# Adaptive Backstepping of Synergistic Hybrid Feedbacks with Application to Obstacle Avoidance

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***Abstract*— In this paper, we show that the hybrid controller that is induced by a Synergistic Lyapunov Function and Feedback (SLFF) pair relative to a compact set, can be extended to the case where the original affine control system is subject to a class of additive disturbances known as matched uncertainties, provided that the estimator dynamics do not add new equilibria to the closed-loop system. We also show that the proposed adaptive hybrid controller is amenable to backstepping. Finally, we apply the proposed hybrid control strategy to the problem of global asymptotic stabilization of a compact set in the presence of an obstacle and we illustrate this application by means of simulation results.**

## I. Introduction

Over the last few years, there has been significant research effort towards the development of new analysis tools for hybrid dynamical systems, i.e., systems whose solutions exhibit both continuous-time and discrete-time behaviors (c.f. [1]). This research effort has fueled the development of novel hybrid control architectures, such as synergistic hybrid control, which we further explore in this paper.

Synergistic hybrid feedback is a hybrid control strategy that is comprised of a collection of potential functions that asymptotically stabilize a given compact set by gradient descent. If, for all equilibria that do not lie within the given compact set, there exists another function in the collection which has a lower value and does share the same equilibria, then it is possible to achieve global asymptotic stabilization of the given compact set through hysteretic switching. This control strategy has been successfully applied to the problem of global asymptotic stabilization of a compact set for systems evolving in compact manifolds, such as pendulum stabilization [2], vector-based rigid body stabilization [3], [4], tracking for a vectored-thrust vehicle [5], rigid-body tracking through unit-quaternion feedback [6] and rotation matrix feedback [7], [8]. Synergistic hybrid feedback provided a solution to global asymptotic stabilization on compact manifolds, thus overcoming the topological obstructions which plagued earlier continuous feedback approaches (c.f. [9]).

Another application which imposes similar limitations to continuous feedback is obstacle avoidance. Obstacle avoidance is an important and longstanding problem that reflects the need to drive mechanical systems from one place to another while avoiding any number of obstacles in its way. Several solutions to this problem have been proposed over the last few decades as highlighted in [10]. In particular, it is possible to find both stochastic [11] as well as determinisc approaches [12] to tackle the obstacle avoidance problem. However, it was shown in [13] that in a spherical state space, there is at least one saddle equilibrium point for each obstacle within the state space, thus precluding global asymptotic stabilization of a setpoint by continuous feedback. To address this limitation a hybrid control solution was proposed in [14]. In this paper, we propose a different hybrid control approach to the same problem using the concept of Synergistic Lyapunov Function and Feedback (SLFF) pairs that was introduced in [15].

The main contributions of this paper are as follows: 1) we provide sufficient conditions for the construction of a SLFF pair for a affine control system that is subject to matched uncertainties from a SLFF pair for the unperturbed system; 2) we develop a backstepping control design that preserves the synergism properties, provided that no new equilibria are added to the system due to the presence of perturbations and; 3) we apply the proposed controller to the problem of global obstacle avoidance. More specifically, in Section III, we present the notion of a SLFF pair and, similar to [15], show how it induces a hybrid controller for a control affine system that renders a given compact set globally asymptotically stable for the closed-loop hybrid system. In Section IV, we demonstrate that, given a SLFF pair for the control affine system, it is possible to modify the controller in order to compensate the effect of bounded matched uncertainties. In Section V, we show that SLFF pairs and the hybrid controllers that they induce are amenable to hybrid backstepping. In Section VI, we show that SLFF pairs can be used to solve the problem of global asymptotic stabilization of a compact set in the presence of an obstacle and we demonstrate this application by means of simulation results in Section VI-A. In Section II, we present some mathematical preliminaries and notation that we use throughout the paper and in Section VII we provide some concluding remarks. The proofs of the results in this paper will appear elsewhere.

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## II. Preliminaries and Notation

### A. Notation

The $n$-dimensional Euclidean space is represented by $\mathbb{R}^n$ and it is equipped with the inner product $\langle u, v\rangle = u^\top v$, defined for each $u, v \in \mathbb{R}^n$ and the norm $|x| := \sqrt{\langle x, x\rangle}$ for each $x \in \mathbb{R}^n$. The $n$-dimensional Euclidean space has the topology generated by a countable basis of open balls of the form $c + \epsilon\mathbb{B} := \{x \in \mathbb{R}^n : |x - c| < \epsilon\}$, where $c \in \mathbb{R}^n$ and $\epsilon > 0$. More generally, given a set $\Omega \subset \mathbb{R}^n$, we define $\Omega + \epsilon\mathbb{B} := \bigcup_{c\in\Omega} c + \epsilon\mathbb{B}$. The operators $\partial S$ and $\overline{S}$ denote the boundary and the closure of a set $S$, respectively. Given a subset $S$ of $X := X_1 \times X_2$, the projection of $S$ onto $X_1$ is represented by $\pi_{X_1}(S) := \{x_1 \in X_1 : (x_1, x_2) \in S \text{ for some } x_2 \in X_2\}$. Similarly, the projection of $S$ onto $X_2$ is denoted by $\pi_{X_2}(S) := \{x_2 \in X_2 : (x_1, x_2) \in S \text{ for some } x_1 \in X_1\}$. The tangent cone to a set $S \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted by $\mathsf{T}_x S$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exists $x_i \in S$, $\tau_i > 0$ with $x_i \to x$ and $\tau_i \to 0^+$ such that $w = \lim_{i\to\infty}(x_i - x)/\tau_i$.

