma\, dB + \left(\mu - \frac{1}{2}|\sigma|^2 - \exp(-a)\right) dt, \qquad a_T = 0. \tag{3.7}$$

Since both μ and σ are bmo processes, we get for a the lower bound

$$-a_t = \mathbb{E}\left[\int_t^T \left(\mu_u - \frac{1}{2}\sigma_u^2 - \exp(-a_u)\right) du \,\middle|\, \mathcal{F}_t\right] \le \mathbb{E}\left[\int_t^T \mu_u\, du \,\middle|\, \mathcal{F}_t\right] \le C,$$

where $C > 0$ is a constant which depends only the bmo-norm of μ. This gives an upper bound on $\exp(-a)$ which in turn, when plugged back into (3.7), implies that $|a|$ is uniformly bounded (by a constant which depends only on the bmo-norms of μ and σ).

The bound on a we just obtained ensures that the equations for Y^i in (3.1) admit unique $(\mathcal{S}^\infty \times \text{bmo})$-solutions (see Zhang [17, Theorem 7.3.3]). The utility maximisation problem

$$\mathbb{E}\left[\int_0^T U^i(c_t)\, dt\right] + \mathbb{E}[U^i(X_T^{\pi,c} + e^i_T)] \longrightarrow \max!$$

over $(c, \pi) \in \mathcal{A}^i_\gamma$ admits a (Leb $\times$ $\mathbb{P}$)-a.e. unique solution, and so we can follow the argument in 1) to show that $\hat{c}^i$ and $\hat{\pi}^i$ introduced in (3.3) achieve equality in (3.5). Finally, similarly as in the proof of 1), the market clearing conditions imply that

$$\mu - \frac{1}{2}\sigma^2 = \bar{\alpha}\mu_e - \frac{1}{2}\sum_{\ell=1}^{L} \kappa^\ell |Z^\ell|^2,$$

which in turn guarantees that $((a, Y), (\sigma, Z))$ solve (3.1). □

Remark 3.3 Careful inspection of the proof of Theorem 3.1 shows that the presence of a traded annuity in the economy allows the individual agent value functions to be decomposed in the form

$$-\exp\left(-\alpha^i \frac{X^i_t}{A_t} - Y^i_t\right), \tag{3.8}$$

in which the annuity appears as a state process. In fact, the form of (3.8) is unchanged even in the presence of additional traded securities; it only requires the annuity to be one of the (possibly multiple) traded securities.

3.2 Existence of an equilibrium

Next we show that under additional assumptions on the problem ingredients – most notably that of a Markovian structure –, the characterisation of Theorem 3.1 can be used to establish the existence of an equilibrium market.

Theorem 3.4 *Under Assumptions* 2.1 *and* 2.3, *the system* (3.1) *admits an* $(\mathcal{S}^\infty \times \mathrm{bmo})$*-solution.*

The BSDE characterisation of Theorem 3.1 immediately implies the main result of the paper.

Theorem 3.5 *Under Assumptions* 2.1 *and* 2.3, *there exist viable market coefficients* $\gamma^* = (\mu^*, \sigma^*)$ *such that* A^{γ^*} *is an equilibrium market.*

Proof of Theorem 3.4 In certain situations, it will be convenient to standardise the notation; so we also write Y^0 for a, Z^0 for σ, and we set

$$g^i(x) = \begin{cases} 0, & i = 0, \\ \alpha^i e^i(T, x), & 1 \le i \le I. \end{cases}$$

The dt-terms in (3.1) define the driver $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{I+1} \times \mathbb{R}^{(I+1)\times d} \to \mathbb{R}^{I+1}$ in the usual way, namely

$$f^0(t, x, y, z) = \bar{\alpha}\mu_e(t, x) - \frac{1}{2}\sum_{\ell=1}^{I} \kappa^\ell |z^\ell|^2 - \exp(-y^0),$$

$$f^i(t, x, y, z) = \frac{1}{2}|z^i|^2 + \exp(-y^0)\big(1 + y^0 + y^i - \alpha^i e^i(t, x)\big), \qquad i = 1, \dots, I.$$

The system (3.1) written in the new notation becomes

$$dY^i_t = f^i(t, \xi_t, Y_t, Z_t)\,dt + Z^i_t\,dB_t, \quad Y^i_T = g^i(\xi_T), \qquad i = 0, \dots, I.$$

Step 1 (truncation). We start by truncating the driver f to obtain a sequence of well-behaved, Lipschitz problems. More precisely, given $N > 0$, we define

$$\iota_N(x) = \max\big(\min(x, N), -N\big) \qquad \text{for } x \in \mathbb{R}$$

and

$$q_N(z) = |z|\,\iota_N(|z|) \qquad \text{for } z \in \mathbb{R}^{1\times d},$$

so that ι_N and q_N are Lipschitz functions with Lipschitz constants 1 and N, respectively. Moreover,

$$|\iota_N(x)| \le N \quad \text{and} \quad |q_N(z)| \le N\,|z|\,.$$

Using the functions defined above, we pose for each $N \in \mathbb{N}$ a truncated version of (3.1),

$$\begin{aligned} da &= \sigma\,dB + \Big(\bar{\alpha}\mu_e - \frac{1}{2}\sum_{\ell=1}^{I}\kappa^\ell q_N(Z^\ell) - e^{-\iota_N(a)}\Big)\,dt, \\ dY^i &= Z^i\,dB + \Big(\frac{1}{2}q_N(Z^i) + e^{-\iota_N(a)}\big(1 + \iota_N(a) + \iota_N(Y^i) - \alpha^i e^i\big)\Big)\,dt, \end{aligned} \tag{BSDE$_N$}$$

with the terminal conditions $Y^i_T = \iota_N(g^i_T)$ and $a_T = 0$. Moreover, we define the driver $(t, x, y, z) \mapsto f^{(N)}(t, x, y, z)$ from the dt-terms in the standard way.

