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On the Incompressible Limit for the Compressible Free-Boundary Euler Equations with Surface Tension in the Case of a Liquid

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Abstract

In this paper we establish the incompressible limit for the compressible free-boundary Euler equations with surface tension in the case of a liquid. Compared to the case without surface tension treated recently in Lindblad and Luo (Commun Pure Appl Math 71:1273–1333, 2018) and Luo (Ann PDE 4(2):1–71, 2018), the presence of surface tension introduces severe new technical challenges, in that several boundary terms that automatically vanish when surface tension is absent now contribute at top order. Combined with the necessity of producing estimates uniform in the sound speed in order to pass to the limit, such difficulties imply that neither the techniques employed for the case without surface tension, nor estimates previously derived for a liquid with surface tension and fixed sound speed, are applicable here. In order to obtain our result, we devise a suitable sound-speed-weighted energy that takes into account the coupling of the fluid motion with the boundary geometry. Estimates are closed by exploiting the full non-linear structure of the Euler equations and invoking several geometric properties of the boundary in order to produce some remarkable cancellations. We stress that we do not assume the fluid to be irrotational.

Contents

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1. Introduction

We consider the motion of a compressible liquid with free surface boundary in $\mathbb{R}^3$. We use the notation $\mathcal{D}_t$ to represent the bounded domain occupied by the fluid at each time t, whose boundary is advected by the fluid. The motion of the fluid is described by the compressible Euler equations

$$\begin{cases} \rho(\partial_t u + \nabla_u u) = -\nabla p, & \text{in } \mathcal{D}, \\ \partial_t \rho + \nabla_u \rho + \rho \operatorname{div} u = 0, & \text{in } \mathcal{D}, \\ p = p(\rho), & \text{in } \mathcal{D}. \end{cases} \tag{1.1}$$

Here, $\mathcal{D} = \cup_{0\leq t\leq T}\{t\} \times \mathcal{D}_t$, $u = u(t, x)$ is the velocity of the fluid, whereas $p = p(t, x)$ and $\rho = \rho(t, x)$ are the pressure and density, respectively. The density is bounded from below away from zero, that is, $\rho \geq$ constant > 0. This condition on the density is what characterizes the fluid as a liquid. The initial and boundary conditions are

$$\begin{cases} \{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, \\ u = u_0, \rho = \rho_0 \quad \text{in } \{0\} \times \mathcal{D}_0, \end{cases} \qquad \begin{cases} (\partial_t + \nabla_u)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \\ p|_{\partial\mathcal{D}} = \sigma\mathcal{H}, \end{cases} \tag{1.2}$$

where $\mathcal{H}$ is the mean curvature of $\partial\mathcal{D}_t$, $\sigma \geq 0$ is a constant, and $T(\partial\mathcal{D})$ is the tangent bundle of $\partial\mathcal{D}$ (the condition $(\partial_t + \nabla_u)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D})$ expresses the fact that the boundary moves with speed equal to the normal component of the velocity). Finally, the equation of state is assumed to be a strictly increasing function of the density, that is,

$$p = p(\rho), \quad p'(\rho) > 0.$$

We shall consider the specific equation of state given by (1.3) in this manuscript. The unknowns in (1)–(1) are u, ρ and $\mathcal{D}_t$, and hence, $\mathcal{H}$ and p are function of the unknowns, and therefore, are not known a priori.

Problem (1)–(1) behaves significantly different depending on whether $\sigma = 0$ or $\sigma > 0$. The former is known as the case without surface tension whereas the latter is the case with surface tension, which is the situation treated in this manuscript. Our goal is to show that, for $\sigma > 0$, the motion of a free-boundary incompressible fluid with surface tension (corresponding to the idealized situation of a constant density fluid) is well-approximated by (1)–(1) when an appropriate notion of compressibility is very small. It is well-known that solutions to the incompressible equations, written in section 1.2 below, cannot be obtained by simply setting ρ to a constant in (1)–(1) (see, for example, [46]). The correct way of setting the incompressible limit is via the fluid's sound speed introduced in section 1.3.

The study of the incompressible limit has a long history in fluid dynamics; see section 1.2. For the case of a motion with free-boundary, the only results we are aware are the recent works [46,48] by Lindblad and the second author, both treating the case $\sigma = 0$. In particular, to the best of our knowledge this is the first proof of the incompressible limit for the free-boundary compressible Euler equations with surface tension, that is, $\sigma > 0$. Despite many new difficulties introduced by the presence of surface tension, which are discussed in section 1.6, it is important to consider the case $\sigma > 0$ because real fluids have surface tension. Thus, this feature has to be incorporated in the construction of more realistic models. We remark that we do *not* assume that the fluid is irrotational.

1.1. Lagrangian Coordinate and the Reference Domain

We introduce Lagrangian coordinates, under which the moving domain becomes fixed. Let Ω be a bounded domain in $\mathbb{R}^3$. Denoting coordinates on Ω by

$y = (y_1, y_2, y_3)$, we define $\eta : [0, T] \times \Omega \to \mathcal{D}$ to be the flow of the velocity u, that is,

$$\partial_t \eta(t, y) = u(t, \eta(t, y)),$$
$$\eta(0, y) = y.$$

We introduce the Lagrangian velocity, density and pressure, respectively, by $v(t, y) := u(t, \eta(t, y))$, $R(t, y) := \rho(t, \eta(t, y))$ and $q(t, y) := p(t, \eta(t, y))$. Therefore,

$$\partial_t \eta = v.$$

For the sake of simplicity and clean notation, here we consider the model case when $\mathcal{D}_0 = \Omega = \mathbb{T}^2 \times (0, 1)$. We set

$$\Gamma_0 := \mathbb{T}^2 \times \{x_3 = 0\}, \quad \Gamma_1 := \mathbb{T}^2 \times \{x_3 = 1\}$$

so that $\Gamma := \partial\Omega = \Gamma_0 \cup \Gamma_1$. Using a partition of unity, as in, for example, [9,40], a general domain can be treated with the same tools we shall present. Choosing Ω as above, however, allows us to focus on the real issues of the problem without being distracted by the cumbersomeness of the partition of the unity. We also note that one might want to consider a situation more akin the finite-depth water waves problem, where the bottom boundary, Γ_0, remains fixed. This case requires only minor modifications from our presentation but, again, we believe that this would be a distraction from the main problem.

Let ∂ be the spatial derivative with respect to the spatial variable y. We introduce the matrix $a = (\partial\eta)^{-1}$. This is well-defined since $\eta(t, \cdot)$ is almost id (that is, the identity diffeomorphism on Ω) whenever t is sufficiently small. Define the cofactor matrix

$$A = Ja,$$

where $J = \det(\partial\eta)$. Then, A satisfies the Piola identity

$$\partial_\mu A^{\mu\alpha} = 0.$$

Here, the summation convention is used for repeated upper and lower indices, and in above and throughout, we adopt the convention that the Greek indices range over 1, 2, 3, while the Latin indices range over 1 and 2.

In terms of v, R, q and a, the system (1)–(1) becomes

$$\begin{cases} R\partial_t v^\alpha + a^{\mu\alpha}\partial_\mu q = 0, & \text{in } [0, T] \times \Omega \\ \partial_t R + Ra^{\mu\alpha}\partial_\mu v_\alpha = 0, & \text{in } [0, T] \times \Omega \\ q = q(R), & \text{in } [0, T] \times \Omega \\ A^{\mu\alpha} N_\mu q + \sigma\sqrt{g}\Delta_g \eta^\alpha = 0, & \text{on } [0, T] \times \Gamma, \\ \eta(0, \cdot) = \text{id}, \quad R(0, \cdot) = R_0 (= \rho_0), \quad v(0, \cdot) = v_0, \end{cases} \tag{1.3}$$

where N is the unit outward normal to Γ, and Δ_g is the Laplacian of the metric g_{ij} induced on $\Gamma(t) = \eta(t, \Gamma)$ by the embedding η, that is,

$$g_{ij} = \partial_i \eta^\mu \partial_j \eta_\mu, \quad \Delta_g(\cdot) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j(\cdot)),$$

where $g = \det g$. Since $\eta(0, \cdot) = \mathrm{id}$, the initial Eulerian and Lagrangian velocities (that is, u_0 and v_0) agree. In addition, we also have $a(0, \cdot) = I$, where I is the identity matrix. Finally, $J = \det(\partial\eta)$ satisfies

$$\partial_t J = J a^{\mu\nu} \partial_\mu v_\nu, \quad [0, T] \times \Omega. \tag{1.4}$$

This, together with the second equation of (1.1), implies

$$RJ = \rho_0, \quad [0, T] \times \Omega, \tag{1.5}$$

and hence the first equation in (1.1) is equivalent to

$$\rho_0 \partial_t v^\alpha + A^{\mu\alpha} \partial_\mu q = 0, \quad \text{in } [0, T] \times \Omega. \tag{1.6}$$

1.2. Background

The study of the motion of a fluid has a long history in mathematics. In particular, the study of free-boundary fluid problems has blossomed over the past decade or so. However, much of this activity has focused on the study of the incompressible free-boundary Euler equations, that is,

$$\begin{cases} \beta \mathfrak{v}_t^\alpha + \mathfrak{a}^{\mu\alpha} \partial_\mu \mathfrak{q} = 0, & \text{in } [0, T] \times \Omega \\ \operatorname{div} \mathfrak{v} = 0, & \text{in } [0, T] \times \Omega \\ \mathfrak{A}^{\mu\alpha} N_\mu \mathfrak{q} + \sigma \sqrt{\mathfrak{g}} \Delta_{\mathfrak{g}} \tilde{\eta}^\alpha = 0, & \text{on } [0, T] \times \Gamma, \end{cases} \tag{1.7}$$

where β is a positive constant corresponding to the fluid's constant density, $\mathfrak{v}$ and $\mathfrak{q}$ are the incompressible Lagrangian velocity and pressure, $\mathfrak{a} = (\partial\tilde{\eta})^{-1}$, $\mathfrak{A} = \det(\partial\tilde{\eta})\mathfrak{a}$, where $\tilde{\eta}$ is the Lagrangian map associated with $\mathfrak{v}$.

It is well-known that for the incompressible equations, $\mathfrak{q}$ is not determined by an equation of state. Rather, it is a Lagrange multiplier enforcing the constraint $\operatorname{div} \mathfrak{v} = 0$. The local well-posedness for the incompressible free-boundary Euler equations has been studied by many authors, see [7,8,11,12,16,17,19,31,41,42,45,47,52, 54–57,65,66,69] and references therein. It is worth mentioning here that when $\mathcal{D}_0$ is unbounded (with finite or infinite depth) and the velocity $\mathfrak{v}_0$ is irrotational (that is, $\operatorname{curl} \mathfrak{v}_0 = 0$, a condition that is preserved by the evolution), this problems is called the water-waves problem, which has received a great deal of attention [4–6,15,24–30,32–35,59,61–63,67,68].

However, the theory of the free-boundary compressible Euler equations is far less developed. It is known that for suitable initial data, the system (1) modeling a liquid admits a local (in time) solution, for example, [9,20,21,43,44,60], and for the gas model, the existence of a local solution was obtained in [10,13,14,36,37,49].

In this paper we study how the solutions to (1.1) and (1.2) are related. Intuitively, one expects that the solution of (1.1) should converge to that of (1.2) when the "compressibility vanishes". The proper way to define this problem is via the fluid's sound speed (see (1.3) below), which corresponds to the speed of propagation of sound waves inside the fluid and captures the fluid's compressibility in that stiffer fluids have larger sound speed.[1]

The incompressible limit problem consists in proving that if a sequence $(v_{0,\kappa}, R_{0,\kappa})$ of *well-prepared* initial data for (1.1) converges to $(\mathfrak{v}_0, \beta)$, where $\mathfrak{v}_0$ is the initial data for the incompressible problem (1.2), and the sound speed at time zero diverges to infinity, then the respective solution (v, R) of (1) converges to $(\mathfrak{v}, \beta)$, where $\mathfrak{v}$ solves (1.2). Here, well-prepared initial data means that, in addition to satisfying the compatibility conditions, the initial data has to be tailored to the above limit (see Theorem 1.4).

The incompressible limit for the compressible Euler equations in a fixed domain (that is, $\mathcal{D}_t = \mathcal{D}_0$ or the whole space) was established by several authors under different assumptions; see [2,3,18,22,23,38,39,50,53] and references therein. In addition, the incompressible limit for the compressible free-boundary Euler equations was solved by Lindblad and the second author in [46] with $\sigma = 0$ in a bounded domain, and by the second author [48] in the same case but with unbounded domain. To our best knowledge, the aforementioned works [46,48] are the only known results in the study of the incompressible limit for equations (1.1). In particular, no result is available for the case with $\sigma > 0$. We will establish a priori estimates for (1.1) that are uniform in the sound speed (see sections 3, 4). In addition, we will construct a sequence of well-prepared data for (1.1) which converges to that of (1.2) when the sound speed tends to infinity (see section 5). As a consequence, we conclude the convergence of the compressible solution to the incompressible one by an Arzelà-Ascoil-type theorem.

1.3. The Sound Speed

Physically, the sound speed is defined as $c = \sqrt{q' \circ R}$. To set up the incompressible limit, it is conveninet to view the sound speed as a paramenter. As in [18,23], we consider a family $\{q_\kappa(R)\}$ parametrized by $\kappa \in [0, \infty)$, where

$$\kappa := q_\kappa'(R)|_{R=\beta}. \tag{1.8}$$

Here, $' = \frac{d}{dR}$, and

$$q_\kappa(R) = c_\gamma \kappa (R^\gamma - \beta^\gamma), \quad c_\gamma = \gamma^{-1} > 0,\ \beta > 0,\ \gamma \geq 1. \tag{1.9}$$

We slightly abuse terminology and call κ the sound speed. In order to consider the incompressible limit, we view the density as a function of the pressure, that is, $R_\kappa = R_\kappa(q) = [(c_\gamma \kappa)^{-1} q + \beta^\gamma]^{1/\gamma}$, and we see that $R_\kappa'(q)$ satisfies

$$\frac{1}{c_0}\mathfrak{R}_\kappa \leq R_\kappa'(q) \leq c_0 \mathfrak{R}_\kappa \tag{1.10}$$

for some fixed constant $c_0 > 0$, where $\mathfrak{R}_\kappa = (c_\gamma \kappa)^{-\frac{1}{\gamma}}$. Also, for $0 \leq k \leq 4$, we have that

$$|R_\kappa^{(k)}(q)| \leq c_0, \quad |R_\kappa^{(k)}(q)| \leq c_0 |R_\kappa'(q)|^k \leq c_0 |R_\kappa'(q)|,$$

[1] This is an experimental fact, see, for example, [64].

$$|q_\kappa^{(k)}(R)| \le c_0 |q_\kappa'(R)| \tag{1.11}$$

hold uniformly in κ.

1.4. The Main Results

Notations. All notations will be defined as they are introduced. In addition, a list of symbols is given at the end of this section for a quick reference.

Definition 1.1. The L^2-based Sobolev spaces are denoted by $H^s(\Omega)$, with the corresponding norm denoted by $||\cdot||_s$; note that $||\cdot||_0 = ||\cdot||_{L^2(\Omega)}$. We denote by $H^s(\Gamma)$ the Sobolev space of functions defined on Γ, with norm $||\cdot||_{s,\Gamma}$.

Theorem 1.2. *Let $\Omega = \mathbb{T}^2 \times (0,1)$ and $v_{0,\kappa}$ be a smooth vector field.*[2] *Let $\rho_{0,\kappa}$ be a smooth function satisfying $\rho_{0,\kappa} \ge c > 0$ and $q_{0,\kappa}$ be the associated pressure given by* (1.3)*. Suppose that for some $\mathfrak{m} \in \mathbb{R}$ such that*

$$||v_{0,\kappa}||_4, ||v_{0,\kappa}||_{4,\Gamma}, ||q_{0,\kappa}||_4, ||q_{0,\kappa}||_{4,\Gamma} \le \mathfrak{m}, \quad \text{for all } \kappa > 0 \tag{1.12}$$

holds. Then there exist a $T > 0$ and a constant $\mathfrak{M}$ such that any smooth solution (v_κ, R_κ) to (1.1) *defined on the time interval $[0,T]$ satisfies*

$$\mathcal{N}(t) \le \mathfrak{M},$$

where

$$\begin{aligned} \mathcal{N} = {} & ||v_\kappa||_4^2 + ||\mathfrak{R}_\kappa \partial_t v_\kappa||_3^2 + ||\mathfrak{R}_\kappa \partial_t^2 v_\kappa||_2^2 + ||\mathfrak{R}_\kappa^{\frac{3}{2}} \partial_t^3 v_\kappa||_1^2 \\ & + ||R_\kappa||_4^2 + ||\partial_t R_\kappa||_3^2 + ||\sqrt{\mathfrak{R}_\kappa} \partial_t^2 R_\kappa||_2^2 + ||\mathfrak{R}_\kappa \partial_t^3 R_\kappa||_1^2 \\ & + ||\partial_t v_\kappa||_2^2 + ||\sqrt{\mathfrak{R}_\kappa} \partial_t^2 v_\kappa||_1^2 + ||\partial_t^2 R_\kappa||_1^2 + E, \end{aligned} \tag{1.13}$$

where E is defined as Definition 3.6.

The next theorem is a direct consequence of Theorem 1.2 *together with the Arzelà-Ascoli theorem.*

Theorem 1.3. *Let $\mathfrak{v}_0 \in H^{6.5}(\Omega)$ be a divergence free vector field and let $\mathfrak{v}$ be the solution to the incompressible free-boundary Euler equations* (1.2) *with data $\mathfrak{v}_0$ defined on a small time interval $[0,T]$. Let $(v_{0,\kappa}, R_{0,\kappa}) \in H^4(\Omega) \times H^4(\Omega)$ be a sequence of initial data for the compressible free-boundary Euler equations* (1.1) *satisfying the compatibility conditions up to order* 3 *(see section* 5.1 *for a statement of the compatibility conditions). Furthermore, assume that $(v_{0,\kappa}, R_{0,\kappa}) \to (\mathfrak{v}_0, \beta)$ in $C^2(\Omega)$ as $\kappa \to \infty$ and that* (1.1) *holds. Let (v_κ, R_κ) be the solution for* (1.1) *with the equation of state* (1.3)*. Then:*

1. *For κ sufficiently large, (v_κ, R_κ) is defined on $[0,T]$.*

[2] By “smooth” we mean “as smooth as necessary for the qualitative arguments (such as integration by parts) to go through.” However, all of our quantitative estimates depend only on the Sobolev norms mentioned in Theorem 1.2.

2. $(v_\kappa, R_\kappa) \to (\mathfrak{v}, \beta)$ *in* $C^0([0, T], C^2(\Omega))$ *after possibly passing to a subsequence.*

Remark. $\mathfrak{v}_0 \in H^{6.5}(\Omega)$ is required so that the initial norms are uniformly bounded. We refer the proof of Theorem 5.1 for details.

Finally, we need the following theorem to show that the data required in Theorem 1.2 and Theorem 1.3 exists:

Theorem 1.4. *Let* $\mathfrak{v}_0 \in H^{6.5}(\Omega)$ *be a divergence free vector field in* Ω*. Then there exists initial data* $(v_{0,\kappa}, R_{0,\kappa}) \in H^4(\Omega) \times H^4(\Omega)$ *satisfying the compatibility conditions up to order* 3 *(see section* 5.1 *for a statement of the compatibility conditions) such that* $(v_{0,\kappa}, R_{0,\kappa}) \to (\mathfrak{v}_0, \beta)$ *in* $C^2(\Omega)$ *as* $\kappa \to \infty$*, and* (1.1) *holds.*

Notation 1.5. *For the sake of clean notations, we will drop the* κ*-indices on* v_κ*,* R_κ*,* q_κ*, that is, we will denote* $(v_\kappa, R_\kappa, q_\kappa) = (v, R, q)$ *when no confusion can arise.*

1.5. On Existence of Solutions

In Theorems 1.2 and 1.3 we have assumed that a solution is given in the stated function spaces, whereas in Theorem 1.4 we showed how to construct initial data for solutions in the corresponding spaces without, however, establishing the existence of solutions. In this section we show that existence of solutions in the spaces we use follow from the existence result of [9], although such an existence result and its corresponding estimates do not suffice to obtain the incompressibe limit, as we also discuss further below.

We begin noticing that given a solution with regularity as in [9], for each fixed κ, the norms appearing in $\mathcal{N}$ are well-defined, where $\mathcal{N}$ is given in equation (1.1). The issue is that the time of existence of the solutions obtained in [9], as well as the a priori bounds in [9], depend on κ, whereas one needs bounds and a time interval that is uniform on κ in order to pass to the limit $\kappa \to \infty$.

The crucial point is that while our estimates hold on a small time interval $[0, T]$, the *smallness of* T does not depend on κ provided that κ is sufficiently large. In a nutshell, the logic to obtain solutions in the spaces where we take the incompressible limit is the following: (i) [9] is used to obtain, for each κ, a solution defined on a time interval $[0, T_\kappa)$; (ii) We apply our estimates to show that the solution from [9] can be controlled on a time interval $[0, T)$ that is uniform on the sound speed κ. This uniform control follows from the use of our weighted-in-κ estimates (that is, the estimates with $\mathfrak{R}_\kappa$-weights) and, in fact, cannot be obtained from the energy used in [9], as we also show below; (iii) A more or less standard continuation argument is then used to obtain that $T_\kappa \geq T$ for all κ sufficiently large. In this way we obtain a family of solutions parametrized by κ and defined on a common time interval. (iv) Our estimates show that on this common time interval the family of solutions converges (up to a subsequence) to the incompressible solution.

Remark. We stress that the uniformity of T on the sound speed κ comes from the fact that we can close our estimate for $\mathcal{N}$ (defined in equation (1.13)) uniformly on κ (for large κ). This can only be done because of the use of $\mathfrak{R}_\kappa$-weights in our energy which is, furthermore, tailored to the incompressible limit.[3] *After* we have obtained such a uniform-in-κ estimate, we can derive further estimates which can in principle depend on κ. In fact, estimates of this type are used below. They are harmless because they are used in arguments that only require *finiteness* of some quantities. However we again insist that the entire argument given below relies on the fact that we are able to derive estimates independent of κ (or, more precisely, independent of κ for all κ sufficiently large).

We will now elaborate on the argument summarized in above. We will present its logic step-by-step, but for the sake of brevity will not write down explicitly many of the estimates involved. After that, we will show that this uniform control of T that we obtained does not follow from the result in [9].

In what follows, denote by E^{CHS} the energy used in [9], that is, equation (1.9) of [9].

Claim I. Continuation criteria. We begin with the following statement: let (v, R) be a solution defined on a time interval $[0, T_*)$ and with regularity given by the norms in E^{CHS}.[4] Set $\mathcal{M} := \sup_{0\leq t<T_*} E^{CHS}(t)$. We claim that if $\mathcal{M} < \infty$, then the solution (v, R) can be continued pass T_*.

Suppose that $\mathcal{M} < \infty$. Because $||\partial_t v||_3$ is controlled by E^{CHS}, the fundamental theorem of calculus combined with $\mathcal{M} < \infty$ shows that $v \in C^0([0, T_*), H^3)$. Let $\{t_\ell\}_{\ell=1}^{\infty}$ be a sequence of times such that $t_\ell \to T_*$. Using again the fundamental theorem of calculus and the triangle inequality, we have

$$||v(t_{\ell+j}) - v(t_\ell)||_3 \leq \mathcal{M}|t_{\ell+j} - t_\ell|,$$

showing that $v(t_\ell)$ is a Cauchy sequence in H^3 so it converges. Since this is true for any sequence $t_\ell \to T_*$, we have that there exists a $v_* \in C^0([0, T_*], H^3)$ that extends $v \in C^0([0, T_*), H^3)$. Moreover, since $v(t_\ell)$ converges to $v_*(T_*)$ in H^3 and is bounded in H^4 (because the H^4-norm is controlled by E^{CHS}), we obtain that in fact $v_*(T_*) \in H^4$. Using the equations of motion, which give $\partial_t R \sim \partial v$, and the fact that $||v||_4$ is controlled by E^{CHS}, we similarly obtain an extension of R to the closed interval. The same argument also gives that the flow η_* of v_*, whose H^5 norm is controlled by E^{CHS} on $[0, T_*)$, satisfies $\eta_*(T_*) \in H^5$. Repeating exactly the same argument for the boundary norms in E^{CHS} (that is, the last sum of (1.9) in [9] and the next-to-the-last term of (1.9) in [9]), we finally conclude that v and R extend to functions on the closed interval $[0, T_*]$ and that $E^{CHS}(T_*) < \infty$. We can now apply Theorem 1.6 of [9], which says that if $E^{CHS}(T_*) < \infty$, then a solution exists on $[T_*, T_* + \varepsilon)$ for some $\varepsilon > 0$.

[3] In particular, as we discuss in this section and in section 1.6, our energy is related to that used in [9], but it also differs from it in important aspects.

[4] To avoid confusion, we stress that by "regularity given by the norms" we mean that the maps belong to the function spaces of the corresponding norms, but we do not mean finiteness of the corresponding energy over the time interval. For example, if we say that v has regularity given by the norms $||v||_s + ||\partial_t v||_{s-1}$, we mean that $v \in H^s$ and $\partial_t v \in H^{s-1}$, but we do not assert that the supremum in t of $||v||_s + ||\partial_t v||_{s-1}$ is finite.

Claim II. Control of E^{CHS} from $\mathcal{N}$ for fixed κ. Let (v, R) be a solution defined on a time interval $[0, \mathcal{T})$ with the regularity given by the norms in E^{CHS}. We will show that if $\sup_{0\leq t<\mathcal{T}} \mathcal{N}(t) < \infty$, then $\sup_{0\leq t<\mathcal{T}} E^{CHS}(t) < \infty$, where $\mathcal{N}$ is the quantity introduced in (1.1).

We begin noticing that the solution (v, R) has enough regularity so that $\mathcal{N}$ is well-defined. Assume that $\mathcal{N}_0 := \sup_{0\leq t<\mathcal{T}} \mathcal{N}(t) < \infty$. This immediately gives that $\sup_{0\leq t<\mathcal{T}} ||\partial_t^k v||_{4-k}$, $k = 0, \ldots, 4$, is bounded in terms of $\mathcal{N}_0$. We remark that this bound, and the ones that follow in this part of the argument, may depend on κ. However, here this is not a problem since we only want to show the finiteness of $\sup_{0\leq t<\mathcal{T}} E^{CHS}(t)$ for fixed κ. Thus, we obtain that all terms in the first sum of (1.9) in [9] are controlled by $\mathcal{N}_0$, except for the case $a = 0$, that is, $||\eta||_5$.

We next bound $||\eta||_5$ by controlling its curl, divergence, and normal component. For the curl, we use the compressible Cauchy invariance, equation (2.7). We note that this requires having $\operatorname{curl} v|_{t=0} \in H^4$, which is true in the assumptions of the existence result of [9] (see equation (4.7) in [9] and the discussion surrounding it). We therefore obtain a bound of $||\operatorname{curl} \eta||_4$ in terms of $\mathcal{N}_0$ and $E^{CHS}(0)$. For the divergence, we apply estimate (34) of [21], except that instead of the $H^{2.5+\delta}$ norm on the LHS, we use the H^4 norm (it is not difficult to see that the same argument as in [21] goes through with the H^4 norm on the LHS). This gives control of $||\operatorname{div} \eta||_4$ in terms of $\mathcal{N}$ and $\operatorname{div} \eta|_{t=0}$, where the latter is smooth in view of the initial condition $\eta(0) = \text{id}$. Thus, we conclude that $||\operatorname{div} \eta||_4$ can be controlled in terms of $\mathcal{N}_0$. Finally, we need to control $\eta \cdot N$ on Γ. For this, we use the boundary condition in (1.3) and apply elliptic estimates. We remark that the coefficients do not have enough regularity for an application of the "standard" elliptic estimates, but we can apply estimates with coefficients in Sobolev spaces (see Theorem 4 and Remark 2 in [51]). Invoking div-curl estimates, we conclude that we can control $||\eta||_5$ in terms of $\mathcal{N}_0$ and $E^{CHS}(0)$.

Remark. The presence of $E^{CHS}(0)$ above comes from the fact that the energy E^{CHS} requires an extra derivative for η, which we do not include in $\mathcal{N}$. In fact, we cannot include this term in $\mathcal{N}$, otherwise we would not be able to close our estimates uniformly in κ, as we discuss in more detail in section 1.6. However, above we showed that if a solution with such extra differentiability is given, then we can update our estimates to control such extra derivative in terms of $\mathcal{N}$, with bounds possibly depending on κ. A similar remark applies to the boundary terms in the last sum of E^{CHS} controlled below, which are more regular in the energy E^{CHS} than in our $\mathcal{N}$.

The term $v_{ttt} \cdot n \in H^1(\Gamma)$ appearing in E^{CHS}, where n is the unit outer normal to the moving boundary, is directly controlled by the boundary term in our energy E which enters in the definition of $\mathcal{N}$ (see Definition 3.6; see also Lemma 2.2 for identities relating n with the projection Π appearing in E) with $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^2 \partial_t^4$ (again, with a bound depending on κ).

Finally, we need to bound the terms in the last sum of E^{CHS} (equation (1.9) of [9]). For this, we time differentiate the boundary condition in (1.3) up to three times, and apply elliptic estimates for operators with coefficients in Sobolev spaces (see above).[5] We also need control of up to three time derivatives of q restricted to Γ in terms of $\mathcal{N}$. Such bounds are immediately available (with constants depending on κ) from the bounds for R and its time derivatives provided by $\mathcal{N}$. We thus conclude that the last sum in E^{CHS} can be controlled in terms of $\mathcal{N}_0$.

From the foregoing, we conclude that $\sup_{0\leq t<\mathcal{T}} E^{CHS}(t) < \infty$ if $\sup_{0\leq t<\mathcal{T}} \mathcal{N}(t) < \infty$, as desired.

Claim III. Uniformity of the interval where the a priori estimates hold. This is just a restatement of Theorem 1.2, but we include it here for clarity of the presentation. The time interval $[0, T]$ in Theorem 1.2 is uniform on κ in the following sense: T has to be chosen sufficiently small, but the smallness of T depends only on a fixed large κ_0. In other words, Theorem 1.2 says that there exists a κ_0 and a $T = T(\kappa_0)$, such that if $\kappa \geq \kappa_0$ and (v, R) is a solution defined on $[0, T)$ and with the regularity given by the norms in $\mathcal{N}$, then $\mathcal{N} \leq \mathfrak{M}$, where the constant $\mathfrak{M}$ in Theorem 1.2 (which is a constant depending on norms of the initial data).

Claim IV. Existence of solutions in the spaces where one takes the incompressible limit. The results of [9], in particular Theorem 1.6, say that given data such that $E^{CHS}(0) < \infty$, there exists a solution (v_κ, R_κ) defined on a short time interval with regularity given by the norms in E^{CHS}. Let $[0, T_\kappa)$ be the maximal interval where the solution (v_κ, R_κ) exists and has the regularity given by the norms in E^{CHS}. We use the subscript $_\kappa$ to indicate that the solution as well as the time interval in principle depend on the sound speed κ. Let T be given by Claim III (that is, by Theorem 1.1) and κ_0 be as in Claim III. We will show that $T_\kappa \geq T$ for all $\kappa \geq \kappa_0$.

Suppose that $T_\kappa < T$. We remark that the solutions (v_κ, R_κ) have enough regularity so that the estimates of our Theorem 1.1 can be applied, that is, all quantities entering in the definition of $\mathcal{N}$ (equation (1.1)) are well defined for the solutions (v_κ, R_κ). Since $T_\kappa < T$, Claim III implies that $\sup_{0\leq t<\mathcal{T}_\kappa} \mathcal{N}_\kappa(t) < \infty$, where we write $\mathcal{N}_\kappa$ to emphasize that this corresponds to the quantity $\mathcal{N}$ for the solution (v_κ, R_κ). By Claim II we then obtain $\sup_{0\leq t<\mathcal{T}} E_\kappa^{CHS}(t) < \infty$, where we write E_κ^{CHS} to emphasize that this corresponds to the energy E^{CHS} for the solution (v_κ, R_κ) (as in Claim II, the resulting bound on $\sup_{0\leq t<\mathcal{T}} E_\kappa^{CHS}(t)$ depends on κ, but only the finiteness of this quantity matters here). By Claim I, the solution (v_κ, R_κ) can be extended pass T_κ, which contradicts the maximality of T_κ.

Thus, we obtained a family of solutions parametrized by κ, $\kappa \geq \kappa_0$ and defined on $[0, T)$; shrinking T a bit if necessary we can consider the close interval $[0, T]$. Moreover, the estimate of Theorem 1.1, $\mathcal{N}(t) \leq \mathfrak{M}$, holds on $[0, T]$ for each solution in this family, with $\mathfrak{M}$ independent of κ in view of Theorem 1.2.