A set-valued map $M : S \rightrightarrows \mathbb{R}^n$ associates a subset of $\mathbb{R}^n$ to each point in $S$, represented by $M(x)$. The graph of a set-valued map $M : S \rightrightarrows \mathbb{R}^n$ is given by $\operatorname{gph} M := \{(x, y) \in S \times \mathbb{R}^n : y \in M(x)\}$, its domain is given by $\operatorname{dom} M = \pi_S(\operatorname{gph} M)$ and its range is $\operatorname{rge} M = \pi_{\mathbb{R}^n}(\operatorname{gph} M)$. A set-valued map $M$ is: outer semicontinuous if $\operatorname{gph} M$ is closed; locally bounded if, for each $x \in \operatorname{dom} M$ there exists a neighborhood $U_x$ of $x$ such that $M(U_x)$ is bounded; upper semicontinuous if $M(x)$ is closed for each $x \in \operatorname{dom} M$, $M$ is outer semicontinuous and locally bounded; lower semicontinuous if, for each $x \in \operatorname{dom} M$, all convergent sequences $x_i \to x$ in $\operatorname{dom} M$ have a subsequence $x_{i(k)}$ such that $M(x_{i(k)})$ converges to $M(x)$; continuous if it is both upper and lower semicontinuous.

Given a differentiable function $F : \mathbb{R}^{m\times n} \times \mathbb{R}^{k\times\ell} \to \mathbb{R}^{p\times q}$, we define

$$\mathcal{D}_X F(X, Y) := \frac{\partial \operatorname{vec}(F)}{\partial \operatorname{vec}(X)^\top}(X, Y)$$

for each $(X, Y) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{k\times\ell}$ for each $X \in \mathbb{R}^{m\times n}$, where $\operatorname{vec}(A) := [e_1^\top A^\top \ \dots \ e_m^\top A^\top]^\top$ for each $A \in \mathbb{R}^{m\times n}$ and $e_i \in \mathbb{R}^m$ is a vector of zeros, except for the $i$-th component, which is 1. If $F : \mathbb{R}^{m\times n} \to \mathbb{R}$, then $\nabla F(X) := \mathcal{D}F(X)^\top$.

### B. Hybrid Systems

A hybrid system $\mathcal{H}$ with state space $\mathbb{R}^n$ is defined as follows:

$$\begin{aligned} \dot{\xi} &\in F(\xi) \quad \xi \in C \\ \xi^+ &\in G(\xi) \quad \xi \in D \end{aligned}$$

where $\xi \in \mathbb{R}^n$ is the state, $C \subset \mathbb{R}^n$ is the flow set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $D \subset \mathbb{R}^n$ denotes the jump set, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the jump map. A solution $\xi$ to $\mathcal{H}$ is parametrized by $(t, j)$, where $t$ denotes ordinary time and $j$ denotes the jump time, and its domain $\operatorname{dom}\xi \subset \mathbb{R}_{\geq 0}\times\mathbb{N}$ is a hybrid time domain: for each $(T, J) \in \operatorname{dom}\xi$, $\operatorname{dom}\xi \cap ([0, T] \times \{0, 1, \dots J\})$ can be written in the form $\cup_{j=0}^{J}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_J$, where $I_j := [t_j, t_{j+1}]$ and the $t_j$'s define the jump times. A solution $\xi$ to a hybrid system is said to be *maximal* if it cannot be extended by flowing nor jumping and *complete* if its domain is unbounded. The projection of solutions onto the $t$ direction is given by $\xi{\downarrow_t}(t) := \xi(t, J(t))$ where $J(t) := \max\{j : (t, j) \in \operatorname{dom}\xi\}$. The distance of a point $\xi \in \mathbb{R}^n$ to a closed set $\mathcal{A} \subset \mathbb{R}^n$ is given by $|\xi|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |y - \xi|$. The definitions of global uniform pre-asymptotic stability and invariance can be found in [1].