For each $N \in \mathbb{N}$, $f^{(N)}$ is continuous in all of its variables, uniformly Lipschitz in both z and y, and $f^{(N)}(t, x, 0, 0)$ is bounded. Assumption 2.1 guarantees that the function $F^{(N)}(t, x, y, z) = -f^{(N)}(t, x, y, z\Sigma^{-1}(t, x))$ has the same properties. Therefore we can apply Proposition 5.1 below to conclude that there exists a solution $(Y^{(N)}, Z^{(N)})$ to (BSDE$_N$) of the form

$$Y^{(N)}_t = v^{(N)}(t, \xi_t), \qquad Z^{(N)}_t = w^{(N)}(t, \xi_t),$$

with $v^{(N)} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{I+1}$ bounded and $w^{(N)} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{(I+1)\times d}$ such that $Z^{(N)}$ is a bmo process. We note that existence for (BSDE$_N$) is also guaranteed by the classical result in Pardoux and Peng [10, Theorem 3.1], but only in the class $\mathcal{S}^2 \times \mathcal{H}^2$ which is too big for our purposes.

Step 2 (uniform estimates). The bounds guaranteed by Proposition 5.1 below all depend on the truncation constant N; so our next task is to explore the special structure of our system and establish bounds in terms of universal quantities. A universal constant, in this proof, will be a quantity that depends on the constants α^i, the time horizon T and the $\mathcal{S}^\infty$-bounds on e^i and μ_e, but not on N. We denote such a constant by C and allow it to change from line to line.

Let $((a^{(N)}, Y^{i,(N)}), (\sigma^{(N)}, Z^{(N)}))$ be the solution to the truncated system from Step 1 above. It follows from the dynamics of $a^{(N)}$ and the fact that $q_N(z) \geq 0$ for all $z \in \mathbb{R}^{1\times d}$ that $a^{(N)} - \int_0^\cdot \bar{\alpha}\mu_e\, dt$ is a supermartingale so that for all $t \in [0, T]$,

$$a^{(N)}_t \geq \mathbb{E}\left[a^{(N)}_T - \int_t^T \bar{\alpha}\mu_e\, dt \middle| \mathcal{F}_t\right] \geq -(T-t)\|\bar{\alpha}\mu_e\|_{\mathcal{S}^\infty}, \qquad \text{i.e., } a^{(N)}_t \geq -C.$$

Next we turn to $Y^{(N)}$ and use the fact that the components of Y are coupled only through a. In this way, we can get uniform bounds on $Y^{i,(N)}$ if we manage to produce a uniform bound on the function of a appearing on the right-hand side of (BSDE$_N$). We start by using the easy-to-check inequality

$$\exp(-x)(1 + |x|) \leq \exp(2x^-) \qquad \text{for all } x \in \mathbb{R}$$

and the fact that $(\iota_N(x))^- \leq (x)^-$ for all x to obtain that for all $t \in [0, T]$,

$$\exp\big(-\iota_N(a^{(N)}_t)\big)\big(1 + |\iota_N(a^{(N)}_t)|\big) \leq C.$$

Moreover, it is readily checked that we can construct a bounded measurable function $\delta^{(N)} : \mathbb{R}^{1\times d} \to \mathbb{R}^d$ such that

$$q_N(z) = z\delta^{(N)}(z) = \sum_{j=1}^d z_j \delta^{j,(N)}(z) \qquad \text{for } z = (z_1, \dots, z_j) \in \mathbb{R}^{1\times d}.$$

Therefore, for each i, there exists a probability measure $\mathbb{P}^i$ $(= \mathbb{P}^{i,N}) \approx \mathbb{P}$ under which the process $\tilde{B}^i = B + \int_0^\cdot \delta^{(N)}(Z^{i,(N)})\, dt$ is a Brownian motion on $[0, T]$. Since $Z^{i,(N)}$

is guaranteed to be in bmo for $\mathbb{P}$, it remains in bmo under the measure $\mathbb{P}^i$ (see Kazamaki [9, Theorem 3.3]). Therefore, the process $\int Z^{i,(N)}\,dB^i$ is a $\mathbb{P}^i$-martingale and we can take the expectation of the ith equation in (BSDE$_N$) with respect to $\mathbb{P}^i$ to obtain

$$\begin{aligned}|Y_t^{i,(N)}| &\le \big|\mathbb{E}^i\big[\iota_N\big(\alpha^i e^i(T)\big)\big|\mathcal{F}_t\big]\big| \\ &\quad + \int_t^T \mathbb{E}^i\big[\exp\big(-\iota_N(a_s^{(N)})\big)\big(1+|\iota_N(a_s^{(N)})|\big)\big|\mathcal{F}_t\big]\,ds \\ &\quad + \int_t^T \mathbb{E}^i\big[\exp\big(-\iota_N(a_s^{(N)})\big)|\iota_N(Y_s^{i,(N)})-\alpha^i e_s^i|\,\big|\mathcal{F}_t\big]\,ds \\ &\le C\bigg(1+\int_t^T \mathbb{E}^i\big[|Y_s^{i,(N)}|\,\big|\mathcal{F}_t\big]\,ds\bigg) \le C\bigg(1+\int_t^T y^i(s)ds\bigg),\end{aligned}$$

where $y^i(t)=\|Y_t^{i,(N)}\|_{L^\infty}$. Here, L^∞ refers to $L^\infty(\Omega,\mathcal{F},\mathbb{P})$, and $\|\cdot\|_{L^\infty}$ is its respective norm. Thus y^i satisfies

$$y^i(t)\le C\bigg(1+\int_t^T y^i(s)\,ds\bigg)\qquad \text{for all } t\in[0,T],$$

for some universal constant C. Gronwall's inequality therefore implies that the value $y^i(0)=\|Y^{i,(N)}\|_{\mathcal{S}^\infty}$ is bounded by another universal constant, and so we conclude that there exists a universal $\mathcal{S}^\infty$-bound on all the $Y^{i,(N)}$.

