Finite Time Stability of Sets for Hybrid Dynamical Systems ⋆

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Abstract

Notions and tools for finite time stability of closed sets for hybrid dynamical systems modeled as hybrid inclusions are introduced. Finite time stability of a closed set is defined as the following two properties: *Lyapunov stability*, namely, the property that solutions that start close to the set stay close to it, and *finite time convergence*. In the latter property, the amount of time required to converge to the set of interest is captured by a settling-time function that depends on *hybrid time*, namely, the time elapsed during flows and the number of jumps (or events) of the hybrid system. Various sufficient conditions for such properties to hold for a given closed set are established. Conditions involving Lyapunov-like functions that strictly decrease during flows, that strictly decrease during jumps, and that strictly decrease along both regimes are proposed – these functions are only required to be locally Lipschitz. A link between (non-finite time) asymptotic stability and the proposed notion is also established. In addition, sufficient conditions for finite time attractivity involving properties at jumps of the system, the second derivative of a Lyapunov-like function, and a property of nested sets are provided. Throughout the paper, examples exercise the results. In addition, an application to finite time parameter estimation is provided and results are applied to it.

Key words: Finite time stability; Lyapunov stability; Hybrid systems.

1 Introduction

Finite time stability is a notion that requires convergence of solutions in finite time. More precisely, a point or, more generally, a closed set $\mathcal{A}$ is finite time stable if the following two properties hold for the distance between any maximal solution and the set of interest: has stable behavior in the Lyapunov sense, that is, solutions that start close to the set stay close to it; and converges to zero in finite time. For a continuous-time system of the form

$$\dot{z} = f(z),$$

the uniform version of finite time stability for a closed set $\mathcal{A}$ is typically defined as follows:

⋆) Every solution $t \mapsto \phi(t)$ to the system satisfies

$$|\phi(t)|_{\mathcal{A}} \leq \beta(|\phi(0)|_{\mathcal{A}}, t) \tag{1}$$

for each t in the domain of definition of ϕ.

In (1), $|\cdot|_{\mathcal{A}}$ denotes the distance to the set $\mathcal{A}$, where β is a class-$\mathcal{GKL}$ function [1]; see, e.g., [19,8]. The $\mathcal{GKL}$ estimate in (1) implies that the Euclidean distance between the solution ϕ and the set $\mathcal{A}$ is upper bounded by a function of their initial distance that decreases to zero in finite time (when the domain of ϕ is long enough). A naturally similar notion can be formulated for discrete-time systems $z^+ = g(z)$, where z^+ denotes the value of the state after discrete updates.

The problem of finite time stability of sets for a class of hybrid dynamical systems is considered in this paper. A hybrid dynamical system is denoted as $\mathcal{H}$, has data (C, F, D, G), and is defined by

$$\begin{aligned} \dot{z} &\in F(z) \quad z \in C, \\ z^+ &\in G(z) \quad z \in D, \end{aligned} \tag{2}$$

where z is the state, F is a set-valued map that defines the continuous dynamics, which are allowed on the subset of the state space C, and G is a set-valued map that defines the discrete behavior, which is allowed on the subset D. A solution to $\mathcal{H}$ is given by a function ϕ that

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[1] A class-$\mathcal{GKL}$ function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is such that, for each fixed $t \geq 0$, $s \mapsto \beta(s,t)$ is strictly increasing and continuous, and $\beta(0,t) = 0$ for all $t \in [0,\infty)$; and, for each fixed $s \geq 0$, $t \mapsto \beta(s,t)$ is continuous and, for some $T \in (0,\infty)$, decreases to zero as t approaches T.

is parametrized by $(t, j) \in [0, \infty) \times \{0, 1, 2, \ldots\}$. Whenever ϕ flows according to the continuous dynamics, t increases with ordinary time, while, whenever ϕ has a jump according to the discrete dynamics, j is incremented by one so as to denote the number of jumps in the solution thus far. For this class of hybrid dynamical systems, the notion of finite time stability, its properties, and sufficient conditions that guarantee it are provided. Due to the combination of continuous and discrete dynamics, an appropriate notion of finite time stability for such systems should allow for the finite time convergence to occur after a finite amount of ordinary time t, a finite number of jumps j, or both. Due to this, the function characterizing the time to converge to the set of interest may depend on both t and j. The presence of the sets C and D in the model (2) further demands solutions to exist for enough (hybrid) time so as to allow them to converge to the set of interest. The notions and results in this paper address all of these challenges.

1.1 Related Work

Significant contributions to the problem of finite time stability in dynamical systems is available in the literature. One of the earliest contributions are the works by Salehi and Ryan [30] and Haimo [17] on finite time controllers. In [30] an optimal feedback controller that converges to the origin in finite time for linear time-invariant systems with Hurwitz system matrix and nonsingular input matrix is provided. In [17], motivated by the need of feedback laws that guarantee convergence to the origin after a finite amount of time, and unlike optimal feedback laws that are given by continuous functions of the state, the notion of *finite time differential equations* in continuous time was introduced. Sufficient conditions involving integrals and auxiliary finite time differential equations are provided for first and second order systems, and, using a comparison principle and Lyapunov functions, extended to nonlinear systems of arbitrary dimension. Building from these results and motivated by the stabilization to the origin of double integrators (see [6]), Bhat and Bernstein formalized the notion of finite time stability of the origin and uniqueness of solutions to such systems in [6] and [8]. The main motivator for the study of uniqueness of solutions is that a system with its origin being finite time stable may not have unique solutions due to the fact that solutions from the origin that go backward in time are nonunique; hence, the right-hand side of the system might not be Lipschitz at the origin. In [8], within a general framework, necessary and sufficient conditions for finite time stability are presented when solutions are unique in forward time; cf. [31,17]. A key contribution of [8] is that continuity of the time to converge function (or settling function) plays a key role in finite time stability of the origin. More recently, Moulay and Perruquetti in [24] address the case when this function is not necessarily continuous. In a follow up article, [26], the authors establish several necessary and sufficient conditions for continuous-time nonautonomous systems using Lyapunov functions. These finite time stability notions and sufficient conditions have been used in multiple applications, including the design of observers [23,21], consensus algorithms for multi-agent systems [20], finite-time converging feedback controllers [30,33], and finite-time parameter estimators [29]. It should be noted that much of the literature on finite time stability pertains to sufficient conditions. It appears that the only rigorous statements about converse Lyapunov theorems for such a notion are in [7,8]. In those articles, Bhat and Bernstein formulate a converse result that requires continuity of the settling-time function. They argue that, in general, one may not be able to relax such assumption and indicate that there are examples with a finite time stable equilibrium point for which it is impossible to find a continuously differentiable Lyapunov function certifying finite time stability.

Another type of finite time stability following the idea in [12] has also been widely researched. Such notion guarantees that, within a finite time interval, all inputs bounded by a prescribed constant result in outputs bounded by another constant. In [4], finite-time control problems are studied. In [3,2], finite time stability for continuous-time linear time-varying systems with jumps is considered. In [5], sufficient conditions for finite time stability of systems with impulsive dynamics are studied under the assumption that the continuous-time dynamics are linear time-varying and impulsive behavior is linear, which restricts its applicability.

The fact that the aforementioned results on finite time stability pertain to either continuous-time or discrete-time systems prevents their use for systems with variables that can change both continuously and, at times, jump, that is, *hybrid dynamical systems*. Hybrid dynamical systems are a powerful class of systems able of capturing the dynamics of a wide range of systems, ranging from mechanical and electrical systems to biological systems, and beyond. While a theory for the study of asymptotic stability of sets, along with its robustness, is available in the literature [14,15], the case of that notion when attractivity occurs in finite (hybrid) time, namely, finite time stability, has not been thoroughly studied for such systems. The hybrid systems literature includes very few – yet important – efforts towards the development of such missing theory. In [27], Nersesov and Haddad introduce sufficient conditions for finite time stability of the origin for a particular class of time-driven hybrid systems, referred to as impulsive dynamical systems. In that work the authors employ scalar and vector Lyapunov functions, being sufficient conditions for the continuous evolution of the impulsive systems their focus – in particular, finite time attractivity due to jumps, or both due to flows and jumps, are not studied in [27]. In [28], Orlov studies finite-time stability of the origin of switched systems that are piecewise continuous and homogeneous, and shows that global asymptotic stability of the origin implies global finite-time stability. In [10], and relying on the results in [8], Chen and coauthors study switched systems and provide conditions to assure finite time stability of the origin. In addition to a notion of finite time stability that does not prioritize neither the continuous nor the discrete dynamics, results on robustness of finite time stability for general hybrid dynamical systems are also not available in the literature.

1.2 Contributions

In this paper, we introduce and study notions of finite time stability for a class of hybrid systems. Finite time stability for closed sets is defined as the property that every maximal solution that starts close to the set stays close to the set (stability) and that, for some neighborhood around the set, solutions converge to it in finite hybrid time (finite time attractivity). The finite time attractivity property is characterized by a function, the settling-time function, that has value equal to zero at the set. Properties with respect to initial conditions and time are proposed. A particular feature of the proposed notions is that when the hybrid system has no jumps, meaning (2) has an empty jump set, then the proposed notions covers those for continuous-time systems. Similarly, these proposed notions cover the case when the hybrid system has no flows. Furthermore, the new notions allow also for solutions that converge after a finite amount of flow time and a finite number of jumps.

After the finite time stability notion is illustrated in examples, we introduce several sufficient conditions guaranteeing it. First, we propose sufficient conditions that guarantee finite time stability of a closed set when, during flows, a Lyapunov function strictly decreases, while at jumps, it is nonincreasing. The proposed conditions imply finite time convergence to the set of interest during flows as long as the domain of the solutions flow for enough amount of time. A dual result for the case when a Lyapunov function strictly decreases during jumps, and remains nonincreasing during flows, is established. The conditions in these two results are combined to formulate Lyapunov conditions guaranteeing strict decrease of a Lyapunov function both along flows and jumps. Unlike results in the literature, and due to the nonsmooth nature of hybrid systems, we allow the Lyapunov functions to be locally Lipschitz on an open subset of the flow set, and continuous everywhere else. Furthermore, the proposed sufficient conditions lead to expressions or bounds on the settling-time function.

As the stability part of the proposed finite time stability notion can be checked using the tools in [15], we explore alternative conditions guaranteeing finite time attractivity. The first result proposes conditions guaranteeing that the set of interest is reached after a jump. This result is shown to be a useful tool to establish finite time attractivity for hybrid systems that have a dead-beat-like property. Under the assumption that the elapsed time between consecutive jumps is uniformly bounded above by a positive constant, a second set of sufficient conditions exploiting properties of the second derivative of a Lyapunov-like function during flows is proposed. In addition, a finite time attractivity result for the case of multiple sets that are finite time attractive with basins of attractions that are nested is introduced.

Building from the tools in [15] for the study of robustness of asymptotically stability, we present conditions assuring robustness to small perturbations of finite time stability of closed sets. The class of perturbations are general enough to include measurement noise, modeling disturbances, and unmodeled hybrid dynamics. Furthermore, we exercise this and earlier results in a concrete application, namely, the estimation of parameters that enter in affine form the dynamics of a continuous-time system. In particular, we show that small perturbations on the dynamics of the affine system do not lead to much different estimates than in the nominal case.

Examples throughout the paper illustrate the results. Our results for general hybrid systems are used to show that a slight variation of the algorithm in [1,18] for finite time estimation of parameters can be shown to induce a finite time stability property of the zero-error estimation set globally.