Existence of initial data compatible with the regularity of E^{CHS}. The constant $\mathfrak{M}$ in Theorem 1.2 depends on norms of the initial data. It will be uniform on κ, for all κ sufficiently large, if the corresponding norms of the initial data are uniform on κ, as assumed in Theorem 1.2. To show that this assumption is not

[5] Using the boundary condition gives control over $v \cdot N$, whereas the corresponding terms in E^{CHS} are $v \cdot n$. But it is not difficult to see that control of the latter follows from control of the former; see the proof of Theorem 4.3.

empty, in Theorem 1.4, we constructed initial data satisfying such uniformity on κ. However, for the existence of solutions given above, we actually need data such that $E^{\text{CHS}}(0) < \infty$. Since $E^{CHS}(0)$ requires more regularity than $(v_0, q_0) \in (H^4(\Omega) \cap H^4(\Gamma)) \times (H^4(\Omega) \cap H^4(\Gamma))$, which is what we stated in Theorem 1.4, we need to explain how data satisfying $E^{CHS}(0) < \infty$ and that is uniform on the sound speed can be obtained. This, however, follows from the proof of Theorem 1.4. Indeed, the data constructed in Theorem 1.4 is regular enough so that $E^{CHS}(0)$ is well-defined for it. However, only the $(H^4(\Omega) \cap H^4(\Gamma))$ norms of this data are needed to be controlled uniformly on κ for Theorem 1.2, so the statement of Theorem 1.4 is restricted to this situation.

E^{CHS} **cannot be controlled uniformly on** κ. The foundation of our result on the incompressible limit is the fact that we can derive estimates that are uniform in κ (for large κ). On the other hand, in order to obtain solutions to the equations of motion with the desired regularity, we relied on [9]. This raises the natural question of whether the incompressible limit could not be obtained directly from the estimates derived in [9]. Here we show that this is not the case, that is, that the energy E^{CHS} cannot be closed uniformly in κ solely within the framework of [9].

The relevant fact is that the energy estimates in [9] are non-uniform in κ and diverge when $\kappa \to \infty$. In particular, the interval of existence obtained in [9] could in principle shrink to zero when $\kappa \to \infty$ (that this is *not* the case is what we showed above using our uniform-in-κ estimates).

Let us now provide details. We will show that Proposition 4.1 in [9] is not uniform in κ. To see this, we take $q = \kappa(R^2 - \beta)$ and so $q'(R) = 2\kappa R$. Plugging the identity $R = \rho_0/J$ to the Euler's equations we obtain

$$\begin{aligned} \rho_o v_t + \kappa A^{\mu\alpha}\partial_\mu(\rho_0^2 J^{-2}) &= 0, \\ R_t + R a^{\mu\alpha}\partial_\mu v_\alpha &= 0. \end{aligned} \tag{1.14}$$

Testing four time-derivatives of (1.5) against $\partial_t^4 v$ in the L^2 and then integrating by parts yields the energy

$$\sup_{t\in[0,T]} ||v_{tttt}(t)||_0^2 + \sup_{t\in[0,T]} ||\sqrt{\kappa}\partial_t^4 J(t)||_0^2 + \sup_{t\in[0,T]} ||v_{ttt}\cdot n(t)||_{1,\Gamma}^2. \tag{1.15}$$

However, to control this energy, we need to control

$$\int_\Omega \kappa\partial_t^4(\rho^2 J^{-2})([\partial_t^4, a^{\mu\alpha}]\partial_\mu v_\alpha).$$

In [9] this term is part of the error term $\mathcal{R}$ and it can be controlled directly by the energy by Hölder's inequality. But there is a mismatch of $\kappa^{1/2}$ between this term and (1.5). In fact, this term is associated with our term $\mathcal{I}_3$ (see section 3.3.3), which requires our $\mathfrak{R}_\kappa$-weighted energy to be controlled.

1.6. Strategy, Organization of the Paper, and Discussion of the Difficulties

In this section we overview the main arguments of the paper, summarize the main difficulties, and explain how they are confronted.

1.6.1. Special Cancellations As mentioned, having $\sigma > 0$ leads to several new difficulties not present when $\sigma = 0$. This can be immediately seen from the boundary terms appearing in the energy estimates (see sections 3.4 and 3.5), since all these terms are proportional to σ and, therefore, automatically vanish when $\sigma = 0$. (Incidentally, we do not set σ to 1 as it is customary but keep it explicit in order to highlight all the terms that would be absent had σ been zero.) Not only are these terms present but, as we discuss below, they are some of the most difficult terms to handle. As a consequence, the methods used in the second author's previous papers to study the problem with $\sigma = 0$ [46,48] cannot be applied when $\sigma > 0$.

At first sight one might think that the surface tension should help with closing a priori estimates since it has a regularizing effect on the boundary. This regularization, however, it is not enough to produce control of the velocity on the boundary. After differentiating the equations with respect to D^k, where D^k is a k^{th} order derivative, possibly mixing space and time derivatives, contracting with $D^k v$ and integrating by parts, one is left with a boundary term that reads, schematically,

$$\int_\Gamma D^k v D^k q \, \mathrm{d}S.$$

It is not difficult to see that we can only hope to control this term by employing the boundary condition so that (again, schematically)

$$\int_\Gamma D^k v D^k q \, \mathrm{d}S \sim \int_\Gamma D^k v D^k (\Delta_g \eta) \, \mathrm{d}S. \tag{1.16}$$

The presence of the boundary Laplacian and the fact that $v = \partial_t \eta$ suggest that we should integrate by parts in space and factor a ∂_t. Although this is the strategy, we end up with a commutator term that is not of lower order. This is because the coefficients of Δ_g involve one derivative of g which, in turn, involves one derivative of η (so that the coefficients depend on as many derivatives of η as the order of the equation). Thus, commuting D^k and Δ_g still leaves a top order term that *cannot* be written as a perfect derivative (in time or space) to be integrated away. Moreover, this top order term does not seem to have any good structure. In fact, one should not expect such term to have a good structure, since differentiating the coefficients of Δ_g corresponds to differentiate g^{ij}, and, thus, to take derivatives of some non-linear combinations of the components g_{ij} and its determinant.

The above difficulties are overcome by observing some remarkable cancellations among the bad top order terms in (1.6.1). Such cancellations are not visible in any way in the expressions that appear by simply manipulating (1.6.1). Rather, they are identified after some judicious and lengthy analysis that relies heavily on some geometric properties, expressed in the form of several geometric identities, of the boundary. The first cancellation appears in (3.5.1.2). The reader can check that the terms that cancel out are top order and that there does not seem to be possible to bound them individually. The second cancellation happens between a term in (3.5.1.2) and (3.5.1.3). This second cancellation is even more remarkable because the terms involved come from completely different parts of $D^k \Delta_g \eta$: one from when all derivatives fall on the coefficient $\sqrt{g} g^{ij}$ of Δ_g, the other from when we integrate one derivative in Δ_g by parts.

We also need a special cancellation for interior terms. This comes from when we take D^k of the first equation in (1.1) and all derivatives fall on a. Since the matrix a already involves one derivative of η, we find terms in $D^{k+1}\eta$, which have one too many derivatives of the Lagrangian map. Exploiting the explicit structure of a, however, we are able to show that, when appropriately grouped, these bad terms cancel each other after some careful integration by parts (see (3.3.2) and what follows).

As this point one may ask if all such cancellations are indeed necessary since a priori estimates for (1.1) have been derived in the literature. The relevant work in this regard is [9]. There, the authors construct initial data where η is everywhere one degree more differentiable than v, and then prove that this extra regularity is propagated by the evolution. They rely on such extra regularity to close the estimates. However, this does not seem possible here because such an extra differentiability is not compatible with the $\mathfrak{R}_\kappa$-weights we need to introduce in order to obtain estimates uniform in the sound speed (see section 1.6.2).

A crucial aspect of all the cancellations mentioned above is that they require the derivatives D^k to contain *at least one time derivative.* As a consequence, only the Sobolev norms of time-derivatives of v on the boundary are controlled from the energy estimates (we remark that the energy *does* involve time derivatives of the variables; it does not seem possible to close the estimates without time-differentiating the equations). To obtain control of non-time differentiated v on the boundary, we rely directly on the boundary condition which, after a time derivative, produces an equation of the form $\Delta_g v = \dots$ which is amenable to elliptic estimates. (One might wonder why we do not take further time derivatives of the boundary condition to obtain estimates for $\partial_t^k v$ on the boundary. The reason is that, as mentioned above, Δ_g does not commute well with derivatives due to the dependence of the coefficients on two derivatives of η, so that we obtain an equation of worsening structure with each derivative.[6] However, for only one time derivative, the resulting equation still has some good structure that can be used to derive estimates.)

1.6.2. $\mathfrak{R}_\kappa$-weighted Estimates Another difficulty to establish the incompressible limit is that one has to derive estimates that are uniform in the sound speed, since the goal is to take the sound speed to infinity. This is substantially different than estimates for (1.1) (with $\sigma > 0$) currently available [9,21]. Establishing the required uniform-in-κ a priori estimate does not seem to be possible solely by the methods used to derive the currently available estimates. In particular, a crucial element to derive such uniform estimates is the use of a non-linear wave equation satisfied by the density, whereas non-uniform-in-κ estimates have been proven without this wave equation. In fact, the known a priori energy bounds rely heavily on the fact that when $\mathfrak{R}_\kappa$ is bounded from below (as κ is bounded from above), $\partial q \approx \partial R$ and

[6] Taking several time derivatives of the boundary condition, in particular, would lead to a source term that can only be bounded with κ-dependent bounds, preventing us from closing the argument uniformly on κ. See section 1.6.2 below for more on the need for uniform bounds. Compare also with the use of the boundary condition to derive estimates for boundary terms in Claim II of section 1.5, where the resulting bounds depend on κ.

$||q||_r \approx ||R||_r$, which is a direct consequence of the equation of state. In particular, the energy used in [21] controls $||\partial_t^k q||_{3-k}$ for free as a lower order term. However, this fact no longer holds when $\mathfrak{R}_\kappa \to 0$. Indeed, since $\partial R = R'\partial q$, $||R||_r$ is merely equivalent to $||\mathfrak{R}_\kappa q||_r$; in other words, we have to take extra effort to control the full Sobolev norms of $\partial_t^k q$. In [46] and [48], where $\sigma = 0$, these norms are controlled by elliptic estimate. This relies on the fact that one is able to control $||q_t||_r$ by the r-th order energy E_r since

$$\partial^r q_t \sim \overline{\partial}^r q_t + \partial^{r-2} q_t + \text{lower order terms},$$

where $\overline{\partial}$ denotes derivatives tangent to the boundary. The first term, $\overline{\partial}^r q_t$, vanishes due to $q|_\Gamma = 0$. However, this method does not work when $\sigma > 0$, which is simply due to the fact that $q \sim \Delta_g \eta$ on Γ, and so $\overline{\partial}^r q_t \sim \overline{\partial}^{r+2} v$ on the boundary which has two derivatives too many.

To resolve the above difficulties, our energy is defined using the $\mathfrak{R}_\kappa$-weighted derivatives $\mathfrak{D}^r$ $(1 \le r \le 4)$, where

$$\mathfrak{D} = \overline{\partial}, \partial_t; \quad \mathfrak{D}^2 = \overline{\partial}^2, \overline{\partial}\partial_t, \sqrt{\mathfrak{R}_\kappa}\partial_t^2; \quad \mathfrak{D}^3 = \overline{\partial}^2\partial_t, \sqrt{\mathfrak{R}_\kappa}(\overline{\partial}\partial_t^2), \mathfrak{R}_\kappa\partial_t^3;$$
$$\mathfrak{D}^4 = \mathfrak{R}_\kappa(\overline{\partial}^3\partial_t), \mathfrak{R}_\kappa(\overline{\partial}^2\partial_t^2), (\mathfrak{R}_\kappa)^{\frac{3}{2}}(\overline{\partial}\partial_t^3), (\mathfrak{R}_\kappa)^2\partial_t^4.$$

The energy $E = E(t)$ is defined by employing these $\mathfrak{R}_\kappa$-weighted derivatives, which is of the form:

$$E = \sum_{1\le\ell\le4} ||\mathfrak{D}^\ell v||^2_{L^2(\Omega)} + \sum_{1\le\ell\le4} \sqrt{\mathfrak{R}_\kappa}||\mathfrak{D}^\ell q||^2_{L^2(\Omega)}$$
$$+ \sigma \sum_{1\le\ell\le4} ||\Pi\overline{\partial}\mathfrak{D}^\ell \eta||^2_{L^2(\Gamma)} + W,$$

where Π is the projection onto the normal to the moving boundary (see Lemma 2.2) and W stands for the energy of the wave equation satisfied by q, which is defined in sections 2.3, 2.4.

The energy estimate for E cannot be closed by itself; in fact, the energy estimate requires control of

$$||v||_4, ||\mathfrak{R}_\kappa v_t||_3, ||\mathfrak{R}_\kappa v_{tt}||_2, ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_1, \tag{1.17}$$

and

$$||R||_4, ||R_t||_3, ||\sqrt{\mathfrak{R}_\kappa}R_{tt}||_2, ||\mathfrak{R}_\kappa R_{ttt}||_1. \tag{1.18}$$

These quantities are not part of the energy since $\mathfrak{D}^\ell$ for $\ell = 1, 2, 3, 4$ do not involve non-tangential derivatives, nor the full tangential spatial derivative $\overline{\partial}^4$. Such missing derivatives, however, cannot be included in the energy because they would lead to the presence of non-tangential derivatives on the boundary. As a consequence, we need to estimate E together with the quantities above in order to close the a priori estimate. This is done with the help of elliptic estimates.

We now schematically show how to get the correct $\mathfrak{R}_\kappa$-weights for our energy, since they are crucial for the desired uniform-in-κ estimates. We differentiate the equations

$$R\partial_t v_\alpha + q'(R)a^{\mu\alpha}\partial_\mu R = 0 \tag{1.19}$$

and

$$\partial_t R + Ra^{\mu\alpha}\partial_\mu v_\alpha = 0 \tag{1.20}$$

with respect to time. Since $R' = R'(q) = \frac{1}{q'(R)}$, equation (1.6.2) implies

$$\partial\partial_t^k R \sim R'\partial_t^{k+1} v; \tag{1.21}$$

in other words, we can trade one (full) spatial derivative on R by one time derivative of v multiplied by R'. On the other hand, in view of the standard div-curl estimate (that is, (A.1) in Appendix), $\partial_t^k v$ is estimated via div $\partial_t^k v$, curl $\partial_t^k v$ and $\partial_t^k v \cdot N$. While in the reference domain $\Omega = \mathbb{T}^2 \times (0, 1)$, $\partial_t^k v \cdot N = \pm\partial_t^k v^3$, which is almost $\Pi\partial_t^k v$, where Π denotes the projection to the normal direction, and hence this can be controlled by E. In addition, curl $\partial_t^k v$ is estimated via Cauchy invariance which can be treated by adapting the method introduced in [21]. Finally, the equation (1.6.2) yields

$$a^{\mu\alpha}\partial_\mu\partial_t^k v_\alpha \sim \partial_t^{k+1} R;$$

in other words, we can estimate div $\partial_t^k v$ using $\partial_t^{k+1} R$. Hence,

$$\partial^4 v \xrightarrow{\text{div}} \partial^3 R_t \xrightarrow{(1.6.2)} R'\partial^2\partial_t^2 v \xrightarrow{\text{div}} R'\partial\partial_t^3 R \xrightarrow{(1.6.2)} (R')^2\partial_t^4 v,$$

where $(R')^2\partial_t^4 v$ is part of E. In addition, we have

$$R'\partial^3\partial_t v \xrightarrow{\text{div}} R'\partial^2\partial_t^2 R \xrightarrow{(1.6.2)} (R')^2\partial\partial_t^3 v \xrightarrow{\text{div}} (R')^2\partial_t^4 R.$$

This algorithm also provides

$$R'\partial^2\partial_t^2 v \xrightarrow{\text{div}} R'\partial\partial_t^3 R \xrightarrow{(1.6.2)} (R')^2\partial_t^4 v,$$
$$(R')^{\frac{3}{2}}\partial\partial_t^3 v \xrightarrow{\text{div}} (R')^{\frac{3}{2}}\partial_t^4 R.$$

Here, $(R')^{\frac{3}{2}}\partial_t^4 R$ can be controlled directly by E since it is equal to $(R')^{\frac{5}{2}}\partial_t^4 q$ up to lower order terms. On the other hand, applying this algorithm starting from $\partial^4 R$, we get

$$\partial^4 R \xrightarrow{(1.6.2)} R'\partial^3\partial_t v \xrightarrow{\text{div}} R'\partial^2\partial_t^2 R \xrightarrow{(1.6.2)} (R')^2\partial\partial_t^3 v \xrightarrow{\text{div}} (R')^2\partial_t^4 R,$$
$$\partial^3\partial_t R \xrightarrow{(1.6.2)} R'\partial^2\partial_t^2 v \xrightarrow{\text{div}} R'\partial\partial_t^3 R \xrightarrow{(v)} (R')^2\partial_t^4 v,$$
$$\sqrt{R'}\partial^2\partial_t^2 R \xrightarrow{(1.6.2)} (R')^{\frac{3}{2}}\partial\partial_t^3 v \xrightarrow{\text{div}} (R')^{\frac{3}{2}}\partial_t^4 R,$$
$$R'\partial\partial_t^3 R \xrightarrow{(1.6.2)} (R')^2\partial_t^4 v.$$

The detailed analysis can be found in section 4. But the above algorithm provides good guideline for the choice of $\mathfrak{R}_\kappa$-weights in (1.6.2) and (1.6.2) using (1.3), as well as in $\mathfrak{D}^r$.

Remark. The condition (1.3) allows us to define the weighted Sobolev norms (for example, (1.1)) with constant $\mathfrak{R}_\kappa$-weights. It is convenient to have constant weights for the boundary estimates in section 3.4 to avoid derivatives falling on R'. In addition, the condition (1.3) allows us to distribute $\mathfrak{R}_\kappa$-weights in order to obtain an uniform control in κ.

The definition of the $\mathfrak{R}_\kappa$-weighted derivative $\mathfrak{D}^r$ allows us to control the highest order (that is, 4th order) mixed norms of q directly by the energy. However, in order to pass to the incompressible limit, we have to control $||v||_4$ directly without $\mathfrak{R}_\kappa$-weights, and this requires the control of $||q_t||_2$. In section 3.2, we control $||q_t||_2$ by the elliptic estimate, which requires the control of $||q_t||_1$ first. This is indeed of lower order but we need to take extra effort to prove that they can be controlled uniformly as $\mathfrak{R}_\kappa \to 0$. In addition, we remark here that in [21], the authors were able to close the a priori energy estimate in H^3. However, in our case, the bound for $||q_t||_1$ require the control of $||v||_4$ and $||\eta||_4$. This is because control of $||\partial q_t||_0^2$ requires integration by parts, which yields $||\overline{\partial}^2\eta||_{1.5,\Gamma}$ and $||\overline{\partial}^2 v||_{1.5,\Gamma}$ at the top order, and these quantities require H^4 control of v, η.

1.6.3. The Initial Data As with the estimates themselves, the initial data has to be constructed uniform in the sound speed in order to allow the passage to the limit $\kappa \to 0$. This was done for $\sigma = 0$ in [46], but that method relied heavily on the fact that q vanishes on the boundary when surface tension is absent. Instead, we employ the method used in [9]: For each $1 \le k \le 3$, the data that satisfies the k-th order compatibility condition is obtained via solving an elliptic equation of order $2k$, which is acquired by time differentiating the boundary condition $q = \sigma\mathcal{H}$ for k times and then restrict at $t = 0$, where the previous $0, \ldots, k-1$-th compatibility conditions are served as the boundary conditions. This construction process allows one to show that the initial data is uniformly bounded for all sound speed κ, so that one can take the limit $\kappa \to \infty$.

1.7. List of Notations

- ∇: Eulerian spatial derivative.
- ∂: Lagrangian spatial derivative.
- $\overline{\partial}$: Tangential spatial derivative. In particular, $\overline{\partial} = (\partial_1, \partial_2)$ in Ω and we will emphasize that these derivatives are tangential by denoting $(\partial_1, \partial_2) = (\overline{\partial}_1, \overline{\partial}_2)$.
- D: Either $\overline{\partial}$ or ∂_t.
- Ω and Γ: The reference domain $(0, 1) \times \mathbb{T}^2$ in Lagrangian coordinate, whose boundary $\partial\Omega = \Gamma$.
- The matrices a and A: $a = (\partial\eta)^{-1}$, and $A = Ja$, where $J = \det(\partial\eta)$.
- κ: The sound speed.
- $\mathfrak{R}_\kappa$: $\mathfrak{R}_\kappa \approx R'_\kappa \to 0$ as $\kappa \to \infty$.
- $||\cdot||_s = ||\cdot||_{H^s(\Omega)}$ and $||\cdot||_{s,\Gamma} = ||\cdot||_{H^s(\Gamma)}$.
- $P(\cdot)$: A smooth function expression in its arguments.
- $\stackrel{L}{=}$: Equality modulo lower order terms that can be controlled appropriately.

2. Preliminary Results

In this section, we give some auxiliary results providing the bounds on the flow map η and the matrix a. In addition, we record several facts, expressions and inequalities that will come in handy in the later sections. These results will be employed in the proof of Theorem 1.2.

Lemma 2.1. *Assume that* $||v||_{L^\infty([0,T],H^4(\Omega))} + ||R||_{L^\infty([0,T],H^4(\Omega))} \le M$. *Let* $p \in [1,\infty)$*, then there exists a sufficiently large constant* $C > 0$*, such that if* $T \in [0, \frac{1}{CM^2}]$ *and* (v,q) *is defined on* $[0,T]$*, the following statements hold:*

1. $||\eta||_4 \le C$.
2. $||a||_3 \le C$.
3. $||a_t||_{L^p(\Omega)} \le C||\partial v||_{L^p(\Omega)}$, *and* $||a_t||_s \le C||\partial v||_s$, $\quad 0 \le s \le 3$.
4. $||\partial_\alpha a_t||_{L^p(\Omega)} \le C||\partial v||_{L^{p_1}}||\partial_\alpha a||_{L^{p_2}} + C||\partial_\alpha \partial v||_{L^p}$, *where* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.
5. $||a_{tt}||_s \le C||\partial v||_s||\partial v||_{L^\infty} + C||\partial v_t||_s$, $\quad 0 \le s \le 2$.
6. $||a_{ttt}||_s \le C||\partial v_t||_s||\partial v||_{L^\infty} + C||\partial v_{tt}||_s$, $\quad 0 \le s \le 1$.
7. $||\partial_t^4 a||_{L^p(\Omega)} \le C||\partial v||_{L^p}||\partial v||_{L^\infty}^2 + C||\partial v_t||_{L^p}||\partial v_t||_{L^\infty} + C||\partial v_{tt}||_{L^p}||\partial v||_{L^\infty} + C||\partial v_{ttt}||_{L^p}$.
8. $J \ge \frac{1}{2}$.
9. *If* ϵ *is sufficiently small and for* $t \in [0, \frac{\epsilon}{CM^2}]$*, we have* $||a^{\alpha\beta} - \delta^{\alpha\beta}||_3 \le \epsilon$*, and* $||a^{\alpha\mu}a^{\beta}_{\mu} - \delta^{\alpha\beta}||_3 \le \epsilon$*. In particular, the form* $a^{\alpha\mu}a^{\beta}_{\mu}$ *is elliptic, that is,* $a^{\alpha\mu}a^{\beta}_{\mu}\xi_\alpha\xi_\beta \ge C^{-1}|\xi|^2$.
10. $C^{-1} \le R \le C$.

Proof. We refer [21] and [31] for the detailed proof. We point out that the proof follows directly from the equations, interpolation, and the fundamental theorem of calculus. □

We record here the explicit form of the matrix a which will be needed:

$$a = J^{-1}\begin{pmatrix} \overline{\partial}_2\eta^2\partial_3\eta^3 - \partial_3\eta^2\overline{\partial}_2\eta^3 & \partial_3\eta^1\overline{\partial}_2\eta^3 - \overline{\partial}_2\eta^1\partial_3\eta^3 & \overline{\partial}_2\eta^1\partial_3\eta^2 - \partial_3\eta^1\overline{\partial}_2\eta^2 \\ \partial_3\eta^2\overline{\partial}_1\eta^3 - \overline{\partial}_1\eta^2\partial_3\eta^3 & \overline{\partial}_1\eta^1\partial_3\eta^3 - \partial_3\eta^1\overline{\partial}_1\eta^3 & \overline{\partial}_1\eta^1\overline{\partial}_1\eta^2 - \overline{\partial}_1\eta^1\partial_3\eta^2 \\ \overline{\partial}_1\eta^2\overline{\partial}_2\eta^3 - \overline{\partial}_2\eta^2\overline{\partial}_1\eta^3 & \overline{\partial}_2\eta^1\overline{\partial}_1\eta^3 - \overline{\partial}_1\eta^1\overline{\partial}_2\eta^3 & \overline{\partial}_1\eta^1\overline{\partial}_2\eta^2 - \overline{\partial}_2\eta^1\overline{\partial}_1\eta^2 \end{pmatrix} \tag{2.1}$$

Moreover, since $A = Ja$, and in view of (2.1), we can write

$$A^{1\alpha} = \epsilon^{\alpha\lambda\tau}\overline{\partial}_2\eta_\lambda\partial_3\eta_\tau, \quad A^{2\alpha} = -\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\eta_\lambda\partial_3\eta_\tau, \quad A^{3\alpha} = \epsilon^{\alpha\lambda\tau}\overline{\partial}_1\eta_\lambda\overline{\partial}_2\eta_\tau. \tag{2.2}$$

Here, $\epsilon^{\alpha\lambda\tau}$ is the fully antisymmetric symbol with $\epsilon^{123} = 1$. This representation will be used to create a special cancellation scheme that leads to control of the energy when all derivatives fall on the cofactor matrix (recall the discussion in section 1.6.1).

We also need some geometric identities to treat the boundary terms in the energy estimate. We record these identities in the next lemma.

Lemma 2.2. *Let n be the outward unit normal to $\eta(\Gamma)$. Let τ be the tangent bundle of $\overline{\eta(\Omega)}$ and ν be the normal bundle of $\eta(\Gamma)$, the canonical projection is given by*

$$\Pi^\alpha_\beta = \delta^\alpha_\beta - g^{kl}\overline{\partial}_k\eta^\alpha\overline{\partial}_l\eta_\beta,$$

and on Γ it holds that

1. $-\Delta_g\eta^\alpha = \mathcal{H}\circ\eta\, n^\alpha\circ\eta$.
2. $n\circ\eta = \frac{a^T N}{|a^T N|}$.
3. $J|a^T N| = \sqrt{g}$.

Above, a^T is the transpose of a. Furthermore, setting $\hat{n} = n\circ\eta$, the following identities hold on Γ:

4. $\Pi^\alpha_\beta = \hat{n}_\beta\hat{n}^\alpha$.
5. $\Pi^\alpha_\lambda\Pi^\lambda_\beta = \Pi^\alpha_\beta$.
6. $\hat{n}_\alpha = \hat{n}_\tau\Pi^\tau_\alpha$.
7. $\sqrt{g}\Delta_g\eta^\alpha = \sqrt{g}g^{ij}\Pi^\alpha_\mu\overline{\partial}^2_{ij}\eta^\mu$.
8. $\partial_t\hat{n}_\mu = -g^{kl}\overline{\partial}_k v^\tau\hat{n}_\tau\overline{\partial}_l\eta_\mu$.
9. $\partial_i\hat{n}_\mu = -g^{kl}\overline{\partial}^2_{ik}\eta^\tau\hat{n}_\tau\overline{\partial}_l\eta_\mu$.
10. $\overline{\partial}_i(\sqrt{g}g^{ik}) = -\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_j\eta^\mu\overline{\partial}_l\eta_\mu$.
11. $\partial_t(\sqrt{g}g^{ij}) = \sqrt{g}(g^{ij}g^{kl} - 2g^{lj}g^{ik})\overline{\partial}_k v^\lambda\overline{\partial}_l\eta_\lambda$.

Proof. These identities are well-known. The interested reader can consult, for example, [21] for their proof. □

The equation of state $q = q(R)$ allows us to control $R'q$ and R interchangeably.

Lemma 2.3. *Suppose $R' := R'(q)$ satisfies (1.3), and let ∂ be either ∂_t or ∂_α, then for each $1 \le r \le 4$, we have:*

$$|R'\partial^r q| \lesssim |\partial^r R| + \sum_{\substack{j_1+\cdots+j_k=r\\ 2\le k\le r}} |\partial^{j_1}R|\cdots|\partial^{j_k}R|. \tag{2.3}$$

Proof. A direct computation yields

$$R'\partial^r q = \partial^r R + \sum_{\substack{j_1+\cdots+j_k=r\\ 2\le k\le r}} C_{j_1,\ldots,j_m,k}R^{(k)}\partial^{j_1}q\cdots\partial^{j_k}q,$$

and invoking (1.3) and the fact $R'\partial q = \partial R$, (2.3) then follows. □

2.1. The Boundary Condition

The identities of Lemma 2.2 imply that the boundary condition

$$A^{\mu\alpha}N_\mu q + \sigma\sqrt{g}\Delta_g\eta^\alpha = 0, \quad \text{on } \Gamma, \tag{2.4}$$

can be expressed in the following equivalent ways:

1. $\sqrt{g}g^{ij}\overline{\partial}^2_{ij}\eta^\alpha - \sqrt{g}g^{ij}g^{ij}\overline{\partial}_k\eta^\alpha\overline{\partial}_l\eta^\mu\overline{\partial}^2_{ij}\eta_\mu = -\frac{1}{\sigma}A^{\mu\alpha}N_\mu q$, where $g^{kl}\overline{\partial}_l\eta^\mu\overline{\partial}^2_{ij}\eta_\mu = \Gamma^k_{ij}$.

2. $\sqrt{g}g^{ij}\Pi^{\alpha}_{\mu}\overline{\partial}^2_{ij}\eta^{\mu} = -\frac{1}{\sigma}A^{\mu\alpha}N_{\mu}q$.
3. $q = -\sigma(A^{3\alpha}\hat{n}_{\alpha})^{-1}\sqrt{g}g^{ij}\hat{n}_{\mu}\overline{\partial}^2_{ij}\eta^{\mu} = -\sigma g^{ij}\hat{n}_{\mu}\overline{\partial}^2_{ij}\eta^{\mu}$, since $(A^{3\alpha}\hat{n}_{\alpha})^{-1}\sqrt{g}$ simplifies to 1.

These identities follow directly from the definition. Interested readers can consult [21] for their proof. The above expressions will be frequently used to deal with the boundary estimates.

2.2. *The Interpolation Inequality*

Besides standard interpolation, we will also use the following interpolation inequality throughout this paper:

Theorem 2.4. *Let $u : \Omega \to \mathbb{R}$ be a H^1 function. Then,*

$$||u||_{L^4(\Omega)} \lesssim ||u||_0^{\frac{1}{2}}||u||_1^{\frac{1}{2}}.$$

Proof. See Theorem 5.8 in [1]. □

2.3. *The Wave Equations of Order 3 or Less*

The second equation in (1.1) can be re-expressed as

$$a^{\mu\alpha}\partial_{\mu}v_{\alpha} = -\frac{R'\partial_t q}{R}, \tag{2.5}$$

where $R' = R'_{\kappa}(q) \sim \mathfrak{R}_{\kappa}$ via assumption (1.3). Identity (2.3) and (1.1) yield, after commuting ∂_t^{r-1} for $1 \leq r \leq 3$ and then $a^{\nu\alpha}\partial_{\nu}$, that

$$JR'\partial_t^{r+1}q - a^{\nu\alpha}A^{\mu}_{\alpha}\partial_{\nu}\partial_{\mu}\partial_t^{r-1}q = \mathcal{F}_r, \tag{2.6}$$

where

$$\begin{aligned}\mathcal{F}_r = &-\sum\nolimits_{\substack{j_1+j_2=r\\ j_1\geq 1}}\big(\partial_t^{j_1}(JR')\big)(\partial_t^{j_2+1}q) + a^{\nu\alpha}(\partial_{\nu}\rho_0)\partial_t^r v_{\alpha}\\ &+\sum\nolimits_{\substack{j_1+j_2=r-1\\ j_1\geq 1}}a^{\nu\alpha}\partial_{\nu}(\partial_t^{j_1}A^{\mu}_{\alpha}\cdot\partial_{\mu}\partial_t^{j_2}q)\\ &-\rho_0\sum\nolimits_{j_1+j_2=r-1}(\partial_t^{j_1+1}a^{\nu\alpha})(\partial_t^{j_2}\partial_{\nu}v_{\alpha}) + a^{\nu\alpha}(\partial_{\nu}A^{\mu}_{\alpha})\partial_{\mu}\partial_t^{r-1}q.\end{aligned} \tag{2.7}$$

The wave equation (2.3) yields an energy identity which is essential when estimating $||q||_2$ and $||q_t||_2$ in section 3.2.