Our next goal is to produce universal bmo-bounds on the processes $Z^{i,(N)}$. This will follow by using the universal boundedness of the Z-free terms in the driver of $Y^{i,(N)}$ from (BSDE$_N$) obtained above in Step 2. Since the ith component of the driver $f^{(N)}$ depends on $Z^{(N)}$ only through $Z^{i,(N)}$, for $1\le i\le I$, we can apply standard exponential-transform estimates. We adapt the argument in Briand and Elie [1, Proposition 2.1] and define

$$\phi(x):=\frac{\exp(2\,|x|)-1-2\,|x|}{4}\qquad \text{for } x\in\mathbb{R},$$

noting that both ϕ and ϕ' are nonnegative and increasing, while $\phi\in C^2(\mathbb{R})$ with $\phi''-2\,|\phi'|=1$. Thus, for any stopping time τ in $[0,T]$, Itô's lemma implies that

$$\begin{aligned}\phi(Y_\tau^{i,(N)}) &\le \mathbb{E}[\phi(Y_T^{i,(N)})|\mathcal{F}_\tau]+\mathbb{E}\bigg[\int_\tau^T |\phi'(Y_s^{i,(N)})|C(1+\|Y^{i,(N)}\|_{\mathcal{S}^\infty})\bigg|\mathcal{F}_\tau\bigg] \\ &\quad +\mathbb{E}\bigg[\int_\tau^T \bigg(\bigg|\frac{1}{2}\phi'(Y_s^{i,(N)})\bigg||Z_s^{i,(N)}|^2-\phi''(Y_s^{i,(N)})|Z_s^{i,(N)}|^2\bigg)ds\,\bigg|\mathcal{F}_\tau\bigg] \\ &\le \phi(\|Y^{i,(N)}\|_{\mathcal{S}^\infty})+C\int_0^T \phi'(\|Y^{i,(N)}\|_{\mathcal{S}^\infty})(1+\|Y^{i,(N)}\|_{\mathcal{S}^\infty})ds \\ &\quad -\mathbb{E}\bigg[\int_\tau^T |Z_s^{i,(N)}|^2ds\,\bigg|\mathcal{F}_\tau\bigg].\end{aligned}$$

Since ϕ is nonnegative, rearranging terms yields

$$\mathbb{E}\left[\int_\tau^T |Z_s^{i,(N)}|^2 ds \middle| \mathcal{F}_\tau\right] \leq \phi(\|Y^{i,(N)}\|_{\mathcal{S}^\infty}) + C\int_0^T \phi'(\|Y^{i,(N)}\|_{\mathcal{S}^\infty})(1+\|Y^{i,(N)}\|_{\mathcal{S}^\infty})ds.$$

The right-hand side admits a universal bound (independent of N and τ), and hence so does the left-hand side.

Finally, we go back to the equation in (BSDE$_N$) satisfied by $a^{(N)}$ and note that the term $\exp(-\iota_N(a^{(N)}))$ is bounded because $(a^{(N)})^-$ is. We can bound $a^{(N)}$ from above in an N-independent manner, by a combination of the bmo-bounds on $Z^{(N)}$ and the sup-norm of μ_e. By taking expectations and using the universal boundedness/ bmo-property of all the other terms, we conclude that $\sigma^{(N)}$ also admits a universal bmo-bound.

Having the universal bounds on $Y^{(N)}$ and $a^{(N)}$, we can remove some of the truncations introduced in (BSDE$_N$). Indeed, for N larger than the largest of the $\mathcal{S}^\infty$-bounds on $Y^{(N)}$ and $a^{(N)}$, we have

$$\iota_N(Y^{i,(N)}) = Y^{i,(N)}, \qquad \iota_N(a^{(N)}) = a^{(N)}.$$

Therefore, there exists a constant N_0 such that for $N \geq N_0$, the processes $(Y^{(N)}, a^{(N)})$ together with $(Z^{(N)}, \sigma^{(N)})$ solve the intermediate system

$$\begin{aligned} da &= \sigma dB + \left(\bar{\alpha}\mu_e - \frac{1}{2}\sum_j^I \kappa^j q_N(Z^j) - \exp(-\iota_{N_0}(a))\right)dt, \\ dY^i &= Z^i dB \\ &\quad + \left(\frac{1}{2}q_N(Z^i) + \exp(-\iota_{N_0}(a))\big(\iota_{N_0}(Y^i) + \iota_{N_0}(a) - \alpha^i e^i + 1\big)\right)dt \end{aligned} \qquad (\text{BSDE}'_N)$$

with the same terminal conditions as (3.1).

Step 3 (Bensoussan–Frehse conditions and the existence of a Lyapunov function). Mere boundedness in $\mathcal{S}^\infty \times$ bmo is not sufficient to guarantee subsequential convergence of the solution $(Y^{(N)}, a^{(N)})$ of the truncated system to a limit which solves (3.1) or (BSDE$'_N$). It has been shown, however, in Xing and Žitković [16, Theorem 2.8], that an additional property – namely the existence of a uniform Lyapunov function – will guarantee such a convergence. The existence of such a function can be deduced from another result of the same paper, Xing and Žitković [16, Proposition 2.11], once its conditions are checked. That proposition states that a uniformly bounded sequence of solutions of a sequence of BSDEs such as (BSDE$'_N$) admits a common Lyapunov function if the structure of its drivers satisfies the so-called Bensoussan–Frehse conditions uniformly in N (see Xing and Žitković [16, Definition 2.10] for the definition). The result of [16, Proposition 2.11] applies here because our system is of upper triangular form when it comes to its quadratic dependence on z. More precisely, the driver of the system (BSDE$'_N$) can be represented as a sum of two functions $f_1^{(N)}$ and $f_2^{(N)}$ given by