The organization of the remainder of this paper is as follows. Section 2 presents preliminaries about the hybrid systems framework used and nonsmooth Lyapunov functions. The finite time stability notion and associated sufficient conditions are in Section 3. A preliminary version of this work was presented in the conference paper [22] without proofs and fewer details. In particular, this paper includes all of the technical proofs, new results (Theorem 3.10, Theorem 3.18, and Proposition 3.21), a new example (Example 3.19), and an entire new section with an application to finite time parameter estimation (Section 4, with Proposition 4.4 and Example 4.5).

2 Preliminaries

2.1 Notation

Given a set $S \subset \mathbb{R}^n$, $\overline{S}$ is the closure of S and is defined by the intersection of all closed sets containing S; $\overline{\text{con}}S$ denotes the closure of the convex hull of S. Given $\nu \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, the Euclidean vector norm $|\nu| := \sqrt{\nu^\top \nu}$, and $[\nu^\top\ w^\top]^\top$ is equivalent to (ν, w). Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is defined as $\operatorname{dom} f := \{x \in \mathbb{R}^m : \ f(x) \text{ is defined}\}$; its range is defined by $\operatorname{rge} f := \{f(x) : x \in \operatorname{dom} f\}$. The right limit of f is defined as $f^+(x) := \lim_{\nu \to 0^+} f(x + \nu)$ if it exists. A function f belongs to $\mathcal{C}^2$ if its derivative is continuously differentiable. For $x \in \mathbb{R}^n$ and $\mathcal{A} \subset \mathbb{R}^n$ closed, $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, denoted by $\alpha \in \mathcal{K}$, if it is zero at zero, continuous, strictly increasing; it is a class-$\mathcal{K}_\infty$ function, denoted by $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and is unbounded; α is positive definite, i.e., $\alpha \in \mathcal{PD}$, if $\alpha(s) > 0\ \forall s > 0$ and $\alpha(0) = 0$. A function $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\varphi \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \varphi(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \varphi(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given a matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{eig}(A)$ is the set of eigenvalues of A. For any $x \in \mathbb{R}$, $\mathrm{ceil}(x)$ denotes the next larger integer of x. The set $\mathcal{SP}^{p\times p}$ contains positive semidefinite matrices with dimension $p \times p$.

2.2 Hybrid Systems

In this paper, a hybrid system $\mathcal{H}$ is defined as in (2) with data (C, F, D, G), where $z \in \mathbb{R}^n$ is the state, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defines the flow map representing the continuous dynamics and C defines the flow set on which F is effective. The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defines the jump map and models the discrete behavior or jumps. The set D defines the jump set, which is the set of points from where jumps are allowed. A solution ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. [2] A solution to $\mathcal{H}$ is called maximal if it is not a truncated version of another solution. It is called complete when its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set of all maximal solutions to $\mathcal{H}$ is denoted by $\mathcal{S}_\mathcal{H}$, and the set of all maximal solutions to $\mathcal{H}$ with initial condition belongs to a set A is denoted by $\mathcal{S}_\mathcal{H}(A)$. A hybrid system $\mathcal{H}$ is said to satisfy the hybrid basic conditions if it satisfies [15, Assumption 6.5]. A set S is said to be forward invariant for $\mathcal{H}$ if for each solution ϕ to $\mathcal{H}$ from $\phi(0, 0) \in S$, then $\mathrm{rge}\,\phi \subset S$.

We refer the reader to [15] for more details on these notions and the hybrid systems framework.

2.3 Nonsmooth Lyapunov Functions

Given a hybrid system $\mathcal{H}$ with data (C, F, D, G), let $V : O \to \mathbb{R}$ be continuous on O and locally Lipschitz on a neighborhood of C. Following [11], the generalized gradient in the sense of Clarke of V at a point $z \in C$, denoted by $\partial V(z)$, is a closed, convex, and nonempty set equal to the convex hull of all limits of the sequence $\nabla V(z_i)$, where z_i is any sequence converging to z that avoids any set with zero Lebesgue measure that contains points at which V is nondifferentiable – since V is locally Lipschitz, ∇V exists almost everywhere. Then, Clarke's generalized directional derivative of V at a point z in the direction of v is given by

$$V^\circ(z, v) = \max_{\zeta\in\partial V(z)} \langle\zeta, v\rangle. \tag{3}$$

[2] A solution to $\mathcal{H}$ is defined in [15, Definition 2.6].

Then, for any solution $t \mapsto z(t)$ to $\dot{z} \in F(z)$,

$$\frac{d}{dt}V(z(t)) \leq V^\circ(z(t), \dot{z}(t)) \tag{4}$$

for almost all t in the domain of definition of the function z, where the derivative $\frac{d}{dt}V(z(t))$ is understood in the standard sense since V is locally Lipschitz. The reader is referred to [11] for more details on the generalized gradient and Clarke's generalized directional derivative.

Following [32], a bound on the increase of the function V along solutions to the hybrid system $\mathcal{H}$ is obtained by defining the function $u_C : O \to [-\infty, +\infty)$ as

$$u_C(z) := \begin{cases} \max\limits_{v\in F(z)} \max\limits_{\zeta\in\partial V(z)} \langle\zeta, v\rangle & \text{if } z \in C \\ -\infty & \text{otherwise} \end{cases} \tag{5}$$

Then, for each solution ϕ to $\mathcal{H}$ and each t at which $\frac{d}{dt}V(\phi(t, j))$ exists, the following bound holds:

$$\frac{d}{dt}V(\phi(t, j)) \leq u_C(\phi(t, j)). \tag{6}$$

Similarly, to obtain a bound on the change in V at jumps, the following quantity is defined:

$$u_D(z) := \begin{cases} \max\limits_{\zeta\in G(z)} V(\zeta) - V(z) & \text{if } z \in D \\ -\infty & \text{otherwise} \end{cases} \tag{7}$$

Then, for any solution ϕ to $\mathcal{H}$ and for any $(t_{j+1}, j), (t_{j+1}, j+1) \in \mathrm{dom}\,\phi$, it follows that

$$V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \leq u_D(\phi(t_{j+1}, j)). \tag{8}$$

Note that when F is a single-valued map, $u_C(z) = V^\circ(z, F(z))$ for each $z \in C$. When G is a single-valued map, $u_D(z) = V(G(z)) - V(z)$ for each $z \in D$.

3 Finite Time Stability

3.1 Finite Time Stability Notions

In this work, inspired from [8], we focus on the following finite time stability notion for hybrid systems $\mathcal{H}$.

Definition 3.1 *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$, a closed set $\mathcal{A} \subset \mathbb{R}^n$, an open neighborhood $\mathcal{N}$ of $\mathcal{A}$, and a function[3] $\mathcal{T} : \mathcal{N} \to [0, \infty)$ called the settling-time function. The closed set $\mathcal{A}$ is said to be*

1) stable *for $\mathcal{H}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N})$ with $|\phi(0, 0)|_\mathcal{A} \leq \delta$, we have $|\phi(t, j)|_\mathcal{A} \leq \varepsilon$ for all $(t, j) \in \mathrm{dom}\,\phi$;*

[3] Or, more precisely, *a functional* determining the amount of time required for a solution ϕ to converge to $\mathcal{A}$.

2) locally finite time attractive *(LFTA) for $\mathcal{H}$ if there exists μ such that $\mathcal{A}+\mu\mathbb{B}\subset\mathcal{N}$ and, for each $\phi\in\mathcal{S}_{\mathcal{H}}(\mathcal{A}+\mu\mathbb{B})$, we have $\sup_{(t,j)\in\operatorname{dom}\phi} t+j\geq\mathcal{T}(\phi)$ and*

$$\lim_{(t,j)\in\operatorname{dom}\phi:t+j\nearrow\mathcal{T}(\phi)}|\phi(t,j)|_{\mathcal{A}}=0; \tag{9}$$

3) locally finite time stable *(LFTS) for $\mathcal{H}$ if it is stable and LFTA for $\mathcal{H}$.*

Remark 3.2 *The global version of the notions in Definition 3.1 is obtained when $\overline{C}\cup D\subset\mathcal{N}$; when LFTA is global, we write FTA. Furthermore, a "pre" version of the finite time attractivity notion, like the one in [15, Definition 3.6], is not pursued here as such a notion would hold for free for maximal solutions that are not complete – in fact, such a "finite-time pre-attractivity" property would always hold for maximal solutions that are not complete by picking $\mathcal{T}$ with large enough values. Since convergence to a set $\mathcal{A}$ that is LFTA occurs in finite time, for each compact set of the basin of attraction there exists a finite (hybrid) time such that every solution from the said compact set converges to $\mathcal{A}$ on or before that time. Due to this, the LFTA notion is uniform on compact sets and in convergence time. Finally, note that for a given $\phi\in\mathcal{S}(\mathcal{N})$ with $\phi(0,0)=\xi$, $\mathcal{T}(\xi)$ can be decomposed as $\mathcal{T}(\xi)=\mathcal{T}^\star(\xi)+\mathcal{J}^\star(\xi)$ for some functions $\mathcal{T}^\star:\mathcal{N}\to\mathbb{R}_{\geq 0}$ and $\mathcal{J}^\star:\mathcal{N}\to\mathbb{N}$, with*

$$\lim_{(t,j)\in\operatorname{dom}\phi:t+j\nearrow\mathcal{T}^\star(\xi)+\mathcal{J}^\star(\xi)}|\phi(t,j)|_{\mathcal{A}}=0.$$

The notions in Definition 3.1 are illustrated in the following examples.

Example 3.3 *Inspired from [25, Example 14], consider the hybrid system $\mathcal{H}=(C,F,D,G)$ with state $z=(x,\tau)\in\mathbb{R}\times[0,1]$ and data given by*[4]

$$\begin{aligned} F(z)&=\begin{bmatrix}-k|x|^\alpha\ sgn(x)\\ 1\end{bmatrix} && z\in C=\mathbb{R}\times[0,1],\\ G(z)&=\begin{bmatrix}-x\\ 0\end{bmatrix} && z\in D=\mathbb{R}\times\{1\},\end{aligned} \tag{10}$$

where $\alpha\in(0,1)$ and $k>0$. Each maximal solution $\phi=(\phi^x,\phi^\tau)$ to $\mathcal{H}$ from $\phi(0,0)=(x_0,\tau_0)$ satisfies

$$\phi(t,0)=\left(|x_0|^{1-\alpha}-k(1-\alpha)t\right)^{\frac{1}{1-\alpha}}\ sgn(x_0) \tag{11}$$

for all $0\leq t\leq\min\left\{1-\tau_0,\frac{|x_0|^{1-\alpha}}{k(1-\alpha)}\right\}$. Let $\bar N$ be such that

$$\frac{|x_0|^{1-\alpha}}{k(1-\alpha)}+\tau_0-2\leq\bar N\leq\frac{|x_0|^{1-\alpha}}{k(1-\alpha)}+\tau_0-1,$$

where $\bar N$ is an integer. If $\bar N\leq -1$, we obtain $\frac{|x_0|^{1-\alpha}}{k(1-\alpha)}\leq 1-\tau_0$. From (11), we have that $\phi(t^\star,0)=0$ where $t^\star=\frac{|x_0|^{1-\alpha}}{k(1-\alpha)}$. Furthermore, $\phi(t,j)=0$ for all $(t,j)\in\operatorname{dom}\phi$ such that $t\geq t^\star$ according to (10). When $\bar N\geq 0$, we have

$$\phi(1-\tau_0,1)=-\left(|x_0|^{1-\alpha}-k(1-\alpha)(1-\tau_0)\right)^{\frac{1}{1-\alpha}}sgn(x_0). \tag{12}$$