## III. Syergistic Lyapunov Functions and Feedback Pairs

In this paper, we address the problem of globally asymptotically stabilizing a given compact set for an affine control system that is subject to exogenous disturbances. In particular, similar to [15], we consider the dynamical system

$$\dot{x} = f(x, q, u, \theta) := \psi_x(x, q) + \psi_u(x, q)u + \psi_\theta(x, q)\theta \quad (1)$$

for each $(x, q, u, \theta) \in \mathcal{X}\times\mathcal{Q}\times\mathbb{R}^m\times\Omega$, where $x \in \mathcal{X}$ denotes the state of the system with $\mathcal{X} \subset \mathbb{R}^n$ closed, $q \in \mathcal{Q}$ with $\mathcal{Q} \subset \mathbb{Z}$ finite represents a logic variable, $u \in \mathbb{R}^m$ represents the input, $\theta \in \Omega$ represents a constant disturbance whose norm is bounded by $\theta_0 \in \mathbb{R}_{\geq 0}$, i.e.

$$\Omega := \{\theta \in \mathbb{R}^\ell : |\theta| \leq \theta_0\}, \quad (2)$$

$\psi_x : \mathcal{X} \times \mathcal{Q} \to \mathbb{R}^n$, $\psi_u : \mathcal{X} \times \mathcal{Q} \to \mathbb{R}^{n\times m}$ and $\psi_\theta : \mathcal{X} \times \mathcal{Q} \to \mathbb{R}^\ell$ are continuous functions satisfying $\psi_\theta(x, q) := \psi_u(x, q)\psi_{\hat{\theta}}(x, q)$ for each $(x, q) \in \mathcal{X} \times \mathcal{Q}$ for some continuously differentiable function $\psi_{\hat{\theta}} : \mathcal{X} \times \mathcal{Q} \to \mathbb{R}^{m\times\ell}$. To meet the desired goal, we resort to the concept of Synergistic Lyapunov Functions and Feedback (SLFF) pairs that was introduced in [15] and which we reproduce next for completeness.

Given a compact set $\mathcal{A} \subset \mathcal{X}\times\mathcal{Q}$, the continuous functions $V : \mathbb{R}^n \times \mathcal{Q} \to \mathbb{R}_{\geq 0}$ and $\kappa : \mathbb{R}^n \times \mathcal{Q} \to \mathbb{R}^m$ form a SLFF pair candidate relative to $\mathcal{A}$ for

$$\begin{cases} \dot{x} = f(x, q, \kappa(x, q), 0) \\ \dot{q} = 0 \end{cases} \qquad (x, q) \in \mathcal{X} \times \mathcal{Q} \quad (3)$$

if:

- (A1) $V$ is continuously differentiable on a neighborhood of $\mathcal{X} \times \mathcal{Q}$;
- (A2) Each sublevel set of $V$, given by
  $$\Omega_V(r) := \{(x, q) \in \mathcal{X} \times \mathcal{Q} : V(x, q) \leq r\},$$
  is compact;
- (A3) $V$ is positive definite relative to $\mathcal{A}$;
- (A4) For all $(x, q) \in \mathcal{X} \times \mathcal{Q}$, $f(x, q, \kappa(x, q), 0)$ belongs to the tangent cone to $\mathcal{X} \times \mathcal{Q}$ at $(x, q)$ and $\langle\nabla V(x, q), f(x, q, \kappa(x, q), 0)\rangle \leq 0$.

A SLFF pair candidate $(V, \kappa)$ relative to $\mathcal{A}$ for (3) induces the hybrid controller with output $\kappa$ and dynamics

$$\dot{q} = 0 \qquad (x, q) \in C := \{(x, q) \in \mathcal{X} \times \mathcal{Q} : \mu_V(x, q) \leq \delta\} \quad (4a)$$

$$q^+ = \varrho_V(x) \quad (x, q) \in D := \{(x, q) \in \mathcal{X} \times \mathcal{Q} : \mu_V(x, q) \geq \delta\} \quad (4b)$$

where $\delta > 0$,

$$\mu_V(x) := V(x, q) - \min_{p\in\mathcal{Q}} V(x, p) \quad (5)$$

for each $(x,q) \in \mathcal{X} \times \mathcal{Q}$ and

$$\varrho_V(x) := \arg\min\{V(x,q) : q \in \mathcal{Q}\} \tag{6}$$

for each $x \in \mathcal{X}$.

The interconnection between (1) and (4) results in the closed-loop hybrid system $\mathcal{H} := (C, F, D, G)$, given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} &= F(x,q) := \begin{bmatrix} f(x,q,\kappa(x,q),\theta) \\ 0 \end{bmatrix} & (x,q) \in C \\ \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} &\in G(x,q) := \begin{bmatrix} x \\ \varrho_V(x) \end{bmatrix} & (x,q) \in D. \end{aligned} \tag{7}$$

Property (A4) guarantees that each sublevel set of $V$ is invariant for the nominal instance of the system (1), i.e., with $\theta = 0$. Properties (A1) through (A3), ensure that the closed-loop hybrid system resulting from the interconnection between (1) and (4) is endowed with the following properties.