$$(f_1^{(N)})^i(t,x,y,z) = \begin{cases} \bar{\alpha}\mu_e(t,x) - \exp(-\iota_{N_0}(y^0)), & i=0, \\ \exp(-\iota_{N_0}(y^0))(\iota_{N_0}(y^i) + \iota_{N_0}(y^0) \\ \qquad\qquad - \alpha^i e^i(t,x) + 1)), & 1 \le i \le I, \end{cases}$$

$$(f_2^{(N)})^i(t,x,y,z) = \begin{cases} -\frac{1}{2}\sum_{l=1}^I \kappa_l q_N(z^l), & i=0, \\ \frac{1}{2} q_N(z^i), & 1 \le i \le I, \end{cases}$$

where the convention that $a = Y^0$ and $\sigma = Z^0$ is used. Therefore, there exists a universal constant C such that for all $0 \le i \le I$, we have

$$|(f_1^{(N)})^i(t,x,y,z)| \le C$$

as well as

$$|(f_2^{(N)})^i(t,x,y,z)| \le C\left(1 + \sum_{j=1}^i |q_N(z^j)|\right) \le C\left(1 + \sum_{j=1}^i |z^j|^2\right).$$

Therefore, $f^{(N)}$ can be split into a subquadratic (in fact bounded) and an upper triangular component, allowing us to conclude that a uniform Lyapunov function for $(f^{(N)})_{N \ge N_0}$ can be constructed.

Step 4 (Passage to a limit). It remains to use Xing and Žitković result [16, Theorem 2.8] to conclude that a subsequence of $(v^{(N)})$ converges towards a continuous function $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{I+1}$ such that $Y_t = v(t,\xi_t)$ and $Z_t = Dv(t,\xi_t)$ solves the limiting system

$$da = \sigma\, dB + \left(\bar{\alpha}\mu_e - \frac{1}{2}\sum_\ell \kappa^\ell |Z^\ell|^2 - \exp\big(-\iota_{N_0}(a)\big)\right) dt,$$

$$dY^i = Z^i\, dB + \left(\frac{1}{2}|Z^i|^2 + \exp\big(-\iota_{N_0}(a)\big)\big(\iota_{N_0}(Y^i) + \iota_{N_0}(a) - \alpha^i e^i + 1\big)\right) dt$$

for $i = 1, \ldots, I$, where $a = Y^0$ and $\sigma = Z^0$ as above. As far as the conditions of [16, Theorem 2.8] are concerned, the most difficult one, the existence of a Lyapunov function, has been settled in Step 3 above. The other conditions – the uniform Hölder-boundedness of the terminal conditions and a priori boundedness – are easily seen to be implied by our assumptions. Finally, since Y is a pointwise limit of a sequence of functions bounded by N_0, the same processes (Y, Z) also solve the original BSDE (3.1) (without truncation at N_0). □

4 On Pareto-optimality of the equilibrium

One of the most important properties of incomplete-market equilibria is that they typically do not produce Pareto-optimal allocations. Our model is no exception; as the main result in this section shows, it describes a genuine incomplete-market equilibrium since it leads to a Pareto-optimal allocation only if the endowment processes e^i satisfy a special pointwise relation.

4.1 Allocations and Pareto-optimality

For two elements $\mathcal{U}_k = (u_k^1, \dots, u_k^I) \in [-\infty, 0)^I$, $k = 1, 2$, we write $\mathcal{U}_1 \leq \mathcal{U}_2$ if $u_1^i \leq u_2^i$ for all $1 \leq i \leq I$. If $\mathcal{U}_1 \leq \mathcal{U}_2$ but $u_1^i < u_2^i$ for some i, we write $\mathcal{U}_1 \lneq \mathcal{U}_2$.

To simplify the notation in the following definition and the sequel, we denote by ν the measure $\mathrm{Leb} + \delta_{\{T\}}$ on $[0, T]$.

Definition 4.1 An I-tuple $\mathcal{A} = (c^1, \dots, c^I)$ of progressively measurable processes is called an *allocation*.

1) The *utility* of the allocation $\mathcal{A} = (c^1, \dots, c^I)$, denoted by $\mathcal{U}(\mathcal{A})$, is the I-tuple $(u^1, \dots, u^I)$, where for $1 \leq i \leq I$,

$$u^i = \mathbb{E}\left[\int_0^T U^i(c_t^i)\,\nu(dt)\right] = \mathbb{E}\left[\int_0^T U^i(c_t^i)\,dt\right] + \mathbb{E}[U^i(c_T^i)] \in [-\infty, 0).$$

2) An allocation $\mathcal{A} = (c^1, \dots, c^I)$ is said to be *feasible* if

$$\sum_i c^i = e + 1 \qquad \nu\text{-a.e.}$$

3) A feasible allocation $\mathcal{A}$ is said to be *Pareto-optimal* if

there is no other feasible allocation $\mathcal{A}'$ such that $\mathcal{U}(\mathcal{A}) \lneq \mathcal{U}(\mathcal{A}')$.

The following lemma is a restatement of the second welfare theorem in our setting. It formulates and proves the well-known characterisation of Pareto-optimality in terms of "marginal rates of substitution" in our setting. We give a short and self-contained proof adapted to our framework.