Theorem 2.5. *For $1 \leq r \leq 3$, let*

$$\begin{aligned}W_r^2 = &\frac{1}{2}\int_{\Omega}\rho_0^{-1}(JR'\partial_t^r q)^2\,\mathrm{d}y + \frac{1}{2}\int_{\Omega}\rho_0^{-1}R'(A^{\nu\alpha}\partial_{\nu}\partial_t^{r-1}q)(A^{\mu}_{\alpha}\partial_{\mu}\partial_t^{r-1}q)\,\mathrm{d}y\\ &+\frac{\sigma}{2}\int_{\Gamma}\mathfrak{R}_{\kappa}\sqrt{g}g^{ij}\Pi^{\alpha}_{\mu}(\overline{\partial}_i\partial_t^r\eta^{\mu})(\overline{\partial}_j\partial_t^r\eta_{\alpha})\,\mathrm{d}S.\end{aligned} \tag{2.8}$$

Then,

$$\sum\nolimits_{1\le r\le 3} W_r^2 \le \epsilon P(\mathcal{N}) + \epsilon(||q||_2^2 + ||q_t||_2^2) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}, \quad t\in[0,T], \tag{2.9}$$

where $T>0$ is sufficiently small.

Proof. See Appendix B. □

2.4. The $\mathfrak{R}_\kappa$-weighted Wave Equations

We consider the following $\mathfrak{R}_\kappa$-weighted derivatives:

$$\mathfrak{R}_\kappa \partial_t^3, \quad \sqrt{\mathfrak{R}_\kappa}\partial_t^2\overline{\partial}, \quad \partial_t\overline{\partial}^2.$$

Writing these derivatives as $\mathfrak{R}_\kappa^\ell D^3$ ($\ell = 1, \frac{1}{2}, 0$), and the identity (2.3) and (1.1) yield, after commuting $\mathfrak{R}_\kappa^\ell D^3$ and then $a^{\nu\alpha}\partial_\nu$, that

$$\mathfrak{R}_\kappa^\ell R' J D^3\partial_t^2 q - \mathfrak{R}_\kappa^\ell a^{\nu\alpha} A^\mu_{\ \alpha}\partial_\nu\partial_\mu D^3 q = \widetilde{\mathcal{F}},$$

where

$$\begin{aligned}\widetilde{\mathcal{F}} = &-\mathfrak{R}_\kappa^\ell[D^3\partial_t, JR']\partial_t q + \mathfrak{R}_\kappa^\ell[D^3,\rho_0]\partial_t(R^{-1}R'\partial_t q)\\ &+\mathfrak{R}_\kappa^\ell a^{\nu\alpha}(\partial_\nu\rho_0)D^3\partial_t v_\alpha + \mathfrak{R}_\kappa^\ell a^{\nu\alpha}\partial_\nu\big([D^3, A^\mu_{\ \alpha}]\partial_\mu q\big) + \mathfrak{R}_\kappa^\ell a^{\nu\alpha}\partial_\nu\big([D^3,\rho_0]\partial_t v_\alpha\big)\\ &-\mathfrak{R}_\kappa^\ell\rho_0[D^3\partial_t, a^{\nu\alpha}]\partial_\nu v_\alpha + \mathfrak{R}_\kappa^\ell a^{\nu\alpha}(\partial_\nu A^\mu_{\ \alpha})\partial_\mu D^3 q.\end{aligned}$$

We need these $\mathfrak{R}_\kappa$-weighted wave equations since their energies yield a better control of certain $\mathfrak{R}_\kappa$-weighted energy terms.

Theorem 2.6. *Let*

$$\begin{aligned}W_4^2 = &\frac{1}{2}\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(JR'D^3\partial_t q)^2\,\mathrm{d}y\\ &+\frac{1}{2}\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}R'(A^{\nu\alpha}\partial_\nu D^3 q)(A^\mu_{\ \alpha}\partial_\mu D^3 q)\,\mathrm{d}y\\ &+\frac{\sigma}{2}\int_\Gamma \mathfrak{R}_\kappa^{2\ell+1}\sqrt{g}g^{ij}\Pi^\alpha_\mu(\overline{\partial}_i D^3\partial_t\eta^\mu)(\overline{\partial}_j D^3\partial_t\eta_\alpha)\,\mathrm{d}S.\end{aligned} \tag{2.10}$$

Then,

$$W_4^2 \le \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}, \quad t\in[0,T],$$

where $T>0$ is sufficiently small.

Proof. See Appendix C. □

Remark. The energy (2.6) yields a better control of q with $1/2$ less $\mathfrak{R}_\kappa$-weights, for example, when $D=\partial_t$, W_4 controls $||\mathfrak{R}_\kappa^2\partial_t^4 q||_0$ and $||\mathfrak{R}_\kappa^{\frac{3}{2}}\partial\partial_t^3 q||_0$. The corresponding terms in E control merely $||\mathfrak{R}_\kappa^{\frac{5}{2}}\partial_t^4 q||_0$ and $||\mathfrak{R}_\kappa^2\partial\partial_t^3 q||_0$. This observation is crucial to control $\mathcal{I}_3$ in section 3.3 when $\mathfrak{D}^4 = \mathfrak{R}_\kappa^{\frac{3}{2}}\overline{\partial}\partial_t^3$ or $\mathfrak{R}_\kappa\overline{\partial}^2\partial_t^2$.

2.5. The Cauchy Invariance

We conclude this section with a compressible version of the Cauchy invariance, which was introduced in [21].

Theorem 2.7. *Let* (v, R) *be a smooth solution to* (1.1). *Then*

$$\epsilon^{\alpha\beta\gamma}\partial_\beta v^\mu \partial_\gamma \eta_\mu = \omega_0^\alpha + \int_0^t \epsilon^{\alpha\beta\gamma} a^{\lambda\mu}\partial_\lambda q \partial_\gamma \eta_\mu \frac{\partial_\beta R}{R^2} \tag{2.11}$$

for $t \in [0, T)$. *Here,* $e^{\alpha\beta\gamma}$ *is the totally antisymmetric symbol with* $\epsilon^{123} = 1$ *and* ω_0 *us the vorticity at* $t = 0$.

3. Energy Estimates

In this section we provide estimates for (v, q) and their time derivatives. We shall make frequent use of the assumptions (1.3)–(1.3) and of the two preliminary lemmas (that is, Lemma 2.1 and Lemma 2.2) in section 2 throughout this section without mentioning them every time.

Notation 3.1. *Let* E *be defined as in Definition* 3.6, *and let*

$$\begin{aligned}\mathcal{P} = P(&||v||_4, ||\mathfrak{R}_\kappa v_t||_3, ||v_t||_2, ||\mathfrak{R}_\kappa v_{tt}||_2, ||\sqrt{\mathfrak{R}_\kappa} v_{tt}||_1, ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_1, ||\mathfrak{R}_\kappa v_{ttt}||_0,\\ &||R||_4, ||R_t||_3, ||\sqrt{\mathfrak{R}_\kappa} R_{tt}||_2, ||R_{tt}||_1, ||\mathfrak{R}_\kappa R_{ttt}||_1, ||\sqrt{\mathfrak{R}_\kappa} R_{ttt}||_0,\\ &||\mathfrak{R}_\kappa \Pi\overline{\partial}^3 v_t||_{0,\Gamma}, ||\mathfrak{R}_\kappa \Pi\overline{\partial}^2 v_{tt}||_{0,\Gamma}, ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} \Pi\overline{\partial} v_{ttt}||_{0,\Gamma}, ||\Pi\overline{\partial}^2 v_t||_{0,\Gamma},\\ &||\sqrt{\mathfrak{R}_\kappa}\Pi\overline{\partial} v_{tt}||_{0,\Gamma})\end{aligned}$$

and $\mathcal{P}_0 = P(||\eta_0||_{7.5}, ||v_0||_4, ||v_0||_{4,\Gamma}, ||q_0||_4, ||q_0||_{4,\Gamma}, ||div\, v_0|_\Gamma||_{3,\Gamma}, ||\Delta v_0|_\Gamma ||_{2,\Gamma})$, *where we abbreviate*

$$||\Pi w||_{0,\Gamma}^2 = \int_\Gamma \Pi^\beta_\mu w^\mu \Pi^\alpha_\beta w_\alpha.$$

Here (and throughout this paper), we use $P(\cdot)$ *to denote a smooth function in its arguments. In addition, we define* $\mathcal{N}$ *to be*

$$\begin{aligned}\mathcal{N}(t) = {}&||v||_4^2 + ||\mathfrak{R}_\kappa v_t||_3^2 + ||\mathfrak{R}_\kappa v_{tt}||_2^2 + ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_1^2 + ||R||_4^2 + ||R_t||_3^2\\ &+ ||\sqrt{\mathfrak{R}_\kappa} R_{tt}||_2^2 + ||\mathfrak{R}_\kappa R_{ttt}||_1^2 + ||v_t||_2^2 + ||\sqrt{\mathfrak{R}_\kappa} v_{tt}||_1^2 + ||R_{tt}||_1 + E.\end{aligned}$$

The rest of this section is devoted to proving:

Theorem 3.2. *(*Energy estimate for E*) For sufficiently large* $\kappa > 0$, *we have*

$$E(t) \le \epsilon P(\mathcal{N}(t)) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}, \tag{3.1}$$

where $t \in [0, T]$ *for some* $T > 0$ *chosen sufficiently small, provided that the a priori assumption*

$$||\partial\eta||_{L^\infty} + ||\partial^2\eta||_{L^\infty} + ||g^{ij}||_{L^\infty} \le M \tag{3.2}$$

holds.

Notation 3.3. *Here and thereafter, we use ϵ to denote a small positive constant which may very from expression to expression. Typically ϵ comes from choosing the time sufficiently small (for example,* Lemma 2.1 (9)) *and the Young's inequality with ϵ. When all estimates are obtained, we can fix ϵ sufficiently small in order to close the estimates.*

3.1. The Energy Identity for the Euler Equations

Notation 3.4. *(Weighted tangential mixed derivatives) We let $\mathfrak{D}^r$, $r = 1, 2, 3, 4$ to be the mixed tangential differential operator defined as*

$$\begin{cases} \mathfrak{D} = \overline{\partial}, \partial_t, \\ \mathfrak{D}^2 = \overline{\partial}^2, \overline{\partial}\partial_t, \sqrt{\mathfrak{R}_\kappa}\partial_t^2, \\ \mathfrak{D}^3 = \overline{\partial}^2\partial_t, \sqrt{\mathfrak{R}_\kappa}(\overline{\partial}\partial_t^2), \mathfrak{R}_\kappa\partial_t^3, \\ \mathfrak{D}^4 = \mathfrak{R}_\kappa(\overline{\partial}^3\partial_t), \mathfrak{R}_\kappa(\overline{\partial}^2\partial_t^2), (\mathfrak{R}_\kappa)^{\frac{3}{2}}(\overline{\partial}\partial_t^3), (\mathfrak{R}_\kappa)^2\partial_t^4. \end{cases}$$

Notation 3.5. *Here and in sequel, we use $\mathcal{R}$ to denote lower order terms whose time integral $\int_0^t \mathcal{R}$ can be controlled by the right hand side of* (3.1).

Definition 3.6. For each fixed $1 \le r \le 4$, let $E = \sum_{r=1,2,3,4}(E_r + W_r^2)$, where

$$E_r = \frac{1}{2}\int_\Omega \rho_0(\mathfrak{D}^r v_\alpha)(\mathfrak{D}^r v^\alpha)\,\mathrm{d}y + \frac{1}{2}\int_\Omega JR'R^{-1}(\mathfrak{D}^r q)^2\,\mathrm{d}y \\ + \frac{\sigma}{2}\int_\Gamma \sqrt{g}g^{ij}\Pi^\alpha_\mu(\overline{\partial}_i\mathfrak{D}^r\eta^\mu)(\overline{\partial}_j\mathfrak{D}^r\eta_\alpha)\,\mathrm{d}S.$$

Here, W_r^2 $(1 \le r \le 4)$ is defined as (2.5) and (2.6), and Π is the normal projection operator defined in Lemma 2.2.

Remark. We use throughout that $||\mathfrak{R}_\kappa^\ell\Pi\overline{\partial}^m\partial_t^l\eta||^2_{0,\Gamma}$ is comparable with the coercive term coming from the boundary part of the energy. We use that g^{ij} is almost the Euclidean metric to make this comparison. For example, in the boundary estimates (section 3.4) we control $||\mathfrak{R}_\kappa^2\Pi\overline{\partial}\partial_t^3 v||^2_{0,\Gamma}$ by E.

The energy defined above is derived by differentiating $\frac{1}{2}\int_\Omega R(\mathfrak{D}^r v_\alpha)(\mathfrak{D}^r v^\alpha)\,\mathrm{d}y$ in time, invoking (1.1), (1.1), (1.1), (1.1), (1.3), (2.3) and the Piola identity

$$\partial_\mu A^{\mu\alpha} = \partial_\mu(Ja^{\mu\alpha}) = 0,$$

which follows from a direct computation using (2.1), so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \rho_0(\mathfrak{D}^r v_\alpha)(\mathfrak{D}^r v^\alpha)\,\mathrm{d}y = -\int_\Omega JR\big(\mathfrak{D}^r v_\alpha\big)\big(\mathfrak{D}^r(a^{\mu\alpha}\frac{\partial_\mu q}{R})\big)\,\mathrm{d}y \\ = -\int_\Omega (\mathfrak{D}^r v_\alpha)\Big(\mathfrak{D}^r(A^{\mu\alpha}\partial_\mu q)\Big)\,\mathrm{d}y + \underbrace{\int_\Omega (\mathfrak{D}^r v_\alpha)\Big([\mathfrak{D}^r, RJ](a^{\mu\alpha}\frac{\partial_\mu q}{R})\Big)\,\mathrm{d}y}_{\mathcal{I}_1}$$

$$
\begin{aligned}
&= \int_\Omega (\mathfrak{D}^r \partial_\mu v_\alpha)\Big(\mathfrak{D}^r (A^{\mu\alpha} q)\Big)\,\mathrm{d}y \underbrace{- \int_\Gamma (\mathfrak{D}^r v_\alpha)\Big(N_\mu \mathfrak{D}^r (A^{\mu\alpha} q)\Big)\,\mathrm{d}y}_{BD} + \mathcal{I}_1 \\
&= \int_\Omega (\mathfrak{D}^r \partial_\mu v_\alpha)(A^{\mu\alpha} \mathfrak{D}^r q)\,\mathrm{d}y + \underbrace{\int_\Omega (\mathfrak{D}^r \partial_\mu v_\alpha)\Big([\mathfrak{D}^r, A^{\mu\alpha}] q\Big)\,\mathrm{d}y}_{\mathcal{I}_2} + BD + \mathcal{I}_1.
\end{aligned}
\tag{3.3}
$$

The term $\int_\Omega (\mathfrak{D}^r \partial_\mu v_\alpha)(A^{\mu\alpha} \mathfrak{D}^r q)\,\mathrm{d}y$ is equal to

$$
\int_\Omega \mathfrak{D}^r (A^{\mu\alpha} \partial_\mu v_\alpha) \mathfrak{D}^r q\,\mathrm{d}y + \underbrace{\int_\Omega \big([\mathfrak{D}^r, A^{\mu\alpha}] \partial_\mu v_\alpha\big) \mathfrak{D}^r q\,\mathrm{d}y}_{\mathcal{I}_3},
$$

where, after invoking (2.3), we obtain

$$
\begin{aligned}
&\int_\Omega \mathfrak{D}^r (A^{\mu\alpha} \partial_\mu v_\alpha) \mathfrak{D}^r q\,\mathrm{d}y = - \int_\Omega \mathfrak{D}^r \Big(\frac{J R' \partial_t q}{R}\Big) \mathfrak{D}^r q\,\mathrm{d}y \\
&\quad = - \int_\Omega J R' R^{-1} (\partial_t \mathfrak{D}^r q) \mathfrak{D}^r q\,\mathrm{d}y + \underbrace{\int_\Omega \big([\mathfrak{D}^r, J R' R^{-1}] \partial_t q\big) \mathfrak{D}^r q\,\mathrm{d}y}_{\mathcal{I}_4}.
\end{aligned}
\tag{3.4}
$$

The first term in the second line of (3.1) is equal to

$$
-\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Omega J R' R^{-1} (\mathfrak{D}^r q)^2\,\mathrm{d}y + \mathcal{R},
$$

where the main term is moved to the left hand side of (3.1).

On the other hand, invoking the boundary condition $A^{\mu\alpha} N_\mu q = -\sigma \sqrt{g} \Delta_g \eta^\alpha$, as well as the seventh identity in Lemma 2.2, BD is equal to

$$
\begin{aligned}
BD &= - \int_\Gamma \mathfrak{D}^r v_\alpha \mathfrak{D}^r (A^{\mu\alpha} N_\mu q)\,\mathrm{d}y \\
&= \sigma \int_\Gamma \mathfrak{D}^r v_\alpha \mathfrak{D}^r (\sqrt{g} \Delta_g \eta^\alpha)\,\mathrm{d}y = \sigma \int_\Gamma \mathfrak{D}^r v_\alpha \mathfrak{D}^r (\sqrt{g} g^{ij} \Pi^\alpha_\mu \overline{\partial}^2_{ij} \eta^\mu)\,\mathrm{d}S \\
&= \sigma \int_\Gamma \sqrt{g} g^{ij} \Pi^\alpha_\mu (\mathfrak{D}^r v_\alpha)(\mathfrak{D}^r \overline{\partial}^2_{ij} \eta^\mu)\,\mathrm{d}S + \underbrace{\sigma \int_\Gamma \mathfrak{D}^r v_\alpha \big([\mathfrak{D}^r, \sqrt{g} g^{ij} \Pi^\alpha_\mu] \overline{\partial}^2_{ij} \eta^\mu\big)\,\mathrm{d}S}_{\mathcal{B}_1}.
\end{aligned}
\tag{3.5}
$$

Integrating by parts the first term in the very last line of (3.1), we have

$$
\begin{aligned}
\sigma \int_\Gamma \sqrt{g} g^{ij} \Pi^\alpha_\mu (\mathfrak{D}^r v_\alpha)(\mathfrak{D}^r \overline{\partial}^2_{ij} \eta^\mu)\,\mathrm{d}S &= -\sigma \int_\Gamma \sqrt{g} g^{ij} \Pi^\alpha_\mu (\overline{\partial}_i \partial_t \mathfrak{D}^r \eta_\alpha)(\overline{\partial}_j \mathfrak{D}^r \eta^\mu)\,\mathrm{d}S \\
&\quad \underbrace{- \sigma \int_\Gamma \overline{\partial}_i (\sqrt{g} g^{ij} \Pi^\alpha_\mu)(\partial_t \mathfrak{D}^r \eta_\alpha)(\overline{\partial}_j \mathfrak{D}^r \eta^\mu)\,\mathrm{d}S}_{\mathcal{B}_2}.
\end{aligned}
\tag{3.6}
$$

The first term on the right hand side of (3.1) is equal to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sigma}{2}\int_\Gamma \sqrt{g}g^{ij}\Pi^\alpha_\mu(\overline{\partial}_i\mathfrak{D}^r\eta^\mu)(\overline{\partial}_j\mathfrak{D}^r\eta_\alpha)\,\mathrm{d}S$$
$$+\frac{1}{2}\underbrace{\sigma\int_\Gamma \partial_t(\sqrt{g}g^{ij}\Pi^\alpha_\mu)(\overline{\partial}_i\mathfrak{D}^r\eta_\alpha)(\overline{\partial}_j\mathfrak{D}^r\eta^\mu)\,\mathrm{d}S}_{\mathcal{B}_3},$$

where the main term is moved to the left hand side of (2.5). Summing things up, we have shown that

$$\frac{\mathrm{d}E_r}{\mathrm{d}t}=\sum\nolimits_{1\le j\le 4}\mathcal{I}_j+\sum\nolimits_{j=1,2,3}\mathcal{B}_j+\mathcal{R}.$$

Thus, Theorem 3.2 follows if the terms $\mathcal{I}_{1,2,3,4}$ and $\mathcal{B}_{1,2,3}$ can be controlled by the right hand side of (3.2), which shall be treated in sections 3.3, 3.4 below. However, before doing this, we need to control $||q||_2$ and $||q_t||_2$.

3.2. Bounds for $||q||_2$ and $||q_t||_2$

Since $\mathfrak{D}^r$ symbolizes both $\mathfrak{R}_\kappa$-weighted and non-$\mathfrak{R}_\kappa$-weighted derivatives, we need to bound $||q||_2$ and $||q_t||_2$ in order to control $\mathcal{I}_3$. Also, the bound for $||q_t||_2$ is required to control $||v||_4$ in section 4. Taking $X=\partial q$ and $X=\partial q_t$, $s=1$, the standard div-curl estimate (A.1) yields that we need to control the lower order terms $||\partial q||_0$ and $||\partial q_t||_0$. We remark here that in the case when $\sigma=0$ (for example, [46]), these terms are controlled via $||\Delta q||_0$ and $||\Delta q_t||_0$, respectively, after integrating by parts and applying the Poincaré's inequality. However, we need to work a bit harder in order to control these quantities when $\sigma>0$.

Notation 3.7. *We write $X\lesssim Y$ to mean $X\le CY$, where $C>0$ is a large constant.*

Notation 3.8. *We are going to identify $\mathcal{P}^n=\mathcal{P}$ ($n\ge 1$) by a slight abuse of notations. Also, when $0\le t<1$, $(\int_0^t\mathcal{P})^n\le t^{n-1}\int_0^t\mathcal{P}^n\le t\int_0^t\mathcal{P}$, via Jensen's inequality.*

Lemma 3.9. *Let $\mathcal{F}_r$ be defined as (2.3). Assuming the a priori assumption (3.2) holds, then for sufficiently large $\kappa>0$ (that is, $\mathfrak{R}_\kappa\ll 1$), we have*

$$||\mathcal{F}_1||_0\lesssim\epsilon\mathcal{N}+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.$$

Proof. First, invoking (1.1) and the assumption (1.3), we have

$$||\partial_t(JR')(\partial_t q)||_0\lesssim\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.$$

Second, invoking Lemma 2.1(1–4), since $\partial_\mu q=R(a^{-1})_{\mu\beta}\partial_t v^\beta$ and $\partial_\nu\rho_0\lesssim\mathfrak{R}_\kappa|\partial_\nu q_0|\le\epsilon|\partial_\nu q_0|$ for sufficiently small $\mathfrak{R}_\kappa$, we get

$$||(\partial_t a^{\nu\alpha})(\partial_\nu v_\alpha)||_0+||a^{\nu\alpha}(\partial_\nu\rho_0)\partial_t v_\alpha||_0+||a^{\nu\alpha}(\partial_\nu A^\mu_\alpha)\partial_\mu q||_0$$
$$\lesssim\epsilon\mathcal{N}+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.$$

□

Lemma 3.10. *Let $\mathcal{F}_r$ be defined as* (2.3)*, Assuming the a priori assumption* (3.2) *holds, then for sufficiently large $\kappa > 0$ (that is, $\mathfrak{R}_\kappa << 1$), we have*

$$||\mathcal{F}_2||_0 \lesssim \epsilon||q_t||_2 + \epsilon(\sqrt{\mathcal{N}} + \mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

Proof. First, there is no problem to control $\sum_{\substack{j_1+j_2=2\\ j_1\geq 1}}||\partial_t^{j_1}(JR')(\partial_t^{j_2+1}q)||_0$ appropriately when $j_1 = 1$ using (1.1) and the assumption (1.3). Moreover, when $j_1 = 2$, one writes $J = \rho_0 R^{-1}$ and then $||\partial_t^2(\rho_0 R^{-1}R')q_t||_0 = ||\rho_0 R'\partial_t^2(R^{-1})q_t||_0$ modulo controllable terms, where

$$\begin{aligned}||\rho_0 R'\partial_t^2(R^{-1})q_t||_0 &\lesssim ||\partial_t^2(R^{-1})||_1^{\frac12}||\partial_t^2(R^{-1})||_0^{\frac12}||R_t||_1^{\frac12}||R_t||_0^{\frac12}\\ &\leq \epsilon(\sqrt{\mathcal{N}} + \mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.\end{aligned}$$

Here, we have applied the interpolation inequality (that is, Theorem 2.4) and the fact $R'\partial_t q = \partial_t R$. Second, invoking Lemma 2.1(1–6) we get

$$\sum_{j_1+j_2=1}||(\partial_t^{j_1+1}a^{\nu\alpha})(\partial_t^{j_2}\partial_\nu v_\alpha)||_0 \lesssim \epsilon\mathcal{N} + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},$$

and since $\partial_\nu A^\mu_{\ \alpha} = O(\epsilon)$ for small time and $\partial_\nu\rho_0 \lesssim \mathfrak{R}_\kappa|\partial_\nu q_0| \leq \epsilon|\partial_\nu q_0|$ for sufficiently small $\mathfrak{R}_\kappa$, we have

$$||a^{\nu\alpha}(\partial_\nu\rho_0)\partial_t^2 v_\alpha||_0 + ||a^{\nu\alpha}(\partial_\nu A^\mu_{\ \alpha})\partial_\mu\partial_t q||_0 \lesssim \epsilon||q_t||_2 + \epsilon\sqrt{\mathcal{N}} + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

Third, since $\partial_\mu q = R(a^{-1})_{\mu\beta}\partial_t v^\beta$, $||a^{\nu\alpha}\partial_\nu(\partial_t a^\mu_{\ \alpha}\cdot\partial_\mu q)||_0$ can be controlled appropriately by interpolation. □

Lemma 3.11. *We have*

$$||\partial q(t,\cdot)||_0^2 + ||\partial q_t(t,\cdot)||_0^2 \leq \epsilon||q_t(t,\cdot)||_2^2 + \epsilon P(\mathcal{N}) + W_3^2 + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P} \tag{3.7}$$

for $t \in [0, T]$ where $T > 0$ is chosen sufficiently small.

Proof. It suffices to consider $||\partial q_t||_0$ only. Integrating by parts yield

$$||\partial q_t||_0^2 = \int_\Omega(\partial_\mu q_t)(\partial^\mu q_t) = -\int_\Omega q_t\Delta q_t + \int_\Gamma(N^\mu\partial_\mu q_t)q_t,$$

and so we need to bound $\int_\Omega q_t\Delta q_t$ and $\int_\Gamma(N^\mu\partial_\mu q_t)q_t$, respectively.
Bound for $\int_\Omega q_t\Delta q_t$: Since $t \in [0, T]$ and $T > 0$ is small, as well as

$$\Delta q_t = (\delta^{\mu\nu} - a^{\mu\alpha}a^\nu_{\ \alpha})\partial_\mu\partial_\nu q_t + a^{\mu\alpha}a^\nu_{\ \alpha}\partial_\mu\partial_\nu q_t$$

Lemma 2.1 implies that

$$\int_\Omega q_t(\Delta q_t) \leq \epsilon||q_t||_2^2 + \int_\Omega q_t(a^{\mu\alpha}a^\nu_{\ \alpha}\partial_\mu\partial_\nu q_t).$$

Now, invoking the wave equation (2.3) and Lemma 3.9, we have

$$\begin{aligned}&\int_\Omega q_t(a^{\mu\alpha}a^{\nu}_{\alpha}\partial_\mu\partial_\nu q_t)\\&\quad=\int_\Omega R'q_tq_{ttt}-\int_\Omega(q_t\mathcal{F}_2)J^{-1}\lesssim||q_t||_0(||R'q_{ttt}||_0+||\mathcal{F}_2||_0)\\&\quad\le||q_t||_0\Big(W_3+\epsilon||q_t||_2+\epsilon(\sqrt{\mathcal{N}}+\mathcal{N})+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}\Big).\end{aligned}$$

On the other hand, since

$$||q_t||_0\le||\partial q_t||_0+\int_\Omega q_t$$

by Poincaré's inequality, if we let $Y=(0,0,y^3)$, then

$$\begin{aligned}||q_t||_0&\le||\partial q_t||_0+\int_\Omega\partial_\mu Y^\mu q=||\partial q_t||_0-\int_\Omega Y^\mu\partial_\mu q_t+\int_\Gamma N_\mu Y^\mu q_t\\&\le C_{\mathrm{vol}\,\Omega}||\partial q_t||_0+\int_\Gamma y^3q_t.\end{aligned}\tag{3.8}$$

To control the last integral $\int_\Gamma y^3q_t$, time differentiating the boundary condition

$$q=-\sigma g^{ij}\hat{n}^\mu\overline{\partial}^2_{ij}\eta_\mu,\quad\text{on }\Gamma$$

gives

$$q_t=-\sigma g^{ij}\hat{n}^\mu\overline{\partial}^2_{ij}v_\mu+\mathcal{R}_{q_t},\quad\text{on }\Gamma\tag{3.9}$$

where $\mathcal{R}_{q_t}$ consists of terms of the form either

$$\sigma g^{ij}g^{kl}(\overline{\partial}_kv^\tau\hat{n}_\tau\overline{\partial}_l\eta_\mu)\overline{\partial}^2_{ij}\eta^\mu\quad\text{or}\quad\sigma(\overline{\partial}^iv_\nu)(\overline{\partial}^j\eta^\nu)\hat{n}_\mu\overline{\partial}^2_{ij}\eta^\mu.$$

Now, invoking Lemma 2.1, Lemma 2.2, and the a priori assumption (3.2), we have

$$\int_\Gamma y^3q_t\lesssim\epsilon\mathcal{N}+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.\tag{3.10}$$

Wrapping these up, we get

$$\int_\Omega q_t\Delta q_t\lesssim\epsilon||q_t||_2^2+\epsilon||\partial q_t||_0^2+\epsilon P(\mathcal{N})+W_3^2+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.$$

Bound for $\int_\Gamma(N^\mu\partial_\mu q_t)q_t$: We have

$$\int_\Gamma(N^\mu\partial_\mu q_t)q_t\le||q_t||_{0,\Gamma}||\partial_3q_t||_{0,\Gamma}\le C(\epsilon^{-1})||q_t||^2_{0,\Gamma}+\epsilon||\partial q_t||^2_{0,\Gamma}.$$

Here, we bound $\epsilon||\partial q_t||^2_{0,\Gamma}$ by $\epsilon||q_t||_2^2$ using the trace lemma, which is part of the right hand side of (3.10). On the other hand, invoking (3.10), we have

$$||q_t||^2_{0,\Gamma}\lesssim\epsilon(\mathcal{N}^2+\mathcal{N})+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.$$

To see this, note that in $||q_t||^2_{0,\Gamma}$, the top order term is $\sqrt{g}g^{ij}\hat{n}^\mu\overline{\partial}^2_{ij}v_\mu$. Using the trace inequality, it suffices to bound $||\sqrt{g}g^{ij}\hat{n}^\mu\partial\overline{\partial}^2_{ij}v_\mu||^2_0$. We control this top order term by the Young's inequality, which leads to the appearance of $\epsilon\mathcal{N}^2$. In addition, the lower order terms are controlled by $\epsilon\mathcal{N}+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}$ using the interpolation.

Hence,

$$\int_\Gamma (N^\mu\partial_\mu q_t)q_t \lesssim \epsilon||q_t||^2_2+\epsilon(\mathcal{N}^2+\mathcal{N})+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.$$

Therefore,

$$\begin{aligned}||\partial q_t||^2_0 &= -\int_\Omega q_t\Delta q_t+\int_\Gamma q_t(N^\mu\partial_\mu q_t)\\ &\lesssim \epsilon||q_t||^2_2+\epsilon(\mathcal{N}+\mathcal{N}^2)+W_3^2+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.\end{aligned}$$

In addition, we are able to control $||\partial q||^2_0$ appropriately by integrating $||\partial q_t||^2_0$ in time, which, together with the estimate for $||\partial q_t||^2_0$, conclude the proof of (3.10). □

In fact, the above proof implies the control for the lowest order norms $||q||_0$ and $||q_t||_0$.

Corollary 3.12. *We have*

$$||q||^2_0+||q_t||^2_0 \lesssim ||\partial q||^2_0+||\partial q_t||^2_0+\epsilon\mathcal{N}+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}. \tag{3.11}$$

Proof. Let $Y=(0,0,y^3)$, the Poincaré's inequality implies

$$||q||_0+||q_t||_0 \lesssim ||\partial q||_0+||\partial q_t||_0+\int_\Omega \partial_\mu Y^\mu q+\int_\Omega \partial_\mu Y^\mu q_t.$$

Now, we proceed as in (3.10)–(3.10) and get

$$\int_\Omega \partial_\mu Y^\mu q+\int_\Omega \partial_\mu Y^\mu q_t \lesssim ||\partial q||_0+||\partial q_t||_0+\epsilon\sqrt{\mathcal{N}}+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P},$$

and hence (3.11) follows after squaring the above estimate. □

Theorem 3.13. *We have*

$$||q(t,\cdot)||^2_2+||q_t(t,\cdot)||^2_2 \lesssim \epsilon P(\mathcal{N})+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}, \tag{3.12}$$

for $t\in[0,T]$ where $T>0$ is chosen sufficiently small.