**Lemma 1.** *Given a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{Q}$ and a SLFF pair candidate relative to $\mathcal{A}$ for* (3)*, the following holds: 1) The function*

$$\nu_V(x) := \min_{q \in \mathcal{Q}} V(x,q) \qquad \forall x \in \mathcal{X}$$

*is continuous; 2) The function $\varrho_V$ in* (6) *is outer semicontinuous and $\varrho_V(x)$ is compact for each $x \in \mathcal{X}$; 3) The function $\mu_V$ in* (5) *is continuous.*

The regularity properties that are presented in Lemma 1 are key in proving that the closed-loop hybrid system (7) satisfies the hybrid basic conditions, given in [1, Assumption 6.5], as proved next.

**Lemma 2.** *Given a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{Q}$ and a SLFF pair candidate relative to $\mathcal{A}$ for* (3)*, the data $(C, F, D, G)$ of the closed-loop hybrid system satisfies the following:*

*(S1) The sets $C$ and $D$ are closed;*

*(S2) The flow map $F$ is outer semicontinuous and locally bounded relative to $C$, and $F(x,q)$ is convex for each $(x,q) \in \mathcal{X} \times \mathcal{Q}$;*

*(S3) The jump map $G$ is outer semicontinuous and locally bounded relative to $D$.*

The hybrid basic conditions ensure that the closed loop system (7) is endowed with robustness to perturbations and small measurement noise, as explained in [1, Chapter 7]. In particular, these conditions ensure that the system is well-posed and allow for the application of invariance principles for hybrid dynamical systems that are at the core of the stability proofs that are presented in this paper. Finally, we prove that each maximal solution to $\mathcal{H}$ in (7) is complete.

**Lemma 3.** *Given a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{Q}$ and a SLFF pair candidate relative to $\mathcal{A}$ for* (3)*, each maximal solution to* (7) *is complete.*

Given a SLFF pair candidate $(V, \kappa)$ relative to $\mathcal{A}$ for (3), we define $\mathcal{E} := \{(x,q) \in \mathcal{X} \times \mathcal{Q} : \langle \nabla V(x,q), F(x,q) \rangle = 0\}$ and let $\Psi \subset \mathcal{E}$ denote the largest weakly invariant subset of

$$\begin{cases} \dot{x} = f(x,q,\kappa(x,q),0) \\ \dot{q} = 0 \end{cases} \qquad (x,q) \in \mathcal{E} \tag{8}$$

Then, the SLFF pair candidate $(V, \kappa)$ is a SLFF pair relative to $\mathcal{A}$ for (3) if

(A5) $\mu_V(x,q) > 0$ for each $(x,q) \in \Psi \backslash \mathcal{A}$.

If there exists $\delta > 0$ such that $\mu_V(x,q) > \delta$ for each $(x,q) \in \Psi \backslash \mathcal{A}$, we say that $(V, \kappa)$ has synergy gap exceeding $\delta$.

Property (A5) implies that it is possible to avoid unwanted equilibria by switching among the available feedback laws and Lyapunov functions as shown in the following theorem.

**Theorem 1** ([15, Theorem 7])**.** *Suppose that $(V, \kappa)$ is a synergistic Lyapunov function and a feedback pair relative to the compact set $\mathcal{A}$ for* (3) *with synergy gap exceeding $\delta$. Then, the compact set $\mathcal{A}$ is globally asymptotically stable for the closed-loop system* (7) *(with $\theta = 0$).*

In the sequel, we show that it is possible to modify a given SLFF pair to compensate for the presence of constant disturbances $\theta \in \Omega$.

## IV. Adaptive Synergistic Hybrid Feedback

In this section, we modify the synergistic hybrid controller in (4) to address the case where $\theta$ in (1) is nonzero (we refer the reader to [17] for an overview of adaptive controller design and backstepping under the influence of model uncertainty). Given a SLFF pair $(V, \kappa)$, let $\hat{\theta} \in \mathbb{R}^\ell$ denote an estimate of the disturbance $\theta$ satisfying

$$\dot{\hat{\theta}} = \Gamma_0 \operatorname{Proj}(\psi_\theta(x,q)^\top \nabla V(x,q), \hat{\theta}), \tag{9}$$

where $\Gamma_0 \in \mathbb{R}^{\ell \times \ell}$ is a positive definite matrix and $\operatorname{Proj} : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ is given by

$$\operatorname{Proj}(\eta, \hat{\theta}) := \begin{cases} \eta & \text{if } \mathrm{p}(\hat{\theta}) \leq 0 \text{ or } \nabla\, \mathrm{p}(\hat{\theta})^\top \eta \leq 0 \\ \left(I_\ell - \frac{\mathrm{p}(\hat{\theta}) \nabla\, \mathrm{p}(\hat{\theta}) \nabla\, \mathrm{p}(\hat{\theta})^\top}{\nabla\, \mathrm{p}(\hat{\theta})^\top \nabla\, \mathrm{p}(\hat{\theta})}\right) \eta & \text{otherwise} \end{cases}$$

for each $(\eta, \hat{\theta}) \in \mathbb{R}^\ell \times \mathbb{R}^\ell$,

$$\mathrm{p}(\hat{\theta}) := \frac{\hat{\theta}^\top \hat{\theta} - \theta_0^2}{\epsilon^2 + 2\epsilon\theta_0}$$

for each $\hat{\theta} \in \mathbb{R}^\ell$ with $\epsilon > 0$ and $\theta_0 > 0$ given in (2). The function $\operatorname{Proj}$ in (9) implements an adaptive feedback law with the following properties (c.f. [18]):