Moreover, after $\bar N+1$ jumps,

$$\begin{aligned}&|\phi(1-\tau_0+\bar N,\bar N+1)|\\ &\quad=\left|\left(|x_0|^{1-\alpha}-k(1-\alpha)(1-\tau_0-\bar N)\right)^{\frac{1}{1-\alpha}}\right|.\end{aligned} \tag{13}$$

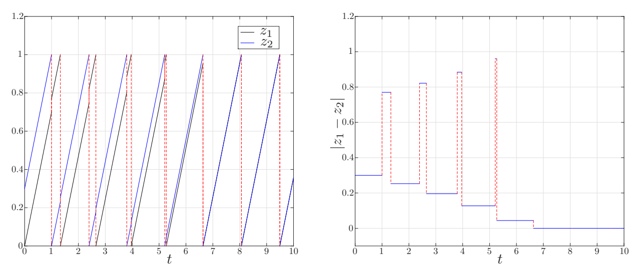
Therefore, using the property that

$$1-\tau_0+\bar N\leq\frac{|x_0|^{1-\alpha}}{k(1-\alpha)}\leq 1-\tau_0+\bar N+1,$$

it implies that ϕ converges to 0 between the $(\bar N+1)$-th jump and the $(\bar N+2)$-th jump. In fact,

$$\phi\left(\frac{|x_0|^{1-\alpha}}{k(1-\alpha)},\bar N+1\right)=0$$

and $\phi(t,j)=0$ for all $(t,j)\in\operatorname{dom}\phi$ such that $t\geq\frac{|x_0|^{1-\alpha}}{k(1-\alpha)}$. Therefore, the set $\{0\}\times[0,1]$ is FTA. △



(a) The projection of the states z_1 and z_2 on the t direction.

(b) The projection of the Euclidean distance between z_1 and z_2 on the t direction

Fig. 1. The trajectories of the states (z_1,z_2) for two fireflies in (14) and the Euclidean distance between them. Parameters used are $\gamma=0.7$, $z_1(0,0)=0$, $z_2(0,0)=0.3$ and $\tilde\varepsilon=0.1$.

Example 3.4 *Consider the model of interacting fireflies in [14, Example 25]. The time of flashes of a firefly is determined by the firefly's internal clock. In between flashes, the internal clock gradually increases at a common rate $\gamma>0$. When it reaches a certain threshold, a flash occurs and the clock is instantly reset to 0. In a group of fireflies, the flash of one firefly affects the internal clock of all other fireflies. That is, when a firefly witnesses a flash*

[4] The function $\operatorname{sgn}:\mathbb{R}\to\{-1,1\}$ is defined as $\operatorname{sgn}(x)=1$ if $x\geq 0$, and $\operatorname{sgn}(x)=-1$ otherwise.

from another firefly, its internal clock instantly increases to a value closer to the threshold. To model the internal clock of the i-th firefly and to simplify the analysis, we consider a normalized clock; namely, the clock, denoted by z_i, takes values in the interval $[0,1]$ and flashes occur when x_i reaches the threshold 1. In between flashes, the clock state flows toward the threshold according to $\dot{z}_i = \gamma$. The resulting hybrid system $\mathcal{H}$ for two fireflies has state $z = (z_1, z_2) \in \mathbb{R}^2$ and data

$$\begin{aligned} F(z) &:= (\gamma, \gamma) && \forall z \in C, \\ G(z) &:= \begin{bmatrix} g((1+\tilde{\varepsilon})z_1) \\ g((1+\tilde{\varepsilon})z_2) \end{bmatrix} && \forall z \in D, \end{aligned} \tag{14}$$

where $C := [0,1] \times [0,1]$ and $D := \{z \in C : \max\{z_1, z_2\} = 1\}$. The parameter $\varepsilon > 0$ represents the effect on the timer of a firefly when another firefly's timer expires, i.e., the timer increases $(1+\tilde{\varepsilon})$ times its current value. The set-valued map g is defined as $g(s) = s$ when $s < 1$, $g(s) = 0$ when $s > 1$ and $g(s) = \{0,1\}$ when $s = 1$. Then, the set of interest is $\mathcal{A} = \{z \in C : z_1 = z_2\}$, which defines the situation when both fireflies flash at the same time, namely, synchronized flashing. It can be shown that the compact set $\mathcal{A}$ is finite time stable for the system $\mathcal{H}$ from any open subsets $\widetilde{\mathcal{N}} \subset \{z \in C : |z_1 - z_2| \neq (1+\tilde{\varepsilon})/(2+\tilde{\varepsilon})\}$ [5] such that $\mathcal{A} \subset \widetilde{\mathcal{N}}$. A rigorous analysis will be carried out in Example 3.12. A simulation is shown in Figure 1. [6] $\triangle$

3.2 Sufficient Conditions for Finite Time Stability

In this section, several sufficient conditions are established for finite time stability of a closed set $\mathcal{A}$ for $\mathcal{H}$.

3.2.1 Sufficient conditions for finite time stability

When the distance of each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N})$ to a closed set $\mathcal{A}$ strictly decreases during flows, the conditions in the following result guarantee finite time stability. The set $\mathcal{N}$ defines an open neighborhood of $\mathcal{A}$.

Theorem 3.5 *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathcal{N} \subset \mathbb{R}^n$ with $\mathcal{N}$ open such that $G(\mathcal{N}) \subset \mathcal{N}$. The set $\mathcal{A}$ is LFTS for $\mathcal{H}$ if there exist a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c_1 > 0, c_2 \in [0,1)$ such that*

1) *for every $\xi \in \mathcal{N} \cap (\overline{C} \cup D) \setminus \mathcal{A}$, each $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$ satisfies $\frac{V^{1-c_2}(\xi)}{c_1(1-c_2)} \leq \sup_{(t,j)\in \operatorname{dom}\phi} t$;*
2) *the function V is positive definite with respect to $\mathcal{A}$ and*

$$u_C(z) \leq -c_1 V^{c_2}(z) \qquad \forall z \in C \cap \mathcal{N}, \tag{15}$$
$$u_D(z) \leq 0 \qquad \forall z \in D \cap \mathcal{N}, \tag{16}$$

where the functions u_C and u_D are defined in (5) and (7), respectively. Furthermore, for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0,0) = \xi$:

a) *the settling-time function $\mathcal{T} : \mathcal{N} \cap (\overline{C} \cup D) \to [0, \infty)$ satisfies $\mathcal{T}(\xi) \leq \mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi)$, where $\mathcal{T}^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$, and $\mathcal{J}^\star(\xi)$ is such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \operatorname{dom}\phi$;*
b) *$|\phi(t,j)|_{\mathcal{A}} = 0$ $\forall (t,j) \in \operatorname{dom}\phi$ such that $t \geq \mathcal{T}^\star(\xi)$.*

Proof Let $\Omega \subset \mathcal{N}$ be a forward invariant set for $\mathcal{H}$ containing an open neighborhood of $\mathcal{A}$; e.g., a small enough sublevel set of V, which is guaranteed to exist by the assumptions. Pick any $\phi \in \mathcal{S}_{\mathcal{H}}(\Omega)$ and note that $\operatorname{rge}\phi \subset \Omega$. Pick any $(t,j) \in \operatorname{dom}\phi$ and let $0 = t_0 \leq t_1 \leq \cdots \leq t_{j+1} = t$ satisfy

$$\operatorname{dom}\phi \cap ([0,t] \times \{0,1,\cdots,j\}) = \textstyle\bigcup_{i=0}^{j} ([t_i, t_{i+1}] \times \{i\}). \tag{17}$$

For each $i \in \{0,1,\ldots,j\}$ and almost all $s \in [t_i, t_{i+1}]$, $\phi(s,i) \in C \cap \mathcal{N}$. Using (6), the condition in (15) implies that, for each $i \in \{0,1,\ldots,j\}$ and for almost all $s \in [t_i, t_{i+1}]$,

$$\frac{d}{ds} V(\phi(s,i)) \leq u_C(\phi(s,j)) \leq -c_1 V^{c_2}(\phi(s,i)), \tag{18}$$

which implies that

$$V^{-c_2}(\phi(s,i)) dV(\phi(s,i)) \leq -c_1 ds \tag{19}$$

when $V(\phi(s,i)) > 0$. Note that the above inequality holds trivially when $V(\phi(s,i)) = 0$. Integrating over $[t_i, t_{i+1}]$ both sides of this inequality yields

$$\frac{1}{1-c_2}\left(V^{1-c_2}(\phi(t_{i+1},i)) - V^{1-c_2}(\phi(t_i,i))\right) \leq -c_1(t_{i+1} - t_i). \tag{20}$$

Similarly, for each $i \in \{1,\cdots,j\}$, $\phi(t_i, i-1) \in D \cap \mathcal{N}$ and

$$V(\phi(t_i,i)) - V(\phi(t_i,i-1)) \leq 0. \tag{21}$$

The two inequalities in (20) and (21) imply that, for each $(t,j) \in \operatorname{dom}\phi$,

$$\frac{1}{1-c_2}\left(V^{1-c_2}(\phi(t,j)) - V^{1-c_2}(\xi)\right) \leq -c_1 t. \tag{22}$$

[5] Solutions from the set $\{z \in C : |z_1 - z_2| = (1+\tilde{\varepsilon})/(2+\tilde{\varepsilon})\}$ do not converge to $\mathcal{A}$.
[6] Code at `https://github.com/HybridSystemsLab/FTSFireflies`

Using $G(\mathcal{N}) \subset \mathcal{N}$, positive definiteness of V, item 1), and the fact that $c_2 \in (0,1)$, we get

$$V^{1-c_2}(\phi(t,j)) \leq V^{1-c_2}(\xi) - c_1(1-c_2)t. \tag{23}$$

Then, it follows that ϕ reaches $\mathcal{A}$ in finite time (t,j). Furthermore, an upper bound for the settling-time function can be computed as

$$\mathcal{T}(\xi) \leq \mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi), \tag{24}$$

where $\mathcal{T}^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$, and $\mathcal{J}^\star(\xi)$ is chosen such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \operatorname{dom}\phi$. Note that $\mathcal{T}^\star(\xi) < \sup_{(t,j)\in\operatorname{dom}\phi} t$, the existence of $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \operatorname{dom}\phi$ is guaranteed. ■

Remark 3.6 *Assumption* 1) *in Theorem 3.5 is satisfied if the domain of each $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N})$ is unbounded in the t direction. A result in a similar spirit, but for small ordinary time asymptotic stability can be found in [16, Proposition 3.2]. Moreover, when the jump set is empty and the flow set is such that $\mathcal{N} \subset C$, $\mathcal{H}$ reduces to a continuous-time system on $\mathcal{N}$, and the result in Theorem 3.5 reduces to a result for continuous-time systems with constraints; see, e.g., [8, Theorem 4.2].*

Remark 3.7 *From Definition 3.1, if a closed set $\mathcal{A}$ is FTS for $\mathcal{H}$ with settling-time function $\mathcal{T} : \mathcal{N} \cap (\overline{C} \cup D) \to \mathbb{R}_{\geq 0}$, where $\mathcal{N}$ is an open neighborhood of $\mathcal{A}$, then the set $\mathcal{A}$ is also pre-asymptotically stable (see [15, Definition 3.6]) for $\mathcal{H}$ with basin of attraction $\mathcal{N}$. However, the reverse implication is not true. When for $\mathcal{K}_\infty$ functions α_1 and α_2, the function V satisfies*

$$\alpha_1(|z|_\mathcal{A}) \leq V(z) \leq \alpha_2(|z|_\mathcal{A})$$

for all $z \in (C \cup D \cup G(D)) \cap \mathcal{N}$ and $\overline{C} \cup D \subset \mathcal{N}$, then Theorem 3.5 guarantees a global uniform version of finite time stability notion in Definition 3.1.