Proof. It suffices to control $||q_t||_2^2$ by the right hand side of (3.12) since the control of $||q||_2^2$ follows from time integrating $||q_t||_2^2$. To control $||q_t||_2^2$, it suffices to consider $||\partial q_t||_1^2$ only thanks to Lemma 3.11 and Corollary 3.12. Now, invoking the div-curl estimate (A.1) with $X = \partial q_t$ and $s = 1$, we have

$$||\partial q_t||_1^2 \lesssim ||\Delta q_t||_0^2 + ||\overline{\partial} q_t||_{0.5,\Gamma}^2 + ||\partial q_t||_0^2.$$

Bound for $||\Delta q_t||_0^2$:Invoking Lemma 2.1, since $t \in [0, T]$ and T is sufficiently small, we have

$$\begin{aligned} ||\Delta q_t||_0^2 &\leq ||a^{\mu\alpha} a^{\nu}_{\ \alpha} \partial_\mu \partial_\nu q_t||_0^2 + ||(\delta^{\mu\nu} - a^{\mu\alpha} a^{\nu}_{\ \alpha})\partial_\mu \partial_\nu q_t||_0^2 \\ &\leq ||a^{\mu\alpha} a^{\nu}_{\ \alpha} \partial_\mu \partial_\nu q_t||_0^2 + \epsilon ||\partial^2 q_t||_0. \end{aligned}$$

Furthermore, the wave equation (2.3) and Lemma 3.10 yield

$$\begin{aligned} ||a^{\mu\alpha} a^{\nu}_{\ \alpha} \partial_\mu \partial_\nu q_t||_0^2 &\leq ||R' q_{ttt}||_0^2 + ||\mathcal{F}_2 J^{-1}||_0^2 \\ &\leq W_3^2 + \epsilon(\mathcal{N} + \mathcal{N}^2) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} + \epsilon ||q_t||_2^2. \end{aligned}$$

Bound for $||\overline{\partial} q_t||_{0.5,\Gamma}^2$: Invoking (3.10) and taking one more tangential derivative, we have

$$\overline{\partial} q_t = \sigma g^{ij} \hat{n}^\mu \overline{\partial}\overline{\partial}_{ij}^2 v_\mu - \sigma g^{ij} g^{kl} (\overline{\partial}_k v^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu) \overline{\partial}\overline{\partial}_{ij}^2 \eta^\mu + \mathcal{R}_{\overline{\partial} q_t},$$

where $\mathcal{R}_{\overline{\partial} q_t}$ consists products of $\overline{\partial}^k \eta$ and $\overline{\partial}^k v$, $k = 1, 2$. To be more specific, $\mathcal{R}_{\overline{\partial} q_t}$ consists terms of the forms

$$\sigma g^{ij} g^{kl} (\overline{\partial}_k \overline{\partial} v^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu) \overline{\partial}_{ij}^2 \eta^\mu, \quad \sigma g^{ij} g^{kl} (\overline{\partial}_k \overline{\partial} \eta^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu) \overline{\partial}_{ij}^2 v^\mu,$$
$$\sigma (\overline{\partial}^i v^\mu)(\overline{\partial}^j \eta_\mu) g^{kl} (\overline{\partial}_k \overline{\partial} \eta^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu) \overline{\partial}_{ij}^2 \eta^\mu, \quad \sigma (\overline{\partial}\overline{\partial}^i \eta^\mu)(\overline{\partial}^j \eta_\mu) g^{kl} (\overline{\partial}_k \overline{\partial} \eta^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu) \overline{\partial}_{ij}^2 v^\mu.$$

Given these, we have

$$||\overline{\partial} q_t||_{0.5,\Gamma}^2 \lesssim \epsilon(\mathcal{N}^2 + \mathcal{N}) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P},$$

by interpolation and the Young's inequality. Here, $\epsilon \mathcal{N}^2$ appears since

$$||\sqrt{g} g^{ij} \hat{n}^\mu \partial \overline{\partial}^3 v_\mu||_0^2 \lesssim \epsilon ||v||_4^4 + ||\sqrt{g} g^{ij} \hat{n}||_2^4 \lesssim \epsilon \mathcal{N}^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P},$$

and we remark here that the interpolation cannot be applied since $\partial \overline{\partial}^3 v$ is of the top order. Wrapping these up and invoking Lemma 3.11 and Corollary 3.12, we get

$$||q_t||_2^2 \lesssim W_3^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} + \epsilon ||q_t||_2^2 + \epsilon(\mathcal{N}^2 + \mathcal{N}),$$

which proves the estimate for $||q_t||_2^2$ by invoking (2.5) and then absorbing $\epsilon ||q_t||_2^2$ to the left hand side. □

Remark. We are unable to control $||\partial q_{tt}||_1$ when surface tension is present. This is due to that the div-curl estimate yields the boundary term $||\overline{\partial} q_{tt}||_{0.5,\Gamma}$, where $\overline{\partial} q_{tt} \sim \overline{\partial}^3 v_t$ on Γ, and hence $||\overline{\partial} q_{tt}||_{0.5,\Gamma}$ yields a loss of derivative. Therefore, one has to define the energy using the $\mathfrak{R}_\kappa$-weighted derivatives and so the corresponding term can then be controlled by the energy.

3.3. Bounds for $\int_0^t \mathcal{I}_{1,2,3,4}$

This section is devoted to control $\int_0^t \mathcal{I}_{1,2,3,4}$. We recall

$$\mathcal{I}_1 = \int_\Omega (\mathfrak{D}^r v_\alpha)\Big([\mathfrak{D}^r, RJ](a^{\mu\alpha}\frac{\partial_\mu q}{R})\Big), \quad \mathcal{I}_2 = \int_\Omega (\mathfrak{D}^r \partial_\mu v_\alpha)\Big([\mathfrak{D}^r, A^{\mu\alpha}]q\Big),$$
$$\mathcal{I}_3 = \int_\Omega \Big([\mathfrak{D}^r, A^{\mu\alpha}]\partial_\mu v_\alpha\Big)\mathfrak{D}^r q, \quad \mathcal{I}_4 = \int_\Omega \Big([\mathfrak{D}^r, JR'R^{-1}]\partial_t q\Big)\mathfrak{D}^r q.$$

Notation 3.14. *In what follows, we use D to denote either $\overline{\partial}$ or ∂_t. This allows us to represent $\mathfrak{D}^r$ as $(\mathfrak{R}_\kappa)^\ell D^r$, where $r = \frac{1}{2}, 1, \frac{3}{2}, 2$.*

3.3.1. Control of $\int_0^t \mathcal{I}_1$ **For non-$\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$**: We recall that there are four mixed derivatives which are not $\mathfrak{R}_\kappa$-weighted, which are $\overline{\partial}, \overline{\partial}^2, \overline{\partial}\partial_t$ and $\overline{\partial}^2\partial_t$. Hence, it suffices to consider only the case when $\mathfrak{D}^r = \overline{\partial}^2\partial_t$. Invoking (1.1) and Theorem 3.13, We have

$$\mathcal{I}_1 = \sum_{\substack{j_1+j_2=2\\ j_1\ge 1}} \int_\Omega (\overline{\partial}^2\partial_t v_\alpha)(\overline{\partial}^{j_1}\rho_0)(\overline{\partial}^{j_2}\partial_t(a^{\mu\alpha}\frac{\partial_\mu q}{R})).$$

Since, to the highest order, the last term on the right hand side is $R^{-1}a^{\mu\alpha}\overline{\partial}\partial q_t$, which can be controlled by invoking Theorem 3.13. Therefore,

$$\int_0^t \mathcal{I}_1 \le \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

The $\epsilon P(N)$ term introduced in Theorem 3.13 does not figure here since $\mathcal{I}_1$ is estimated under the time integral.
For $\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$: It suffices to consider derivatives of the form $(\mathfrak{R}_\kappa)^\ell D^{r-2}\overline{\partial}\partial_t$, where $\ell = \frac{1}{2}, 1, \frac{3}{2}$ and $r \le 4$, since otherwise $\mathcal{I}_1$ would be 0 due to (1.1).

$$\mathcal{I}_1 = \sum_{j_1+j_2=r-2} \int_\Omega (\mathfrak{R}_\kappa)^{2\ell}(D^{r-2}\overline{\partial}\partial_t v_\alpha)(\overline{\partial}D^{j_1}\rho_0)(D^{j_2}\partial_t(a^{\mu\alpha}\frac{\partial_\mu q}{R})).$$

We henceforth adopt

Notation 3.15. *We use $\stackrel{L}{=}$ to denote equality modulo lower order terms that can be controlled, that is, $A \stackrel{L}{=} B$ mean $A = B +$ error terms, where the "error terms" can be controlled by the bound of B plus $\mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}$.*

Invoking (1.3) and (1.3) at $t = 0$ lead to

$$\int_0^t \mathcal{I}_1 \stackrel{L}{=} \sum_{j_1+j_2=r-2}\int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell+1}(D^{r-2}\overline{\partial}\partial_t v_\alpha)(\overline{\partial}D^{j_1}p_0)(D^{j_2}\partial_t(a^{\mu\alpha}\frac{\partial_\mu q}{R}))$$
$$= \int_0^t \int_\Omega (\overline{\partial}D^{j_1}p_0)\Big((\mathfrak{R}_\kappa)^{\ell+\frac{1}{2}}D^{r-2}\overline{\partial}\partial_t v_\alpha\Big)\Big((\mathfrak{R}_\kappa)^{\ell+\frac{1}{2}}D^{j_2}\partial_t(a^{\mu\alpha}\frac{\partial_\mu q}{R})\Big) \le \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

Remark. The above expression yields a slightly better bound for $D^{r-2}\overline{\partial}\partial_t v$, since $\mathcal{P}$ requires only $||(R')^\ell D^{r-2}\overline{\partial}\partial_t v||_0$.

3.3.2. Control of $\int_0^t \mathcal{I}_2$ For each r, $\int_0^t \mathcal{I}_2$ contains a term which is of the order $r+1$, that is,

$$\int_0^t \mathfrak{T} = \int_\Omega (\mathfrak{D}^r \partial_\mu v_\alpha)(\mathfrak{D}^r A^{\mu\alpha})q.$$

There is no problem to control $\mathfrak{T}$ when $r \le 2$, and when $r = 3$, we need to put extra effort to control $\mathfrak{T}$ when $\mathfrak{D}^3 = \overline{\partial}^2 \partial_t$ since there are terms which cannot be controlled directly without $\mathfrak{R}_\kappa$-weights, and one needs to integrate by parts in (tangential) spatial derivative and time derivative, respectively. On the other hand, when $r = 4$, this term is of above the top order, but it can be controlled using one of the special cancellations referred to in section 1.6.1, as we now show.

For non-$\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$: As mentioned above, we consider only the case when $r = 3$ and $\mathfrak{D}^3 = \overline{\partial}^2\partial_t$. In this case,

$$\mathfrak{T} = \int_\Omega (\overline{\partial}^2 \partial_\mu \partial_t v_\alpha)(\overline{\partial}^2 \partial_t A^{\mu\alpha})q.$$

Although this term is of the correct order, $\overline{\partial}^2\partial_\mu\partial_t v$ cannot be controlled without $\mathfrak{R}_\kappa$-weights. Hence, we integrate by parts with respect to the tangential derivative and get

$$\begin{aligned}\mathfrak{T} &= -\Big(\int_\Omega (\overline{\partial}\partial_\mu\partial_t v_\alpha)(\overline{\partial}^3\partial_t A^{\mu\alpha})q + \int_\Omega (\overline{\partial}\partial_\mu\partial_t v_\alpha)(\overline{\partial}^2\partial_t A^{\mu\alpha})\overline{\partial}q\Big)\\ &\le ||\overline{\partial}\partial v_t||_0 ||\overline{\partial}^3\partial_t A||_0 ||q||_{L^\infty} + ||\overline{\partial}\partial v_t||_0 ||\overline{\partial}^2\partial_t A||_{L^4} ||\overline{\partial}q||_{L^4}.\end{aligned}$$

Here, one adapts Theorem 3.13 to control $||q||_2$. Integrating with respect to time, we obtain

$$\int_0^t \mathfrak{T} \lesssim \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

We next consider $\mathcal{I}_2 - \mathfrak{T}$. All terms involved in $\mathcal{I}_2 - \mathfrak{T}$ can be controlled straightforwardly after integrating by part with respect to $\overline{\partial}$ thanks to Theorem 3.13, except for

$$\int_\Omega (\overline{\partial}^2\partial_\mu\partial_t v_\alpha)(\partial_t A^{\mu\alpha})(\overline{\partial}^2 q).$$

This is due to that integrating by part in $\overline{\partial}$ yields $\overline{\partial}^3 q$ which cannot be controlled without $\mathfrak{R}_\kappa$-weights. To deal with this issue, we consider

$$\int_0^t\int_\Omega (\overline{\partial}^2\partial_\mu\partial_t v_\alpha)(\partial_t A^{\mu\alpha})(\overline{\partial}^2 q).$$

Integrating by part in time, we get

$$\begin{aligned}&\int_\Omega (\overline{\partial}^2\partial_\mu v_\alpha)(\partial_t A^{\mu\alpha})(\overline{\partial}^2 q)\Big|_0^t - \int_0^t\int_\Omega (\overline{\partial}^2\partial_\mu v_\alpha)(\partial_t^2 A^{\mu\alpha})(\overline{\partial}^2 q)\\ &- \int_0^t\int_\Omega (\overline{\partial}^2\partial_\mu v_\alpha)(\partial_t A^{\mu\alpha})(\overline{\partial}^2\partial_t q).\end{aligned}$$

The last two terms are bounded by $\mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}$ thanks to Theorem 3.13, while the pointwise term at $t = 0$ by $\mathcal{P}_0$. The pointwise term at t is bounded by

$$||\eta||_3||\overline{\partial}^2 q||_0||\overline{\partial}^2 v||_1^{\frac{1}{2}}||\overline{\partial}^2 v||_2^{\frac{1}{2}}||v||_1^{\frac{1}{2}}||v||_2^{\frac{1}{2}} \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},$$

which is controlled by the right hand side of (3.2).

For $\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$: $\mathcal{I}_2$ contains a term above the top order, that is,

$$\mathfrak{T} = \int_\Omega (\mathfrak{D}^r \partial_\mu v_\alpha)(\mathfrak{D}^r A^{\mu\alpha})q.$$

This term is controlled using the aforementioned special cancellation (see section 1.6.1). For $\mathfrak{R}_\kappa$-weighted derivatives, it suffices to consider only the case when $r = 4$, that is, the derivatives are of the form $(\mathfrak{R}_\kappa)^\ell D^3\partial_t$, for $\ell = 1, \frac{3}{2}, 2$. Then the "tricky" term to be bounded is

$$\mathcal{L} = \int_0^t \mathfrak{T} = \int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell}(D^3\partial_t\partial_\mu v_\alpha)(D^3\partial_t A^{\mu\alpha})q. \tag{3.13}$$

In view of (2.1), expanding the index μ in (3.3.2), we have

$$\begin{aligned}
\mathcal{L} = &\underbrace{\int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2 D^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_1 D^3\partial_t v_\alpha}_{L_1} + \underbrace{\int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2 \eta_\lambda \partial_3 D^3 v_\tau \overline{\partial}_1 D^3\partial_t v_\alpha}_{L_2}\\
&\underbrace{-\int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2 D^3\partial_t v_\alpha}_{L_3} \underbrace{-\int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 \eta_\lambda \partial_3 D^3 v_\tau \overline{\partial}_2 D^3\partial_t v_\alpha}_{L_4}\\
&\underbrace{\int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 v_\lambda \overline{\partial}_2\eta_\tau \partial_3 D^3\partial_t v_\alpha}_{L_5} + \underbrace{\int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 \eta_\lambda \overline{\partial}_2 D^3 v_\tau \partial_3 D^3\partial_t v_\alpha}_{L_6} + L_{low},
\end{aligned} \tag{3.14}$$

where L_{low} are lower order terms, which are all of the form

$$\begin{aligned}
&\sum_{\substack{j_1+j_2=3\\ j_1,j_2\le 2}} \int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q(\partial D^{j_1} v)(\partial D^{j_2}\eta)(\partial D^3\partial_t v)\\
&= \sum_{\substack{j_1+j_2=3\\ j_1,j_2\le 2}} \Big(\int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q(\partial D^{j_1} v)(\partial D^{j_2}\eta)(\partial D^3 v)\\
&\quad - \int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q_t(\partial D^{j_1} v)(\partial D^{j_2}\eta)(\partial D^3 v)\\
&\quad - \int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q(\partial D^{j_1} v_t)(\partial D^{j_2}\eta)(\partial D^3 v)\\
&\quad \int_0^t \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q(\partial D^{j_1} v)(\partial D^{j_2} v)(\partial D^3 v)\Big).
\end{aligned}$$

Invoking Theorem 3.13, it is easy to see that the last three terms are controlled by $\mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}$, while the pointwise term at t is treated similar to (3.3.2)–(3.3.2), after distributing correct amount of $\mathfrak{R}_\kappa$-weights to each term. We omit the detail here. But it is worth noting that there are more than enough $\mathfrak{R}_\kappa$-weights for the pointwise term since there is one time derivative less.

Next, integrating by part in time in L_3, we find

$$-\int_0^t\int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2 D^3\partial_t v_\alpha \stackrel{L}{=} -\int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2 D^3 v_\alpha$$
$$\int_0^t\int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 \partial_t v_\lambda \partial_3\eta_\tau \overline{\partial}_2 D^3 v_\alpha.$$

Adding L_1, we get

$$\begin{aligned}
L_1 + L_3 &\stackrel{L}{=} \int_0^t\int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2 D^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_1 D^3\partial_t v_\alpha \\
&\quad + \int_0^t\int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3\partial_t v_\lambda \partial_3\eta_\tau \overline{\partial}_2 D^3 v_\alpha \\
&\quad - \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2 D^3 v_\alpha|_0^t \\
&= -\int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2 D^3 v_\alpha|_0^t = L_{13},
\end{aligned}$$

since first and the second term cancels with each other by the antisymmetry of $\epsilon^{\alpha\lambda\tau}$. Similarly, we have

$$\begin{aligned}
L_4 + L_6 &\stackrel{L}{=} L_{46} = \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 \eta_\lambda \overline{\partial}_2 D^3 v_\tau \partial_3 D^3 v_\alpha|_0^t, \\
L_2 + L_5 &\stackrel{L}{=} L_{25} = \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 v_\lambda \overline{\partial}_2\eta_\tau \partial_3 D^3 v_\alpha|_0^t.
\end{aligned}$$

Bounds for L_{13}, L_{46} **and** L_{25} Since L_{13} is pointwise in t, it suffices to consider

$$\int_\Omega (\mathfrak{R}_\kappa)^{2\ell} q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 D^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2 D^3 v_\alpha|_t$$

only, since the other part is controlled directly by $\mathcal{P}_0$. In addition, since D^3 corresponds to ∂_t^3, $\overline{\partial}\partial_t^2$, $\overline{\partial}^2\partial_t$ and $\overline{\partial}^3$, associated with weights $(\mathfrak{R}_\kappa)^2$, $(\mathfrak{R}_\kappa)^{\frac{3}{2}}$, $\mathfrak{R}_\kappa$ and $\mathfrak{R}_\kappa$, respectively, we have:

$$\begin{aligned}
&\int_\Omega (\mathfrak{R}_\kappa)^4 q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1 \partial_t^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2 \partial_t^3 v_\alpha|_t \\
&\quad\le \int_\Omega (\mathfrak{R}_\kappa) q\epsilon^{\alpha\lambda\tau}((\mathfrak{R}_\kappa)^{\frac{3}{2}}\overline{\partial}_1 \partial_t^3 v_\lambda)\partial_3\eta_\tau((\mathfrak{R}_\kappa)^{\frac{3}{2}}\overline{\partial}_2 \partial_t^3 v_\alpha)|_t \\
&\quad\le ||R||_2||\eta||_3||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_1^2 \le \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},
\end{aligned}$$

where we have used $||\mathfrak{R}_\kappa q||_2 \lesssim ||R'q||_2 = ||R||_2$. Similarly, we have

$$\int_\Omega (\mathfrak{R}_\kappa)^3 q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\overline{\partial}\partial_t^2 v_\lambda \partial_3\eta_\tau \overline{\partial}_2\overline{\partial}\partial_t^2 v_\alpha|_t + \int_\Omega (\mathfrak{R}_\kappa)^2 q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\overline{\partial}^2\partial_t v_\lambda \partial_3\eta_\tau \overline{\partial}_2\overline{\partial}^2\partial_t v_\alpha|_t$$

$$+\int_{\Omega}(\mathfrak{R}_\kappa)^2 q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\overline{\partial}^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2\overline{\partial}^3 v_\alpha|_t \le \epsilon P(\mathcal{N})+\mathcal{P}_0+\mathcal{P}\int_0^t \mathcal{P}.$$

Moreover, this method can be adapted to control L_{46} and L_{25}, and we omit the details. Therefore,

$$(L_1+L_3)+(L_4+L_6)+(L_2+L_5)\le \epsilon P(\mathcal{N})+\mathcal{P}_0+\mathcal{P}\int_0^t \mathcal{P}.$$

Now, we complete the treatment of $\mathcal{I}_2$ by estimating the rest of the terms, that is, $\mathcal{I}_2-\mathfrak{T}$, for $\mathfrak{R}_\kappa$-weighted forth order derivatives. Expressing

$$\mathcal{I}_2-\mathfrak{T}=\int_{\Omega}(\mathfrak{R}_\kappa)^{2\ell}(D^3\partial_t\partial_\mu v_\alpha)\Big(D^3\partial_t(A^{\mu\alpha}q)-A^{\mu\alpha}D^3\partial_t q-(D^3\partial_t A^{\mu\alpha})q\Big),$$

and similarly, to the non-$\mathfrak{R}_\kappa$-weighted case, we consider $\int_0^t \mathcal{I}_2-\mathfrak{T}$ and integrate by part in time to get

$$\int_0^t \mathcal{I}_2-\mathfrak{T}\stackrel{L}{=}\int_{\Omega}(\mathfrak{R}_\kappa)^{2\ell}(D^3\partial_\mu v_\alpha)\Big(D^3\partial_t(A^{\mu\alpha}q)-A^{\mu\alpha}D^3\partial_t q-(D^3\partial_t A^{\mu\alpha})q\Big)\Big|_0^t$$
$$-\int_0^t\int_{\Omega}(\mathfrak{R}_\kappa)^{2\ell}(D^3\partial_\mu v_\alpha)\partial_t\Big(D^3\partial_t(A^{\mu\alpha}q)-A^{\mu\alpha}D^3\partial_t q-(D^3\partial_t A^{\mu\alpha})q\Big).$$

First, it is easy to check that

$$\int_0^t\int_{\Omega}(\mathfrak{R}_\kappa)^{2\ell}(D^3\partial_\mu v_\alpha)\partial_t\Big(D^3\partial_t(A^{\mu\alpha}q)-A^{\mu\alpha}D^3\partial_t q-(D^3\partial_t A^{\mu\alpha})q\Big)\le\int_0^t \mathcal{P},$$

Second, for the pointwise terms at t, it suffices to consider the case when $D^3=\partial_t^3$ and $\ell=2$, since the bounds for the other (easier) cases follow from the same method. There are three terms, that is,

$$\int_{\Omega}(\mathfrak{R}_\kappa)^4(\partial_t^3\partial_\mu v_\alpha)(\partial_t^3 A)\partial_t q,\quad \int_{\Omega}(\mathfrak{R}_\kappa)^4(\partial_t^3\partial_\mu v_\alpha)(\partial_t^2 A)\partial_t^2 q,\quad \int_{\Omega}(\mathfrak{R}_\kappa)^4(\partial_t^3\partial_\mu v_\alpha)(\partial_t A)\partial_t^3 q. \tag{3.15}$$

These terms are treated as

$$\begin{aligned}&\int_{\Omega}(\mathfrak{R}_\kappa)^4(\partial_t^3\partial_\mu v_\alpha)(\partial_t^3 A)\partial_t q\\ &\quad\approx\int_{\Omega}(\mathfrak{R}_\kappa)^4(\partial_t^3\partial_\mu v_\alpha)\Big((\partial v_{tt})(\partial\eta)+(\partial v_t)(\partial v)\Big)\partial_t q\\ &\quad\lesssim \|(\mathfrak{R}_\kappa)^{\frac32}v_{ttt}\|_1\|(\mathfrak{R}_\kappa)^{\frac32}v_{tt}\|_1^{\frac12}\|(\mathfrak{R}_\kappa)^{\frac32}v_{tt}\|_2^{\frac12}\|\eta\|_3\|\mathfrak{R}_\kappa q_t\|_0^{\frac12}\|\mathfrak{R}_\kappa q_t\|_1^{\frac12}\\ &\qquad+\|(\mathfrak{R}_\kappa)^{\frac32}v_{ttt}\|_1\|\mathfrak{R}_\kappa v_t\|_1^{\frac12}\|\mathfrak{R}_\kappa v_t\|_2^{\frac12}\|(\mathfrak{R}_\kappa)^{\frac12}v\|_3\|\mathfrak{R}_\kappa q_t\|_0^{\frac12}\|\mathfrak{R}_\kappa q_t\|_1^{\frac12}\\ &\quad\lesssim\epsilon P(\mathcal{N})+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P},\end{aligned}$$

and

$$\int_{\Omega}(\mathfrak{R}_\kappa)^4(\partial_t^3\partial_\mu v_\alpha)(\partial_t^2 A)\partial_t^2 q=\int_{\Omega}(\mathfrak{R}_\kappa)^4(\partial_t^3\partial_\mu v_\alpha)\Big((\partial v)^2+(\partial v_t)(\partial\eta)\Big)\partial_t^2 q$$

$$\lesssim ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_1 \Big(||v||_2 ||\mathfrak{R}_\kappa v||_3 + ||\mathfrak{R}_\kappa v_t||_1^{\frac{1}{2}} ||\mathfrak{R}_\kappa v_t||_2^{\frac{1}{2}} ||\eta||_3 \Big) ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} q_{tt}||_0^{\frac{1}{2}} ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} q_{tt}||_1^{\frac{1}{2}}$$

$$\lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

Finally, we have

$$\int_\Omega (\mathfrak{R}_\kappa)^4 (\partial_t^3 \partial_\mu v_\alpha)(\partial_t A)\partial_t^3 q = \int_\Omega (\mathfrak{R}_\kappa)^4 (\partial_t^3 \partial_\mu v_\alpha)(\partial v)(\partial \eta)\partial_t^3 q$$
$$\lesssim ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_1 ||v||_1^{\frac{1}{2}} ||v||_2^{\frac{1}{2}} ||\eta||_3 ||(\mathfrak{R}_\kappa)^{\frac{5}{2}} q_{ttt}||_0^{\frac{1}{2}} ||(\mathfrak{R}_\kappa)^{\frac{5}{2}} q_{ttt}||_1^{\frac{1}{2}}$$
$$\lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}. \tag{3.16}$$

3.3.3. Control of $\int_0^t \mathcal{I}_3$

For non-$\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$: Expressing these derivatives as D^r where $r \le 3$, we have

$$\mathcal{I}_3 = \sum_{\substack{j_1+j_2=r \\ j_1 \ge 1}} \int_\Omega (D^{j_1} A^{\mu\alpha})(\partial_\mu D^{j_2} v_\alpha)(D^r q)$$
$$\le \sum_{\substack{j_1+j_2=r \\ j_1 \ge 1}} ||(D^{j_1} A^{\mu\alpha})(\partial_\mu D^{j_2} v_\alpha)||_0 ||D^r q||_0,$$

and so $\int_0^t \mathcal{I}_3 \le \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}$ in light of Theorem 3.13.

For $\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$: It suffices to consider only the case when $r = 4$, that is, the derivatives are of the form $(\mathfrak{R}_\kappa)^\ell D^3 \partial_t$, for $\ell = 1, \frac{3}{2}, 2$. Now,

$$\mathcal{I}_3 = \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} (\partial_t A^{\mu\alpha})(D^3 \partial_\mu v_\alpha)(D^3 \partial_t q)$$
$$+ \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} (D A^{\mu\alpha})(D^2 \partial_t \partial_\mu v_\alpha)(D^3 \partial_t q)$$
$$+ \int_\Omega (\mathfrak{R}_\kappa)^{2\ell} (\partial_t D^3 A^{\mu\alpha})(\partial_\mu v_\alpha)(D^3 \partial_t q) + \text{error terms},$$

where the main term is equal to

$$\int_\Omega (\partial_t A^{\mu\alpha})\Big((\mathfrak{R}_\kappa)^{\ell-\frac{1}{2}} D^3 \partial_\mu v_\alpha\Big)\Big((\mathfrak{R}_\kappa)^{\ell+\frac{1}{2}} D^3 \partial_t q\Big)$$
$$+ \int_\Omega (\overline{\partial} A^{\mu\alpha})\Big((\mathfrak{R}_\kappa)^{\ell} D^2 \partial_t \partial_\mu v_\alpha\Big)\Big((\mathfrak{R}_\kappa)^{\ell} D^3 \partial_t q\Big)$$
$$+ \int_\Omega (\partial_\mu v_\alpha)\Big((\mathfrak{R}_\kappa)^{\ell-\frac{1}{2}} \partial_t D^3 A^{\mu\alpha}\Big)\Big((\mathfrak{R}_\kappa)^{\ell+\frac{1}{2}} D^3 \partial_t q\Big) = \mathcal{I}_{3,1} + \mathcal{I}_{3,2} + \mathcal{I}_{3,3},$$

where $\mathcal{I}_{3,2}$ does not appear when $\mathfrak{D}^4 = \mathfrak{R}_\kappa^2 \partial_t^4$.

$\int_0^t \mathcal{I}_{3,1} + \mathcal{I}_{3,3}$ can be controlled directly by $\mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}$. For $\mathcal{I}_{3,2}$, one requires the wave energy (2.6) to control $||(\mathfrak{R}_\kappa)^\ell D^3 \partial_t q||_0$ when D^3 contains at least one ∂_t, and (2.3) to control this term when $D^3 = \overline{\partial}^3$ (that is, $\mathfrak{R}_\kappa \overline{\partial}^3 \partial_t q \sim \overline{\partial}^3 \partial_t R$), and so $\int_0^t \mathcal{I}_{3,2}$ can be controlled appropriately by $\mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}$.[7] Furthermore, the (time

[7] This is explained in the remark after Theorem 2.6.

integrated) error terms are of the form

$$\begin{aligned}&\sum_{\substack{j_1+j_2+j_3=3\\ j_1+j_2\geq 1}}\int_0^t\int_\Omega (\mathfrak{R}_\kappa)^{2\ell}(\partial D^{j_1}\eta)(\partial D^{j_2}v)(\partial D^{j_3}v)(D^4 q)\\ &\quad=\sum_{\substack{j_1+j_2+j_3=3\\ j_1+j_2\geq 1}}\int_0^t\int_\Omega \Big((\mathfrak{R}_\kappa)^{\ell-\frac12}(\partial D^{j_1}\eta)(\partial D^{j_2}v)(\partial D^{j_3}v)\Big)\Big((\mathfrak{R}_\kappa)^{\ell+\frac12}D^4 q\Big)\\ &\quad\leq \mathcal{P}_0+\mathcal{P}\int_0^t \mathcal{P}.\end{aligned}$$

3.3.4. Control of $\int_0^t \mathcal{I}_4$ $\mathcal{I}_4$ is the easiest one to control among the other $\mathcal{I}$ terms. This is due to the assumption (1.3), which implies that there are "sufficient" $\mathfrak{R}_\kappa$-weights that can be distributed for all terms. In addition to this, we can also use the fact $DR = R'Dq$ to get an extra $\mathfrak{R}_\kappa$-weights if necessary.

For non-$\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$: By (1.3) and since $r\leq 3$, invoking Theorem 3.13, we have

$$\int_0^t \mathcal{I}_4 \stackrel{L}{=} \sum_{\substack{j_1+j_2=r\\ j_1\geq 1}}\int_0^t\int_\Omega \mathfrak{R}_\kappa\Big(D^{j_1}(\rho_0 R^{-2})\Big)(D^{j_2}\partial_t q)(D^r q)\leq \mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.$$

For $\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$: For $\ell=\frac12, 1, \frac32, 2$, we have

$$\begin{aligned}\int_0^t \mathcal{I}_4 &\stackrel{L}{=} \sum_{\substack{j_1+j_2=r\\ j_1\geq 1}}\int_0^t\int_\Omega (\mathfrak{R}_\kappa)^{2\ell+1}\Big(D^{j_1}(\rho_0 R^{-2})\Big)(D^{j_2}\partial_t q)(D^r q)\\ &=\sum_{\substack{j_1+j_2=r\\ j_1\geq 1}}\int_0^t\int_\Omega \Big((\mathfrak{R}_\kappa)^{\ell+\frac12}\Big(D^{j_1}(\rho_0R^{-2})\Big)(D^{j_2}\partial_t q)\Big)\Big((\mathfrak{R}_\kappa)^{\ell+\frac12}D^r q\Big)\\ &\leq \mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P},\end{aligned}$$

where the fact $DR = R'Dq$ is used if $j_1 = 1$.