(P1) Each solution $t \mapsto \hat{\theta}(t)$ to

$$\dot{\hat{\theta}} = \Gamma_0 \operatorname{Proj}(\eta(t), \hat{\theta}),$$

from $\hat{\theta} \in \Omega + \epsilon\overline{\mathbb{B}}$ with input $t \mapsto \eta(t)$ satisfies $\operatorname{rge} \hat{\theta} \subset \Omega + \epsilon\overline{\mathbb{B}}$;

(P2) We have that

$$(\theta - \hat{\theta})^\top \operatorname{Proj}(\eta, \hat{\theta}) \geq (\theta - \hat{\theta})^\top \eta$$

for each $(\eta, \hat{\theta}) \in \mathbb{R}^\ell \times \mathbb{R}^\ell$.

Given a SLFF pair relative to $\mathcal{A}$ for (3), denoted by $(V, \kappa)$, we define

$$\begin{aligned} V_0(x,q,\hat{\theta}) &:= V(x,q) + \frac{1}{2}(\theta - \hat{\theta})^\top \Gamma_0^{-1} (\theta - \hat{\theta}) \\ \kappa_0(x,q,\hat{\theta}) &:= \kappa(x,q) - \psi_{\hat{\theta}}(x,q)\hat{\theta} \end{aligned}$$

for each $(x,q,\hat{\theta}) \in \mathcal{S}_0 := \mathcal{X} \times \mathcal{Q} \times (\Omega + \epsilon\overline{\mathbb{B}})$. Following the controller design that was introduced in Section III and setting $u = \kappa_0(x,q)$, we obtain the closed-loop hybrid system $\mathcal{H}_0 := (C_0, F_0, D_0, G_0)$ given by

$$(\dot{x}, \dot{q}, \dot{\hat{\theta}}) = F_0(x,q,\hat{\theta}) \qquad (x,q,\hat{\theta}) \in C_0 \tag{11a}$$

$$(x^+, q^+, \hat{\theta}^+) \in G_0(x,q,\hat{\theta}) \qquad (x,q,\hat{\theta}) \in D_0 \tag{11b}$$

where $C_0 := \{(x,q,\hat{\theta}) \in \mathcal{S}_0 : \mu_{V_0}(x,q,\hat{\theta}) \leq \delta\}$, $D_0 := \{(x,q,\hat{\theta}) \in \mathcal{S}_0 : \mu_{V_0}(x,q,\hat{\theta}) \geq \delta\}$ and

$$F_0(x,q,\hat{\theta}) := \begin{bmatrix} f(x,q,\kappa_0(x,q),\theta) \\ 0 \\ \Gamma_0 \operatorname{Proj}(\psi_\theta(x,q)^\top \nabla V(x,q), \hat{\theta}) \end{bmatrix} \quad \forall (x,q,\hat{\theta}) \in C_0 \tag{12a}$$

$$G_0(x,q,\hat{\theta}) := \begin{bmatrix} x \\ \varrho_{V_0}(x,\hat{\theta}) \\ \hat{\theta} \end{bmatrix} \qquad \forall (x,q,\hat{\theta}) \in D_0. \tag{12b}$$

We show next that the functions $(V_0, \kappa_0)$ in (10) form a SLFF pair candidate relative to the compact set

$$\mathcal{A}_0 := \mathcal{A} \times \{\theta\}. \tag{13}$$

**Proposition 1.** *The pair $(V_0, \kappa_0)$ in (10) is a SLFF pair candidate relative to $\mathcal{A}_0$ for*

$$(\dot{x}, \dot{q}, \dot{\hat{\theta}}) = F_0(x,q,\hat{\theta}) \qquad (x,q,\hat{\theta}) \in \mathcal{S}_0. \tag{14}$$

Since $(V_0, \kappa_0)$ is a SLFF pair candidate relative to $\mathcal{A}_0$ for (14), we have in particular that $V_0$ is nonincreasing along solutions to the closed-loop system (11). However, showing that $(V_0, \kappa_0)$ is a SLFF pair relative to $\mathcal{A}_0$ for (14) requires further assumptions on the nature of the SLFF pair $(V, \kappa)$ and the function $\psi_\theta$, as shown next.