In the following example, the conditions in Theorem 3.5 are exercised.

Example 3.8 *Consider the system in Example 3.3, the function $V : \mathbb{R} \times [0,1] \to \mathbb{R}_{\geq 0}$ given by $V(z) = \frac{1}{2}x^2$ for each $z \in C$, and the compact set $\mathcal{A} = \{0\} \times [0,1]$. We have that, for each $z \in C$,*

$$\langle \nabla V(z), F(z) \rangle = -k|x|^{1+\alpha} = -2^{\frac{1+\alpha}{2}} k V(z)^{\frac{1+\alpha}{2}}.$$

Then, condition (15) *is satisfied with $\mathcal{N} = \mathbb{R} \times \mathbb{R}$, $c_1 = 2^{\frac{1+\alpha}{2}} k > 0$ and $c_2 = \frac{1+\alpha}{2} \in (0,1)$. Moreover, for all $z \in D$, $V(G(z)) - V(z) = 0$, which verifies the condition in* (16)*. Note that the condition in item* 1) *follows since every maximal solution to $\mathcal{H}$ in* (10) *is complete (with its domain of definition unbounded in the t direction); e.g., by applying [15, Proposition 6.10]. Therefore, by Theorem 3.5, the set $\{0\} \times [0,1]$ is LFTS.* △

Inspired by Example 3.4, when the distance of a solution $\phi \in \mathcal{S}_\mathcal{H}$ to a closed set $\mathcal{A}$ strictly decreases at jumps, we can establish the following result.

Theorem 3.9 *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathcal{N} \subset \mathbb{R}^n$ with $\mathcal{N}$ open such that $G(\mathcal{N}) \subset \mathcal{N}$. The set $\mathcal{A}$ is LFTS for $\mathcal{H}$ if there exist a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c > 0$ such that*

1) *for every $\xi \in \mathcal{N} \cap (\overline{C} \cup D) \setminus \mathcal{A}$, each $\phi \in \mathcal{S}_\mathcal{H}(\xi)$ satisfies*

$$\operatorname{ceil}\left(\frac{V(\xi)}{c}\right) \leq \sup_{(t,j)\in\operatorname{dom}\phi} j;$$

2) *the function V is positive definite with respect to $\mathcal{A}$ such that*

$$\begin{aligned} u_C(z) &\leq 0 && \forall z \in C \cap \mathcal{N}, \\ u_D(z) &\leq -\min\{c, V(z)\} && \forall z \in D \cap \mathcal{N}. \end{aligned} \tag{25}$$

where the functions u_C and u_D are defined in (5) *and* (7)*, respectively. Furthermore, for each $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0,0) = \xi$:*

a) *the settling-time function $\mathcal{T} : \mathcal{N} \cap (\overline{C} \cup D) \to [0,\infty)$ satisfies*
$$\mathcal{T}(\xi) \leq \mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi),$$
where $\mathcal{J}^\star(\xi) = \operatorname{ceil}\left(\frac{V(\xi)}{c}\right)$ and $\mathcal{T}^\star(\xi)$ is such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \operatorname{dom}\phi$ and $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi) - 1) \in \operatorname{dom}\phi$;

b) *$|\phi(t,j)|_\mathcal{A} = 0$ $\forall (t,j) \in \operatorname{dom}\phi$ such that $j \geq \mathcal{J}^\star(\xi)$.*

Proof Let $\Omega \subset \mathcal{N}$ be a forward invariant set for $\mathcal{H}$ containing an open neighborhood of $\mathcal{A}$; e.g., a small enough sublevel set of V. Pick any $\phi \in \mathcal{S}_\mathcal{H}(\Omega)$ and note that $\operatorname{rge}\phi \subset \Omega$. Pick any $(t,j) \in \operatorname{dom}\phi$ and let $0 = t_0 \leq t_1 \leq \cdots \leq t_{j+1} = t$ satisfy (17). For each $i \in \{0,1,\ldots,j\}$ and almost all $s \in [t_i, t_{i+1}]$, $\phi(s,i) \in C$. Using (6), the condition in (25) implies that, for each $i \in \{0,1,\ldots,j\}$ and for almost all $s \in [t_i, t_{i+1}]$, $\frac{dV(\phi(s,i))}{ds} \leq 0$. Integrating over $[t_i, t_{i+1}]$ both sides of this inequality yields

$$V(\phi(t_{i+1}, i)) - V(\phi(t_i, i)) \leq 0. \tag{26}$$

Similarly, by using (8) and (25), for each $i \in \{1, \cdots, j\}$, $\phi(t_i, i-1) \in D$ and

$$V(\phi(t_i, i)) - V(\phi(t_i, i-1)) \leq -\min\{c, V(\phi(t_i, i-1))\}. \tag{27}$$

The two inequalities in (26) and (27) imply that, for each $(t,j)\in\operatorname{dom}\phi$,

$$V(\phi(t,j)) - V(\xi) \le -\sum_{i=1}^{j}\min\{c, V(\phi(t_i,i-1))\}.$$

Proceeding as in the proof of Theorem 3.5, ϕ converges to $\mathcal{A}$ in finite time. Furthermore, an upper bound for the settling-time function can be computed as $\mathcal{T}(\xi) \le \mathcal{T}^\star(\xi)+\mathcal{J}^\star(\xi)$, where $\mathcal{J}^\star(\xi)=\operatorname{ceil}\left(\frac{V(\xi)}{c}\right)$ and $\mathcal{T}^\star(\xi)$ is such that

$$(\mathcal{T}^\star(\xi),\mathcal{J}^\star(\xi)),(\mathcal{T}^\star(\xi),\mathcal{J}^\star(\xi)-1)\in\operatorname{dom}\phi.$$

Note that for $\mathcal{J}^\star(\xi) < \sup_{(t,j)\in\operatorname{dom}\phi} j$, the existence of $(\mathcal{T}^\star(\xi),\mathcal{J}^\star(\xi))\in\operatorname{dom}\phi$ is guaranteed. ∎

The following result combines the conditions in Theorem 3.5 and in Theorem 3.9 to arrive to strict conditions for finite time stability of a closed set. By combining the conditions in those results, a tighter bound on the settling-time function can be obtained. A proof can be formulated by combining the arguments in the proofs of Theorem 3.5 and Theorem 3.9.

Theorem 3.10 *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $\mathcal{A}\subset\mathcal{N}\subset\mathbb{R}^n$ with $\mathcal{N}$ open such that $G(\mathcal{N})\subset\mathcal{N}$. The set $\mathcal{A}$ is LFTS for $\mathcal{H}$ if there exist a continuous function $V:\mathcal{N}\to\mathbb{R}_{\ge 0}$, locally Lipschitz on an open neighborhood of $C\cap\mathcal{N}$, and $c_1,c_3>0, c_2\in[0,1)$ such that item 1) in Theorem 3.5 and item 1) in Theorem 3.9 are satisfied, the function V is positive definite with respect to $\mathcal{A}$ and*

$$u_C(z) \le -c_1 V^{c_2}(z) \qquad \forall z\in C\cap\mathcal{N}, \qquad (28)$$
$$u_D(z) \le -\min\{c_3, V(z)\} \qquad \forall z\in D\cap\mathcal{N}, \qquad (29)$$

where the functions u_C and u_D are defined in (5) and (7), respectively. Furthermore, for each $\phi\in\mathcal{S}_\mathcal{H}(\mathcal{N}\cap(\overline{C}\cup D))$ with $\phi(0,0)=\xi$:

a) the settling-time function $\mathcal{T}:\mathcal{N}\cap(\overline{C}\cup D)\to[0,\infty)$ satisfies $\mathcal{T}(\xi)\le\min_{i\in\{1,2\}}\{\mathcal{T}_i^\star(\xi)+\mathcal{J}_i^\star(\xi)\}$, where $\mathcal{T}_1^\star(\xi)=\frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$, $\mathcal{J}_1^\star(\xi)$ is such that $(\mathcal{T}_1^\star(\xi),\mathcal{J}_1^\star(\xi))\in\operatorname{dom}\phi$, $\mathcal{J}_2^\star(\xi)=\operatorname{ceil}\left(\frac{V(\xi)}{c_3}\right)$, and $\mathcal{T}_2^\star(\xi)$ is such that $(\mathcal{T}_2^\star(\xi),\mathcal{J}_2^\star(\xi))\in\operatorname{dom}\phi$ and $(\mathcal{T}_2^\star(\xi),\mathcal{J}_2^\star(\xi)-1)\in\operatorname{dom}\phi$;

b) $|\phi(t,j)|_\mathcal{A}=0$ for all $(t,j)\in\operatorname{dom}\phi$ such that $t\ge\mathcal{T}_1^\star(\xi)$ or $j\ge\mathcal{J}_2^\star(\xi)$.

The following result provides conditions for finite time stability of a set that is already asymptotically stable.

Theorem 3.11 *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $\mathcal{A}\subset\mathcal{N}\subset\mathbb{R}^n$ with $\mathcal{N}$ open such that $G(\mathcal{N})\subset\mathcal{N}$. The set $\mathcal{A}$ is LFTS to $\mathcal{H}$ if*

1) the set $\mathcal{A}$ is uniformly asymptotically stable[7] with basin of attraction including $\mathcal{N}$;
2) there exists a neighborhood $U\subset\mathcal{N}$ of $\mathcal{A}$ such that:
2.1) for every $\phi\in\mathcal{S}_\mathcal{H}(U\cap(\overline{C}\cup D))$, $(t,1)\in\operatorname{dom}\phi$ for some $t\in\mathbb{R}_{\ge 0}$;
2.2) $G((D\cap U)\setminus\mathcal{A})\subset\mathcal{A}$.

Proof Pick any $(t,j)\in\operatorname{dom}\phi$ and let $0=t_0\le t_1\le\cdots\le t_{j+1}=t$ satisfy (17). For each $i\in\{0,1,\dots,j\}$ and almost all $s\in[t_i,t_{i+1}]$, $\phi(s,i)\in C$. Since the set $\mathcal{A}$ is asymptotically stable, there exists $\varphi\in\mathcal{KL}$ such that

$$|\phi(t,j)|_\mathcal{A}\le\varphi(|\xi|_\mathcal{A},t+j). \qquad (30)$$

Then, for each $\varepsilon>0$, there exists $(\mathcal{T}(\xi),\mathcal{J}(\xi))\in\operatorname{dom}\phi$ such that $|\phi(t,j)|_\mathcal{A}<\varepsilon$ for all $(t,j)\in\operatorname{dom}\phi$ and $t+j\ge\mathcal{T}(\xi)+\mathcal{J}(\xi)$. If $\phi(\mathcal{T}(\xi),\mathcal{J}(\xi))\in\mathcal{A}$, then the claim follows; if not, from assumption in item 2.1), we obtain that there exists $t'\ge\mathcal{T}(\xi)$ such that $(t',\mathcal{J}(\xi)+1)\in\operatorname{dom}\phi$. Using the assumption in item 2.2), we get

$$\phi(\mathcal{T}(\xi),\mathcal{J}(\xi)+1)\in\mathcal{A}$$

and $|\phi(\mathcal{T}(\xi),\mathcal{J}(\xi)+1)|_\mathcal{A}=0$. The stability part follows from the fact that $\mathcal{A}$ is asymptotically stable for $\mathcal{H}$. ∎

In the following example, the results of Theorem 3.11 are applied.