3.4. Control of $\int_0^t \mathcal{B}$ for non-$\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$

This section is devoted to control the boundary terms

$$\begin{aligned}\mathcal{B}_1 &= \sigma\int_\Gamma (\mathfrak{D}^r v_\alpha)\big([\mathfrak{D}^r, \sqrt{g}g^{ij}\Pi^\alpha_\mu]\overline{\partial}^2_{ij}\eta^\mu\big)\,dS,\\ \mathcal{B}_2 &= -\sigma\int_\Gamma \overline{\partial}_i(\sqrt{g}g^{ij}\Pi^\alpha_\mu)(\partial_t\mathfrak{D}^r\eta_\alpha)(\overline{\partial}_j\mathfrak{D}^r\eta^\mu)\,dS,\\ \mathcal{B}_3 &= \frac12\sigma\int_\Gamma \partial_t(\sqrt{g}g^{ij}\Pi^\alpha_\mu)(\overline{\partial}_i\mathcal{D}^r\eta_\alpha)(\overline{\partial}_j\mathcal{D}^r\eta^\mu)\,dS,\end{aligned}$$

which appears in the energy estimate when $\mathfrak{D}^r$ is non-$\mathfrak{R}_\kappa$-weighted. The $\mathfrak{R}_\kappa$-weighted cases are treated in section 3.5.

We recall that if $\mathfrak{D}^r$ is non-$\mathfrak{R}_\kappa$-weighted, then $r \leq 3$, that is, the corresponding term is of lower order. Because of this, it would be suffice to consider the case when $\mathfrak{D}^r = \overline{\partial}^2\partial_t$. Now, since $\Pi^\alpha_\mu = \hat{n}_\mu\hat{n}^\alpha$, we have

$$\mathcal{B}_1 = \sigma \sum\nolimits_{\substack{j_1+j_2=3\\ j_1\geq 1}} \int_\Gamma (\overline{\partial}^2\partial_t v_\alpha) D^{j_1}(\sqrt{g}g^{ij}\hat{n}_\mu\hat{n}^\alpha)(D^{j_2}\overline{\partial}^2_{ij}\eta^\mu)\,\mathrm{d}S,$$
$$\mathcal{B}_2 = \sigma \int_\Gamma \overline{\partial}_i(\sqrt{g}g^{ij}\Pi^\alpha_\mu)(\partial_t\overline{\partial}^2 v_\alpha)(\overline{\partial}_j\overline{\partial}^2 v^\mu)\,\mathrm{d}S,$$
$$\mathcal{B}_3 = \frac{1}{2}\sigma \int_\Gamma \partial_t(\sqrt{g}g^{ij}\Pi^\alpha_\mu)(\overline{\partial}_i\overline{\partial}^2 v_\alpha)(\overline{\partial}_j\overline{\partial}^2 v^\mu)\,\mathrm{d}S$$

Invoking Lemma 2.2, we get

$$\partial_t(\sqrt{g}g^{ij}\hat{n}_\mu\hat{n}^\alpha) = Q(\overline{\partial}\eta)\overline{\partial}v, \quad \overline{\partial}(\sqrt{g}g^{ij}\hat{n}_\mu\hat{n}^\alpha) = Q(\overline{\partial}\eta)\overline{\partial}^2\eta,$$

where Q is a rational function, and hence

$$\overline{\partial}\partial_t(\sqrt{g}g^{ij}\hat{n}_\mu\hat{n}^\alpha) = Q(\overline{\partial}\eta, \overline{\partial}v)\overline{\partial}^2\eta + Q(\overline{\partial}\eta, \overline{\partial}v)\overline{\partial}^2 v,$$
$$\overline{\partial}^2(\sqrt{g}g^{ij}\hat{n}_\mu\hat{n}^\alpha) = Q(\overline{\partial}\eta, \overline{\partial}^2\eta)\overline{\partial}^3\eta,$$
$$\overline{\partial}^2\partial_t(\sqrt{g}g^{ij}\hat{n}_\mu\hat{n}^\alpha) = Q(\overline{\partial}\eta, \overline{\partial}v, \overline{\partial}^2\eta, \overline{\partial}^2 v)(\overline{\partial}^3\eta + \overline{\partial}^3 v).$$

In light of these, we have

$$\int_0^t \mathcal{B}_2 = \sigma \int_0^t \int_\Gamma Q(\overline{\partial}\eta)\overline{\partial}^2\eta(\partial^2\partial_t v)(\partial^3 v)\,\mathrm{d}S \leq \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},$$

via $(H^{-\frac{1}{2}}, H^{\frac{1}{2}})$ duality. Moreover, $\int_0^t \mathcal{B}_3$ is treated similarly. On the other hand,

$$\mathcal{B}_1 \stackrel{L}{=} \sigma \int_\Gamma (\overline{\partial}^2\partial_t v)Q(\overline{\partial}\eta)(\overline{\partial}v)(\overline{\partial}^4\eta) + \sigma \int_\Gamma (\overline{\partial}^2\partial_t v)Q(\overline{\partial}\eta, \overline{\partial}v, \overline{\partial}^2\eta, \overline{\partial}^2 v)(\overline{\partial}^3\eta + \overline{\partial}^3 v)(\overline{\partial}^2\eta)$$
$$+ \sigma \int_\Gamma (\overline{\partial}^2\partial_t v)Q(\overline{\partial}\eta, \overline{\partial}^2\eta)(\overline{\partial}^3\eta)(\overline{\partial}^2 v) + \sigma \int_\Gamma (\overline{\partial}^2\partial_t v)Q(\overline{\partial}\eta, \overline{\partial}v)(\overline{\partial}^2 v + \overline{\partial}^2\eta)(\overline{\partial}^3\eta).$$

The last three terms can be controlled in a routine fashion.

However, $\sigma \int_\Gamma (\overline{\partial}^2\partial_t v)Q(\overline{\partial}\eta)(\overline{\partial}v)(\overline{\partial}^4\eta)$ cannot be controlled directly since $(H^{-\frac{1}{2}}, H^{\frac{1}{2}})$ duality requires the control $||v_t||_3$ which is not part of $\mathcal{P}$, and so we consider

$$\sigma \int_0^t \int_\Gamma (\overline{\partial}^2\partial_t v)Q(\overline{\partial}\eta)(\overline{\partial}v)(\overline{\partial}^4\eta),$$

and then integrate by parts in t. This yields

$$\sigma \int_0^t \int_\Gamma (\overline{\partial}^2\partial_t v)Q(\overline{\partial}\eta)(\overline{\partial}v)(\overline{\partial}^4\eta) = \sigma \int_\Gamma (\overline{\partial}^2 v)Q(\overline{\partial}\eta)(\overline{\partial}v)(\overline{\partial}^4\eta)|_0^t 1$$
$$- \sigma \int_0^t \int_\Gamma (\overline{\partial}^2 v)Q(\overline{\partial}\eta, \overline{\partial}v)(\overline{\partial}^4\eta) - \sigma \int_0^t \int_\Gamma (\overline{\partial}^2 v)Q(\overline{\partial}\eta)(\overline{\partial}v_t)(\overline{\partial}^4\eta)$$
$$- \sigma \int_0^t \int_\Gamma (\overline{\partial}^2 v)Q(\overline{\partial}\eta)(\overline{\partial}v)(\overline{\partial}^4 v).$$

The last three term on the right hand side can be controlled directly by $\mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}$ via $(H^{-\frac{1}{2}}, H^{\frac{1}{2}})$ duality. Moreover, the pointwise term is bounded by

$$\begin{aligned}
&\mathcal{P}_0 + \sigma Q(||\eta||_4)||(\overline{\partial}^2 v)(\overline{\partial} v)||_1 \\
&\lesssim \mathcal{P}_0 + \sigma Q(||\eta||_4)(||v||_3^{\frac{1}{2}}||v||_4^{\frac{1}{2}}||v||_1^{\frac{1}{2}}||v||_2^{\frac{1}{2}} + ||v||_2||v||_3) \\
&\lesssim \epsilon\mathcal{N} + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.
\end{aligned}$$

3.5. Control of $\mathcal{B}$ for $\mathfrak{R}_\kappa$-weighted $\mathfrak{D}^r$

Here we show how to control $\mathcal{B}$ when $\mathfrak{D}^2 = \sqrt{\mathfrak{R}_\kappa}\partial_t^2$, $\mathfrak{D}^3 = \sqrt{\mathfrak{R}_\kappa}(\overline{\partial}\partial_t^2)$, $\mathfrak{D}^3 = \mathfrak{R}_\kappa\partial_t^3$, $\mathfrak{D}^4 = \mathfrak{R}_\kappa(\overline{\partial}^3\partial_t)$, $\mathfrak{D}^4 = \mathfrak{R}_\kappa(\overline{\partial}^2\partial_t^2)$, $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^{\frac{3}{2}}(\overline{\partial}\partial_t^3)$ and $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^2\partial_t^4$.

3.5.1. Case $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^2\partial_t^4$ The ensuing calculations produce a series of terms. In what follows we focus on the most delicate ones, in particular those leading to special cancellations. The remaining terms will either be of lower order or can be controlled by arguments similar to the ones presented for the aforementioned main terms. Therefore, all such remainders are collected and estimated at the very end in section 3.5.1.4. We note that certain cancellations are only visible after a series of manipulations have been made, requiring us to keep track of the explicit form of most terms in our calculations.

The following remark will be used throughout below. In view of identity Lemma 2.2–6, we have $\hat{n}^\alpha\overline{\partial}^m\partial_t^k v_\alpha = \hat{n}_\tau\Pi^{\tau\alpha}\overline{\partial}^m\partial_t^k v_\alpha$, so that an estimate for $\hat{n}\cdot\overline{\partial}^m\partial_t^k v$ can controlled by $\Pi\overline{\partial}^m\partial_t^k v$.

We shall also need the following identity

$$\partial_t v^\alpha\overline{\partial}_l\eta_\alpha = -\frac{J}{\rho_0}\overline{\partial}_l q, \quad \text{on } \Gamma, \tag{3.17}$$

which is obtained upon contracting the first equation in (1.1) with $\overline{\partial}_l\eta_\alpha$, using the definition of a, and (1.1).

3.5.1.1 Estimate for $\int_0^t \mathcal{B}_3$ with $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^2\partial_t^4$

Using $\mathfrak{D}^4 = \mathfrak{R}_\kappa^2\partial_t^4$ in $\mathcal{B}_3$ gives

$$\begin{aligned}
\mathcal{B}_3 &= \frac{1}{2}\sigma\int_\Gamma \partial_t(\sqrt{g}g^{ij}\Pi_\mu^\alpha)\overline{\partial}_i(\mathfrak{R}_\kappa^2\partial_t^3 v_\alpha)\overline{\partial}_j(\mathfrak{R}_\kappa^2\partial_t^3 v^\mu)\,\mathrm{d}S \\
&= \frac{1}{2}\sigma\int_\Gamma \partial_t(\sqrt{g}g^{ij})\mathfrak{R}_\kappa^4\Pi_\mu^\alpha\overline{\partial}_i\partial_t^3 v_\alpha\overline{\partial}_j\partial_t^3 v^\mu\,\mathrm{d}S \\
&\quad + \sigma\int_\Gamma \sqrt{g}g^{ij}\mathfrak{R}_\kappa^4\partial_t\Pi_\lambda^\alpha\Pi_\mu^\lambda\overline{\partial}_i\partial_t^3 v_\alpha\overline{\partial}_j\partial_t^3 v^\mu\,\mathrm{d}S \\
&= \mathcal{B}_{31} + \mathcal{B}_{32},
\end{aligned}$$

where we used Lemma 2.2–5. We can immediately estimate

$$||\mathcal{B}_{31}|| \leq \mathcal{P}||\Pi\mathfrak{R}_\kappa^2\overline{\partial}\partial_t^3 v||_{0,\Gamma}.$$

For $\mathcal{B}_{32}$, use Lemma 2.2–4 to find

$$\begin{aligned}\mathcal{B}_{32} &= \sigma \int_\Gamma \sqrt{g} g^{ij} \mathfrak{R}_\kappa^4 \partial_t \hat{n}^\alpha \hat{n}_\lambda \Pi_\mu^\lambda \overline{\partial}_i \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &\quad + \sigma \int_\Gamma \sqrt{g} g^{ij} \mathfrak{R}_\kappa^4 \hat{n}^\alpha \partial_t \hat{n}_\lambda \Pi_\mu^\lambda \overline{\partial}_i \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &= \mathcal{B}_{321} + \mathcal{B}_{322}.\end{aligned}$$

We have

$$||\mathcal{B}_{322}|| \leq \mathcal{P}||\Pi\mathfrak{R}_\kappa^2\overline{\partial}\partial_t^3 v||_{0,\Gamma}.$$

Using Lemma 2.2–8 we can write

$$\mathcal{B}_{321} = -\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \hat{n}_\lambda \hat{n}_\tau \overline{\partial}_k v^\tau \overline{\partial}_l \eta^\alpha \Pi_\mu^\lambda \overline{\partial}_i \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S.$$

From (3.5.1) we have

$$\overline{\partial}_l \eta^\alpha \overline{\partial}_i \partial_t^3 v_\alpha = -\frac{J}{\rho_0} \overline{\partial}_i \overline{\partial}_l \partial_t^2 q + [\overline{\partial}_i \partial_t^2, -\frac{J}{\rho_0} \overline{\partial}_l] q - [\overline{\partial}_i \partial_t^2, \overline{\partial}_l \eta_\alpha \partial_t] v^\alpha.$$

Thus,

$$\begin{aligned}\mathcal{B}_{321} &= \sigma \int_\Gamma \frac{J}{\rho_0} \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \hat{n}_\lambda \hat{n}_\tau \overline{\partial}_k v^\tau \overline{\partial}_i \overline{\partial}_l \partial_t^2 q \Pi_\mu^\lambda \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &\quad - \sigma \int_\Gamma \frac{1}{\rho_0} \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \hat{n}_\lambda \hat{n}_\tau \overline{\partial}_k v^\tau ([\overline{\partial}_i \partial_t^2, -\frac{J}{\rho_0} \overline{\partial}_l] q) \Pi_\mu^\lambda \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &\quad + \sigma \int_\Gamma \frac{1}{\rho_0} \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \hat{n}_\lambda \hat{n}_\tau \overline{\partial}_k v^\tau ([\overline{\partial}_i \partial_t^2, \overline{\partial}_l \eta_\alpha \partial_t] v^\alpha) \Pi_\mu^\lambda \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &= \mathcal{B}_{3211} + \mathcal{B}_{3212} + \mathcal{B}_{3213}.\end{aligned}$$

Integrating $\overline{\partial}_i$ by parts in $\mathcal{B}_{3211}$,

$$\begin{aligned}\mathcal{B}_{3211} &= -\sigma \int_\Gamma \frac{J}{\rho_0} \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \hat{n}_\lambda \hat{n}_\tau \overline{\partial}_k v^\tau \overline{\partial}_l \partial_t^2 q \Pi_\mu^\lambda \overline{\partial}_i \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &\quad - \sigma \int_\Gamma \overline{\partial}_i (\frac{J}{\rho_0} \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \hat{n}_\lambda \hat{n}_\tau \overline{\partial}_k v^\tau \overline{\partial}_l \partial_t^2 q \Pi_\mu^\lambda) \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S.\end{aligned}$$

From section 2.1, item 3, we have

$$\overline{\partial}_l \partial_t^2 q = -\sigma g^{mn} \hat{n}_\beta \overline{\partial}_m \overline{\partial}_n \overline{\partial}_l \partial_t v^\beta - [\overline{\partial}_l \partial_t^2, \sigma g^{mn} \hat{n}_\beta \overline{\partial}_m \overline{\partial}_n] \eta^\beta,$$

so that

$$\mathcal{B}_{3211} = \sigma^2 \int_\Gamma \frac{J}{\rho_0} \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \hat{n}_\lambda \hat{n}_\tau \overline{\partial}_k v^\tau g^{mn} \hat{n}_\beta \overline{\partial}_m \overline{\partial}_n \overline{\partial}_l \partial_t v^\beta \Pi_\mu^\lambda \overline{\partial}_i \overline{\partial}_j \partial_t^3 v^\mu$$

$$
\begin{aligned}
&+\sigma\int_{\Gamma}\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{ij}g^{kl}\hat{n}_\lambda\hat{n}_\tau\overline{\partial}_k v^\tau[\overline{\partial}_l\partial_t^2,\sigma g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n]\eta^\beta\Pi_\mu^\lambda\overline{\partial}_i\overline{\partial}_j\partial_t^3 v^\mu\,\mathrm{d}S\\
&-\sigma\int_{\Gamma}\overline{\partial}_i(\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{ij}g^{kl}\hat{n}_\lambda\hat{n}_\tau\overline{\partial}_k v^\tau\overline{\partial}_l\partial_t^2 q\Pi_\mu^\lambda)\overline{\partial}_j\partial_t^3 v^\mu\,\mathrm{d}S\\
&=\mathcal{B}_{32111}+\mathcal{B}_{32112}+\mathcal{B}_{32113}.
\end{aligned}
$$

In $\mathcal{B}_{32111}$, we use Lemma 2.2–6 and factor a ∂_t from ∂_t^3 to obtain

$$
\begin{aligned}
\mathcal{B}_{32111}&=\sigma^2\partial_t\int_{\Gamma}\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{ij}g^{kl}\hat{n}_\tau\overline{\partial}_k v^\tau g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n\overline{\partial}_l\partial_t v^\beta\hat{n}_\mu\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu\\
&\quad-\sigma^2\int_{\Gamma}\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{ij}g^{kl}\hat{n}_\tau\overline{\partial}_k v^\tau g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n\overline{\partial}_l\partial_t^2 v^\beta\hat{n}_\mu\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu\\
&\quad-\sigma^2\int_{\Gamma}\partial_t(\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{ij}g^{kl}\hat{n}_\tau\overline{\partial}_k v^\tau g^{mn}\hat{n}_\beta\hat{n}_\mu)\overline{\partial}_m\overline{\partial}_n\overline{\partial}_l\partial_t v^\beta\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu\\
&=\mathcal{B}_{321111}+\mathcal{B}_{321112}+\mathcal{B}_{321113}.
\end{aligned}
$$

For the first term, that is, $\mathcal{B}_{321111}$, we have

$$
\int_0^t\mathcal{B}_{321111}\leq\mathcal{P}_0+\mathfrak{R}_\kappa^{\frac{1}{2}}\mathcal{P}(\mathfrak{R}_\kappa||\Pi\mathfrak{R}_\kappa\overline{\partial}^3\partial_t v||_{0,\Gamma})(||\Pi\mathfrak{R}_\kappa^{\frac{3}{2}}\overline{\partial}^2\partial_t^2 v||_{0,\Gamma}).
$$

Using Young's inequality and the fact that $\mathfrak{R}_\kappa$ can be made very small for large κ, we can bound the right-hand side by $\mathcal{P}_0+\epsilon P(\mathcal{N})+\epsilon\mathcal{N}$.

For $\mathcal{B}_{321112}$, write

$$
\begin{aligned}
&g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n\overline{\partial}_l\partial_t^2 v^\beta\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu\\
&\quad=\overline{\partial}_l(g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n\partial_t^2 v^\beta)\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu\\
&\qquad-[\overline{\partial}_l,g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n\partial_t^2]v^\beta\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu\\
&\quad=\frac{1}{2}\overline{\partial}_l(\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu)^2\\
&\qquad-[\overline{\partial}_l,g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n\partial_t^2]v^\beta\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu,
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathcal{B}_{321112}&=-\frac{1}{2}\sigma^2\int_{\Gamma}\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{kl}\hat{n}_\tau\overline{\partial}_k v^\tau\overline{\partial}_l(\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu)^2\\
&\quad+\sigma^2\int_{\Gamma}\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{kl}\hat{n}_\tau\overline{\partial}_k v^\tau[\overline{\partial}_l,g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n\partial_t^2]v^\beta\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu.
\end{aligned}
$$

Integarting $\overline{\partial}_l$ by parts in the first integral,

$$
\begin{aligned}
\mathcal{B}_{321112}&=\frac{1}{2}\sigma^2\int_{\Gamma}\overline{\partial}_l(\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{kl}\hat{n}_\tau\overline{\partial}_k v^\tau)(\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu)^2\\
&\quad+\sigma^2\int_{\Gamma}\frac{J}{\rho_0}\mathfrak{R}_\kappa^4\sqrt{g}g^{kl}\hat{n}_\tau\overline{\partial}_k v^\tau[\overline{\partial}_l,g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n\partial_t^2]v^\beta\hat{n}_\mu g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^2 v^\mu\\
&=\mathcal{B}_{3211121}+\mathcal{B}_{3211122}.
\end{aligned}
$$

Writing

$$\mathcal{B}_{3211121} = \mathfrak{R}_\kappa \frac{1}{2}\sigma^2 \int_\Gamma \overline{\partial}_l \left(\frac{J}{\rho_0} \sqrt{g} g^{kl} \hat{n}_\tau \overline{\partial}_k v^\tau \right) \left(\hat{n}_\mu g^{ij} \mathfrak{R}_\kappa^{\frac{3}{2}} \overline{\partial}_i \overline{\partial}_j \partial_t^2 v^\mu \right)^2,$$

we have

$$\mathcal{B}_{3211121} \leq \epsilon P(\mathcal{N}).$$

This concludes the estimate for the most delicate terms in $\int_0^t \mathcal{B}_3$. The remaining terms in $\mathcal{B}_3$, that is, $\mathcal{B}_{3211122}$, $\mathcal{B}_{321113}$, $\mathcal{B}_{32113}$, $\mathcal{B}_{32112}$, $\mathcal{B}_{3212}$, and $\mathcal{B}_{3213}$, are treated in section 3.5.1.4 below.

3.5.1.2 Estimate for $\int_0^t \mathcal{B}_2$ with $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^2 \partial_t^4$ We now move to estimate $\mathcal{B}_2$:

$$\begin{aligned}
\mathcal{B}_2 &= -\sigma \int_\Gamma \overline{\partial}_i (\sqrt{g} g^{ij} \Pi_\mu^\alpha)(\mathfrak{R}_\kappa^2 \partial_t^4 v_\alpha)(\mathfrak{R}_\kappa^2 \overline{\partial}_j \partial_t^3 v^\mu)\, \mathrm{d}S \\
&= -\sigma \int_\Gamma \overline{\partial}_i (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \mathfrak{R}_\kappa^2 \partial_t^4 v_\alpha \mathfrak{R}_\kappa^2 \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\
&\quad - \sigma \int_\Gamma \sqrt{g} g^{ij} \overline{\partial}_i \Pi_\mu^\alpha \mathfrak{R}_\kappa^2 \partial_t^4 v_\alpha \mathfrak{R}_\kappa^2 \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\
&= \mathcal{B}_{21} + \mathcal{B}_{22}.
\end{aligned} \tag{3.18}$$

We show below that $\mathcal{B}_{21}$ exactly cancels with a term coming from $\mathcal{B}_1$. Here we move to estimate $\mathcal{B}_{22}$. Using Lemma 2.2–4,

$$\begin{aligned}
\mathcal{B}_{22} &= -\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \overline{\partial}_i \hat{n}_\mu \hat{n}^\alpha \partial_t^4 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\
&\quad - \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha \partial_t^4 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\
&= \mathcal{B}_{221} + \mathcal{B}_{222}.
\end{aligned}$$

We use Lemma 2.2–9 to write

$$\mathcal{B}_{221} = \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \overline{\partial}_i \overline{\partial}_k \eta^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu \hat{n}^\alpha \partial_t^4 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S.$$

From (3.5.1) we have

$$\overline{\partial}_l \eta_\mu \overline{\partial}_j \partial_t^3 v^\mu = -\frac{J}{\rho_0} \overline{\partial}_j \overline{\partial}_l \partial_t^2 q + [\overline{\partial}_j \partial_t^2, -\frac{J}{\rho_0} \overline{\partial}_l] q - [\overline{\partial}_j \partial_t^2, \overline{\partial}_l \eta_\mu \partial_t] v^\mu,$$

whence

$$\begin{aligned}
\mathcal{B}_{221} &= -\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_i \overline{\partial}_k \eta^\tau \hat{n}_\tau \hat{n}^\alpha \partial_t^4 v_\alpha \overline{\partial}_j \overline{\partial}_l \partial_t^2 q \, \mathrm{d}S \\
&\quad + \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \overline{\partial}_i \overline{\partial}_k \eta^\tau \hat{n}_\tau \hat{n}^\alpha \partial_t^4 v_\alpha [\overline{\partial}_j \partial_t^2, -\frac{J}{\rho_0} \overline{\partial}_l] q \, \mathrm{d}S \\
&\quad - \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} g^{kl} \overline{\partial}_i \overline{\partial}_k \eta^\tau \hat{n}_\tau \hat{n}^\alpha \partial_t^4 v_\alpha [\overline{\partial}_j \partial_t^2, \overline{\partial}_l \eta_\mu \partial_t] v^\mu \, \mathrm{d}S
\end{aligned}$$

$$= \mathcal{B}_{2211} + \mathcal{B}_{2212} + \mathcal{B}_{2213}.$$

In $\mathcal{B}_{2211}$, we factor a ∂_t in $\partial_t^4 v_\alpha$ to obtain

$$\begin{aligned}\mathcal{B}_{2211} &= -\sigma\,\partial_t \int_\Gamma \mathfrak{R}_\kappa^4 \frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha \partial_t^3 v_\alpha \overline{\partial}_j\overline{\partial}_l\partial_t^2 q\,\mathrm{d}S \\ &\quad + \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t(\frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha\overline{\partial}_j\overline{\partial}_l\partial_t^2 q)\partial_t^3 v_\alpha\,\mathrm{d}S \\ &= \mathcal{B}_{22111} + \mathcal{B}_{22112}.\end{aligned}$$

For $\mathcal{B}_{22111}$, we integrate $\overline{\partial}_j$ by parts to produce

$$\int_0^t \mathcal{B}_{22111} \le \mathcal{P}(||\mathfrak{R}_\kappa^2\Pi\overline{\partial}\partial_t^3 v||_{0,\Gamma})(||\mathfrak{R}_\kappa^2\overline{\partial}\partial_t^2 q||_{0,\Gamma}) + \int_0^t \mathcal{P},$$

where

$$\begin{aligned}||\mathfrak{R}_\kappa^2\Pi\overline{\partial}\partial_t^3 v||_{0,\Gamma}||\mathfrak{R}_\kappa^2\overline{\partial}\partial_t^2 q||_{0,\Gamma} &\lesssim \tilde{\epsilon}||\mathfrak{R}_\kappa^2\Pi\overline{\partial}\partial_t^3 v||_{0,\Gamma} + \mathfrak{R}_\kappa^{\frac{1}{2}}\tilde{\epsilon}^{-1}||\mathfrak{R}_\kappa^{\frac{3}{2}}\overline{\partial}\partial_t^2 q||_1 \\ &\lesssim \epsilon(||\mathfrak{R}_\kappa^2\Pi\overline{\partial}\partial_t^3 v||_{0,\Gamma} + ||\mathfrak{R}_\kappa^{\frac{3}{2}}\overline{\partial}\partial_t^2 q||_1) \lesssim \epsilon P(\mathcal{N}),\end{aligned}$$

after choosing $\mathfrak{R}_\kappa$ sufficiently small and replacing q by R.

For $\mathcal{B}_{22112}$, write

$$\begin{aligned}\mathcal{B}_{22112} &= \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t(\frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha)\overline{\partial}_j\overline{\partial}_l\partial_t^2 q\,\partial_t^3 v_\alpha\,\mathrm{d}S \\ &\quad + \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha\overline{\partial}_j\overline{\partial}_l\partial_t^3 q\,\partial_t^3 v_\alpha\,\mathrm{d}S \\ &= \mathcal{B}_{221121} + \mathcal{B}_{221122}.\end{aligned}$$

The term $\mathcal{B}_{221121}$ can be handled with integration by parts with respect to $\overline{\partial}_j$ (it yields a term in $||\Pi\mathfrak{R}_\kappa^2\overline{\partial}\partial_t^3 v||_{0,\Gamma}$). For $\mathcal{B}_{221122}$, we use section 2.1, item 3, to write

$$\begin{aligned}\mathcal{B}_{221122} &= -\sigma^2 \int_\Gamma \mathfrak{R}_\kappa^4 \frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha\overline{\partial}_j\overline{\partial}_l(g^{mn}\hat{n}_\beta\partial_m\partial_n\partial_t^2 v^\beta)\partial_t^3 v_\alpha\,\mathrm{d}S \\ &\quad - \sigma^2 \int_\Gamma \mathfrak{R}_\kappa^4 \frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha(\overline{\partial}_j\overline{\partial}_l[\partial_t^3, g^{mn}\hat{n}_\beta\overline{\partial}_m\overline{\partial}_n]\eta^\beta)\partial_t^3 v_\alpha \\ &= \mathcal{B}_{2211221} + \mathcal{B}_{2211222}.\end{aligned}$$

Integrating by parts $\overline{\partial}_l$ in $\mathcal{B}_{2211221}$,

$$\begin{aligned}\mathcal{B}_{2211221} &= \sigma^2 \int_\Gamma \mathfrak{R}_\kappa^4 \frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha\overline{\partial}_j(g^{mn}\hat{n}_\beta\partial_m\partial_n\partial_t^2 v^\beta)\overline{\partial}_l\partial_t^3 v_\alpha\,\mathrm{d}S \\ &\quad + \sigma^2 \int_\Gamma \overline{\partial}_l(\mathfrak{R}_\kappa^4 \frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha)\overline{\partial}_j(g^{mn}\hat{n}_\beta\partial_m\partial_n\partial_t^2 v^\beta)\partial_t^3 v_\alpha\,\mathrm{d}S \\ &= \sigma^2 \int_\Gamma \mathfrak{R}_\kappa^4 \frac{J}{\rho_0}\sqrt{g}g^{ij}g^{kl}\overline{\partial}_i\overline{\partial}_k\eta^\tau \hat{n}_\tau\hat{n}^\alpha\overline{\partial}_j(g^{mn}\hat{n}_\beta\partial_m\partial_n\partial_t^2 v^\beta)\overline{\partial}_l\partial_t^3 v_\alpha\,\mathrm{d}S\end{aligned}$$

$$+\sigma^2 \int_\Gamma \mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_l \overline{\partial}_i \overline{\partial}_k \eta^\tau \hat{n}_\tau \hat{n}^\alpha \overline{\partial}_j (g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta) \partial_t^3 v_\alpha \, \mathrm{d}S$$
$$+\sigma^2 \int_\Gamma \overline{\partial}_l (\mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \hat{n}_\tau \hat{n}^\alpha) \overline{\partial}_i \overline{\partial}_k \eta^\tau \overline{\partial}_j (g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta) \partial_t^3 v_\alpha \, \mathrm{d}S,$$

and then integrating by parts $\overline{\partial}_i$ on the second integral,

$$\begin{aligned}
\mathcal{B}_{2211221} &= \sigma^2 \int_\Gamma \mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_i \overline{\partial}_k \eta^\tau \hat{n}_\tau \hat{n}^\alpha \overline{\partial}_j (g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta) \overline{\partial}_l \partial_t^3 v_\alpha \, \mathrm{d}S \\
&\quad - \sigma^2 \int_\Gamma \mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_l \overline{\partial}_k \eta^\tau \hat{n}_\tau \hat{n}^\alpha \overline{\partial}_i \overline{\partial}_j (g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta) \partial_t^3 v_\alpha \, \mathrm{d}S \\
&\quad - \sigma^2 \int_\Gamma \mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_l \overline{\partial}_k \eta^\tau \hat{n}_\tau \hat{n}^\alpha \overline{\partial}_j (g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta) \overline{\partial}_i \partial_t^3 v_\alpha \, \mathrm{d}S \\
&\quad - \sigma^2 \int_\Gamma \overline{\partial}_i (\mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \hat{n}_\tau \hat{n}^\alpha) \overline{\partial}_l \overline{\partial}_k \eta^\tau \overline{\partial}_j (g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta) \partial_t^3 v_\alpha \, \mathrm{d}S \\
&\quad + \sigma^2 \int_\Gamma \overline{\partial}_l (\mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \hat{n}_\tau \hat{n}^\alpha) \overline{\partial}_i \overline{\partial}_k \eta^\tau \overline{\partial}_j (g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta) \partial_t^3 v_\alpha \, \mathrm{d}S \\
&= \mathcal{B}_{22112211} + \mathcal{B}_{22112212} + \mathcal{B}_{22112213} + \mathcal{B}_{22112214} + \mathcal{B}_{22112215}.
\end{aligned} \tag{3.19}$$

Note that the first and third terms, that is, $\mathcal{B}_{22112211}$ and $\mathcal{B}_{22112213}$, cancel each other in view of the following identity, which can be verified by inspection,

$$\sum_{i,k,l=1}^{2} (g^{ij} g^{kl} - g^{ik} g^{lj}) = 0.$$

For the second term, $\mathcal{B}_{22112212}$, integrate $\overline{\partial}_j \overline{\partial}_j$ by parts:

$$\begin{aligned}
\mathcal{B}_{22112212} &= -\sigma^2 \int_\Gamma \mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_l \overline{\partial}_k \eta^\tau \hat{n}_\tau g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta \hat{n}^\alpha \overline{\partial}_i \overline{\partial}_j \partial_t^3 v_\alpha \, \mathrm{d}S \\
&\quad - \sigma^2 \int_\Gamma \overline{\partial}_j (\mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_l \overline{\partial}_k \eta^\tau \hat{n}_\tau) g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta \hat{n}^\alpha \overline{\partial}_i \partial_t^3 v_\alpha \, \mathrm{d}S \\
&\quad - \sigma^2 \int_\Gamma \overline{\partial}_i (\mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_l \overline{\partial}_k \eta^\tau \hat{n}_\tau \hat{n}^\alpha) \overline{\partial}_j (g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta) \partial_t^3 v_\alpha \, \mathrm{d}S \\
&= \mathcal{B}_{221122121} + \mathcal{B}_{221122122} + \mathcal{B}_{221122123}.
\end{aligned}$$

Factoring a ∂_t from $\overline{\partial}_i \overline{\partial}_j \partial_t^3 v_\alpha$ in $\mathcal{B}_{221122121}$, we find

$$\begin{aligned}
\mathcal{B}_{221122121} &= -\frac{1}{2}\sigma^2 \int_\Gamma \mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_l \overline{\partial}_k \eta^\tau \hat{n}_\tau g^{mn} \hat{n}_\beta \partial_m \partial_n \partial_t^2 v^\beta \hat{n}^\alpha \overline{\partial}_i \overline{\partial}_j \partial_t^2 v_\alpha \, \mathrm{d}S \\
&\quad + \frac{1}{2}\sigma^2 \int_\Gamma \partial_t (\mathfrak{R}^4_\kappa \frac{J}{\rho_0} \sqrt{g} g^{ij} g^{kl} \overline{\partial}_l \overline{\partial}_k \eta^\tau \hat{n}_\tau g^{mn} \hat{n}_\beta \hat{n}^\alpha) \partial_m \partial_n \partial_t^2 v^\beta \overline{\partial}_i \overline{\partial}_j \partial_t^2 v_\alpha \, \mathrm{d}S \\
&= \mathcal{B}_{2211221211} + \mathcal{B}_{2211221212}.
\end{aligned}$$

The first term, $\mathcal{B}_{2211221211}$, can be estimated by $\epsilon P(\mathcal{N})$. Here, the small number ϵ comes form estimating $\overline{\partial}_l \overline{\partial}_k \eta^\tau$ in L^∞ and using that $\eta(0)$ is the identity diffeomorphism.