**Theorem 2.** *Given a SLFF pair $(V, \kappa)$ relative to a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{Q}$ for (3) with synergy gap exceeding $\delta$, let $\Psi$ denote the largest weakly invariant subset of (8) and let $\Psi_0$ denote the largest weakly invariant subset of*

$$(\dot{x}, \dot{q}, \dot{\hat{\theta}}) = F_0(x,q,\hat{\theta}) \qquad (x,q,\hat{\theta}) \in \mathcal{E}_0$$

*with $\mathcal{E}_0 := \{(x,q,\hat{\theta}) \in \mathcal{S}_0 : \langle \nabla V_0(x,q,\hat{\theta}), F_0(x,q,\hat{\theta})\rangle = 0\}$. If the projection of $\Psi_0 \backslash \mathcal{A}_0$ onto $\mathcal{X} \times \mathcal{Q}$ is a subset of $\Psi \backslash \mathcal{A}$, i.e.,*

$$\pi_{\mathcal{X}\times\mathcal{Q}}(\Psi_0 \backslash \mathcal{A}_0) \subset \Psi \backslash \mathcal{A}, \tag{15}$$

*then $(V_0, \kappa_0)$ in (10) is a SLFF pair relative to $\mathcal{A}_0$ for (14) with synergy gap exceeding $\delta$.*

It follows directly from [15, Theorem 7] that, under the assumptions of Theorem 2, the set $\mathcal{A}_0$ in (13) is globally asymptotically stable for the closed-loop hybrid system (11). Next, we demonstrate that synergism is a property that can also be preserved through backstepping.

## V. Adaptive Backstepping of Synergistic Feedbacks

Consider the following dynamical system

$$\dot{x} = f(x,q,u,\theta)$$
$$\dot{\hat{\theta}} = \Gamma_0 \operatorname{Proj}(\upsilon(\zeta), \hat{\theta})$$
$$\dot{u} = \tau$$

with $\zeta := (x,q,\hat{\theta},u) \in \mathcal{S}_1 := \mathcal{X} \times \mathcal{Q} \times (\Omega + \epsilon\overline{\mathbb{B}}) \times \mathbb{R}^m$, which is obtained from (1) by adding the estimator dynamics with

$$\upsilon(\zeta) := \psi_\theta(x,q)^\top \nabla V(x,q) + \psi_\theta(x,q)^\top \mathcal{D}_x(\kappa_0(x,q,\hat{\theta}))^\top \Gamma_1^{-1}(u - \kappa_0(x,q,\hat{\theta}))$$

for each $\zeta \in \mathcal{S}_1$, and $u$ is considered as a state of the dynamical system with $\tau \in \mathbb{R}^m$ as the new input.

Given a compact subset $\mathcal{A}$ of $\mathcal{X} \times \mathcal{Q}$ and a SLFF pair $(V, \kappa)$ relative to $\mathcal{A}$ for (3), the main goal of this section is the construction of a SLFF pair $(V_1, \kappa_1)$ relative to the compact set

$$\mathcal{A}_1 := \{\zeta \in \mathcal{S}_1 : (x,q,\hat{\theta}) \in \mathcal{A}_0,\ u = \kappa_0(x,q,\hat{\theta})\}.$$

In this direction, we define

$$V_1(\zeta) := V_0(x,q,\hat{\theta}) + \frac{1}{2}(u - \kappa_0(x,q,\hat{\theta}))^\top \Gamma_1^{-1}(u - \kappa_0(x,q,\hat{\theta})) \tag{16a}$$

$$\kappa_1(\zeta) := -\psi_{\hat{\theta}}(x,q)\Gamma_0 \operatorname{Proj}(\upsilon(\zeta),\hat{\theta}) - k_u(u - \kappa_0(x,q,\hat{\theta})) - \Gamma_1 \psi_u(x,q)^\top \nabla V(x,q) + \mathcal{D}_x(\kappa_0(x,q,\hat{\theta}))f(x,q,u,\hat{\theta}) \tag{16b}$$

for each $(x,q,\hat{\theta},u) \in \mathcal{S}_1$, under the additional assumption that $\kappa$ is continuously differentiable. Following the controller design that was introduced in Section III and setting $\tau = \kappa_1(\zeta)$, we obtain the closed-loop hybrid system $\mathcal{H}_1 := (C_1, F_1, D_1, G_1)$ given by

$$\dot{\zeta} = F_1(\zeta) \qquad \zeta \in C_1 := \{\zeta \in \mathcal{S}_1 : \mu_{V_1}(\zeta) \leq \delta\} \tag{17a}$$
$$\zeta^+ = G_1(\zeta) \qquad \zeta \in D_1 := \{\zeta \in \mathcal{S}_1 : \mu_{V_1}(\zeta) \geq \delta\} \tag{17b}$$

where $\delta > 0$ and

$$F_1(\zeta) := \begin{bmatrix} f(x,q,u,\theta) \\ 0 \\ \Gamma_0 \operatorname{Proj}(\upsilon(\zeta),\hat{\theta}) \\ \kappa_1(\zeta) \end{bmatrix} \qquad \forall \zeta \in C_1 \tag{18a}$$

$$G_1(\zeta) := \begin{bmatrix} x \\ \varrho_{V_1}(x,\hat{\theta},u) \\ \hat{\theta} \\ u \end{bmatrix} \qquad \forall \zeta \in D_1. \tag{18b}$$

We are able to prove the following result using arguments similar to those of Theorem 2.