Example 3.12 *Consider the system in Example 3.4. To show the FTS property of the set $\mathcal{A}$, let $k=\frac{\tilde\varepsilon}{2+\tilde\varepsilon}$ and consider the function*

$$V(z):=\min\{|z_1-z_2|,1+k-|z_1-z_2|\}\quad\forall z\in\mathcal{X}, \qquad (31)$$

where

$$\mathcal{X}:=\left\{z\in\mathbb{R}^2: V(z)<\tfrac{1+k}{2}\right\}=\left\{z\in\mathbb{R}^2:|z_1-z_2|\neq\tfrac{1+k}{2}\right\}.$$

This function V is continuously differentiable on the open set $\mathcal{X}\setminus\mathcal{A}$ and it is Lipschitz on $\mathcal{X}$. Following [14, Example 25], let $m^\star=\frac{1+k}{2}$ and $m\in(0,m^\star)$, $K_m=\{z\in C\cup D: V(z)\le m\}$, and define $C_m=C\cap K_m$ and $D_m=D\cap K_m$. By definition of V, it follows that

$$\langle\nabla V(z),F(z)\rangle=0\quad\forall z\in C_m\setminus\mathcal{A}. \qquad (32)$$

Now consider $z\in D_m$. Since V is symmetric on the variables z_1 and z_2, without loss of generality, consider the case $z=(1,z_2)$, where $z_2\in[0,1]\setminus\{1/(2+\tilde\varepsilon)\}$. Then,

$$V(z)=\min\{1-z_2,k+z_2\}, \qquad (33)$$
$$V(G(z))=\min\{g((1+\tilde\varepsilon)z_2),1+k-g((1+\tilde\varepsilon)z_2)\}. \qquad (34)$$

[7] A closed set $\mathcal{A}$ is said to be uniformly asymptotically stable with basin of attraction $\mathcal{B}_\mathcal{A}$ if there exists a class-$\mathcal{KL}$ function β such that every solution ϕ to $\mathcal{H}$ from $\mathcal{B}_\mathcal{A}$ is complete and satisfies $|\phi(t,j)|_\mathcal{A}\le\beta(|\phi(0,0)|_\mathcal{A},t+j)$ for all $(t,j)\in\operatorname{dom}\phi$.

When $g((1+\tilde{\varepsilon})z_2) = (1+\tilde{\varepsilon})z_2$, there are two cases

- *if $z_2 < 1/(2+\tilde{\varepsilon})$, $V(z) = k+z_2 > (1+\tilde{\varepsilon})z_2 \geq V(G(z))$;*
- *if $z_2 > 1/(2+\tilde{\varepsilon})$, $V(z) = 1-z_2 \geq V(G(z))$.*

Therefore, the set $\mathcal{A}$ is globally asymptotically stable for the system $\mathcal{H}_m = (C_m, F, D_m, G)$ and using [15, Proposition 6.10], every maximal solution to $\mathcal{H}_m$ is complete. Furthermore, given $\tilde{\varepsilon} > 0$, for $\varepsilon = \tilde{\varepsilon}/(1+\tilde{\varepsilon})$ and picking m such that $(\mathcal{A}+\varepsilon\mathbb{B}) \cap C \subset C_m$, we have that for all $z \in D_m \cap (\mathcal{A}+\varepsilon\mathbb{B})$,

$$G(z) = 0 \in \mathcal{A}. \tag{35}$$

Then, it follows from Theorem 3.11 that $\mathcal{A}$ is finite time stable for the system $\mathcal{H}_m = (C_m, F, D_m, G)$ with $\mathcal{N} = \{z \in C \cup D : V(z) < m\}$. △

3.2.2 Sufficient conditions for finite time attractivity

In this section, two sufficient conditions guaranteeing global finite time attractivity are presented and illustrated in examples. These results are followed by sufficient conditions for local finite time attractivity exploiting a property of the second derivative of a Lyapunov function and a property of nested sets.

Next, a result similar to that of Theorem 3.9 is established when maximal solutions to $\mathcal{H}$ converge to $\mathcal{A}$ through jumps.

Proposition 3.13 *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$ and a closed nonempty set $\mathcal{A} \subset \mathbb{R}^n$. If a nonempty set $\widetilde{\mathcal{A}}$ is finite time attractive for $\mathcal{H}$, there exists $\delta > 0$ such that $\widetilde{\mathcal{A}} + \delta\mathbb{B} \subset G^{-1}(\mathcal{A})$, and no flows from the set $(\widetilde{\mathcal{A}} + \delta\mathbb{B}) \setminus \mathcal{A}$ are possible, then the set $\mathcal{A}$ is FTA, where $G^{-1}(\mathcal{A}) := \{z \in D : G(z) \subset \mathcal{A}\}$.*

Remark 3.14 *A sufficient condition guaranteeing that no flows from the set $(\widetilde{\mathcal{A}} + \delta\mathbb{B}) \setminus \mathcal{A}$ are possible is when the flow set C is closed and $F(z) \cap T_C(z) = \emptyset$ for all $z \in C \cap ((\widetilde{\mathcal{A}} + \delta\mathbb{B}) \setminus \mathcal{A})$.*

The following corollary considers the situation when the dynamics of a hybrid system $\mathcal{H}$ are linear in one of the state components and its solutions exhibit dwell-time behavior.

Corollary 3.15 *Consider a hybrid system $\mathcal{H}$ with state $z = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $n_1 > 0, n_2 > 0$, and the closed set $\mathcal{A} = \{0\} \times \mathbb{R}^{n_2} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The set $\mathcal{A}$ is FTA for $\mathcal{H}$ if the following holds:*

1) *the flow map and jump map are single valued and their x_1 components are linear, i.e., $F(z) = (Ax_1, f(x_2))$ for all $z \in C$ and $G(z) = (Bx_1, g(x_2))$ for all $z \in D$ with $A \in \mathbb{R}^{n_1 \times n_1}$, $B \in \mathbb{R}^{n_1 \times n_1}$, $f : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, and $g : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$;*
2) *for each $\phi \in \mathcal{S}_\mathcal{H}$, $\sup_{(t,j)\in \operatorname{dom}\phi} j \geq n_1 + 1$, where n_1 is the dimension of x_1 component, and the flow time between every two consecutive jumps after the first jump are identical, i.e., there exists $\gamma > 0$ such that $t_{j+1} - t_j = \gamma$ for all $j \in \mathbb{N} \setminus \{0\}$ and $j \leq n_1 + 1$;*
3) *the matrix $B\exp(A\gamma)$ is nilpotent, where A, B come from item 1) and γ from item 2).*

Furthermore, for each $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0,0) = \xi$:

a) *there exists a settling-time function $\mathcal{T} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ satisfying $\mathcal{T}(\xi) \leq \mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi)$, where $\mathcal{J}^\star(\xi) = n_1 + 1$ and $\mathcal{T}^\star(\xi)$ is such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \operatorname{dom}\phi$ and $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi) - 1) \in \operatorname{dom}\phi$;*
b) *$|\phi(t,j)|_\mathcal{A} = 0$ $\forall (t,j) \in \operatorname{dom}\phi$ such that $j \geq \mathcal{J}^\star(\xi)$.*

Proof Let $\phi = (\phi^{x_1}, \phi^{x_2}) \in \mathcal{S}_\mathcal{H}$. Pick any $(t,j) \in \operatorname{dom}\phi$ and let $0 = t_0 \leq t_1 \leq \cdots \leq t_{j+1} = t$ satisfy (17). For each $i \in \{0, 1, \ldots, j\}$ and almost all $s \in [t_i, t_{i+1}]$, $\phi(s,i) \in C$ and

$$\phi^{x_1}(s,i) = \exp(A(s-t_i))\phi^{x_1}(t_i, i). \tag{36}$$

Similarly, for each $i \in \{1, \cdots, j\}$, $\phi(t_i, i-1) \in D$ and

$$\phi^{x_1}(t_i, i) = B\phi^{x_1}(t_i, i-1). \tag{37}$$

Therefore, using (36) and (37), for any $(t,j) \in \operatorname{dom}\phi$, we get,

$$\phi^{x_1}(t,j) = \exp(A(t-t_j))(B\exp(A\gamma))^{\max\{0,j-1\}}B \exp(A\tau_0))\phi^{x_1}(0,0),$$

where τ_0 is the time it takes to reach the first jump (may not equal to γ). Since the matrix $B\exp(A\tau) \in \mathbb{R}^{n_1 \times n_2}$ is nilpotent, $(B\exp(A\tau))^{n_1} = 0$. Therefore, the finite time convergence is reached within $n_1 + 1$ jumps. ■

Remark 3.16 *The second component x_2 of the state in the system in Corollary 3.15 can be arbitrary, but it would typically be involved in a mechanism that guarantees that the property in item 2) holds. Due to this, x_2 may include variables that behave like a timer. If the hybrid system $\mathcal{H}$ with linear flow and jump dynamics in Corollary 3.15 is such that $C = \emptyset$, then, the result in Corollary 3.15 is similar to the results about deadbeat convergence for discrete-time systems with constraints; see, e.g., [13].*

Example 3.17 *Consider a hybrid system with state $z = (x_1, x_2)$, $x_1 = (x_{11}, x_{12})$ and $z \in \mathcal{X} := \mathbb{R}^2 \times [0,1]$, its* *data $\mathcal{H} = (C, f, D, g)$ is given by*

$$
\begin{aligned}
\dot{z} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad z \in C \\
z^+ &= \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z \in D
\end{aligned} \tag{38}
$$

where $C = \{z \in \mathcal{X} : x_2 \in [0, 1]\}$, $D = \{z \in \mathcal{X} : x_2 = 0\}$,

$$
G = \frac{1}{5} \begin{bmatrix} 2\cos(1) - \sin(1) & -\cos(1) - 2\sin(1) \\ 4\cos(1) - 2\sin(1) & -2\cos(1) - 4\sin(1) \end{bmatrix}. \tag{39}
$$

Consider the set $\mathcal{A} = \{0\} \times \{0\} \times [0, 1]$ and a solution $\phi = (\phi^{x_1}, \phi^{x_2}) \in \mathcal{S}_{\mathcal{H}}$. Then, $\phi^{x_1}(t, 0) = \exp(At)\phi^{x_1}(0, 0)$ for all $t \in [0, 1 - \phi^{x_2}(0, 0)]$, where

$$
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
$$

Furthermore, after three jumps,

$$
\begin{aligned}
&\phi^{x_1}(3 - \phi^{x_2}(0, 0), 3) \\
&\qquad = (G \exp(A))^2 G \exp(A(1 - \phi^{x_2}(0, 0)))\phi^{x_1}(0, 0).
\end{aligned}
$$

Note that $G \exp(A)$ is a nilpotent matrix, i.e., all eigenvalues are located at zero. In fact, since

$$
\exp(A) = \begin{bmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{bmatrix},
$$

which is an invertible matrix, we have that $G = G_0(\exp(A))^{-1}$ for any given nilpotent matrix G_0. By letting

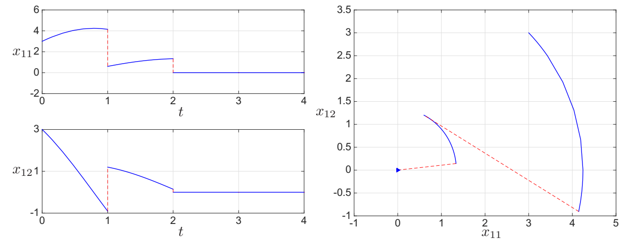
$$
G_0 = \begin{bmatrix} 2/5 & -1/5 \\ 4/5 & -2/5 \end{bmatrix},
$$

we obtain (39). Then, $(G \exp(A))^2 = 0$. Therefore, the solution ϕ converges to $\mathcal{A}$ within 3 jumps. Furthermore, since the time between two consecutive jumps is equal to one, $\mathcal{A}$ is finite time stable to $\mathcal{H}$ with $T = 3$. A simulation is shown in Figure 2, where finite time convergence is approached within 2 jumps.[8] *Note that the trajectory shown in Figure 2 reaches $\mathcal{A}$ after the second jump due to the fact that $G \exp(A) G$ is zero in this example.* △

Next, we establish a result when the second derivative of a Lyapunov function is strictly concave.