Now we move to $\mathcal{B}_{222}$. Factoring a ∂_t from $\partial_t^4 v_\alpha$, we find

$$\begin{aligned}\mathcal{B}_{222} &= -\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha \partial_t^4 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &= -\sigma \partial_t \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &\quad + \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^4 v^\mu \, \mathrm{d}S \\ &\quad + \sigma \int_\Gamma \partial_t (\mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha) \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S.\end{aligned}$$

Integrating $\overline{\partial}_j$ by parts in the second integral,

$$\begin{aligned}\mathcal{B}_{222} &= -\sigma \partial_t \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S \\ &\quad - \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha \overline{\partial}_j \partial_t^3 v_\alpha \partial_t^4 v^\mu \, \mathrm{d}S \\ &\quad + \sigma \int_\Gamma \partial_t (\mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha) \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^3 v^\mu \, \mathrm{d}S. \\ &\quad - \sigma \int_\Gamma \overline{\partial}_j (\mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}^\alpha) \partial_t^3 v_\alpha \partial_t^4 v^\mu \, \mathrm{d}S \\ &= \mathcal{B}_{2221} + \mathcal{B}_{2222} + \mathcal{B}_{2223} + \mathcal{B}_{2224}.\end{aligned}$$

Note that $\mathcal{B}_{2222} = \mathcal{B}_{221}$, so this term is estimated as above. The term $\mathcal{B}_{2221}$ can, after time integration, be estimated using Young's inequality and interpolation.

With exception of $\mathcal{B}_{21}$, which, as said, involves a special cancellation showed below, this concludes the estimate of the most delicate terms in $\int_0^t \mathcal{B}_2$. The remaining terms $\mathcal{B}_{2212}$, $\mathcal{B}_{2213}$, $\mathcal{B}_{2211222}$, $\mathcal{B}_{22112214}$, $\mathcal{B}_{22112215}$, $\mathcal{B}_{221122122}$, $\mathcal{B}_{221122123}$, $\mathcal{B}_{2211221212}$, $\mathcal{B}_{2223}$, and $\mathcal{B}_{2224}$, are treated in section 3.5.1.4 below.

3.5.1.3 Estimate for $\int_0^t \mathcal{B}_1$ We now move to estimate $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^2 \partial_t^4$

We now move to estimate $\mathcal{B}_1$:

$$\begin{aligned}\mathcal{B}_1 &= \sigma \int_\Gamma (\mathfrak{R}_\kappa^2 \partial_t^4 v_\alpha) \big([\mathfrak{R}_\kappa^2 \partial_t^4, \sqrt{g} g^{ij} \Pi_\mu^\alpha] \overline{\partial}_{ij}^2 \eta^\mu \big) \, \mathrm{d}S \\ &= 4\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t (\sqrt{g} g^{ij} \Pi_\mu^\alpha) \overline{\partial}_i \overline{\partial}_j \partial_t^3 \eta^\mu \partial_t^4 v_\alpha \\ &\quad + 6\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t^2 (\sqrt{g} g^{ij} \Pi_\mu^\alpha) \overline{\partial}_i \overline{\partial}_j \partial_t^2 \eta^\mu \partial_t^4 v_\alpha \\ &\quad + 4\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t^3 (\sqrt{g} g^{ij} \Pi_\mu^\alpha) \overline{\partial}_i \overline{\partial}_j \partial_t \eta^\mu \partial_t^4 v_\alpha \\ &\quad + \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t^4 (\sqrt{g} g^{ij} \Pi_\mu^\alpha) \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\ &= \mathcal{B}_{11} + \mathcal{B}_{12} + \mathcal{B}_{13} + \mathcal{B}_{14}.\end{aligned}$$

We have

$$
\begin{aligned}
\mathcal{B}_{14} &= \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \partial_t^4 \Pi_\mu^\alpha \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad + 4\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t (\sqrt{g} g^{ij}) \partial_t^3 \Pi_\mu^\alpha \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad + 6\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t^2 (\sqrt{g} g^{ij}) \partial_t^2 \Pi_\mu^\alpha \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad + 4\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t^3 (\sqrt{g} g^{ij}) \partial_t \Pi_\mu^\alpha \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad + \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_t^4 (\sqrt{g} g^{ij}) \Pi_\mu^\alpha \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&= \mathcal{B}_{141} + \mathcal{B}_{142} + \mathcal{B}_{143} + \mathcal{B}_{144} + \mathcal{B}_{145}.
\end{aligned}
$$

Using Lemma 2.2–4, we have

$$
\begin{aligned}
\mathcal{B}_{141} &= \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}^\alpha \partial_t^4 \hat{n}_\mu \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad + 4\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \partial_t \hat{n}^\alpha \partial_t^3 \hat{n}_\mu \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad + 6\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \partial_t^2 \hat{n}^\alpha \partial_t^2 \hat{n}_\mu \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad + 4\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \partial_t^3 \hat{n}^\alpha \partial_t \hat{n}_\mu \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad + \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \partial_t^4 \hat{n}^\alpha \hat{n}_\mu \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&= \mathcal{B}_{1411} + \mathcal{B}_{1412} + \mathcal{B}_{1413} + \mathcal{B}_{1414} + \mathcal{B}_{1415}.
\end{aligned}
$$

From Lemma 2.2–8 we have

$$
\partial_t^4 \hat{n}_\mu = -g^{kl} \overline{\partial}_k \partial_t^3 v^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu - [\partial_t^3, g^{kl} \hat{n}_\tau \overline{\partial}_l \eta_\mu \overline{\partial}_k] v^\tau,
$$

and thus

$$
\begin{aligned}
\mathcal{B}_{1411} &= -\sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}^\alpha g^{kl} \overline{\partial}_k \partial_t^3 v^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&\quad - \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \sqrt{g} g^{ij} \hat{n}^\alpha [\partial_t^3, g^{kl} \hat{n}_\tau \overline{\partial}_l \eta_\mu \overline{\partial}_k] v^\tau \overline{\partial}_i \overline{\partial}_j \eta^\mu \partial_t^4 v_\alpha \\
&= \mathcal{B}_{14111} + \mathcal{B}_{14112}.
\end{aligned}
$$

We now invoke Lemma 2.2–10, to replace $\sqrt{g} g^{ij} g^{kl} \overline{\partial}_i \overline{\partial}_j \eta^\mu \overline{\partial}_l \eta_\mu$ in $\mathcal{B}_{14111}$ by $-\overline{\partial}_i (\sqrt{g} g^{ik})$, obtaining

$$
\mathcal{B}_{14111} = \sigma \int_\Gamma \mathfrak{R}_\kappa^4 \partial_i (\sqrt{g} g^{ik}) \overline{\partial}_k \partial_t^3 v^\tau \hat{n}_\tau \hat{n}^\alpha \partial_t^4 v_\alpha. \tag{3.20}
$$

We see that this term exactly cancels $\mathcal{B}_{21}$, as mentioned earlier. The other terms in the estimate of $\int_0^t \mathcal{B}_1$ are treated in section 3.5.1.4.

3.5.1.4 Remainders in $\int_0^t \mathcal{B}$ with $\mathfrak{D}^4 = \mathfrak{R}_\kappa^2 \partial_t^4$

Above we have showed how to control the most delicate terms in the estimate for $\int_0^t \mathcal{B}$ when $\mathfrak{D}^4 = \mathfrak{R}_\kappa^2 \partial_t^4$. In particular, we have showed how some top order terms, which seemingly cannot be individually bounded, cancel out when taken together. Now we consider the remaining terms, which we list here for the reader's convenience. They are, for $\mathcal{B}_3$,

$$\mathcal{B}_{3211122},\ \mathcal{B}_{321113},\ \mathcal{B}_{32113},\ \mathcal{B}_{32112},\ \mathcal{B}_{3212},\ \text{and}\ \mathcal{B}_{3213}$$

from section 3.5.1.3; for $\mathcal{B}_2$

$$\mathcal{B}_{2212},\ \mathcal{B}_{2213},\ \mathcal{B}_{2211222},\ \mathcal{B}_{22112214},\ \mathcal{B}_{22112215},$$
$$\mathcal{B}_{221122122},\ \mathcal{B}_{221122123},\ \mathcal{B}_{2211221212},\ \mathcal{B}_{2223},\ \text{and}\ \mathcal{B}_{2224}$$

from section 3.5.1.2; for $\mathcal{B}_1$

$$\mathcal{B}_{11},\ \mathcal{B}_{12},\ \mathcal{B}_{13},\ \mathcal{B}_{142},\ \mathcal{B}_{143},\ \mathcal{B}_{144},\ \mathcal{B}_{145},\ \mathcal{B}_{1412},\ \mathcal{B}_{1413},\ \mathcal{B}_{1414},\ \mathcal{B}_{1415},\ \text{and}\ \mathcal{B}_{14112} \tag{3.21}$$

from section 3.5.1.3. Not all these terms are immediately of lower order, but they can be estimated using the same kind of ideas that have already been employed. Therefore, it suffices to briefly indicate how this is done.

The terms $\mathcal{B}_{3212}$, $\mathcal{B}_{3213}$, $\mathcal{B}_{32112}$, $\mathcal{B}_{321113}$, and $\mathcal{B}_{3211122}$ can be bounded directly. The term $\mathcal{B}_{32113}$ is bounded upon replacing q by R and estimating in routine fashion.

The terms $\mathcal{B}_{2212}$ and $\mathcal{B}_{2213}$ can be estimated with integration by parts in time. The terms $\mathcal{B}_{2211222}$, $\mathcal{B}_{22112214}$, $\mathcal{B}_{22112215}$, $\mathcal{B}_{221122122}$, $\mathcal{B}_{2211221212}$ $\mathcal{B}_{2223}$, and $\mathcal{B}_{2224}$ can be estimated directly. The term $\mathcal{B}_{221122123}$ requires integration by parts in space and then using arguments similar to above, with one extra step: after integrating $\overline{\partial}_j$ by parts, we obtain a term with four derivatives of η. This term, however, has the form $\hat{n}_\tau g^{ij} \overline{\partial}^2 \overline{\partial}_i \overline{\partial}_j \eta^\tau$, which allows us to use section 2.1, item 3, to eliminate two derivatives of η. (Alternatively, we can use elliptic estimates for equations with Sobolev coefficients, as, for example, Theorem 4 and Remark 2 in [51]).

The terms listed in (3.5.1.4) are again handled by a repetition of ideas used above (without requiring special cancellations). In particular, Lemma 2.2–8 is used heavily and Lemma 2.2–11 is employed to estimate $\mathcal{B}_{145}$.

Combining these observations with the estimates of section 3.5.1.1, 3.5.1.2, and 3.5.1.1, we finally obtain

$$\int_0^t (\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3) \leq \mathcal{P}_0 + \epsilon \mathcal{N} + \epsilon P(\mathcal{N}) + \mathcal{P} \int_0^t \mathcal{P},\ \text{when}\ \mathfrak{D}^4 = \mathfrak{R}_\kappa^2 \partial_t^4.$$

3.5.2. Estimate of the Remaining Weighed Boundary Terms It remains to carry out control of $\int_0^t (\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3)$ when $\mathfrak{D}^2 = \sqrt{\mathfrak{R}_\kappa}\partial_t^2$, $\mathfrak{D}^3 = \sqrt{\mathfrak{R}_\kappa}(\overline{\partial}\partial_t^2)$, $\mathfrak{D}^3 = \mathfrak{R}_\kappa \partial_t^3$, $\mathfrak{D}^4 = \mathfrak{R}_\kappa(\overline{\partial}^3 \partial_t)$, $\mathfrak{D}^4 = \mathfrak{R}_\kappa(\overline{\partial}^2 \partial_t^2)$, and $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^{\frac{3}{2}}(\overline{\partial}\partial_t^3)$. These cases are treated in an almost identical fashion as the case $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^2 \partial_t^4$ from section 3.5.1. In this regard, we note that a crucial requirement to carry these estimates is that $\mathfrak{D}$ contains at least one time derivative, which is the case for all the $\mathfrak{R}_\kappa$-weighted derivatives we need to consider.[8] We therefore conclude

$$\int_0^t (\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3) \leq \mathcal{P}_0 + \epsilon\mathcal{N} + \epsilon P(\mathcal{N}) + \mathcal{P}\int_0^t \mathcal{P},$$

for $\mathfrak{D}^2 = \sqrt{\mathfrak{R}_\kappa}\partial_t^2$, $\mathfrak{D}^3 = \sqrt{\mathfrak{R}_\kappa}(\overline{\partial}\partial_t^2)$, $\mathfrak{D}^3 = \mathfrak{R}_\kappa\partial_t^3$, $\mathfrak{D}^4 = \mathfrak{R}_\kappa(\overline{\partial}^3\partial_t)$, $\mathfrak{D}^4 = \mathfrak{R}_\kappa(\overline{\partial}^2\partial_t^2)$, $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^{\frac{3}{2}}(\overline{\partial}\partial_t^3)$, and $\mathfrak{D}^4 = (\mathfrak{R}_\kappa)^2\partial_t^4$.

4. Closing the Estimate

In this section, we prove:

Theorem 4.1. *Let $\mathcal{N}(t)$ and $\mathcal{P}(t)$ be defined as Notation 3.1, then for sufficiently large κ (that is, $\mathfrak{R}_\kappa \ll 1$), we have:*

$$\mathcal{N}(t) \leq C(M)\left(\epsilon P(\mathcal{N}(t)) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}\right), \quad t \in [0, T],$$

where $T > 0$ is chosen sufficiently small, provided that:

$$||\partial\eta||_{L^\infty} + ||\partial^2\eta||_{L^\infty} \leq M, \tag{4.1}$$

$$||g^{ij}||_{L^\infty} + ||\Gamma^k_{ij}||_{L^\infty} \leq M, \tag{4.2}$$

hold a priori for some large constant M.

Since the energy estimate for E is established in the previous section (that is, Theorem 3.2), we only need to show

$$\begin{aligned} &||v||_4^2 + ||\mathfrak{R}_\kappa v_t||_3^2 + ||\mathfrak{R}_\kappa v_{tt}||_2^2 + ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_1^2 \\ &\quad + ||R||_4^2 + ||R_t||_3^2 + ||\sqrt{\mathfrak{R}_\kappa}R_{tt}||_2^2 + ||\mathfrak{R}_\kappa R_{ttt}||_1^2 \\ &\quad + ||v_t||_2^2 + ||\sqrt{\mathfrak{R}_\kappa}v_{tt}||_1^2 + ||\mathfrak{R}_\kappa v_{ttt}||_0^2 + ||R_{tt}||_1 + ||\sqrt{\mathfrak{R}_\kappa}R_{ttt}||_0 \\ &\leq C(M)\left(\epsilon P(\mathcal{N}(t)) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}\right). \end{aligned} \tag{4.3}$$

This is proved via an iterated argument using div-curl estimate (A.1). It suffices to consider the first line in (4.1), since the second line consists lower order terms and can be treated by the same method. Taking $X = v$ and $s = 4$, (A.1) yields

$$||v||_4^2 \lesssim ||div\, v||_3^2 + ||curl\, v||_3^2 + ||v^3||_{3.5,\partial}^2 + ||v||_0^2. \tag{4.4}$$

On the other hand, taking $X = \mathfrak{R}_\kappa\partial_t v$ and $s = 3$, we have:

$$||\mathfrak{R}_\kappa v_t||_3^2 \lesssim ||\mathfrak{R}_\kappa div\, v_t||_2^2 + ||\mathfrak{R}_\kappa curl\, v_t||_2^2 + ||\mathfrak{R}_\kappa v_t^3||_{2.5,\Gamma}^2 + ||\mathfrak{R}_\kappa v_t||_0^2.$$

[8] Incidentally, this is why an estimate for the normal component of v with no time derivatives has to be obtained in a different way, see section 4.1.

Similarly, by taking $X = R' v_{tt}$, $s = 2$ *and* $X = (R')^{\frac{3}{2}} v_{ttt}$, $s = 1$, *we get*

$$||\mathfrak{R}_\kappa v_{tt}||_2^2 \lesssim ||\mathfrak{R}_\kappa div\, v_{tt}||_1^2 + ||\mathfrak{R}_\kappa curl\, v_{tt}||_1^2 + ||\mathfrak{R}_\kappa v_{tt}^3||_{1.5,\Gamma}^2 + ||\mathfrak{R}_\kappa v_{tt}||_0^2, \tag{4.5}$$

$$||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_1^2 \lesssim ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} div\, v_{ttt}||_0^2 + ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} curl\, v_{ttt}||_0^2 + ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}^3||_{0.5,\Gamma}^2 + ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}||_0^2, \tag{4.6}$$

respectively. In light of (4.1)–(4.1), *in order to estimate* v *and its time derivative, we need to bound* $div\, \partial_t^k v$, $curl\, \partial_t^k v$ *and* $\partial_t^k v^3$, *for* $k = 0, 1, 2, 3$, *respectively.*

4.1. Bounds for the Curl and the Boundary Term of v

In this section we prove:

Theorem 4.2.

$$||curl\, v||_3^2 + ||\mathfrak{R}_\kappa curl\, v_t||_2^2 + ||\mathfrak{R}_\kappa curl\, v_{tt}||_1^2 + ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} curl\, v_{ttt}||_0^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \tag{4.7}$$

Proof. The proof is almost identical to section 4 of [21], and so we omit the details. The only modification is that the weights $\mathfrak{R}_\kappa$ or $(\mathfrak{R}_\kappa)^{\frac{3}{2}}$ are used to compensate $q'(R) \sim \mathfrak{R}_\kappa^{-1}$, which allows us to get an uniform control. □

On the other hand, we have:

Theorem 4.3.

$$||v^3||_{3.5,\Gamma}^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \tag{4.8}$$

and

$$||\mathfrak{R}_\kappa v_t^3||_{2.5,\Gamma}^2 \lesssim \epsilon \mathcal{N} + ||\mathfrak{R}_\kappa \Pi \overline{\partial}^3 v_t||_{0,\Gamma}^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \tag{4.9}$$

$$||\mathfrak{R}_\kappa v_{tt}^3||_{1.5,\Gamma}^2 \lesssim \epsilon \mathcal{N} + ||\mathfrak{R}_\kappa \Pi \overline{\partial}^2 v_{tt}||_{0,\Gamma}^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \tag{4.10}$$

$$||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}^3||_{0.5,\Gamma}^2 \lesssim \epsilon \mathcal{N} + ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} \Pi \overline{\partial}^3 v_{ttt}||_{0,\Gamma}^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \tag{4.11}$$

Proof. For any vector field X, the following identity allows one to compare $(\Pi \overline{\partial} X)^3$ and $\overline{\partial} X^3$:

$$(\Pi \overline{\partial} X)^3 = \Pi_\lambda^3 \overline{\partial} X^\lambda = \overline{\partial} X^3 - g^{kl} \overline{\partial}_k \eta^3 \overline{\partial}_l \eta_\lambda \overline{\partial} X^\lambda. \tag{4.12}$$

Invoking (4.3), let $X = \mathfrak{R}_\kappa^{\frac{3}{2}} \partial_t^3 v$ and then taking $H^{-0.5}(\Gamma)$ norm yields

$$||\mathfrak{R}_\kappa^{\frac{3}{2}} \overline{\partial} \partial_t^3 v^3||_{-0.5,\Gamma}^2 \lesssim ||\mathfrak{R}_\kappa^{\frac{3}{2}} \Pi \overline{\partial} \partial_t^3 v||_{0,\Gamma}^2 + ||g^{kl} \overline{\partial}_k \eta^3 \overline{\partial}_l \eta_\lambda||_{1.5,\Gamma}^2 ||\mathfrak{R}_\kappa^{\frac{3}{2}} \partial_t^3 v^\lambda||_{0.5,\Gamma}^2.$$

We add $||\mathfrak{R}_\kappa^{\frac{3}{2}}\partial_t^2 v^3||^2_{-0.5,\Gamma}$ to both sides, use the fact that $||\mathfrak{R}_\kappa^{\frac{3}{2}}\partial_t^2 v^3||^2_{-0.5,\Gamma}$ $+ ||\mathfrak{R}_\kappa^{\frac{3}{2}}\overline{\partial}\partial_t^3 v^3||^2_{-0.5,\Gamma}$ is equivalent to $||\mathfrak{R}_\kappa^{\frac{3}{2}}\partial_t^2 v^3||^2_{0.5,\Gamma}$, invoke $\overline{\partial}_k\eta^3 = \int_0^t \overline{\partial}_k v^3$ (which is true since $\eta^3(0) = 1$), to conclude (4.3), where the term $||\mathfrak{R}_\kappa^{\frac{3}{2}}\partial_t^2 v^3||^2_{-0.5,\Gamma}$ on the right hand side is estimated using interpolation, Young's inequality, and the fundamental theorem of calculus.

Similarly, using (4.3) with $X = \mathfrak{R}_\kappa\overline{\partial}\partial_t^2 v$ and $X = \mathfrak{R}_\kappa\overline{\partial}^2\partial_t v$, estimating in $H^{-0.5}(\Gamma)$ yields (4.3) and (4.3), respectively. Now, we need to control $||v^3||_{3.5,\Gamma}$. This cannot be controlled using the above method since $||\Pi\overline{\partial}^4 v||^2_{0,\Gamma}$ is not part of the energy E. Nevertheless, we recall the boundary condition

$$\sqrt{g}\Delta_g\eta^\alpha = \sqrt{g}g^{ij}\overline{\partial}^2_{ij}\eta^\alpha - \sqrt{g}g^{ij}\Gamma^k_{ij}\overline{\partial}_k\eta^\alpha = -\sigma^{-1}A^{\mu\alpha}N_\mu q, \quad \text{on } \Gamma \tag{4.13}$$

where $\Gamma^k_{ij} = g^{kl}\overline{\partial}_l\eta^\mu\overline{\partial}^2_{ij}\eta_\mu$. Time differentiating (4.3) with $\alpha = 3$ gives:

$$\begin{aligned}\sqrt{g}g^{ij}\overline{\partial}^2_{ij}v^3 - \sqrt{g}g^{ij}\Gamma^k_{ij}\overline{\partial}_k v^3 &= -\partial_t(\sqrt{g}g^{ij})\overline{\partial}^2_{ij}\eta^3 - \partial_t(\sqrt{g}g^{ij}\Gamma^k_{ij})\overline{\partial}_k\eta^3\\ &- \sigma^{-1}\partial_t A^{\mu3}N_\mu q - \sigma^{-1}A^{\mu3}N_\mu\partial_t q\end{aligned} \tag{4.14}$$

holds on Γ. Because $g^{ij} \in H^{2.5}(\Gamma)$ and $\Gamma^k_{ij} \in H^{1.5}(\Gamma)$, invoking the elliptic estimate for rough coefficients (see, e.g, Theorem 4 and Remark 2 in Milani [51]), we obtain:

$$\begin{aligned}||v^3||^2_{3.5,\Gamma} \lesssim_M\ &||\partial_t(\sqrt{g}g^{ij})\overline{\partial}^2_{ij}\eta^3||^2_{1.5,\Gamma} + ||\partial_t(\sqrt{g}g^{ij}\Gamma^k_{ij})\overline{\partial}_k\eta^3||^2_{1.5,\Gamma}\\ &+ ||\partial_t A^{\mu3}N_\mu q||^2_{1.5,\Gamma} + ||A^{\mu3}N_\mu\partial_t q||^2_{1.5,\Gamma},\end{aligned}$$

which can be controlled appropriately by the right hand side of (4.3), where the last two terms can be controlled by with the help of Theorem 3.13. □

4.2. Bounds for v, R and Their Time Derivatives

Let $k = 1, 2, 3$, commuting ∂_t^k to the second equation of (1.1), we get

$$\begin{aligned}\partial^\alpha\partial_t^k v_\alpha =&(\delta^{\mu\alpha} - a^{\mu\alpha})\partial_\mu\partial_t^k v_\alpha - \sum_{\substack{j_1+j_2=k\\ j_1\geq 1}} R^{-1}\partial_t^{j_1}(Ra^{\mu\alpha})(\partial_\mu\partial_t^{j_2}v_\alpha)\\ &- R^{-1}\partial_t^{k+1}R.\end{aligned} \tag{4.15}$$

In addition, the first equation of (1.1) can be re-written as

$$R'R\partial_t v^\alpha + a^{\mu\alpha}\partial_\mu R = 0.$$

Commuting ∂_t^k to this equation and invoking (1.3), we get

$$\begin{aligned}\partial^\alpha\partial_t^k R =& (\delta^{\mu\alpha} - a^{\mu\alpha})\partial_\mu\partial_t^k R - R'R\partial_t^{k+1}v^\alpha\\ &- \sum_{\substack{j_1+j_2=k\\ j_1\geq 1}}[(\partial_t^{j_1}a^{\nu\alpha})(\partial_\mu\partial_t^{j_2}R) + \big(\partial_t^{j_1}(R'R)\big)(\partial_t^{j_2+1}v^\alpha)].\end{aligned} \tag{4.16}$$

When $k = 3$, multiplying $(R')^{\frac{3}{2}}$ and then taking L^2 norm on both sides of (4.2), we get

$$||(R')^{\frac{3}{2}}\partial^\alpha\partial_t^3 v_\alpha||_0 \le \epsilon||(R')^{\frac{3}{2}}\partial_t^3 v_\alpha||_1 + C\sum_{\substack{j_1+j_2=3\\ j_1\ge 1}}||(R')^{\frac{3}{2}}\partial_t^{j_1}(Ra^{\mu\alpha})(\partial_\mu\partial_t^{j_2}v_\alpha)||_0 + C||(R')^{\frac{3}{2}}R_{tttt}||_0,$$

where we have used Lemma 2.1(9)(10). The term

$$\sum_{\substack{j_1+j_2=3\\ j_1\ge 1}}||(R')^{\frac{3}{2}}\partial_t^{j_1}(Ra^{\mu\alpha})(\partial_\mu\partial_t^{j_2}v_\alpha)||_0$$

is of lower order and can be controlled appropriately. Squaring and using Theorem 3.2, we have

$$||(\mathfrak{R}_\kappa)^{\frac{3}{2}}\operatorname{div} v_{ttt}||_0^2 \lesssim ||(R')^{\frac{3}{2}}\operatorname{div} v_{ttt}||_0^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

Now, in view of (4.1), invoking (4.2), (4.3) and Theorem 3.2 gives

$$||(\mathfrak{R}_\kappa)^{\frac{3}{2}}v_{ttt}||_1^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}. \tag{4.17}$$

We now move to estimate $||\mathfrak{R}_\kappa R_{ttt}||_1$. Invoking (4.2) for $k = 3$, multiplying R' on both sides and taking L^2 norm, we have

$$||R'R_{ttt}||_1 \lesssim \epsilon||R'R_{ttt}||_1 + ||(R')^2 v_{tttt}||_0 + \epsilon\mathcal{N} + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

Here, $\epsilon\mathcal{N}$ appears when controlling the error term of (4.2).[9] Squaring this provides

$$||\mathfrak{R}_\kappa R_{ttt}||_1^2 \lesssim ||R'R_{ttt}||_1^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}, \tag{4.18}$$

where Theorem 3.2 is also used.

Next, we estimate $||\mathfrak{R}_\kappa \operatorname{div} v_{tt}||_1$. Invoking (4.2) with $k = 2$, multiplying R' and then applying H^1 norm on both sides, we get

$$||R'\partial^\alpha\partial_t^2 v_\alpha||_1 \le \epsilon||R'v_{tt}||_2 + C\sum_{\substack{j_1+j_2=2\\ j_1\ge 1}}||R'\partial_t^{j_1}(Ra^{\mu\alpha})(\partial_\mu\partial_t^{j_2}v_\alpha)||_1 + C||R'R_{ttt}||_1.$$

Using (4.2), squaring the above estimate leads to

$$||\mathfrak{R}_\kappa \operatorname{div} v_{tt}||_1^2 \lesssim ||R'\operatorname{div} v_{tt}||_1^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$

[9] Specifically, $\epsilon\mathcal{N}$ is required to control $||R'(\partial_t^3 a^{\nu\alpha})(\partial_\nu R)||_0$. This term involves $||R'(\partial v_{tt})(\partial R)||_0$ at the top order, which is bounded by $||\mathfrak{R}_\kappa\partial v_{tt}||_1^{\frac{1}{2}}||\mathfrak{R}_\kappa\partial v_{tt}||_0^{\frac{1}{2}}||\partial R||_1^{\frac{1}{2}}||\partial R||_0^{\frac{1}{2}} \le \epsilon(\sqrt{\mathcal{N}} + \mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}$.

In light of (4.1), the above bound for $||\mathfrak{R}_\kappa \text{div}\, v_{tt}||_1^2$, together with (4.2), (4.3) and Theorem 3.2, gives

$$||\mathfrak{R}_\kappa v_{tt}||_2^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}. \tag{4.19}$$

Furthermore, invoking (4.2) for $k = 2$, multiplying $\sqrt{R'}$ and taking H^1 norm and squaring, we get:

$$||\sqrt{R'}R_{tt}||_2^2 \lesssim \epsilon||\sqrt{R'}R_{tt}||_2^2 + ||(R')^{\frac{3}{2}}v_{ttt}||_1^2 + \epsilon\mathcal{N} + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},$$

which implies, after invoking (4.2), that

$$||\sqrt{\mathfrak{R}_\kappa}R_{tt}||_2^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}. \tag{4.20}$$

In addition, this allow us to continue this procedure to get an estimate for $R'\text{div}\, v_t$; let $X = R'\partial_t v$ and $s = 3$ in (4.2), we gets

$$||R'\text{div}\, v_t||_2 \lesssim \epsilon||R'v_t||_3 + ||R'R_{tt}||_2 + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},$$

squaring, and invoking (4.2), (4.3) and (4.2) gives

$$||\mathfrak{R}_\kappa v_t||_3^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}. \tag{4.21}$$

Now, invoking (4.2) for $k = 1$, squaring and taking H^2 norm yields

$$||R_t||_3^2 \lesssim \epsilon||R_t||_3^2 + ||R'v_{tt}||_2 + \epsilon\mathcal{N} + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P} \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}, \tag{4.22}$$

as a consequence of (4.2).

Finally, the above procedure yields

$$||\text{div}\, v||_3 \lesssim \epsilon||v||_4 + ||R_t||_3,$$

and hence

$$||v||_4^2 \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},$$

via (4.2), (4.3) and (4.2). Moreover, we have:

$$||R||_4^2 \lesssim \epsilon||R||_4^2 + ||R'v_t||_3 + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P} \lesssim \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},$$

via (4.2).

4.3. The Continuity Argument, Proof of Theorem 1.2

Recovering the a priori assumptions: We need to control the left hand side of (4.1)–(4.1) by $\epsilon P(\mathcal{N})+\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}$. The control for (4.1) is a direct consequence of the Sobolev embedding , that is,

$$||\partial\eta||_{L^\infty}+||\partial^2\eta||_{L^\infty}\lesssim||\eta||_4\le\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}.$$

This also controls the left hand side of (4.1) by the definition of g^{ij} and Γ^k_{ij}.