**Theorem 3.** *Given a SLFF pair $(V, \kappa)$ relative to a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{Q}$ for (3) with synergy gap exceeding $\delta$ and $\kappa$ continuously differentiable on an open neighborhood of $\mathcal{X} \times \mathcal{Q}$, if (15) holds, then $(V_1, \kappa_1)$ is a SLFF pair relative to $\mathcal{A}_1$ for*

$$\dot{\zeta} = F_1(\zeta) \qquad \zeta \in \mathcal{S}_1 \tag{19}$$

*with synergy gap exceeding $\delta$.*

It follows from Theorem 3 and Theorem 1 that $\mathcal{A}_1$ is globally asymptotically stable for (19). In Theorems 2 and 3, we demonstrate that a given SLFF pair for the unperturbed system (3) generates SLFF pairs for (14) and (21), respectively, under the additional assumption (15). In the next section, we apply the proposed controllers to global asymptotic stabilization of a compact set in the presence of an obstacle.

## VI. Synergistic Artificial Potential Functions for Global Obstacle Avoidance

In this section, we present a solution to the problem of global asymptotic stabilization of a compact subset $\mathcal{A}_x$ of $\mathbb{R}^n$ for (1) in the presence of an obstacle using the synergistic hybrid feedback strategy of Sections IV and V. In particular, we consider that an obstacle is represented by a compact subset $\mathcal{N}$ of $\mathbb{R}^n$ that we remove from the state space.

In this direction, let $\mathcal{Q} \subset \mathbb{Z}$ be finite set and $\{\mathcal{M}_q\}_{q\in\mathcal{Q}}$ denote a collection of closed subsets of $\mathbb{R}^n$ satisfying

$$\bigcap_{q\in\mathcal{Q}} \mathcal{M}_q = \mathcal{N} \tag{20a}$$

$$\forall q \in \mathcal{Q} \quad \mathcal{M}_q \cap \mathcal{A}_x = \emptyset. \tag{20b}$$

For each $q \in \mathcal{Q}$, let $V_q : \mathbb{R}^n \backslash \mathcal{M}_q \to \mathbb{R}_{\geq 0}$ denote a proper indicator of $\mathcal{A}_x$ on its domain, i.e., each $V_q$ is a continuous function that is positive definite relative to $\mathcal{A}_x$ and $V_q(x_i) \to \infty$ when $i \to \infty$ if either $|x_i| \to \infty$ or the sequence $\{x_i\}_{i=0}^{\infty}$ approaches $\mathcal{M}_q$. Each function $V_q$ is commonly referred to as an artificial potential function (see e.g. [19]).

Defining $V(x,q) := V_q(x)$ for each $(x,q) \in \mathbb{R}^n \times \mathcal{Q}$, it follows from the previous assumptions that each sublevel set of $V$ is compact and that $V$ is positive definite relative to $\mathcal{A} := \mathcal{A}_x \times \mathcal{Q}$. Crucially, it follows from the construction of $V$ that none of its sublevel sets include the obstacle $\mathcal{N}$, i.e.,

$$\pi_{\mathbb{R}^n}(\Omega_V(r)) \cap \mathcal{N} = \emptyset \text{ for all } r \geq 0. \tag{21}$$

Therefore, if each $V_q$ is continuously differentiable and if, for each $q \in \mathcal{Q}$, there exists a continuously differentiable function $\kappa : \mathbb{R}^n \backslash \mathcal{M}_q \to \mathbb{R}^m$ such that (A4) and (A5) hold, then $(V, \kappa)$ is a SLFF pair relative to $\mathcal{A}$ for (3). In particular, this means that the controller design of Section III can be used for global asymptotic stabilization of a compact set $\mathcal{A}$ for (3). In addition, if condition (15) holds, then any of controller design strategies that are presented in Sections IV and V can be applied.

**Remark 1.** *Note that, since* (23) *holds by construction, the state space $\mathcal{X}$ can be taken as a sublevel set of $V$. Due to the properties of $V$, it is always possible to encompass solutions whose initial conditions lie arbitrarily close to $\mathcal{N}$.*

In the next section, we present a particular SLFF pair that allows for global asymptotic stabilization of the origin of a simple integrator on $\mathbb{R}^2$ in the presence of an obstacle.

### A. Numerical Example

Let us consider the dynamical system

$$\dot{x} = f(x,q,u,\theta) := u + \theta \tag{22}$$

where $x \in \mathbb{R}^2$ denotes the state of the system, $u \in \mathbb{R}^2$ denotes the input and $\theta \in \Omega$ denotes a constant disturbance. In this section, we make use of synergistic hybrid feedback to design a controller that globally asymptotically stabilizes the origin of (24), hence $\mathcal{A}_x = \{0\}$, given the presence of an obstacle that is represented by the compact set

$$\mathcal{N} := c + r\overline{\mathbb{B}}$$

where $c := (c_1, c_2) = (1, 0)$ and $r := 1/2$.