Theorem 3.18 *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ with*

[8] Code at `https://github.com/HybridSystemsLab/FTSNilpotency`



(a) The projections of trajectories of x_{11} and x_{12} on the t direction.

(b) Phase plot of x_{11} and x_{12}, the origin is denoted by the blue triangle.

Fig. 2. The trajectories of components x_{11}, x_{12} of solutions to (38). Initial condition is $z(0, 0) = (3, 3, 1)$.

single-valued maps F and G, and a closed set $\mathcal{A} \subset \mathcal{N} \subset \mathbb{R}^n$ with $\mathcal{N}$ open such that $G(\mathcal{N}) \subset \mathcal{N}$. Suppose that for each $\xi \in \mathcal{N} \cap (\overline{C} \cup D)$, each $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$ satisfies

1) $\sup_{(t,j) \in \operatorname{dom} \phi} t = \infty$,
2) *there exists $\underline{\gamma} > 0$ such that $t_{j+1} - t_j \geq \underline{\gamma}$ for all $j \geq 1$, where $(t_j, j), (t_{j+1}, j) \in \operatorname{dom} \phi$ (i.e., the elapsed time between consecutive jumps is uniformly bounded below by a positive constant).*

Moreover, suppose there exists a function $V \in \mathcal{C}^2$ such that V is positive definite with respect to $\mathcal{A}$. Denoting the function $\nu : \mathbb{R}^n \to \mathbb{R}$ by $\nu(z) := \langle \nabla V(z), F(z) \rangle$, consider the following conditions:

3) *for some $c_1 > 0$,*

$$
\langle \nabla \nu(z), F(z) \rangle + c_1 \leq 0 \quad \forall z \in C \cap \mathcal{N}, \tag{40}
$$
$$
V(G(z)) - V(z) \leq 0 \quad \forall z \in D \cap \mathcal{N}, \tag{41}
$$

4) *there exists $M : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0, 0) = \xi$, $\nu(\phi(t, j)) \leq M(\xi)$ for all $(t, j) \in \operatorname{dom} \phi$ such that $\phi(t, j) \in \overline{C} \cap \mathcal{N}$,*
5) *for any $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$, $J = \sup_{(t,j) \in \operatorname{dom} \phi} j < \infty$,*
6) *for any $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$, $J = \sup_{(t,j) \in \operatorname{dom} \phi} j = \infty$, and there exists $\overline{\gamma}$ such that $t_{j+1} - t_j \leq \overline{\gamma}$ for all $j \geq 1$ and $(t_j, j), (t_{j+1}, j) \in \operatorname{dom} \phi$ (i.e., the elapsed time between consecutive jumps is uniformly bounded above by a positive constant),*
7) *for any $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0, 0) = \xi$, $t_1 = \sup_{(t,0) \in \operatorname{dom} \phi} t$ satisfies $t_1 \geq \mathcal{T}^\star(\xi)$, where*

$$
\mathcal{T}^\star(\xi) = \frac{\dot{V}(\xi) + \sqrt{\dot{V}^2(\xi) + 2c_1 V(\xi)}}{c_1}. \tag{42}
$$

Then, we have the following properties:

a) *when items 1)-3), 4), 5) hold, the set $\mathcal{A}$ is LFTA and, for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0, 0) = \xi$ such* *that* $\operatorname{rge}\phi \subset \mathcal{N}$*, the settling-time function satisfies*

$$\mathcal{T}^\star(\xi) = t_J + \frac{M(\xi)+\sqrt{M(\xi)^2+2(V(\xi)+M(\xi)t_J)c_1}}{c_1}; \quad (43)$$

b) when items 1)-3), 4), 6) *hold and, for each* $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N} \cap (\overline{C} \cup D))$ *with* $\phi(0,0) = \xi$ *such that* $\operatorname{rge}\phi \subset \mathcal{N}$*,*

$$\frac{1}{2}c_1\underline{\gamma}^2 - M(\xi)\overline{\gamma} > 0,$$

the set $\mathcal{A}$ *is LFTA, and the settling-time function satisfies*

$$\mathcal{J}^\star(\xi) = \frac{V(\xi) + c_1\underline{\gamma}^2}{\frac{1}{2}c_1\underline{\gamma}^2 - M(\xi)\overline{\gamma}};$$

c) when items 1)-3), 7) *hold, the set* $\mathcal{A}$ *is LFTA and the settling-time function for a given* $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N} \cap (\overline{C} \cup D))$ *with* $\phi(0,0) = \xi$ *such that* $\operatorname{rge}\phi \subset \mathcal{N}$*, satisfies* $\mathcal{T}(\xi) \le \mathcal{T}^\star(\xi)$ *where* $\mathcal{T}^\star$ *is given in* (42)*. If* $\mathcal{T}(\xi) = \mathcal{T}^\star(\xi)$*,* $\mathcal{A} \cap D \neq \emptyset$*.*

Proof Let $\Omega \subset \mathcal{N}$ be a forward invariant set for $\mathcal{H}$ containing an open neighborhood of $\mathcal{A}$; e.g., a small enough sublevel set of V. Pick any $\phi \in \mathcal{S}_\mathcal{H}(\Omega)$ and note that $\operatorname{rge}\phi \subset \Omega$. Pick any $(t,j) \in \operatorname{dom}\phi$ and let $0 = t_0 \le t_1 \le \cdots \le t_{j+1} = t$ satisfy (17). For each $i \in \{0,1,\ldots,j\}$ and almost all $s \in [t_i, t_{i+1}]$, $\phi(s,i) \in C \cap \mathcal{N}$. Then, using (40), we have that, for each $i \in \{0,1,\ldots,j\}$ and for almost all $s \in [t_i, t_{i+1}]$,

$$\frac{d^2}{ds^2}V(\phi(s,i)) + c_1 \le 0. \quad (44)$$

Integrating over $[t_i, t_{i+1}]$ both sides of this inequality twice yields

$$\begin{aligned} &V(\phi(t_{i+1},i)) - V(\phi(t_i,i)) \\ &\quad \le \dot V(\phi(t_i,i))(t_{i+1}-t_i) - \frac{1}{2}c_1(t_{i+1}-t_i)^2 \\ &\quad \le M(\xi)(t_{i+1}-t_i) - \frac{1}{2}c_1(t_{i+1}-t_i)^2. \end{aligned} \quad (45)$$

where we used the property in item 4). Similarly, for each $i \in \{1,\cdots,j\}$, $\phi(t_i, i-1) \in D \cap \mathcal{N}$, and thus

$$V(\phi(t_i,i)) - V(\phi(t_i,i-1)) \le 0 \quad \forall i \in \{1,\cdots,j\}. \quad (46)$$

Using inequalities (45) and (46), it implies that, $\forall j \in \mathbb{N}$,

$$\begin{aligned} &V(\phi(t,j)) - V(\xi) \\ &\le M(\xi)t - \frac{1}{2}c_1\max\{(j-1),0\}\underline{\gamma}^2 - \frac{1}{2}c_1(t-t_j)^2. \end{aligned} \quad (47)$$

The inequality (47) leads to the following cases:

- when item 5) holds, we have

  $$V(\phi(t,j)) \le V(\xi) + M(\xi)t_j + M(\xi)(t-t_j) - \tfrac{1}{2}c_1(t-t_j)^2.$$

  Then, using the the fact V is nonnegative, it follows that the resulting $\mathcal{T}^\star(\xi)$ is equal to the one in (43) since item 1) guarantees that $t \ge \mathcal{T}^\star(\xi)$ satisfies $(t,j) \in \operatorname{dom}\phi$ for some j. Therefore, item *a*) holds;
- when item 6) holds, then, we have for $j \ge 1$,

  $$V(\phi(t,j)) \le V(\xi) + c_1\underline{\gamma}^2 - (j+1)\left(\frac{1}{2}c_1\underline{\gamma}^2 - M(\xi)\overline{\gamma}\right).$$

  Note that by assumption in item *b*), $\frac{1}{2}c_1\underline{\gamma}^2 - \overline{\gamma}M(\xi) > 0$. Therefore, there exists $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \operatorname{dom}\phi$ such that

  $$\begin{aligned} &V(\phi(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi))) \\ &\le V(\xi) + c_1\underline{\gamma}^2 - (\mathcal{J}^\star(\xi)+1)\left(\frac{1}{2}c_1\underline{\gamma}^2 - M(\xi)\overline{\gamma}\right) \le 0. \end{aligned}$$

  In particular, $\mathcal{J}^\star(\xi) = \frac{V(\xi)+c_1\underline{\gamma}^2}{\frac{1}{2}c_1\underline{\gamma}^2 - M(\xi)\overline{\gamma}}$. Therefore, item *b*) holds;
- when item 7) holds, then, from (44), we have, for $(t,0) \in \operatorname{dom}\phi$,

  $$V(\phi(t,0)) \le V(\xi) + \dot V(\xi)t - \frac{1}{2}c_1 t^2. \quad (48)$$

  From (48), we obtain that

  $$\mathcal{T}^\star(\xi) = \frac{\dot V(\xi) + \sqrt{\dot V^2(\xi) + 2c_1V(\xi)}}{c_1}, \quad (49)$$

  which leads to item *c*). ∎

Example 3.19 *Consider the bouncing ball system [15, Example 1.1]* $\mathcal{H}$ *with data*

$$\mathcal{H}: \begin{cases} \dot x = F(x) := \begin{bmatrix} x_2 \\ -\chi \end{bmatrix} & x \in C, \\ x^+ = G(x) := \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} & x \in D, \end{cases} \quad (50)$$

where $C = \{x \in \mathbb{R}^2 : x_1 \ge 0\}$*,* $D = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \le 0\}$*. The parameter* $\chi > 0$ *denotes the gravity. See [15, Example 1.1] for more details on this model. Consider the set* $\mathcal{A} = \{x \in C \cup D : x_1 = 0\}$ *(denotes the ground) and a smooth function* $V(x) = x_1$*. Then,* $\nu(x) = x_2\ \forall x \in \mathbb{R}^2$*. Moreover,* $\langle \nabla\nu(x), F(x)\rangle + \chi \le 0$ *for all* $x \in C$*, and* $V(G(x)) - V(x) = 0$ *for all* $x \in D$*. Furthermore, for a maximal solution* $\phi = (\phi_1, \phi_2)$ *to* $\mathcal{H}$ *from* $\phi(0,0) = (\xi_1, \xi_2)$*, it follows that* $t_1 = \frac{\xi_2 + \sqrt{\xi_2^2 + 2\chi\xi_2}}{\chi} = \mathcal{T}^\star(\phi(0,0))$*. Therefore, items* 3) *and* 7) *in Theorem 3.18 hold with*

$\mathcal{N} = \mathbb{R}^2$. *By item (c) of Theorem 3.18, the set $\mathcal{A}$ is FTA for $\mathcal{H}$. Note that $\mathcal{A}$ is not stable since a maximal solution from $\mathcal{A}$ with nonzero x_2 component will leave $\mathcal{A}$.* $\triangle$

Our last sufficient condition pertains to finite time convergence of solutions from closed sets. To establish such condition, we use the following notion.