Estimates at $t=0$:

As we have seen that $\mathcal{P}$ involves quantities involving time derivatives, and so one needs to show that these quantities can be controlled by $\mathcal{P}_0$. More precisely, we show

$$\begin{aligned}&||\mathfrak{R}_\kappa v_t(0)||_3+||\mathfrak{R}_\kappa v_{tt}(0)||_2+||(\mathfrak{R}_\kappa)^{\frac{3}{2}}v_{ttt}(0)||_1\\&\quad+||R_t(0)||_3+||\sqrt{\mathfrak{R}_\kappa}R_{tt}(0)||_2+||\mathfrak{R}_\kappa R_{ttt}(0)||_1\\&||v_t(0)||_2+||\sqrt{\mathfrak{R}_\kappa}v_{tt}(0)||_2+||\mathfrak{R}_\kappa v_{ttt}(0)||_0\\&\quad+||R_{tt}(0)||_1+||\sqrt{\mathfrak{R}_\kappa}R_{ttt}(0)||_0\le\mathcal{P}_0.\end{aligned}$$

This estimate is straightforward, that is, we use (1.1) to obtain $||\mathfrak{R}_\kappa v_t(0)||_3\le||\rho_0^{-1}\partial q(0)||_3\lesssim\mathcal{P}_0$. Moreover, we use (4.2) with $k=0$ at $t=0$ to obtain $||R_t(0)||_3\le||\rho_0^{-1}\mathrm{div}\,v(0)||_3\le\mathcal{P}_0$. The other quantities in (4.3) can be controlled similarly. In addition, we also need

$$||\mathfrak{R}_\kappa v_t(0)||_{3,\Gamma}+||\mathfrak{R}_\kappa v_{tt}(0)||_{2,\Gamma}+||(\mathfrak{R}_\kappa)^{\frac{3}{2}}v_{ttt}(0)||_{1,\Gamma}\le\mathcal{P}_0.$$

To control $||\mathfrak{R}_\kappa v_t(0)||_{3,\Gamma}$, we use (4.2) to obtain $R'v_t^i(0)=-\delta^{ij}\partial_jR(0)$, which implies $||\mathfrak{R}_\kappa v_t^i(0)||_{3,\Gamma}\le||R(0)||_{4,\Gamma}\le\mathcal{P}_0$. On the other hand, we control the normal component $v_t^3(0)$ using the elliptic estimate. Time differentiating (4.3) and then restricting at $t=0$ yield

$$\overline{\Delta}v_t^3(0)=-\sigma^{-1}q_{tt}(0)+F,$$

where F satisfies $||\mathfrak{R}_\kappa F||_{1,\Gamma}\le\mathcal{P}_0$. From the elliptic theory, the control of $||\mathfrak{R}_\kappa v_t^3(0)||_{3,\Gamma}$ requires the control of $||\mathfrak{R}_\kappa q_{tt}(0)||_{1,\Gamma}$ and hence $||R'q_{tt}(0)||_{1,\Gamma}$. Invoking the wave equation (2.3), this is bounded by $||\Delta q(0)||_{1,\Gamma}+||\mathcal{F}_1||_{1,\Gamma}$. There is no problem to control $||\mathcal{F}_1||_{1,\Gamma}$ by $\mathcal{P}_0$ in light of (2.3). Furthermore, invoking the compatibility condition in section 5, that is, $q_0=\sigma\overline{\Delta}\eta_0^3$, one controls $||\Delta q_0||_{1,\Gamma}$ by $||\eta_0^3||_{5.5}$.

The estimates for $||\mathfrak{R}_\kappa v_{tt}(0)||_{2,\Gamma}$, $||(\mathfrak{R}_\kappa)^{\frac{3}{2}}v_{ttt}(0)||_{1,\Gamma}$ are treated in a similar way, upon time differentiating more times and proceeding as above. We omit the details, but explain the estimates up to the highest order in an expository way. First, to control the tangential component, we use (4.2) and (4.2) to get

$$R'v_{tt}^i(0)\sim\delta^{ij}\overline{\partial}_jR_t(0)\sim\delta^{ij}\overline{\partial}_j\partial_\alpha v^\alpha(0),$$

$$(R')^{\frac{3}{2}} v_{ttt}^i(0) \sim \sqrt{R'}\delta^{ij}\overline{\partial}_j R_{tt}(0) \sim \sqrt{R'}\delta^{ij}\overline{\partial}_j \Delta q_0 \sim \sqrt{R'}\delta^{ij}\overline{\partial}_j \Delta\overline{\Delta}\eta_0^3,$$

where $\sim$ means up to controllable terms. This yields that

$$||\mathfrak{R}_\kappa v_{tt}^i(0)||_{2,\Gamma}, \quad ||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}^i(0)||_{1,\Gamma}$$

are controlled by $||\text{div } v_0||_{3,\Gamma}$ and $||\eta_0^3||_{6.5}$, respectively. Second, to control the normal component, time-differentiating (4.3) two times and restricting at $t = 0$ yields $\overline{\Delta} v_{tt}^3(0) \sim q_{ttt}(0)$. Therefore, from the elliptic theory, the control of $||\mathfrak{R}_\kappa v_{tt}^3(0)||_{2,\Gamma}$ requires that of $||\mathfrak{R}_\kappa q_{ttt}(0)||_{0,\Gamma}$ and hence $||\Delta q_t(0)||_{0,\Gamma}$, in light of the wave equation. Invoking the compatibility condition $q_t(0) \sim \overline{\Delta} v_0^3$, $||\Delta q_t(0)||_{0,\Gamma}$ is controlled by $||\Delta\overline{\Delta} v_0^3||_{0,\Gamma}$. On the other hand, time-differentiating (4.3) three times and restricting at $t = 0$ yields $\overline{\Delta} v_{ttt}^3(0) \sim q_{tttt}(0)$. Therefore, from the elliptic theory, the control of $||(\mathfrak{R}_\kappa)^{\frac{3}{2}} v_{ttt}^3(0)||_{1,\Gamma}$ requires that of $||(\mathfrak{R}_\kappa)^{\frac{3}{2}} q_{tttt}(0)||_{-1,\Gamma}$ and hence $||\sqrt{\mathfrak{R}_\kappa}\Delta q_{tt}(0)||_{-1,\Gamma}$. Invoking the compatibility conditions $q_{tt}(0) \sim \overline{\Delta}\partial_3 q(0)$ and $q(0) \sim \overline{\Delta}\eta_0^3$, we have that $||\sqrt{R'}\Delta q_{tt}(0)||_{-1,\Gamma}$ is bounded by $||\eta_0^3||_{6.5}$.

Hence, Theorem 4.1 implies

$$\mathcal{N}(t) \lesssim \epsilon P(\mathcal{N}(t)) + P(\mathcal{N}(0)) + P(\mathcal{N}(t)) \int_0^t P(\mathcal{N}(s))\, \mathrm{d}S.$$

Invoking the continuity-boostrap argument in [58], this implies that there exists $\mathfrak{M} > 0$ such that

$$\mathcal{N}(t) \leq \mathfrak{M}, \quad \text{whenever } t \in [0, T], \tag{4.23}$$

for some $T > 0$.

4.4. Passing to the Incompressible Limit, Proof of Theorem 1.3

Proof for statement 1:

This is standard since we have an uniform a priori estimate.

Proof for statement 2:

The bound (4.3) implies that $||v_\kappa||_4 + ||R_\kappa||_4 \leq \sqrt{\mathfrak{M}}$ uniformly as $\kappa \to \infty$. Therefore, by the Sobolev embedding, we have:

$$\sum\nolimits_{\ell\leq 2}\Big(||\partial^\ell v_\kappa||_{L^\infty(\Omega)} + ||\partial^\ell R_\kappa||_{L^\infty(\Omega)}\Big) \leq \sqrt{\mathfrak{M}}.$$

This yields that for each fixed $t \in [0, T]$, v_κ and R_κ are uniformly bounded and equicontinuous in $C^2(\Omega)$, which implies the convergence of v_κ and R_κ in $C^2(\Omega)$. Moreover, $v_\kappa \to \mathfrak{v}$ since $a^{\mu\alpha}\partial_\mu(v_\kappa)_\alpha \to 0$ in $L^\infty(\Omega)$, which is a consequence of $||\partial_t q_\kappa||_2$ being bounded independent of κ and $R'_\kappa \to 0$ as $\kappa \to \infty$.

5. The Initial Data

5.1. The Compatibility Conditions

The compatibility conditions for the initial data are necessary for construction of solutions, as well as for passing the solution to the incompressible limit. We recall that since

$$q = \sigma g^{ij} \hat{n}_\mu \overline{\partial}^2_{ij} \eta^\mu, \quad \text{on } \Gamma,$$

we have

$$q|_{t=0} = \left(\sigma g^{ij} \hat{n}_\mu \overline{\partial}^2_{ij} \eta^\mu\right)\Big|_{t=0} := H_0, \quad \text{on } \Gamma,$$

which is the zero-th order compatibility condition. In addition, for each $j \geq 1$, the j-th order compatibility reads as

$$\partial_t^j q|_{t=0} = \partial_t^j \left(\sigma g^{ij} \hat{n}_\mu \overline{\partial}^2_{ij} \eta^\mu\right)\Big|_{t=0} := H_j, \quad \text{on } \Gamma. \tag{5.1}$$

Our goal is to construct $(\mathbf{v}_0, \mathbf{q}_0)$ that verifies the compatibility condition (5.1) for $j = 0, 1, 2, 3$. We shall focus on the case when $\Omega = \mathbb{T}^2 \times (0, 1)$, whose boundary Γ is flat. Our method can easily be generalized to more general domains.

5.2. Formal Construction

We shall describe our method formally which serves as a good guideline for readers. Since

$$q \sim \overline{\Delta} \eta^3, \quad \text{on } \Gamma,$$

we get

$$q_t \sim \overline{\Delta} v^3, \quad q_{tt} \sim \overline{\Delta} v_t^3, \quad q_{ttt} \sim \overline{\Delta} v_{tt}^3, \quad \text{on } \Gamma,$$

after taking time derivatives. Moreover, since the Euler equations imply

$$v_t \sim \partial q, \quad q_t \sim \kappa \operatorname{div} v,$$

we have

$$q_{tt} \sim \overline{\Delta} \partial_3 q, \quad q_{ttt} \sim \overline{\Delta} \partial_3 q_t \sim \kappa \overline{\Delta} \partial_3 \operatorname{div} v, \quad \text{on } \Gamma.$$

For each $\ell = 0, 1, 2, 3$, we obtain the ℓ-th order compatibility condition after restricting the above expression at $t = 0$, that is,

$$\begin{aligned} q|_{t=0} &\sim \overline{\Delta} \eta_0^3, \quad \text{on } \Gamma, \\ q_t|_{t=0} &\sim \overline{\Delta} v_0^3, \quad \text{on } \Gamma, \\ q_{tt}|_{t=0} &\sim \overline{\Delta} \partial_3 q_0, \quad \text{on } \Gamma, \\ q_{ttt}|_{t=0} &\sim \kappa \overline{\Delta} \partial_3 \operatorname{div} v, \quad \text{on } \Gamma. \end{aligned}$$

On the other hand, since

$$q_t \sim \kappa \mathrm{div}\, v, \quad q_{tt} \sim \kappa \mathrm{div}\, v_t \sim \kappa \Delta q, \quad q_{ttt} \sim \kappa \Delta q_t \sim \kappa^2 \Delta \mathrm{div}\, v,$$

then

$$q_0 \sim \overline{\Delta} \eta_0^3, \quad \text{on } \Gamma, \tag{5.2}$$

$$\mathrm{div}\, v_0 \sim \kappa^{-1} \overline{\Delta} v_0^3, \quad \text{on } \Gamma, \tag{5.3}$$

$$\Delta q_0 \sim \kappa^{-1} \overline{\Delta} \partial_3 q_0, \quad \text{on } \Gamma, \tag{5.4}$$

$$\Delta \mathrm{div}\, v_0 \sim \kappa^{-1} \overline{\Delta} \partial_3 \mathrm{div}\, v_0, \quad \text{on } \Gamma. \tag{5.5}$$

In other words, the first order compatibility condition (that is, (5.1) when $j = 1$), is expressed in v_0, and the second order compatibility condition is expressed in q_0, and finally the third order compatibility condition is expressed in v_0 again.

To construct initial data that satisfies the compatibility conditions up to order 3, our first step is to obtain $(\mathbf{u}_0, \mathbf{p}_0)$ that satisfies the (5.2). This is easy, since we can simply let $\mathbf{u}_0$ to be velocity for the incompressible case, that is, $\mathbf{u}_0 = u_0$, and $\mathbf{p}_0$

$$\begin{aligned} -\Delta \mathbf{p}_0 &= (\partial_\mu \mathbf{u}_0^\nu)(\partial_\nu \mathbf{u}_0^\mu), \quad \text{in } \Omega, \\ \mathbf{p}_0 &= \overline{\Delta} \eta_0^3, \quad \text{on } \Gamma. \end{aligned} \tag{5.6}$$

Our next step is to construct a velocity vector field $\mathbf{w}_0 = (\mathbf{w}_0^1, \mathbf{w}_0^2, \mathbf{w}_0^3)$ that satisfies (5.2). To achieve this, we set $\mathbf{w}_0^1 = \mathbf{u}_0^1$ and $\mathbf{w}_0^2 = \mathbf{u}_0^2$, while we define $\mathbf{w}_0^3$ via solving

$$\begin{aligned} \Delta^2 \mathbf{w}_0^3 &= \Delta^2 \mathbf{u}_0^3, \quad \text{in } \Omega, \\ \mathbf{w}_0^3 &= \mathbf{u}_0^3, \quad \partial_3 \mathbf{w}_0^3 \sim \kappa^{-1} \overline{\Delta} \mathbf{u}_0^3 - \partial_1 \mathbf{u}_0^1 - \partial_2 \mathbf{u}_0^2, \quad \text{on } \Gamma. \end{aligned} \tag{5.7}$$

We now construct $\mathbf{q}_0$ that satisfies (5.2). We define $\mathbf{q}_0$ by the solution of

$$\begin{aligned} \Delta^3 \mathbf{q}_0 &= 0, \quad \text{in } \Omega, \\ \mathbf{q}_0 &= \mathbf{p}_0, \quad \partial_3 \mathbf{q}_0 = \partial_3 \mathbf{p}_0, \quad \Delta \mathbf{q}_0 \sim \kappa^{-1} \overline{\Delta} \partial_3 \mathbf{p}_0, \quad \text{on } \Gamma. \end{aligned} \tag{5.8}$$

Finally, we need to construct $\mathbf{v}_0$ using (5.2). To achieve this, we set $\mathbf{v}_0^1 = \mathbf{u}_0^1$, $\mathbf{v}_0^2 = \mathbf{u}_0^2$, and we define $\mathbf{v}_0^3$ by solving

$$\begin{aligned} &\Delta^4 \mathbf{v}_0^3 = \Delta^4 \mathbf{w}_0^3, \quad \text{in } \Omega, \\ &\mathbf{v}_0^3 = \mathbf{w}_0^3, \quad \partial_3 \mathbf{v}_0^3 \sim \kappa^{-1} \overline{\Delta} \mathbf{w}_0^3 - \partial_1 \mathbf{w}_0^1 - \partial_2 \mathbf{w}_0^2, \quad \text{on } \Gamma, \\ &\partial_3^2 \mathbf{v}_0^3 \sim \kappa^{-1} \partial_3 \overline{\Delta} \mathbf{w}_0^3 - \partial_3 \partial_1 \mathbf{w}_0^1 - \partial_3 \partial_2 \mathbf{w}_0^2, \quad \text{on } \Gamma, \\ &\Delta \partial_3 \mathbf{v}_0^3 \sim \kappa^{-1} \overline{\Delta} \partial_3 \mathrm{div}\, \mathbf{w}_0 - \Delta \partial_1 \mathbf{w}_0^1 - \Delta \partial_2 \mathbf{w}_0^2, \quad \text{on } \Gamma. \end{aligned} \tag{5.9}$$

Remark. In fact, $\overline{\Delta} \eta_0^3 = 0$ on the boundary of the reference domain $\mathbb{T}^2 \times (0, 1)$. But that we do not use this condition exactly because we want to keep the regularity of each argument as it should hold for the general domain.

Theorem 5.1. *Let* $\mathrm{u}_0 \in H^{6.5}(\Omega)$ *be a divergence free vector field in* Ω *and* $\boldsymbol{p}_0$ *be the associated pressure. Then there exists initial data* $(\boldsymbol{v}_0, \boldsymbol{q}_0) = (\boldsymbol{v}_0^\kappa, \boldsymbol{q}_0^\kappa)$ *satisfying the compatibility conditions up to order 3, that is, (5.2)–(5.2), such that* $\boldsymbol{v}_0^\kappa \to \boldsymbol{u}_0$ *in* $C^2(\Omega)$ *and* $\operatorname{div} \boldsymbol{v}_0^\kappa \to 0$ *in* $C^1(\Omega)$ *as* $\kappa \to \infty$*, and* $\mathcal{P}_0$ *is uniformly bounded for all* κ.

Proof. $(\mathbf{v}_0, \mathbf{q}_0)$ verifies (5.2)–(5.2) follows automatically from our construction. Since $\mathbf{p}_0$ satisfies the elliptic equation (5.2), for $s \geq 4$, we have

$$||\mathbf{p}_0||_s \lesssim ||\Delta \mathbf{p}_0||_{s-2} + ||\mathbf{p}_0||_{s-0.5,\Gamma}, \tag{5.10}$$

which requires $||\mathbf{u}_0||_{s-1}$ and $||\eta_0||_{s+2}$ to control. Moreover, by the poly-harmonic estimate applied to (5.2) we have

$$\begin{aligned} ||\mathbf{q}_0||_s &\lesssim ||\Delta \mathbf{q}_0||_{s-2.5,\Gamma} + ||\partial_3 \mathbf{q}_0||_{s-1.5,\Gamma} + ||\mathbf{q}_0||_{s-0.5,\Gamma} \\ &\lesssim \kappa^{-1} ||\overline{\Delta}\partial_3 \mathbf{p}_0||_{s-2} + ||\partial_3 \mathbf{p}_0||_{s-1} + ||\mathbf{p}_0||_s. \end{aligned}$$

Invoking (5.1), this requires $||\mathbf{u}_0||_s$ and $||\eta_0||_{s+3}$ to control. On the other hand, invoking (5.2) and the poly-harmonic estimate, we get

$$\begin{aligned} ||\mathbf{w}_0^3||_s &\lesssim ||\Delta^2 \mathbf{u}_0^3||_{s-4} + ||\partial_3 \mathbf{w}_0^3||_{s-1.5,\Gamma} + ||\mathbf{w}_0^3||_{s-0.5,\Gamma} \\ &\lesssim ||\Delta^2 \mathbf{u}_0^3||_{s-4} + \kappa^{-1}||\overline{\Delta}\mathbf{u}_0^3||_{s-1} + ||\partial_1 \mathbf{w}_0^1||_{s-1} + ||\partial_2 \mathbf{w}_0^2||_{s-1} + ||\mathbf{u}_0^3||_s, \end{aligned}$$

which needs $||\mathbf{u}_0^3||_{s+1}$ to control. In addition, since $\mathbf{w}_0^i = \mathbf{u}_0^i$, one controls $||\mathbf{w}_0||_s$ via $||\mathbf{u}_0||_{s+1}$. Moreover, invoking (5.2) and the poly-harmonic estimate, we get

$$\begin{aligned} ||\mathbf{v}_0^3||_s &\lesssim ||\Delta^4 \mathbf{u}_0^3||_{s-8} + ||\Delta\partial_3 \mathbf{v}_0^3||_{s-3.5,\Gamma} + ||\partial_3^2 \mathbf{v}_0^3||_{s-2.5,\Gamma} + ||\partial_3 \mathbf{v}_0^3||_{s-1.5,\Gamma} + ||\mathbf{v}_0^3||_{s-0.5,\Gamma} \\ &\lesssim ||\Delta^4 \mathbf{u}_0^3||_{s-8} + \kappa^{-1}||\overline{\Delta}\partial_3 \operatorname{div} \mathbf{w}_0||_{s-3} + ||\Delta\partial_1 \mathbf{w}_0^1||_{s-3} + ||\Delta\partial_2 \mathbf{w}_0^2||_{s-3} \\ &\quad + \kappa^{-1}||\partial_3\overline{\Delta}\mathbf{w}_0^3||_{s-2} + ||\partial_3\partial_1 \mathbf{w}_0^1||_{s-2} + ||\partial_3\partial_2 \mathbf{w}_0^2||_{s-2} \\ &\quad + \kappa^{-1}||\overline{\Delta}\mathbf{w}_0^3||_{s-1} + ||\partial_1 \mathbf{w}_0^1||_{s-1} + ||\partial_2 \mathbf{w}_0^2||_{s-1} + ||\mathbf{w}_0^3||_s, \end{aligned}$$

which requires $||\mathbf{w}_0^3||_{s+1}$ and hence $||\mathbf{u}_0^3||_{s+2}$ to control. Once again, since $\mathbf{v}_0^i = \mathbf{u}_0^i$, one controls $||\mathbf{v}_0||_s$ through $||\mathbf{u}_0||_{s+2}$.

Next, since (5.2) implies

$$\begin{aligned} &\Delta^2(\mathbf{w}_0^3 - \mathbf{u}_0^3) = 0, \quad \text{in } \Omega, \\ &\mathbf{w}_0^3 - \mathbf{u}_0^3 = 0, \quad \partial_3(\mathbf{w}_0^3 - \mathbf{u}_0^3) \sim \kappa^{-1}\overline{\Delta}\mathbf{u}_0^3, \quad \text{on } \Gamma, \end{aligned}$$

we have that $||\mathbf{w}_0^3 - \mathbf{u}_0^3||_s \to 0$ as $\kappa \to \infty$, and hence $\mathbf{w}_0 \to \mathbf{u}_0$ in $H^s(\Omega)$ as $\kappa \to \infty$. Similarly, (5.2) implies $\mathbf{v}_0 \to \mathbf{w}_0$ in $H^s(\Omega)$ as $\kappa \to \infty$, and so we conclude that $\mathbf{v}_0 \to \mathbf{u}_0$ in $H^s(\Omega)$ as $\kappa \to \infty$. Furthermore, because $s \geq 4$ and $\mathbf{v}_0$ is uniformly bounded in H^s, we have that $\mathbf{v}_0 \to \mathbf{u}_0$ in $C^2(\Omega)$ thanks to Arzelà-Ascoli and $\operatorname{div} \mathbf{v}_0 \to \operatorname{div} \mathbf{u}_0 = 0$ in $C^1(\Omega)$.

Finally, we recall that $\mathcal{P}_0$ consists of

$$||\mathbf{v}_0||_4, ||\mathbf{v}_0||_{4,\Gamma}, ||\mathbf{q}_0||_4, ||\mathbf{q}_0||_{4,\Gamma}, ||\operatorname{div} \mathbf{v}_0|_\Gamma||_{3,\Gamma}, ||\Delta \mathbf{v}_0|_\Gamma||_{2,\Gamma}),$$

which can all be controlled by $||\mathbf{u}_0||_{s+2} = ||\mathrm{u}_0||_{s+2}$ and $||\eta_0||_{s+3}$ with $s = 4.5$. □

Remark. The initial data constructed in Theorem 5.1 is given in terms of the initial pressure $\mathbf{q}_0$ instead of the initial density R_0. This is because the boundary condition is more easily stated in terms of q and we need to make sure that the quantities $||\mathbf{q}_0||_4$ and $||\mathbf{q}_0||_{4,\Gamma}$ are bounded uniformly in κ, but we can compute R_0 through the equation of states $R = R(q)$, that is, $R_0 = [(c_\gamma \kappa)^{-1}\mathbf{q}_0 + \beta]^{1/\gamma}$.

The rest of this section is devoted to provide detailed construction, and for the sake of simple expositions, we assume the equation of state is taken to be

$$q(R) = \kappa(R-1).$$

This allows us to exchange q and R in an explicit way. Also, throughout the rest of this section, we shall use Q to denote a rational function.

5.3. Construction for $(\boldsymbol{u}_0, \boldsymbol{p}_0, \Omega)$ *that Satisfies (5.1) While* $j = 0$

Let $\mathbf{u}_0 = \mathfrak{v}_0$, where $\mathfrak{v}_0$ is the data for the incompressible Euler equations. Since $H_0 = \sigma\overline{\Delta}\eta_0^3$, we define $\mathbf{p}_0$ by solving

$$\begin{cases} -\Delta\mathbf{p}_0 = (\partial_\mu \mathbf{u}_0^\nu)(\partial_\nu \mathbf{u}_0^\mu), & \text{in } \Omega, \\ \mathbf{p}_0 = H_0, & \text{on } \Gamma. \end{cases}$$

5.4. Construction for $\mathbf{w}_0$ *that Satisfies (5.1) While* $j = 1$

We next consider the first order compatibility condition, that is, $\partial_t q|_{t=0} = H_1$. Since

$$\partial_t\left(\sigma g^{ij}\hat{n}_\mu \overline{\partial}^2_{ij}\eta^\mu\right) = \sigma g^{ij}\hat{n}_\mu \overline{\partial}^2_{ij} v^\mu + \sigma Q(\hat{n}, \overline{\partial}\eta, \overline{\partial} v)\overline{\partial}^2\eta, \tag{5.11}$$

we have that

$$H_1 = \sigma\overline{\Delta} v_0^3 + \sigma Q(\overline{\partial}\eta_0, \overline{\partial} v_0)\overline{\partial}^2\eta_0.$$

On the other hand, since $\partial_t q = -R\kappa a^{\mu\alpha}\partial_\mu v_\alpha$, (5.1) with $j = 1$ becomes

$$\operatorname{div} v_0 = \kappa^{-1}(\kappa^{-1}q_0 + 1)H_1, \quad \text{on } \Gamma,$$

and so

$$\partial_3 v_0^3 = \kappa^{-1}(\kappa^{-1}q_0 + 1)H_1 - \partial_1 v_0^1 - \partial_2 v_0^2, \quad \text{on } \Gamma.$$

Furthermore, this suggests that $\mathbf{w}_0$ should be constructed as follows: let $\mathbf{w}_0 = (\mathbf{u}_0^1, \mathbf{u}_0^2, \mathbf{w}_0^3)$, where $\mathbf{w}_0^3$ solves

$$\begin{cases} \Delta^2\mathbf{w}_0^3 = \Delta^2\mathbf{u}_0^3, & \text{in } \Omega, \\ \mathbf{w}_0^3 = \mathbf{u}_0^3, & \text{on } \Gamma, \\ \partial_3\mathbf{w}_0^3 = \kappa^{-1}\sigma(\kappa^{-1}\mathbf{p}_0 + 1)\overline{\Delta}\mathbf{u}_0^3 - \kappa^{-1}\sigma(\kappa^{-1}\mathbf{p}_0 + 1)Q(\overline{\partial}\eta_0, \overline{\partial}\mathbf{u}_0)\overline{\partial}^2\eta_0 \\ -\partial_1\mathbf{u}_0^1 - \partial_2\mathbf{u}_0^2, & \text{on } \Gamma. \end{cases}$$

5.5. Construction for $\mathbf{q}_0$ that Satisfies (5.1) While $j = 2$

The second order compatibility condition reads $\partial_t^2 q|_{t=0} = H_2$, and we need to express this in terms of η_0, v_0 and q_0, which yields a system satisfied by $\mathbf{p}_0$. Invoking (5.4), we have

$$\partial_t^2\Big(\sigma g^{ij}\hat{n}_\mu \overline{\partial}^2_{ij}\eta^\mu\Big) =\sigma g^{ij}\hat{n}_\mu \overline{\partial}^2_{ij} v_t^\mu + \sigma Q(\hat{n}, \overline{\partial}\eta, \overline{\partial}v)\overline{\partial}^2 v \\ + \sigma Q(\hat{n}, \overline{\partial}\eta, \overline{\partial}v)\overline{\partial}^2\eta(\overline{\partial}v_t + 1). \tag{5.12}$$

In addition, since $Rv_t^\mu + a^{\nu\mu}\partial_\nu q = 0$, we get for $s = 1, 2$ that

$$\overline{\partial}^s(v_t^\mu) = -R^{-1}a^{\nu\mu}\overline{\partial}^s\partial_\nu q - \sum_{1\le k\le s}\overline{\partial}^k(R^{-1}a^{\nu\mu})\overline{\partial}^{s-k}\partial_\nu q.$$

This, together with (5.5) and the equation of state $R = \kappa^{-1}q + 1$ imply

$$H_2 = H_2(\eta_0, p_0, v_0) = -\sigma(\kappa^{-1}q_0 + 1)^{-1}\overline{\Delta}\partial_3 q_0 + \sigma Q(\overline{\partial}\eta_0, \overline{\partial}v_0)\overline{\partial}^2 v_0 \\ - \sigma Q\Big((\kappa^{-1}q_0 + 1)^{-1}, \kappa^{-1}\overline{\partial}q_0, \partial\eta_0, \overline{\partial}\partial\eta_0\Big)\overline{\partial}\partial q_0 \\ - \sigma Q\Big((\kappa^{-1}q_0 + 1)^{-1}, \partial\eta_0, \kappa^{-1}\partial q_0, \kappa^{-1}\overline{\partial}^2 q_0\Big)\partial q_0 \\ + \sigma Q\Big((\kappa^{-1}q_0 + 1)^{-1}, \overline{\partial}\eta_0, \overline{\partial}\partial\eta_0, \overline{\partial}v_0, \partial q_0\Big)(\overline{\partial}\partial q_0 + \overline{\partial}^2\eta_0). \tag{5.13}$$

On the other hand, the continuity equation implies $Ra^{\mu\alpha}\partial_\mu v_\alpha = -\kappa^{-1}\partial_t q$, and hence

$$-\kappa^{-1}\partial_t^2 q = \partial_t(Ra^{\mu\alpha})\partial_\mu v_\alpha + Ra^{\mu\alpha}\partial_\mu\partial_t v_\alpha \\ = \partial_t(Ra^{\mu\alpha})\partial_\mu v_\alpha - Ra^{\mu\alpha}\partial_\mu(R^{-1}a^\nu{}_\alpha\partial_\nu q) \\ = -a^{\mu\alpha}a^\nu{}_\alpha\partial_\mu\partial_\nu q - Ra^{\mu\alpha}\partial_\mu(R^{-1}a^\nu{}_\alpha)\partial_\nu q + \partial_t(Ra^{\mu\alpha})\partial_\mu v_\alpha. \tag{5.14}$$

Restricting the above identity to the boundary Γ and then taking $t = 0$, we get

$$\kappa^{-1}\partial_t^2 q|_{t=0} = \Delta q_0 - Q\Big((\kappa^{-1}q_0 + 1)^{-1}, \partial\eta_0, \partial^2\eta_0, \overline{\partial}v_0, \kappa^{-1}\partial q_0\Big)\partial q_0 \\ + Q(\kappa^{-1}q_0, \partial\eta_0, \partial v_0)\partial v_0. \tag{5.15}$$

Invoking (5.5) and (5.5), we are able to rewrite (5.1) when $j = 2$ as

$$\Delta q_0 = Q\Big((\kappa^{-1}q_0 + 1)^{-1}, \partial\eta_0, \partial^2\eta_0, \overline{\partial}v_0, \kappa^{-1}\partial q_0\Big)\partial q_0 \\ - Q(\kappa^{-1}q_0, \partial\eta_0, \partial v_0)\partial v_0 + \kappa^{-1}H_2(\eta_0, p_0, v_0). \tag{5.16}$$

This yields that $\mathbf{q}_0$ should solve

$$\begin{cases} \Delta^3\mathbf{q}_0 = 0, & \text{in } \Omega, \\ \mathbf{q}_0 = \mathbf{p}_0, & \text{on } \Gamma, \\ \frac{\partial\mathbf{q}_0}{\partial N} = \partial_3\mathbf{q}_0 = \partial_3\mathbf{p}_0 = \frac{\partial\mathbf{p}_0}{\partial N}, & \text{on } \Gamma, \\ \Delta\mathbf{q}_0 = \varphi, & \text{on } \Gamma. \end{cases}$$

Here,

$$\varphi = Q\Big((\kappa^{-1}\mathbf{p}_0 + 1)^{-1}, \partial\eta_0, \partial^2\eta_0, \overline{\partial}\mathbf{w}_0, \kappa^{-1}\partial\mathbf{p}_0\Big)\partial\mathbf{p}_0 \\ - Q(\kappa^{-1}\mathbf{p}_0, \partial\eta_0, \partial\mathbf{w}_0)\partial\mathbf{w}_0 + \kappa^{-1}H_2(\eta_0, \mathbf{p}_0, \mathbf{w}_0),$$

which is obtained from (5.5).