Lemma 4.2 *Suppose that $\mathcal{A}$ is a feasible allocation with $\mathcal{U}(\mathcal{A}) \in (-\infty, 0)^I$. Then the following two statements are equivalent*:

1) *$\mathcal{A}$ is Pareto-optimal.*

2) *The process $\alpha^i c^i - \alpha^j c^j$ is ν-a.e. constant, for all pairs $1 \leq i, j \leq I$.*

Proof "2) ⇒ 1)" Suppose that $\mathcal{A}_k = (c_k^1, \dots, c_k^I)$, $k = 1, 2$, are two feasible allocations with utility vectors $\mathcal{U}(\mathcal{A}_k) = (u_k^1, \dots, u_k^I) \in (-\infty, 0)^I$. We assume that 2) holds for $\mathcal{A}_1$ so that there exist constants $m^i > 0$ such that

$$m^i \exp(-\alpha^i c_1^i) = m^j \exp(-\alpha^j c_1^j) \qquad \nu\text{-a.e., for all } i, j. \tag{4.1}$$

Thanks to the concavity of the utility function U^i and the fact that its derivative is $\exp(-\alpha^i \cdot)$, we have for all i the ν-a.e. inequality

$$m^i U^i(c_2^i) \leq m^i U^i(c_1^i) + m^i \exp(-\alpha^i c_1^i)(c_2^i - c_1^i).$$

Feasibility of both allocations together with (4.1) implies that

$$\sum_i m^i u_2^i \le \sum_i m^i u_1^i.$$

Consequently, we cannot have $\mathcal{U}(\mathcal{A}_1) \lneqq \mathcal{U}(\mathcal{A}_2)$ which in turn implies that $\mathcal{A}_1$ is Pareto-optimal.

"1) ⇒ 2)" We argue by contradiction and assume that $\mathcal{A}$ is a feasible allocation with $\mathcal{U}(\mathcal{A}) \in (-\infty, 0)^I$ for which 2) fails for some pair of indices $i_1 \neq i_2$. The idea is to construct a better allocation by transferring some consumption good from i_1 to i_2 and vice versa, while keeping everything else the same. There is no loss of generality in assuming that $i_1 = 1, i_2 = 2$ and $I = 2$.

The failure of 2) implies that there exists a bounded progressively measurable process δ such that

$$\mathbb{E}\bigg[\int_0^T \exp(-\alpha^1 c_t^1)\delta_t\,\nu(dt)\bigg] > 0 > \mathbb{E}\bigg[\int_0^T \exp(-\alpha^2 c_t^2)\delta_t\,\nu(dt)\bigg]. \tag{4.2}$$

For any $\varepsilon \in (-1, 1) \setminus \{0\}$ and $i = 1, 2$, we have

$$\frac{1}{|\varepsilon|}|U^i(c^i + \varepsilon\delta) - U^i(c^i)| \le \exp(-\alpha^i c^i)\bigg|\frac{e^{-\alpha^i \varepsilon\delta} - 1}{\alpha^i \varepsilon}\bigg|.$$

Since δ is bounded, the dominated convergence theorem implies that

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\bigg[\int_0^T \frac{1}{\varepsilon}\big(U^i(c_t^i + \varepsilon\delta_t) - U^i(c_t^i)\big)\,\nu(dt)\bigg] = \mathbb{E}\bigg[\int_0^T \exp(-\alpha^i c_t^i)\delta_t\,\nu(dt)\bigg]$$

for $i = 1, 2$. Thanks to (4.2), for a small enough value of $\varepsilon_0 > 0$, we have

$$\mathbb{E}\bigg[\int_0^T U^1(c_t^1 + \varepsilon_0\delta_t)\,\nu(dt)\bigg] > \mathbb{E}\bigg[\int_0^T U^1(c_t^1)\,\nu(dt)\bigg],$$
$$\mathbb{E}\bigg[\int_0^T U^2(c_t^2 - \varepsilon_0\delta_t)\,\nu(dt)\bigg] > \mathbb{E}\bigg[\int_0^T U^2(c_t^2)\,\nu(dt)\bigg].$$

Therefore, the utility of the feasible allocation $(c^1 + \varepsilon_0\delta, c^2 - \varepsilon_0\delta)$ strictly dominates that of $\mathcal{A} = (c^1, c^2)$, implying that $\mathcal{A}$ cannot be Pareto-optimal. □

4.2 Equilibrium allocations and conditions for Pareto-optimality

As in Remark 3.2, 2), each pair $\gamma = (\mu, \sigma)$ of equilibrium market coefficients comes with a unique set of strategies $\hat{c}^i$, $\hat{\pi}^i$ for the agents, as well as related gains processes $\hat{X}^i$, $1 \le i \le I$. These ingredients can be used to construct a feasible allocation $\mathcal{A}(\gamma) = (c^1, \dots, c^I)$ as follows: We set

$$c_t^i = \hat{c}_t^i \text{ for } t \in [0, T) \quad \text{and} \quad c_T^i = \hat{X}_T^i + e_T^i = \pi_T^i + e_T^i.$$

Theorem 4.3 *Under Assumption 2.3, let γ be a pair of equilibrium market coefficients with associated annuity price process A, agent strategies $\hat{\pi}^i, \hat{c}^i$ for $1 \le i \le I$ and allocation $\mathcal{A}(\gamma)$. We define*

$$D_t = \frac{1}{A_t} \exp(-\int_0^t \frac{1}{A_s} ds), \qquad E^i = D_T e^i_T + \int_0^T D_t e^i_t \, dt$$

for $1 \le i \le I$. Then the following statements are equivalent:

1) *The random variables $\alpha^i E^i - \alpha^j E^j$ are constant a.s., for all $1 \le i, j \le I$.*