Definition 3.20 *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and closed sets $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}^n$. Let $\mathcal{T} : \mathcal{N} \to \mathbb{R}_{\geq 0}$ with $\mathcal{N} \subset \mathbb{R}^n$ open and such that $\mathcal{A}_1 \subset \mathcal{N}$. The closed set $\mathcal{A}_2$ is said to be* finite time attractive from $\mathcal{A}_1$ *(FTA from $\mathcal{A}_1$) for $\mathcal{H}$ if for every solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{A}_1)$,*

$$\lim_{t+j \nearrow \mathcal{T}(\phi(0,0))} |\phi(t,j)|_{\mathcal{A}_2} = 0. \tag{51}$$

Now, we are ready to present the following cascade-like result.

Proposition 3.21 *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and closed sets $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{k^\star} \subset \mathbb{R}^n$, $k^\star \in \mathbb{N}$. Suppose that*

1) *the set $\mathcal{A}_0$ is FTA for $\mathcal{H}$ with the settling-time function $\mathcal{T}_0 : \mathcal{N}_0 \to \mathbb{R}_{\geq 0}$, $\mathcal{N}_0 \subset \mathbb{R}^n$ open and such that $\overline{C} \cup D \subset \mathcal{N}_0$ and for any $\phi \in \mathcal{S}_{\mathcal{H}}$ with $\phi(0,0) = \xi$, $\sup_{(t,j)\in \operatorname{dom}\phi}(t+j) > \mathcal{T}_0(\xi)$,*
2) *for each $i \in \{1, 2, \ldots, k^\star\}$, $\mathcal{A}_i$ is FTA from $\mathcal{A}_{i-1}$ with the settling-time function $\mathcal{T}_i : \mathcal{N}_i \to \mathbb{R}_{\geq 0}$, $\mathcal{N}_i \subset \mathbb{R}^n$ open and such that $(\overline{C} \cup D) \cap \mathcal{A}_i \subset \mathcal{N}_i$, and for any $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{A}_{i-1})$ with $\phi(0,0) = \xi$, $\sup_{(t,j)\in \operatorname{dom}\phi}(t+j) > \mathcal{T}_i(\xi)$.*

Then, the set $\mathcal{A}_{k^\star}$ is globally finite time attractive.

Proof Let $\phi \in \mathcal{S}_{\mathcal{H}}$ and $\phi(0,0) = \xi_0$. Since $\mathcal{A}_0$ is FTA for $\mathcal{H}$, by item 1), there exists $\mathcal{T}_0(\xi_0) = \mathcal{T}_0^\star(\xi_0) + \mathcal{J}_0^\star(\xi_0)$ such that $\lim_{t+j\nearrow \mathcal{T}_0(\xi_0)} |\phi(t,j)|_{\mathcal{A}_0} = 0$ and $\phi(\mathcal{T}_0^\star(\xi_0), \mathcal{J}_0^\star(\xi_0)) \in \mathcal{A}_0$. Using the assumption that $\mathcal{A}_1$ is FTA from $\mathcal{A}_0$ for $\mathcal{H}$ and item 2), there exists $\mathcal{T}_1(\xi_1) = \mathcal{T}_1^\star(\xi_1) + \mathcal{J}_1^\star(\xi_1)$ such that $(\mathcal{T}_1^\star(\xi_1) + \mathcal{T}_0^\star(\xi_0), \mathcal{J}_1^\star(\xi_1) + \mathcal{J}_0^\star(\xi_0)) \in \operatorname{dom}\phi$ and

$$\lim_{t+j\nearrow \mathcal{T}_1^\star(\xi_1)+\mathcal{T}_0^\star(\xi_0)+\mathcal{J}_1^\star(\xi_1)+\mathcal{J}_0^\star(\xi_0)} |\phi(t,j)|_{\mathcal{A}_1} = 0.$$

where $\xi_1 = \phi(\mathcal{T}_0^\star(\phi(0,0)), \mathcal{J}_0^\star(\phi(0,0)))$. By recursively using the property that $\mathcal{A}_i$ is FTA from $\mathcal{A}_{i-1}$ for $i \in \{1, 2, \ldots, k^\star\}$ and item 2), we obtain that

$$\lim_{t+j\nearrow \sum_{i=0}^{k^\star}(\mathcal{T}_i^\star(\xi_i)+\mathcal{J}_i^\star(\xi_i))} |\phi(t,j)|_{\mathcal{A}_{k^\star}} = 0, \tag{52}$$

where $\xi_i = \phi(\sum_{\ell=0}^{i-1} \mathcal{T}_\ell^\star(\xi_\ell), \sum_{\ell=0}^{i-1} \mathcal{J}_\ell^\star(\xi_\ell))$ for $i \in \{1, 2, 3, \ldots, k^\star\}$, and $\xi_0 = \phi(0,0)$. Therefore, $\mathcal{A}_{k^\star}$ is globally finite time attractive. ■

Remark 3.22 *Using results from [15, Lemma 7.20], one can show that if $\mathcal{H}$ satisfies the hybrid basic conditions and a compact set $\mathcal{A}$ is FTS for $\mathcal{H}$ then the compact set $\mathcal{A}$ is semiglobally practically robustly $\mathcal{KL}$ pre-asymptotically stable for $\mathcal{H}$.*

4 Application to Finite Time Parameter Estimation

In this section, we show an application of the proposed finite time stability notion in Definition 3.1 and results in Section 3 for the design of a finite time parameter estimation algorithm with robustness. As in [1,18], the systems considered are of the form

$$\dot{x} = f(x) + g(x)\theta \tag{53}$$

where $x \in \mathbb{R}^n$ is the state which can be measured, and $\theta \in \mathbb{R}^p$ is the unknown parameter vector. Following [1,18], a parameter estimation algorithm for system (53) leads to a hybrid system $\mathcal{H}_p$ with state $z = (x, \hat{x}, \hat{\theta}, L, Q, \eta, \Gamma) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^{n\times p} \times \mathcal{SP}^{p\times p} \times \mathbb{R}^n \times \mathbb{R}^p$ and data [9]

$$\begin{aligned} F(z) &:= \begin{bmatrix} f(x) + g(x)\theta \\ f(x) + g(x)\hat{\theta} + K(x - \hat{x}) + Lh(z) \\ h(z) \\ g(x) - KL \\ L^\top L \\ -K\eta \\ L^\top(L\hat{\theta} + x - \hat{x} - \eta) \end{bmatrix} && \forall z \in C \\ G(z) &:= (x, x, Q^{-1}\Gamma, 0, 0, 0, 0) && \forall z \in D \end{aligned} \tag{54}$$

where $\hat{x}$ is the estimate of x, $\hat{\theta}$ is the estimate of θ, L, Q, η, Γ are auxiliary state variables, and $h(z) = \Omega(L^\top + g(x)^\top)(x - \hat{x})$. The parameters of the algorithm are $\varepsilon > 0$, $K > 0$, $\Omega = \Omega^\top > 0$. The flow set is $C = \{z \in \mathcal{X} : \det(Q) \leq \varepsilon\}$, and the jump set is $D = \{z \in \mathcal{X} : \det(Q) \geq \varepsilon\}$. To make the jump map well-defined as in [1,18], the following persistency of excitation condition is imposed.

Assumption 4.1 *The function g is differentiable and there exist $\sigma_1, \sigma_2 > 0$ such that for any $t_0 \geq 0$ and any solution $t \mapsto \phi_x(t)$ to (53),*

$$\int_{t_0}^{t_0+\sigma_1} g^\top(\phi_x(s)) g(\phi_x(s)) ds \geq \sigma_2 I. \tag{55}$$

[9] The states L and Q can be vectorized to yield a vector z, but to simplify notation we omit that.

Remark 4.2 *Under the persistency of excitation property of g in Assumption 4.1, for each solution ϕ to the hybrid system with data as in (54), it can be shown that there exists $T > 0$ such that the Q component of ϕ, namely, ϕ_Q, is invertible at $(T, 0)$, i.e.,*

$$\phi_Q(T,0) = \int_0^T \phi_L^\top(\tau,0)\phi_L(\tau,0)d\tau \tag{56}$$

satisfies $\det(\phi_Q(T,0)) \neq 0$. *In fact, let $\phi \in \mathcal{S}_\mathcal{H}$ with $\phi = (\phi_x, \phi_{\hat{x}}, \phi_{\hat{\theta}}, \phi_L, \phi_Q, \phi_\eta, \phi_\Gamma)$. Under Assumption 4.1 and following the proof of [18, Theorem 3], we have the following property: there exist $\sigma_1, \sigma_2 > 0$ such that*

$$\int_0^{\sigma_1} \phi_L^\top(s,0)\phi_L(s,0)ds \geq \sigma_2 I$$

Moreover, it follows that for any $m > 0$,

$$\int_0^{m\sigma_1} \phi_L^\top(s,0)\phi_L(s,0)ds \geq m\sigma_2 I.$$

By picking $m^\star, \varepsilon > 0$ such that $(m^\star\sigma_1, 0) \in \operatorname{dom}\phi$ and $(m^\star\sigma_2)^p \geq \varepsilon$, we have

$$\begin{aligned}\phi_Q(m^\star\sigma_1,0) &= \int_0^{m^\star\sigma_1} \phi_L^\top(\tau,0)\phi_L(\tau,0)d\tau + \phi_Q(0,0)\\ &\geq m^\star\sigma_2 I.\end{aligned}$$

Then, using [18, Lemma 2], it follows that

$$\det(\phi_Q(m^\star\sigma_1,0)) \geq \det(m^\star\sigma_2 I) = (m^\star\sigma_2)^p \geq \varepsilon \tag{57}$$

Remark 4.3 *Note that the model in [1] does not have recursive jumps and that the model in [18] has a slightly different flow map. In particular, in [18], the term $Lh(z)$ in F involved in the right-hand side for $\hat{x}$ is in the right-hand side of η instead.*

Proposition 4.4 *Suppose Assumption 4.1 holds for* (53). *Consider the parameter estimation algorithm leading to the hybrid system with data as in* (54) *with*[10] $K = K_1 + \frac{1}{4}g(x)\Omega g^\top(x)$, $K_1 > \frac{1}{4}I$, $\tau_{\max} > 0$, *and* $\varepsilon > 0$ *such that* $\det(\phi_Q(\tau_{\max}, 0)) = \det\left(\int_0^{\tau_{\max}} \phi_L^\top(\tau,0)\phi_L(\tau,0)d\tau\right) = \varepsilon$ *for all maximal solutions with zero initial conditions of the components L, Q, η, and Γ. Then, the set $\mathcal{A}_1 = \{z \in \mathcal{X} : \hat{x} = x, \eta = 0, \hat{\theta} = \theta\}$ is globally finite time stable for $\mathcal{H}$. Furthermore, the settling-time function for a given $\phi \in \mathcal{S}_\mathcal{H}$ with $\phi(0,0) = \xi \in \mathcal{X}$ satisfies $\mathcal{T}(\xi) \leq \tau_{\max} + 1$.*

Proof Given the choices of $K, K_1, \tau_{\max}$ and ε such that the assumption is satisfied, let $\phi \in \mathcal{S}_\mathcal{H}$ with $\phi = (\phi_x, \phi_{\hat{x}}, \phi_{\hat{\theta}}, \phi_L, \phi_Q, \phi_\eta, \phi_\Gamma)$. Under Assumption 4.1, and according to the arguments in Remark 4.2, there exists $\overline{T} > 0$ such that $\det(\phi_Q(\overline{T}, 0)) = \varepsilon$ is satisfied. Therefore, $\phi(\overline{T}, 0) \in D$ and it follows that the set $\mathcal{A}_0 = \{z \in \mathcal{X} : L = 0, Q = 0, \eta = 0, \Gamma = 0\}$ is globally finite time attractive for $\mathcal{H}$.