5.6. *Construction for* $\mathbf{v}_0$ *that Satisfies (5.1) While* $j = 3$

Our last step is to construct $\mathbf{v}_0$ that satisfies third order compatibility condition, that is, $\partial_t^3 q|_{t=0} = H_3$ on Γ. Similar to what has been done for the previous cases when $j = 0, 1, 2$, we shall first compute the compatibility condition explicitly. Invoking (5.5), as well as $v_t^\mu = -R^{-1}a^{\nu\mu}\partial_\mu q$ and $\partial_t q = -\kappa R a^{\mu\alpha}\partial_\mu v_\alpha$, we have

$$\begin{aligned}
\partial_t^3\Big(\sigma(g^{ij}\hat{n}_\mu\overline{\partial}_{ij}^2\eta^\mu\Big) &= -\partial_t\Big(\sigma(g^{ij}\hat{n}_\mu\overline{\partial}_{ij}^2(R^{-1}a^{\nu\mu}\partial_\nu q)\Big) \\
&\quad + \sigma Q(\hat{n}, \overline{\partial}\eta, \overline{\partial}v)\overline{\partial}^2 v_t + \sigma Q(g, \hat{n}, \overline{\partial}\eta, \overline{\partial}v, \overline{\partial}v_t)\overline{\partial}^2 v \\
&\quad + \sigma Q(\hat{n}, \overline{\partial}\eta, \overline{\partial}v, \overline{\partial}v_t)(\overline{\partial}v_{tt} + \overline{\partial}^2\eta) \\
&= -\sigma g^{ij}\hat{n}_\mu\overline{\partial}_{ij}^2\partial_t(R^{-1}a^{\nu\mu}\partial_\nu q) + \sigma Q(\hat{n}, \overline{\partial}\eta, \overline{\partial}v)\overline{\partial}_{ij}^2(R^{-1}a^{\nu\mu}\partial_\nu q) \\
&\quad + \sigma Q(\hat{n}, R^{-1}, \overline{\partial}R^{-1}, \overline{\partial}^2 R^{-1}, \overline{\partial}v, \partial\eta, \overline{\partial}\partial\eta, \overline{\partial}^2\partial\eta, \partial q, \overline{\partial}\partial q)\overline{\partial}^2\partial q \\
&\quad + \sigma Q(\hat{n}, R^{-1}, \overline{\partial}R^{-1}, \overline{\partial}v, \partial\eta, a, \overline{\partial}\partial\eta, \partial q, \overline{\partial}\partial q)\overline{\partial}^2 v \\
&\quad + \sigma\kappa Q(\hat{n}, R^{-1}, \partial R^{-1}, \partial v, \partial^2 v, \partial\eta, \overline{\partial}\partial\eta, \partial q, \overline{\partial}\partial q)(a^{\mu\alpha}\partial_\mu\overline{\partial}\partial v_\alpha + \overline{\partial}^2\eta),
\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
\sigma g^{ij}\hat{n}_\mu\overline{\partial}_{ij}^2\partial_t(R^{-1}a^{\nu\mu}\partial_\nu q) &= \sigma g^{ij}\hat{n}_\mu R^{-1}a^{\nu\mu}\partial_\nu\overline{\partial}_{ij}^2 q_t \\
&\quad + \sigma Q(\hat{n}, R, \overline{\partial}R, \overline{\partial}^2 R, \partial v, \partial\eta, \overline{\partial}\partial\eta, \overline{\partial}^2\partial\eta, \partial q, \overline{\partial}\partial q, \overline{\partial}^2\partial q)(\partial q_t + \overline{\partial}\partial q_t) \\
&= -\kappa\sigma g^{ij}\hat{n}_\mu a^{\nu\mu}\partial_\nu\overline{\partial}_{ij}^2(a^{\alpha\beta}\partial_\alpha v_\beta) \\
&\quad + \kappa\sigma\sum\nolimits_{k=1,2,3} Q(\hat{n}, \partial^k R)\Big(\partial^{3-k}(a^{\alpha\beta}\partial_\alpha v_\beta)\Big) \\
&\quad + \kappa\sigma Q(\hat{n}, R, \partial R, \partial^2 R, \partial v, \partial^2 v, \partial^3 v, \partial\eta, \overline{\partial}\partial\eta, \overline{\partial}^2\partial\eta, \partial q, \overline{\partial}\partial q, \overline{\partial}^2\partial q)\Big(a^{\alpha\beta}\partial_\alpha \\
&\quad (\overline{\partial}\partial v_\beta + \partial v_\beta)\Big).
\end{aligned} \tag{5.18}$$

Restricting (5.6) and (5.6) at $t = 0$, we get

$$\begin{aligned}
H_3 = H_3(\eta_0, q_0, v_0) &= -\kappa\sigma\partial_3\overline{\Delta}\mathrm{div}\, v_0 - \sigma\sum_{\ell=1,2,3}(\partial^\ell q_0)(\partial^{3-\ell}\mathrm{div}\, v_0) \\
&\quad + \kappa\sigma Q\Big((\kappa^{-1}q_0+1)^{-1}, \partial v_0, \partial^2 v_0, \partial^3 v_0, \partial\eta_0, \partial^2\eta_0, \partial^3\eta_0, \partial q_0, \partial^2 q_0, \overline{\partial}^2\partial q_0\Big) \\
&\quad \sum_{\ell=1,2}\partial^\ell\mathrm{div}\, v_0.
\end{aligned} \tag{5.19}$$

Next, invoking (5.5), we obtain

$$
\begin{aligned}
\kappa^{-1}q_{ttt} &= \partial_t\Big(a^{\mu\alpha}a^{\nu}_{\ \alpha}\partial_\mu\partial_\nu q + Ra^{\mu\alpha}\partial_\mu(R^{-1}a^{\nu}_{\ \alpha})\partial_\nu q - \partial_t(Ra^{\mu\alpha})\partial_\mu v_\alpha\Big)\\
&= a^{\mu\alpha}a^{\nu}_{\ \alpha}\partial_\mu\partial_\nu q_t + Ra^{\mu\alpha}\partial_\mu(R^{-1}a^{\nu}_{\ \alpha})\partial_\nu q_t\\
&\quad + Q(R, R^{-1}, \partial R^{-1}, \partial\eta, \overline{\partial}\partial\eta, v, \partial v)\partial^2 q\\
&= -R\kappa a^{\mu\alpha}a^{\nu}_{\ \alpha}\partial_\mu\partial_\nu(a^{\beta\gamma}\partial_\beta v_\gamma)\\
&\quad - 2\kappa a^{\mu\alpha}a^{\nu}_{\ \alpha}(\partial_\mu R)\partial_\nu(a^{\beta\gamma}\partial_\beta v_\gamma)\\
&\quad - \kappa a^{\mu\alpha}a^{\nu}_{\ \alpha}(\partial_\mu\partial_\nu R)(a^{\beta\gamma}\partial_\beta v_\gamma)\\
&\quad + Q(R, R^{-1}, \partial R^{-1}, \partial\eta, \overline{\partial}\partial\eta)\partial(a^{\beta\gamma}\partial_\beta v_\gamma)\\
&\quad + Q(R, \partial R, R^{-1}, \partial R^{-1}, \partial\eta, \overline{\partial}\partial\eta)a^{\beta\gamma}\partial_\beta v_\gamma\\
&\quad + Q(R, R^{-1}, \partial R^{-1}, \partial\eta, \overline{\partial}\partial\eta, v, \partial v)\partial^2 q. \qquad (5.20)
\end{aligned}
$$

Restricting (5.6) to the boundary Γ and then taking $t = 0$, we have

$$
\begin{aligned}
\kappa^{-1}q_{ttt}|_{t=0} &= -\kappa R_0\Delta\mathrm{div}\, v_0 - \sum\nolimits_{\ell=1,2} 2(\partial^\ell q_0)(\partial^{2-\ell}\mathrm{div}\, v_0)\\
&+ Q\Big((\kappa^{-1}q_0+1)^{-1}, \kappa^{-1}q_0, \partial v_0, \partial\eta_0, \partial^2\eta_0\Big)\sum\nolimits_{\ell=0,1}\partial^\ell \mathrm{div}\, v_0\\
&+ Q\Big((\kappa^{-1}q_0+1)^{-1}, \kappa^{-1}q_0, v_0, \partial v_0, \partial\eta_0, \partial^2\eta_0\Big)\partial^2 q_0.
\end{aligned}
$$

Invoking (5.6), the compatibility condition $q_{ttt}|_{t=0} = H_3$ can then be re-expressed as

$$
\Delta\mathrm{div}\, v_0 = \psi(\eta_0, q_0, v_0)
$$

where

$$
\begin{aligned}
\psi(\eta_0, q_0, v_0) &= -\kappa^{-1}(\kappa^{-1}q_0+1)\sum\nolimits_{\ell=1,2} 2(\partial^\ell q_0)(\partial^{2-\ell}\mathrm{div}\, v_0)\\
&+ \kappa^{-1}Q\Big((\kappa^{-1}q_0+1)^{-1}, \kappa^{-1}q_0, \partial v_0, \partial\eta_0, \partial^2\eta_0\Big)\sum\nolimits_{\ell=0,1}\partial^\ell \mathrm{div}\, v_0\\
&+ \kappa^{-1}Q\Big((\kappa^{-1}q_0+1)^{-1}, \kappa^{-1}q_0, v_0, \partial v_0, \partial\eta_0, \partial^2\eta_0\Big)\partial^2 q_0\\
&- \kappa^{-2}(\kappa^{-1}q_0+1)^{-1}H_3(\eta_0, q_0, v_0).
\end{aligned}
$$

This implies that $\mathbf{v}_0 = (\mathbf{v}_0^1, \mathbf{v}_0^2, \mathbf{v}_0^3)$ should be constructed such that $\mathbf{v}_0^1 = \mathbf{u}_0^1$ and $\mathbf{v}_0^2 = \mathbf{u}_0^2$, whereas $\mathbf{v}_0^3$ solves

$$
\begin{cases}
\Delta^4\mathbf{v}_0^3 = \Delta^4\mathbf{w}_0^3, \quad \text{in } \Omega,\\
\mathbf{v}_0^3 = \mathbf{w}_0^3, \quad \text{on } \Gamma,\\
\partial_3\mathbf{v}_0^3 = \kappa^{-1}\sigma(\kappa^{-1}\mathbf{q}_0+1)\overline{\Delta}\mathbf{w}_0^3 - \kappa^{-1}\sigma(\kappa^{-1}\mathbf{q}_0+1)Q(\overline{\partial}\eta_0, \overline{\partial}\mathbf{w}_0)\overline{\partial}^2\eta_0\\
-\partial_1\mathbf{w}_0^1 - \partial_2\mathbf{w}_0^2, \quad \text{on } \Gamma,\\
\partial_3^2\mathbf{v}_0^3 = \partial_3\Big(\kappa^{-1}\sigma(\kappa^{-1}\mathbf{q}_0+1)\overline{\Delta}\mathbf{w}_0^3 - \kappa^{-1}\sigma(\kappa^{-1}\mathbf{q}_0+1)Q(\overline{\partial}\eta_0, \overline{\partial}\mathbf{w}_0)\overline{\partial}^2\eta_0\\
-\partial_1\mathbf{w}_0^1 - \partial_2\mathbf{w}_0^2\Big), \quad \text{on } \Gamma,\\
\Delta\partial_3\mathbf{v}_0^3 = \psi(\eta_0, \mathbf{q}_0, \mathbf{u}_0) - \Delta\partial_1\mathbf{w}_0^1 - \Delta\partial_2\mathbf{w}_0^2, \quad \text{on } \Gamma.
\end{cases}
$$

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Appendix

A Basic Estimates

Theorem A.1. (Standard div-curl estimates) *Let X be a vector field on Ω with sufficiently regular boundary Γ. Define $div\, X = \partial_j X^j$ and $(curl\, X)_{ij} = \partial_i X_j - \partial_j X_i$, then for $1 \le s \le 4$, we have*

$$||X||_s \lesssim ||div\, X||_{s-1} + ||curl\, X||_{s-1} + ||X \cdot N||_{s-0.5,\Gamma} + ||X||_0, \tag{A.1}$$
$$||X||_s \lesssim ||div\, X||_{s-1} + ||curl\, X||_{s-1} + ||X \cdot \mathcal{T}||_{s-0.5,\Gamma} + ||X||_0, \tag{A.2}$$

where N is the outward unit normal to Γ, whereas $\mathcal{T}$ is the unit vector which is tangent to Γ.

Proof. We refer [47] for the detailed proof. □

B The Energy Identity for the Wave Equations of Order 3

We recall that for $r = 1, 2, 3$, the wave equation reads as

$$JR'\partial_t^{r+1} q - a^{\nu\alpha} A^{\mu}_{\alpha} \partial_\nu \partial_\mu \partial_t^{r-1} q = \mathcal{G}_r + \mathcal{S}_r,$$

where

$$\begin{aligned}\mathcal{G}_r = &-\sum_{\substack{j_1+j_2=r\\ j_1\ge 1}} \left(\partial_t^{j_1}(JR')\right)(\partial_t^{j_2+1} q) + a^{\nu\alpha}(\partial_\nu \rho_0)\partial_t^r v_\alpha \\ &+ \sum_{\substack{j_1+j_2=r-1\\ j_1\ge 1}} a^{\nu\alpha}\partial_\nu(\partial_t^{j_1} A^{\mu}_{\alpha} \cdot \partial_\mu \partial_t^{j_2} q) \\ &- \rho_0 \sum_{j_1+j_2=r-1} (\partial_t^{j_1+1} a^{\nu\alpha})(\partial_t^{j_2}\partial_\nu v_\alpha).\end{aligned}$$

and

$$\mathcal{S}_r = a^{\nu\alpha}(\partial_\nu A^{\mu}_{\alpha})\partial_\mu \partial_t^{r-1} q.$$

Theorem B.1. *For $r = 1, 2, 3$, let*

$$W_r^2 = \frac{1}{2}\int_\Omega \rho_0^{-1}(JR'\partial_t^r q)^2\,\mathrm{d}y + \frac{1}{2}\int_\Omega \rho_0^{-1}R'(A^{\nu\alpha}\partial_\nu\partial_t^{r-1}q)(A^\mu_{\ \alpha}\partial_\mu\partial_t^{r-1}q)\,\mathrm{d}y$$
$$+ \frac{\sigma}{2}\int_\Gamma \mathfrak{R}_\kappa\sqrt{g}g^{ij}\Pi^\alpha_\mu(\overline{\partial}_i\partial_t^r\eta^\mu)(\overline{\partial}_j\partial_t^r\eta_\alpha)\,\mathrm{d}S.$$

Then,

$$\sum\nolimits_{r\le 3} W_r^2 \le \epsilon P(\mathcal{N}) + \epsilon(||q||_2^2 + ||q_t||_2^2) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}, \quad t\in[0,T],$$

where $T > 0$ is sufficiently small.

Proof of Theorem B.1. It suffices to consider the case when $r = 3$. Invoking (1.1) and (1.3), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}&\int_\Omega \rho_0^{-1}(JR'\partial_t^3 q)^2\,\mathrm{d}y\\ &= \int_\Omega \rho_0^{-1}(JR'\partial_t^3 q)(a^{\nu\alpha}A^\mu_{\ \alpha}\partial_\nu\partial_\mu\partial_t^2 q)\,\mathrm{d}y\\ &\quad + \int_\Omega \rho_0^{-1}(JR'\partial_t^3 q)(\mathcal{G}_3 + \mathcal{S}_3)\,\mathrm{d}y + \mathcal{R}\end{aligned} \tag{B.1}$$

where $\mathcal{R}$ consists of error terms that are generated when ∂_t falls on either J or R', which we have no problem to control. In addition,

$$\begin{aligned}&\int_\Omega \rho_0^{-1}(JR'\partial_t^3 q)(a^{\nu\alpha}A^\mu_{\ \alpha}\partial_\nu\partial_\mu\partial_t^2 q)\,\mathrm{d}y\\ &= \int_\Omega \rho_0^{-1}(R'\partial_t^3 q)(A^{\nu\alpha}\partial_\nu)(A^\mu_{\ \alpha}\partial_\mu\partial_t^2 q)\,\mathrm{d}y - \int_\Omega \rho_0^{-1}(JR'\partial_t^3 q)\mathcal{S}_3.\end{aligned} \tag{B.2}$$

The last term in (B.1) cancels with the corresponding term in (B.1), which is essential since $||\mathcal{S}_3||_0$ cannot be controlled uniformly when $R' \to 0$. Moreover, the first term on the right hand side of (B.1) is treated as

$$\begin{aligned}&\int_\Omega \rho_0^{-1}(R'\partial_t^3 q)(A^{\nu\alpha}\partial_\nu)(A^\mu_{\ \alpha}\partial_\mu\partial_t^2 q)\,\mathrm{d}y\\ &= -\int_\Omega \rho_0^{-1}R'(A^{\nu\alpha}\partial_\nu\partial_t^3 q)(A^\mu_{\ \alpha}\partial_\mu\partial_t^2 q)\,\mathrm{d}y\\ &\quad + \int_\Gamma \rho_0^{-1}R'(A^{\nu\alpha}N_\nu\partial_t^3 q)(A^\mu_{\ \alpha}\partial_\mu\partial_t^2 q)\,\mathrm{d}S + \mathcal{R}.\end{aligned} \tag{B.3}$$

The first term on the right hand side of (B.1) is equal to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \rho_0^{-1}R'(A^{\nu\alpha}\partial_\nu\partial_t^2 q)(A^\mu_{\ \alpha}\partial_\mu\partial_t^2 q)\,\mathrm{d}y + \mathcal{R},$$

and hence moved to the left. In addition,

$$\int_\Gamma \rho_0^{-1}R'(A^{\nu\alpha}N_\nu\partial_t^3 q)(A^\mu_{\ \alpha}\partial_\mu\partial_t^2 q)\,\mathrm{d}S$$

$$
\begin{aligned}
&= \int_\Gamma \rho_0^{-1} R' \partial_t^3 (A^{\nu\alpha} N_\nu q) \partial_t^2 (A^\mu_{\ \alpha} \partial_\mu q)\, \mathrm{d}S \\
&\quad \underbrace{- \int_\Gamma \rho_0^{-1} R' \partial_t^3 (A^{\nu\alpha} N_\nu q)(\partial_t A^\mu_{\ \alpha})(\partial_\mu \partial_t q)}_{\mathcal{WB}_1} \\
&\quad \underbrace{- \sum_{\substack{j_1+j_2=3 \\ j_1 \geq 1}} \int_\Gamma \rho_0^{-1} R' \partial_t^2 (A^\mu_{\ \alpha} \partial_\mu q)(\partial_t^{j_1} A^{\nu\alpha})(N_\nu \partial_t^{j_2} q)}_{\mathcal{WB}_2} \\
&\quad + \underbrace{\sum_{\substack{j_1+j_2=3 \\ j_1 \geq 1}} \int_\Gamma \rho_0^{-1} R' (\partial_t A^\mu_{\ \alpha})(\partial_\mu \partial_t q)(\partial_t^{j_1} A^{\nu\alpha})(N_\nu \partial_t^{j_2} q)}_{\mathcal{WB}_3},
\end{aligned}
$$

which is due to

$$
\begin{aligned}
A^{\nu\alpha} N_\nu \partial_t^3 q &= \partial_t^3 (A^{\nu\alpha} N_\nu q) - \sum_{\substack{j_1+j_2=3 \\ j_1 \geq 1}} (\partial_t^{j_1} A^{\nu\alpha}) N_\nu \partial_t^{j_2} q, \\
A^\mu_{\ \alpha} \partial_\mu \partial_t^2 q &= \partial_t^2 (A^\mu_{\ \alpha} \partial_\mu q) - (\partial_t A^\mu_{\ \alpha}) \partial_\mu \partial_t q.
\end{aligned}
$$

Next, invoking (1.1), (1.3) and (2.1), the main boundary term is equal to

$$
\begin{aligned}
&\sigma \int_\Gamma \mathfrak{R}_\kappa \sqrt{g} g^{ij} \Pi^\alpha_\mu (\partial_t^3 \overline{\partial}^2_{ij} \eta^\mu)(\partial_t^4 \eta_\alpha) \\
&\qquad + \underbrace{\sigma \sum_{\substack{j_1+j_2=3 \\ j_1 \geq 1}} \int_\Gamma \mathfrak{R}_\kappa (\partial_t^{j_1} \sqrt{g} g^{ij} \Pi^\alpha_\mu)(\overline{\partial}^2_{ij} \partial_t^{j_2} \eta^\mu)(\partial_t^3 v_\alpha)}_{\mathcal{WB}_4} \\
&\quad = -\sigma \int_\Gamma \mathfrak{R}_\kappa \sqrt{g} g^{ij} \Pi^\alpha_\mu (\partial_t^3 \overline{\partial}_i \eta^\mu)(\overline{\partial}_j \partial_t^4 \eta_\alpha) + \mathcal{WB}_4 \\
&\qquad \underbrace{- \sigma \int_\Gamma \mathfrak{R}_\kappa \overline{\partial}_j (\sqrt{g} g^{ij} \Pi^\alpha_\mu)(\partial_t^3 \overline{\partial}_i \eta^\mu)(\partial_t^3 v_\alpha)}_{\mathcal{WB}_5}.
\end{aligned}
$$

The first term on the last line is equal to

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t} \frac{\sigma}{2} \int_\Gamma \mathfrak{R}_\kappa \sqrt{g} g^{ij} \Pi^\alpha_\mu (\partial_t^3 \overline{\partial}_i \eta^\mu)(\overline{\partial}_j \partial_t^3 \eta_\alpha) \\
&\quad + \underbrace{\frac{\sigma}{2} \int_\Gamma \mathfrak{R}_\kappa \partial_t (\sqrt{g} g^{ij} \Pi^\alpha_\mu)(\partial_t^3 \overline{\partial}_i \eta^\mu)(\overline{\partial}_j \partial_t^3 \eta_\alpha)}_{\mathcal{WB}_6},
\end{aligned}
$$

where the main term is moved to the left, and this completes the construction for (2.5).

The proof of Theorem B.1 requires the bound for $\int_0^t ||\mathcal{G}_3||_0$ and $\sum_{1 \leq j \leq 6} \int_0^t \mathcal{WB}_j$. There is no problem to control $\int_0^t ||\mathcal{G}_3||_0$. In addition, using the duality, we have

$$
\mathcal{WB}_1 \lesssim P(||v||_3, ||\eta||_3) ||R' \partial_t^3 (A^{3\alpha} q)||_0 ||\partial_t q||_2,
$$

and

$$\mathcal{WB}_2 \lesssim P(||v||_3, ||\eta||_3)\Big(||(\sqrt{R'}\partial_t^2(A^\mu_{\ \alpha}\partial_\mu q)||_0(\sqrt{R'}\partial\partial_t^2 v)||_0||q||_2 + ||(\sqrt{R'}\partial_t^2(A^\mu_{\ \alpha}\partial_\mu q)||_0(\sqrt{R'}\partial\partial_t v)||_1||\partial_t q||_2 + ||\sqrt{R'}\partial_t^2(A^\mu_{\ \alpha}\partial_\mu q)||_0||\sqrt{R'}q_{tt}||_1\Big).$$

Therefore, $\int_0^t \mathcal{WB}_1 + \mathcal{WB}_2$ can be controlled appropriately. Moreover, $\int_0^t \mathcal{WB}_3 + \mathcal{WB}_6$ is controlled in a routine fashion. On the other hand, $\int_0^t \mathcal{WB}_4 + \mathcal{WB}_5$ is treated in [21], where the $\mathfrak{R}_\kappa$-weights are incorporated so that the estimates in [21] can go through. □

C The Energy Identity for $\mathfrak{R}_\kappa$-weighted Wave Equations

We recall that the $\mathfrak{R}_\kappa$-weighted wave equation reads:

$$\mathfrak{R}_\kappa^\ell R' J D^3\partial_t^2 q - \mathfrak{R}_\kappa^\ell a^{\nu\alpha}A^\mu_{\ \alpha}\partial_\nu\partial_\mu D^3 q = \widetilde{\mathcal{G}}_4 + \widetilde{\mathcal{S}}_4,$$

where

$$\widetilde{\mathcal{G}}_4 = -\mathfrak{R}_\kappa^\ell[D^3\partial_t, JR']\partial_t q + \mathfrak{R}_\kappa^\ell[D^3, \rho_0]\partial_t(R^{-1}R'\partial_t q) + \mathfrak{R}_\kappa^\ell a^{\nu\alpha}(\partial_\nu\rho_0)D^3\partial_t v_\alpha + \mathfrak{R}_\kappa^\ell a^{\nu\alpha}\partial_\nu\big([D^3, A^\mu_{\ \alpha}]\partial_\mu q\big) + \mathfrak{R}_\kappa^\ell a^{\nu\alpha}\partial_\nu\big([D^3, \rho_0]\partial_t v_\alpha\big) - \mathfrak{R}_\kappa^\ell\rho_0[D^3\partial_t, a^{\nu\alpha}]\partial_\nu v_\alpha,$$

and

$$\widetilde{\mathcal{S}}_4 = \mathfrak{R}_\kappa^\ell a^{\nu\alpha}(\partial_\nu A^\mu_{\ \alpha})\partial_\mu D^3 q.$$

Here, $\ell = 1$ when $D^3 = \partial_t^3$, $\ell = \frac{1}{2}$ when $D^3 = \partial_t^2\overline{\partial}$ and $\ell = 0$ when $D^3 = \partial_t\overline{\partial}^2$.

Theorem C.1. *Let*

$$W_4^2 = \frac{1}{2}\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(JR'D^3\partial_t q)^2\,\mathrm{d}y + \frac{1}{2}\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}R'(A^{\nu\alpha}\partial_\nu D^3 q)(A^\mu_{\ \alpha}\partial_\mu D^3 q)\,\mathrm{d}y + \frac{\sigma}{2}\int_\Gamma \mathfrak{R}_\kappa^{2\ell+1}\sqrt{g}g^{ij}\Pi^\alpha_\mu(\overline{\partial}_i D^3\partial_t\eta^\mu)(\overline{\partial}_j D^3\partial_t\eta_\alpha)\,\mathrm{d}S.$$

Then,

$$W_4^2 \le \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}, \quad t\in[0,T],$$

where $T > 0$ is sufficiently small.

Proof of Theorem C.1. Invoking (1.1) and (1.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(JR'D^3\partial_t q)^2\,\mathrm{d}y$$
$$=\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(JR'D^3\partial_t q)(a^{\nu\alpha}A^\mu_{\ \alpha}\partial_\nu\partial_\mu D^3 q)\,\mathrm{d}y$$
$$+\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(JR'D^3\partial_t q)(\widetilde{\mathcal{G}}_4+\widetilde{\mathcal{S}}_4)\,\mathrm{d}y+\mathcal{R}, \tag{C.1}$$

where $\mathcal{R}$ consists error terms that are generated when ∂_t falls on either J or R', which we have no problem to control. In addition,

$$\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(JR'D^3\partial_t q)(a^{\nu\alpha}A^\mu_{\ \alpha}\partial_\nu\partial_\mu D^3 q)\,\mathrm{d}y$$
$$=\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(R'D^3\partial_t q)(A^{\nu\alpha}\partial_\nu)(A^\mu_{\ \alpha}\partial_\mu D^3 q)\,\mathrm{d}y$$
$$-\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(JR'D^3\partial_t q)\widetilde{\mathcal{S}}_4. \tag{C.2}$$

The last term in (C.1) cancels with the corresponding term in (C.1), which is essential since $||\widetilde{\mathcal{S}}_3||_0$ cannot be controlled uniformly when $R'\to 0$. Moreover, the first term on the right hand side of (C.1) is treated as

$$\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}(R'D^3\partial_t q)(A^{\nu\alpha}\partial_\nu)(A^\mu_{\ \alpha}\partial_\mu D^3 q)\,\mathrm{d}y$$
$$=-\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}R'(A^{\nu\alpha}\partial_\nu D^3\partial_t q)(A^\mu_{\ \alpha}\partial_\mu D^3 q)\,\mathrm{d}y$$
$$+\int_\Gamma \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}R'(A^{\nu\alpha}N_\nu D^3\partial_t q)(A^\mu_{\ \alpha}\partial_\mu D^3 q)\,\mathrm{d}S+\mathcal{R}. \tag{C.3}$$

The first term on the right hand side of (C.1) is equal to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}R'(A^{\nu\alpha}\partial_\nu D^3 q)(A^\mu_{\ \alpha}\partial_\mu D^3 q)\,\mathrm{d}y+\mathcal{R},$$

and hence moved to the left. In addition,

$$\int_\Gamma \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}R'(A^{\nu\alpha}N_\nu D^3\partial_t q)(A^\mu_{\ \alpha}\partial_\mu D^3 q)\,\mathrm{d}S$$
$$=\int_\Gamma \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}R'D^3\partial_t(A^{\nu\alpha}N_\nu q)D^3(A^\mu_{\ \alpha}\partial_\mu q)\,\mathrm{d}S$$
$$\underbrace{-\int_\Gamma \rho_0^{-1}\mathfrak{R}_\kappa^{2\ell}R'D^3\partial_t(A^{\nu\alpha}N_\nu q)\big([D^3,A^\mu_{\ \alpha}]\partial_\mu q\big)}_{\widetilde{\mathcal{WB}}_1}$$
$$\underbrace{-\int_\Gamma \rho_0^{-1}R'\mathfrak{R}_\kappa^{2\ell}D^3(A^\mu_{\ \alpha}\partial_\mu q)\big([D^3\partial_t,A^{\nu\alpha}]N_\nu q\big)}_{\widetilde{\mathcal{WB}}_2}$$

$$+\underbrace{\int_\Gamma \rho_0^{-1} R' \mathfrak{R}_\kappa^{2\ell}\big([D^3\partial_t, A^{\nu\alpha}]N_\nu q\big)\big([D^3, A^\mu_{\ \alpha}]\partial_\mu q\big)}_{\widetilde{\mathcal{WB}}_3},$$

which is due to

$$A^{\nu\alpha}N_\nu D^3\partial_t q = D^3\partial_t(A^{\nu\alpha}N_\nu q) - [D^3\partial_t, A^{\nu\alpha}]N_\nu q,$$
$$A^\mu_{\ \alpha}\partial_\mu D^3 q = D^3(A^\mu_{\ \alpha}\partial_\mu q) - [D^3, A^\mu_{\ \alpha}]\partial_\mu q.$$

Next, invoking (1.1), (1.3) and (2.1), the main boundary term is equal to

$$\sigma\int_\Gamma \mathfrak{R}_\kappa^{2\ell+1}\sqrt{g}g^{ij}\Pi^\alpha_\mu(D^3\partial_t\overline{\partial}^2_{ij}\eta^\mu)(D^3\partial_t^2\eta_\alpha)$$
$$+\underbrace{\int_\Gamma \mathfrak{R}_\kappa^{2\ell+1}[D^3\partial_t, \sqrt{g}g^{ij}\Pi^\alpha_\mu](\overline{\partial}^2_{ij}\eta^\mu)(D^3\partial_t v_\alpha)}_{\widetilde{\mathcal{WB}}_4}+\mathcal{R}$$
$$= -\sigma\int_\Gamma \mathfrak{R}_\kappa^{2\ell+1}\sqrt{g}g^{ij}\Pi^\alpha_\mu(D^3\partial_t\overline{\partial}_i\eta^\mu)(\overline{\partial}_j D^3\partial_t^2\eta_\alpha) + \widetilde{\mathcal{WB}}_4$$
$$\underbrace{-\sigma\int_\Gamma \mathfrak{R}_\kappa^{2\ell+1}\overline{\partial}_j(\sqrt{g}g^{ij}\Pi^\alpha_\mu)(D^3\partial_t\overline{\partial}_i\eta^\mu)(D^3\partial_t v_\alpha)}_{\widetilde{\mathcal{WB}}_5}+\mathcal{R}.$$

The first term on the last line is equal to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sigma}{2}\int_\Gamma \mathfrak{R}_\kappa^{2\ell+1}\sqrt{g}g^{ij}\Pi^\alpha_\mu(D^3\partial_t\overline{\partial}_i\eta^\mu)(\overline{\partial}_j D^3\partial_t\eta_\alpha)$$
$$+\underbrace{\frac{\sigma}{2}\int_\Gamma \mathfrak{R}_\kappa^{2\ell+1}\partial_t(\sqrt{g}g^{ij}\Pi^\alpha_\mu)(D^3\partial_t\overline{\partial}_i\eta^\mu)(\overline{\partial}_j D^3\partial_t\eta_\alpha)}_{\widetilde{\mathcal{WB}}_6},$$

where the main term is moved to the left, and this completes the construction for (2.5).

The proof of Theorem C.1 requires the bound for $\int_0^t ||\widetilde{\mathcal{G}}_4||_0$ and $\sum_{1\le j\le 6}\int_0^t \widetilde{\mathcal{WB}}_j$. First, $\int_0^t \widetilde{\mathcal{WB}}_1 + \widetilde{\mathcal{WB}}_2$ can be controlled similar to $\int_0^t \mathcal{WB}_1 + \mathcal{WB}_2$ in the previous section, after distributing correct $\mathfrak{R}_\kappa$-weights. Second, the control of $\int_0^t ||\widetilde{\mathcal{G}}_4||_0$ and $\int_0^t \widetilde{\mathcal{WB}}_3$ can be done in a routine fashion. Finally, $\int_0^t \widetilde{\mathcal{WB}}_4 + \widetilde{\mathcal{WB}}_5 + \widetilde{\mathcal{WB}}_6$ is treated similar to $\int_0^t \mathcal{B}$ in section 3.4. □

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