2) *The allocation $\mathcal{A}(\gamma)$ is Pareto-optimal.*

Proof We associate to γ the agent strategies $(\hat{\pi}^i, \hat{c}^i)_{1 \le i \le I}$ as well as the processes A, $a = \log A$, $\hat{X}^i$, Y^i and Z^i, $1 \le i \le I$, as explained in Remark 3.2, 2). We also record the following identity which follows directly from (3.3) and is used in both parts of the proof: we have

$$\alpha^i \hat{c}^i_t = \alpha^i \hat{c}^i_0 + a_t - a_0 + \int_0^t Z^i_s \, dB_s + \int_0^t \left(\frac{1}{2} |Z^i_s|^2 + 1/A_s \right) ds. \tag{4.3}$$

"1) ⇒ 2)" Assumption 1) implies that there exist constants h^i such that

$$E^i = h^i + \kappa^i E \text{ a.s.}, \qquad \text{where } E = D_T e_T + \int_0^T D_t e_t \, dt.$$

The self-financing conditions of Definition 2.5 imply that given a fixed admissible consumption process c, its (only) financing portfolio process π admits the dynamics $d\pi = \frac{1}{A}(e^i - c + \pi)\, dt$ with $\pi_0 = \pi^i_0$. Setting $L_t = \exp(-\int_0^t 1/A_s \, ds)$ so that $D = L/A$, we obtain

$$d(\pi_t L_t) = \frac{L_t}{A_t}(e_t - c_t)\, dt. \tag{4.4}$$

Together with $L_0 = 1$ and $A_T = 1$, (4.4) yields a relationship of the form

$$D_T(e^i_T + \pi_T) + \int_0^T D_t c_t \, dt = \pi^i_0 + D_T e^i_T + \int_0^T D_t e^i_t \, dt = \pi^i_0 + E^i. \tag{4.5}$$

Moreover, it follows that the agent i can produce a pair (c, π) with prescribed values for c and π_T if and only if these values satisfy the budget constraint (4.5). It also follows that the optimisation problem faced by agent i depends on the process e^i and the initial holding π^i_0 only through the sum $\pi^i_0 + E^i$. In particular, thanks to the uniqueness of the optimal consumption process $\hat{c}^i$, the same $\hat{c}^i$ (and therefore, by Lemma 4.2, the same verdict on Pareto-optimality) will be obtained if we replace each e^i by the process $\kappa^i e$ and adjust the initial holding π^i_0 by the constant h^i.

In this new market, all agents have identical random endowments; so by uniqueness, the associated processes Y^i defined in (3.1) agree. Consequently, the processes Z^i are also identical, and we can use (4.3) to conclude that the processes $\alpha^i \hat{c}^i$ agree up to a constant $(\text{Leb} \otimes \mathbb{P})$-a.e. By continuity, the same is true for $t = T$ a.s., and we can use Lemma 4.2 to conclude that the allocation $\mathcal{A}$ is Pareto-optimal.

"2) ⇒ 1)" The Pareto-optimality assumption 2) implies via Lemma 4.2 that the processes $\alpha^i \hat{c}^i$ agree up to an additive constant ν-a.e. Representation (4.3) implies that the processes Z^i, $1 \le i \le I$, coincide $(\text{Leb} \otimes \mathbb{P})$-a.e. Thus the difference $\Delta = Y^i - Y^j$ admits the dynamics

$$d\Delta = \frac{1}{A}\big(\Delta - (\alpha^i e^i - \alpha^j e^j)\big)\,dt, \quad \Delta_T = \alpha^i e^i_T - \alpha^j e^j_T \qquad \text{a.s.} \tag{4.6}$$

As in the proof of "1) ⇒ 2)", (4.6) solves to give

$$D_T(\alpha^i e^i_T - \alpha^j e^j_T) + \int_0^T D_t(\alpha^i e^i_t - \alpha^j e^j_t)\,dt = Y^i_0 - Y^j_0 \qquad \text{a.s.},$$

which immediately implies 1). □

5 Bounded solutions of Lipschitz quasilinear systems

The main result of this section, Proposition 5.1, collects some results on systems of parabolic equations with Lipschitz nonlinearities on derivatives up to the first order. We suspect that these results may be well known to PDE specialists, but we were unable to find a precise reference under the same set of assumptions in the literature and therefore decided to include a fairly self-contained proof.

In the sequel, D denotes the derivative operator with respect to all spatial variables, i.e., all variables except t. For $d, J \in \mathbb{N}$ and $\beta \ge 0$, we define the following three Banach spaces, where we always use Lebesgue measure on $\mathbb{R}^d$:

1) $\mathbb{L}^\infty = \mathbb{L}^\infty(\mathbb{R}^d; \mathbb{R}^J)$ or $\mathbb{L}^\infty = \mathbb{L}^\infty(\mathbb{R}^d; \mathbb{R}^{J\times d})$, depending on the context;

2) $W^{1,\infty} = W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^J)$, with the norm $\|U\|_{W^{1,\infty}} = \|U\|_{\mathbb{L}^\infty} + \|DU\|_{\mathbb{L}^\infty}$;

3) $\mathbb{L}^1_\beta = \mathbb{L}^1_\beta([0,T); W^{1,\infty})$; this is the Banach space of Borel-measurable functions $u : [0,T] \to W^{1,\infty}$ endowed with the exponentially weighted norm

$$\|u\|_{\mathbb{L}^1_\beta} = \int_0^T e^{-\beta(T-t)}\|u(t,\cdot)\|_{W^{1,\infty}}\,dt.$$

The infinitesimal generator of the state process ξ is given by

$$\mathcal{A}u(t,x) = Du(t,x)\Lambda(t,x) + \frac{1}{2}\,\mathrm{Tr}\big(D^2u(t,x)\Sigma(t,x)\Sigma^T(t,x)\big)$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$.