Now, consider $\phi^\star = (\phi_x^\star, \phi_{\hat{x}}^\star, \phi_{\hat{\theta}}^\star, \phi_L^\star, \phi_Q^\star, \phi_\eta^\star, \phi_\Gamma^\star) \in \mathcal{S}_\mathcal{H}(\mathcal{A}_1)$. Due to the dynamics of the state component Q, $\phi^\star$ also reaches the jump set in finite time. In particular, $\phi^\star(\tau_{\max}, 0) \in D$. At the jump, according to the jump map, by using the relationship[11] $\eta = x - \hat{x} - L(\theta - \hat{\theta})$, we have that

$$\phi_\Gamma^\star(\tau_{\max},0) = \int_0^{\tau_{\max}} \phi_L^\star(s,0)^\top \phi_L^\star(s,0)ds\,\theta. \tag{58}$$

Therefore, we obtain $\phi_{\hat{\theta}}^\star(\tau_{\max}, 1) = \theta$. Furthermore, according to the construction of the jump map G, we have $\phi_x^\star(\tau_{\max}, 1) = \phi_{\hat{x}}^\star(\tau_{\max}, 1)$ and $\phi_\eta^\star(\tau_{\max}, 1) = 0$. Thus, the set $\mathcal{A}_1$ is finite time attractive from $\mathcal{A}_0$ for $\mathcal{H}$.

To show that the set $\mathcal{A}_1$ is stable, for any given $\varepsilon > 0$, let $\delta > 0$ be such that for any

$$\phi = (\phi_x, \phi_{\hat{x}}, \phi_{\hat{\theta}}, \phi_L, \phi_Q, \phi_\eta, \phi_\Gamma) \in \mathcal{S}_\mathcal{H}$$

with $|\phi(0,0)|_{\mathcal{A}_1} \leq \delta$, the following holds:

$$|\phi(t,0)|_{\mathcal{A}_1} \leq \varepsilon \quad \forall (t,0) \in \operatorname{dom}\phi. \tag{59}$$

Note that there exists such δ since the set $\mathcal{A}_0$ is globally finite time attractive for $\mathcal{H}$, hence solution components $\phi_L, \phi_Q, \phi_\eta, \phi_\Gamma$ do not have finite time escape behavior. In fact, the inequality (59) can be checked by computing the finite time reachable set from initial conditions that belongs to the δ-ball of the set $\mathcal{A}_1$.

Now, denote $\phi_e = \phi_x - \phi_{\hat{x}}$ and $\phi_{\tilde{\theta}} = \phi_\theta - \phi_{\hat{\theta}}$, then, $|\phi|_{\mathcal{A}_1}^2 = \frac{1}{2}(\phi_e^\top\phi_e + \phi_{\tilde{\theta}}^\top \Omega^{-1}\phi_{\tilde{\theta}} + \phi_\eta^\top\phi_\eta)$, where $\Omega = \Omega^\top > 0$. Then[12], for almost all $(t, j) \in \operatorname{dom}\phi$ such that $j \geq 1$,

$$\frac{d}{dt}|\phi|_{\mathcal{A}_1}^2 \leq -\phi_e^\top K_1\phi_e - \phi_\eta^\top\left(K_1 - \frac{1}{4}I\right)\phi_\eta \tag{60}$$

where we used the property $\phi_\eta = \phi_e - \phi_L\phi_{\tilde{\theta}}$, which holds after the first jump, and $K = K_1 + \frac{1}{4}g\Omega g^\top$. Moreover,

[10] The arguments in K are dropped for simplicity.

[11] The state component η satisfies $\eta = x - \hat{x} - L(\theta - \hat{\theta})$ during flows after the first jump. In fact, for solutions with initial conditions in $G(D)$ and during flow, $\dot{\eta} = -K\eta$, let $\eta_1 = x - \hat{x} - L(\theta - \hat{\theta})$ then $\dot{\eta}_1 = g(x)(\theta - \hat{\theta}) - K(x - \hat{x}) - Lh(z) - L(-h(z)) - (g(x) - KL)(\theta - \hat{\theta}) = -K\eta_1$. Furthermore, for maximal solutions with initial conditions in $\mathcal{A}_1$, $\eta(0,0) = 0 = \eta_1(0,0)$.

[12] For simplicity, the solution argument (t, j) is omitted.

for each $(t, j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$,

$$|\phi(t, j+1)|^2_{\mathcal{A}_1} - |\phi(t, j)|^2_{\mathcal{A}_1} = -|\phi(t, j)|^2_{\mathcal{A}_1} \leq 0. \quad (61)$$

Then, under the conditions in the statement, integrating both sides of the inequality (60) and using (61) leads to $|\phi(t, j)|_{\mathcal{A}_1} \leq |\phi(t_1, 1)|_{\mathcal{A}_1} \leq |\phi(t_1, 0)|_{\mathcal{A}_1}$, where $(t_1, 0) \in \operatorname{dom} \phi$ is such that $(t_1, 1) \in \operatorname{dom} \phi$. Furthermore, by using (59), it follows that $|\phi(t, j)|_{\mathcal{A}_1} \leq \varepsilon$. Therefore, the set $\mathcal{A}_1$ is stable. Since the set $\mathcal{A}_0$ is FTA for $\mathcal{H}$, then, by applying Proposition 3.21 with $k^\star = 1$, the set $\mathcal{A}_1$ is globally finite time stable for $\mathcal{H}$. ■

Example 4.5 *Consider the following frequency estimation problem: given a signal $t \mapsto \zeta(t) = \Upsilon \sin(\omega t)$, where $\Upsilon > 0$ is the magnitude and $\omega > 0$ is the frequency, globally estimate ω from measurements of ζ. To formulate this problem as the problem of estimating a parameter (θ) of a model as in* (53)*, let $x = (x_1, x_2)$ be such that $x_1 = \zeta$ and $\dot{x}_1 = x_2$, $t \mapsto \zeta(t)$ is the first component to the solution to* (62)*, where Υ is related to the initial condition. This is an exosystem/generator approach for the generation of $t \mapsto \zeta(t)$. Note that a similar problem was studied in [9]. Then, we obtain the following parametric form*

$$\dot{x} = Ax + g(x)\theta, \quad (62)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}, \quad (63)$$

and $\theta = \omega^2 - 1$. Then, for any $t_0 \geq 0$, $\sigma_1 = \frac{\pi}{\omega}$ and $\sigma_2 = \frac{\pi \Upsilon^2}{2\omega}$, it can be checked that

$$\int_{t_0}^{t_0+\sigma_1} g(x(s))^\top g(x(s)) ds = \frac{\pi \Upsilon^2}{2\omega},$$

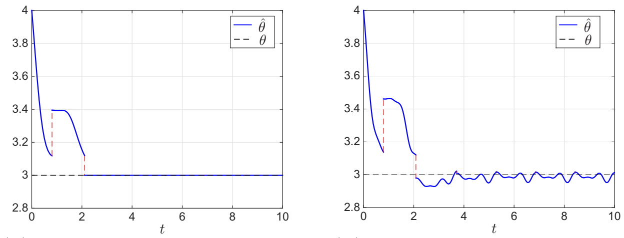
which verifies Assumption 4.1. Then, a parameter estimation algorithm for estimating θ can be designed following the construction in (54)*. A simulation*[13] *is shown in Figure 3(a), where the parameters are $\Upsilon = 1$, $\omega = 2$, $\varepsilon = 0.3$, $\Omega = 1$, $K_1 = 1$ and the initial conditions are* $x(0,0) = (2,2)$, $\phi_{\hat{x}}(0,0) = (1,1)$, $\phi_{\hat{\theta}}(0,0) = 4$, $\phi_L(0,0) = (0.1, 0.1)$, $\phi_Q(0,0) = 0$, $\phi_\eta(0,0) = (0.1, 0.2)$, $\phi_\Gamma(0,0) = 0.1$*, the estimate $\hat{\theta}$ converges to θ after two jumps.*

Table 1
Summary of the robustness property with respecting to different values of ρ

ρ	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
ε	0.03	0.07	0.12	0.16	0.21	0.26	0.32	0.37

To study the robustness property of finite time stability,

[13] Code at `https://github.com/HybridSystemsLab/FTSParameter`



(a) The projection of the estimate $\hat{\theta}$ on the t direction ($\theta = 3$).

(b) The projection of the estimate $\hat{\theta}$ on the t direction under perturbations d_2^x on the system model (62).

Fig. 3. Simulations of the closed-loop system between (62) and (54), showing $\hat{\theta}$ converging to θ in finite time.

consider the following perturbed model

$$\dot{x} = Ax + g(x)\theta + d_2^x, \quad (64)$$

where d_2^x is an unknown perturbation on the state x. A simulation is shown in Figure 3(b) with same set of parameters, and the perturbation signal $t \mapsto d_2^x(t)$ is given by $d_2^x(t) = \rho(\sin(10t), \cos(10t))$ with $\rho = 1$. In fact, for the frequency estimation algorithm, since $g(x)$ in (63) *is continuous, the resulting flow map $\tilde{F}$ is continuous. Furthermore, $Q \in \mathbb{R}_{\geq 0}$ and Q^{-1} is continuous on D. Therefore, hybrid basic conditions are satisfied and thus Remark 3.22 can be applied. Its performance with difference values of ρ is summarized in Table 1, where $\rho \approx 10\varepsilon$.* △

5 Conclusion

A notion of finite time stability consisting of both stability and finite time attractivity was proposed for hybrid systems. Sufficient conditions guaranteeing the new notion were proposed. These conditions conveniently isolate the properties needed when finite time convergence occurs via flows or via jumps. Conditions for robustness of the new notion to perturbations, though generic, rely on the regularity of the data of the hybrid system so as to preserve the finite time stability property semiglobally and practically. Current efforts are focused on establishing robustness in terms of ISS and extending the results to networked hybrid systems.

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