Proposition 5.1 *Suppose that the Borel-measurable functions $g : \mathbb{R}^d \to \mathbb{R}^J$ and $F : [0,T] \times \mathbb{R}^{1\times d} \times \mathbb{R}^J \times \mathbb{R}^{J\times d} \to \mathbb{R}^J$ are such that for some M and for all t, x, y_1, y_2, z_1, z_2, we have*

(a) $|g(x)| \le M$;
(b) $|F(t,x,0,0)| \le M$;
(c) $|F(t,x,y_2,z_2) - F(t,x,y_1,z_1)| \le M(|y_2 - y_1| + |z_2 - z_1|)$.

Moreover, suppose the functions Λ *and* Σ *satisfy the conditions of Assumption* 2.1 (*with the constant* K). *Then the following hold*:

1) *The PDE system*

$$u_t + \mathcal{A}u + F(\cdot, \cdot, u, Du) = 0, \qquad u(T, \cdot) = g \tag{5.1}$$

admits a weak solution u *on* $[0, T]$. *Moreover,* $u(t, \cdot) \in W^{1,\infty}$ *for all* $t \in [0, T)$ *and there exists a constant* $C = C(J, d, M, T, K) \in [0, \infty)$ *such that*

$$\|u(t, \cdot)\|_{\mathbb{L}^\infty} \leq C \text{ for all } t \in [0, T] \quad \text{and} \quad \int_0^T \|Du(t, \cdot)\|_{\mathbb{L}^\infty}\, dt \leq C.$$

2) *Let* u *denote a solution of* (5.1) *as in* 1) *and let* $(\xi_t)_{t\in[0,T]}$ *be a strong solution of the SDE*

$$d\xi_t = \Lambda(t, \xi_t)\, dt + \Sigma(t, \xi_t)\, dB_t.$$

Then the pair (Y, Z) *defined by* $Y_t = u(t, \xi_t)$ *and* $Z_t = Du(t, \xi_t)\Sigma(t, \xi_t)$ *is an* $(\mathcal{S}^\infty \times \text{bmo})$-*solution to the system*

$$dY^i_t = -F^i\big(t, \xi_t, Y_t, Z_t\Sigma^{-1}(t, \xi_t)\big)\, dt + Z^i_t\, dB_t \tag{5.2}$$

with terminal conditions $Y^i_T = g^i(\xi_T)$, $i = 1, \ldots, I$.

Proof Throughout the proof, C denotes a constant which may depend on J, d, M, T or K, but not on β, t, s or x, and can change from line to line; we call such a constant universal. The assumptions on F imply that uniformly in t and for all $U, V \in W^{1,\infty}$, we have

$$\|F(t, \cdot, U, DU) - F(t, \cdot, V, DV)\|_{\mathbb{L}^\infty} \leq C\|U - V\|_{W^{1,\infty}}$$

and

$$\|F(t, \cdot, U, DU)\|_{\mathbb{L}^\infty} \leq C(1 + \|U\|_{W^{1,\infty}}). \tag{5.3}$$

Let $p(t, x; s, x')$ denote a fundamental solution associated to the operator $\frac{\partial}{\partial t} + \mathcal{A}$, i.e., $(t, x) \mapsto p(t, x, s, x')$ solves

$$p_t + \mathcal{A}p = 0 \qquad \text{for } t, x \in [0, s) \times \mathbb{R}^d$$

classically and satisfies the boundary condition

$$\lim_{t \nearrow s} \int_{\mathbb{R}^d} \psi(x) p(t, x, s, x')\, dx = \psi(x')$$

for each bounded and continuous ψ. We refer the reader to Friedman [7, Theorem 1.6.10 and the discussion preceding it] for the existence of a positive fundamental solution under the conditions of Assumption 2.1. Moreover, equations (6.12) and (6.13) of Friedman [7, Chap. 1, Sect. 6] state that there exist universal constants $C, \lambda > 0$ such that for $t < s$ and all x, x', we have

$$
\begin{aligned}
|p(t,x,s,x')| &\leq C\varphi_\lambda(t,x,s,x'),\\
|\partial_{x_k} p(t,x,s,x')| &\leq C\frac{1}{\sqrt{s-t}}\varphi_\lambda(t,x,s,x')
\end{aligned}
\tag{5.4}
$$

for all $k = 1, \dots, d$, where

$$
\varphi_\lambda(t,x;s,x') = \frac{1}{(2\pi\lambda^2(s-t))^{d/2}} \exp\left(-\frac{1}{2\lambda^2(s-t)}|x'-x|^2\right)
$$

is the scaled heat kernel (which is itself a fundamental solution associated to the operator $\frac{\partial}{\partial t} + \frac{1}{2}\lambda^2\Delta$). These properties allow us to define for each $u \in \mathbb{L}^1_\beta$ the function $\Phi[u] : [0,T] \times \mathbb{R}^d \to \mathbb{R}^J$ by

$$
\Phi[u](t,x) = \int_{\mathbb{R}^d}\int_t^T F\big(s,x',u(s,x'),Du(s,x')\big)p(t,x;s,x')\,ds\,dx'. \tag{5.5}
$$

Estimate (5.3) guarantees that $\Phi[u]$ is well defined with $\Phi[u](t,\cdot) \in \mathbb{L}^\infty$. The Gaussian bounds in (5.4) imply that one can pass the derivative under the integral sign to obtain

$$
\partial_{x_k}\Phi[u](t,x) = \int_{\mathbb{R}^d}\int_t^T F\big(s,x',u(s,x'),Du(s,x')\big)\partial_{x_k} p(t,x;s,x')\,ds\,dx'. \tag{5.6}
$$

Consequently, $t \mapsto \Phi[u](t,\cdot)$ is an a.e. defined measurable map from $[0,T]$ to $W^{1,\infty}$ for each $u \in \mathbb{L}^1_\beta$. To bound the norm of $\Phi[u]$, we start with the estimate, fuelled by (5.4),

$$
\begin{aligned}
\|\Phi[u](t,\cdot)\|_{W^{1,\infty}} &\leq C\int_t^T \big(1 + \|u(s,\cdot)\|_{W^{1,\infty}}\big)\\
&\qquad \times \int_{\mathbb{R}^d}\left(p(t,x;s,x') + \sum_{k=1}^J |\partial_{x_k} p(t,x;s,x')|\right)dx'\,ds\\
&\leq C\int_t^T \frac{1}{\sqrt{s-t}}\big(1 + \|u(s,\cdot)\|_{W^{1,\infty}}\big)\,ds.
\end{aligned}
$$

Furthermore, (5.5) and (5.6) imply that

$$
\begin{aligned}
\|\Phi[u]\|_{\mathbb{L}^1_\beta} &\leq C\int_0^T e^{\beta(t-T)}\int_t^T \frac{1}{\sqrt{s-t}}\big(1 + \|u(s,\cdot)\|_{W^{1,\infty}}\big)\,ds\,dt\\
&= C\int_0^T \big(1 + \|u(s,\cdot)\|_{W^{1,\infty}}\big)\,ds\int_0^s \frac{1}{\sqrt{s-t}}e^{\beta(t-T)}\,dt\\
&= \frac{C\sqrt{\pi}}{\sqrt{\beta}}\int_0^T \big(1 + \|u(s,\cdot)\|_{W^{1,\infty}}\big)\,\mathrm{Erf}(\sqrt{\beta s})\,ds\\
&\leq \frac{C}{\sqrt{\beta}}\big(1 + \|u\|_{\mathbb{L}^1_\beta}\big),
\end{aligned}
$$

recalling that we allow the constant C to change from line to line. A similar computation also yields

$$\|\Phi[u]-\Phi[v]\|_{\mathbb{L}^1_\beta}\le\frac{C}{\sqrt{\beta}}\|u-v\|_{\mathbb{L}^1_\beta}. \tag{5.7}$$

Next, for $g\in\mathbb{L}^\infty$, we define

$$\Psi[g](t,x)=\int_{\mathbb{R}^d}g(x')p(t,x;T,x')\,dx'$$

so that as above,

$$\|\Psi[g](t,\cdot)\|_{W^{1,\infty}}\le\frac{C}{\sqrt{T-t}}\|g\|_{\mathbb{L}^\infty},\qquad \|\Psi[g]\|_{\mathbb{L}^1_\beta}\le\frac{C}{\sqrt{\beta}}\|g\|_{\mathbb{L}^\infty}$$

and $\Psi[g]\in\mathbb{L}^1_\beta$ for each $g\in\mathbb{L}^\infty$. Therefore, the function

$$\Gamma[u]=\Phi[u]+\Psi[g]$$

maps $\mathbb{L}^1_\beta$ into $\mathbb{L}^1_\beta$ and (5.7) implies that it is Lipschitz with constant $C/\sqrt{\beta}$. Since C does not depend on β, we can turn Γ into a contraction by choosing a large enough β and conclude that Γ admits a unique fixed point $u\in\mathbb{L}^1_\beta$. The integral representations in (5.5) and (5.6) allow us to conclude that u and Du are continuous functions on $[0,T)\times\mathbb{R}^d$. Moreover, thanks to the Markov property of ξ, we have

$$u(t,\xi_t)=\mathbb{E}\left[g(\xi_T)+\int_t^T f(s,\xi_s)\,ds\middle|\mathcal{F}_t\right]\qquad\text{a.s.,}$$

where

$$f(s,x)=F\big(s,x,u(s,x),Du(s,x)\big)\in\mathbb{R}^J.$$

Since $\|f(t,\cdot)\|_{\mathbb{L}^\infty}\le C(1+\|u(t,\cdot)\|_{W^{1,\infty}})$ for all t, the map $t\mapsto\|u(t,\cdot)\|_{W^{1,\infty}}$ belongs to $\mathbb{L}^1_\beta$, and (stripped of its norm) the space $\mathbb{L}^1_\beta$ does not depend on the choice of β. Therefore,

$$\|u(t,\xi_t)-\mathbb{E}[g(\xi_T)|\mathcal{F}_t]\|_{\mathbb{L}^\infty}\le\int_t^T\|f(s,\cdot)\|_{\mathbb{L}^\infty}\,ds\longrightarrow 0\qquad\text{as } t\to T.$$

Since g is bounded, we have $\|u(t,\cdot)\|_{\mathbb{L}^\infty}\le C$ for all t. Moreover, the martingale $\mathbb{E}[g(\xi_T)|\mathcal{F}_t]$, $0\le t\le T$, admits a continuous modification; so the process Y defined by

$$Y_t=\begin{cases}u(t,\xi_t), & t<T,\\ g(\xi_T), & t=T,\end{cases}$$

is a.s. continuous. This allows us to conclude furthermore that $Y_t+\int_0^t f(s,\xi_s)$,

$0 \le t \le T$, is a continuous modification of the martingale

$$M_t = \mathbb{E}\bigg[g(\xi_T) + \int_0^T f(s, \xi_s)\, ds \bigg| \mathcal{F}_t \bigg],$$

making Y a semimartingale. To show that (Y, Z) as in the statement indeed solves (5.2), we need to argue that $M_t - M_0$ must be of the form $\int_0^t Du(s, \xi_s)\Sigma(s, \xi_s)\, dB_s$. This can be proved by approximation as in the proof of Xing and Žitković [16, Lemma 4.4].

The last step is to argue that (Y, Z) is an $(\mathcal{S}^\infty \times \text{bmo})$-solution. The function u is uniformly bounded, so it suffices to establish the bmo-property of Z. This can be bootstrapped from the boundedness of Y by applying Itô's formula to the bounded processes $\exp(cY^i)$, $i = 1, \dots, J$, for a large enough constant c. A similar argument is already presented in the proof of Theorem 3.4; so we skip the details. □

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