Following the control design strategy that is outlined in Section VI, we define $\mathcal{Q} := \{-1, 1\}$ and

$$\mathcal{M}_q := \mathcal{N} \cup \{(x_1,x_2) \in \mathbb{R}^2 : qx_2 \geq 0, x_1 = c_1\}. \tag{22}$$

It is straightforward to verify that $\{\mathcal{M}_q\}_{q\in\mathcal{Q}}$ satisfies (22). Next, we define $\phi : \mathcal{S} \to (-\pi,\pi) \times \{R \in \mathbb{R} : R > r\}$ as follows:

$$\phi(x,q) := \begin{bmatrix}\phi_1(x,q)\\ \phi_2(x,q)\end{bmatrix} := \begin{bmatrix}\operatorname{atan2}(q(x_1 - c_1), c_2 - x_2)\\ |x - c|\end{bmatrix} \tag{23}$$

for each $(x,q) \in \mathcal{S}$ with

$$\mathcal{S} := \{(x,q) \in \mathbb{R}^2 \times \mathcal{Q} : x \in \mathbb{R}^2 \backslash \mathcal{M}_q\} \tag{24}$$

which is an smooth function with smooth inverse where

$$\operatorname{atan2}(y_2,y_1) := \begin{cases} \arctan\left(\frac{y_2}{y_1}\right) & \text{if } y_1 > 0 \\ \arctan\left(\frac{y_2}{y_1}\right) + \pi & \text{if } y_1 < 0,\ y_2 > 0 \\ \arctan\left(\frac{y_2}{y_1}\right) - \pi & \text{if } y_1 < 0,\ y_2 < 0 \\ \pi/2 & \text{if } y_1 = 0,\ y_2 > 0 \\ -\pi/2 & \text{if } y_1 = 0,\ y_2 < 0 \end{cases}$$

for each $(y_1,y_2) \in \{(y_1,y_2) \in \mathbb{R}^2 : y_2 = 0,\ y_1 \leq 0\}$ is the four quadrant tangent inverse. Using (25), we define

$$U(\alpha,R) := \frac{(\alpha + \pi/2)^2}{2(\alpha+\pi)(\pi-\alpha)} + \frac{(R - |c|)^2}{2(R-r)}$$

for each $(\alpha,R) \in (-\pi,\pi) \times \{R \in \mathbb{R} : R > r\}$, and

$$V_q(x) := U(\phi(x,q)) \tag{25}$$

for each $x \in \mathbb{R}^2 \backslash \mathcal{M}_q$, which is a smooth proper indicator of $\mathcal{A}_x$ on $\mathbb{R}^2 \backslash \mathcal{M}_q$. To see this, note that, if $\{\alpha_i\}_{i\in\mathbb{N}}$ is a convergent sequence to either $\pi$ or $-\pi$, then

$$\frac{(\alpha_i + \pi/2)^2}{2(\alpha_i+\pi)(\pi-\alpha_i)} \to +\infty \text{ as } i \to \infty.$$

Moreover, we also have that

$$\frac{(R_i - |c|)^2}{2(R_i - r)} \to +\infty \text{ as } i \to \infty$$

if $\{R_i\}_{i\in\mathbb{N}}$ is a convergent sequence to $r$, hence $V_q(\xi_i) \to +\infty$ for each convergent sequence $\{\xi_i\}_{i\in\mathbb{N}}$ to $\mathcal{M}_q$. To verify that $V_q$ is positive definite, note that

$$\begin{aligned} V_q(x) = 0 &\iff \phi(x,q) = (-\pi/2, |c|) \\ &\iff x = 0. \end{aligned}$$

Using the gradient-based feedback rule

$$\kappa(x,q) := -\nabla V_q(x) \tag{26}$$

for each $(x,q) \in \mathcal{S}$, we have that

$$\langle \nabla V_q(x), \kappa(x,q)\rangle \leq -|\nabla V_q(x)|^2$$

for each $(x,q) \in \mathcal{S}$.

**Proposition 2.** *Given $r > 0$, let $V(x,q) := V_q(x)$ for each $(x,q) \in \mathcal{S}$ and $\mathcal{X} := \Omega_V(r)$ with $V_q$ and $\mathcal{S}$ given in* (27) *and* (26)*, respectively. Then, the functions $V$ and $\kappa$ given in* (27) *and* (28)*, respectively, form a SLFF pair relative to $\mathcal{A} := \mathcal{A}_x \times \mathcal{Q} = \{0\} \times \{-1,1\}$ for* (24) *with $\theta = 0$. Moreover, condition* (15) *also holds.*

Proposition 2 allows for the application of the controller design strategies that are described in Sections IV and V.

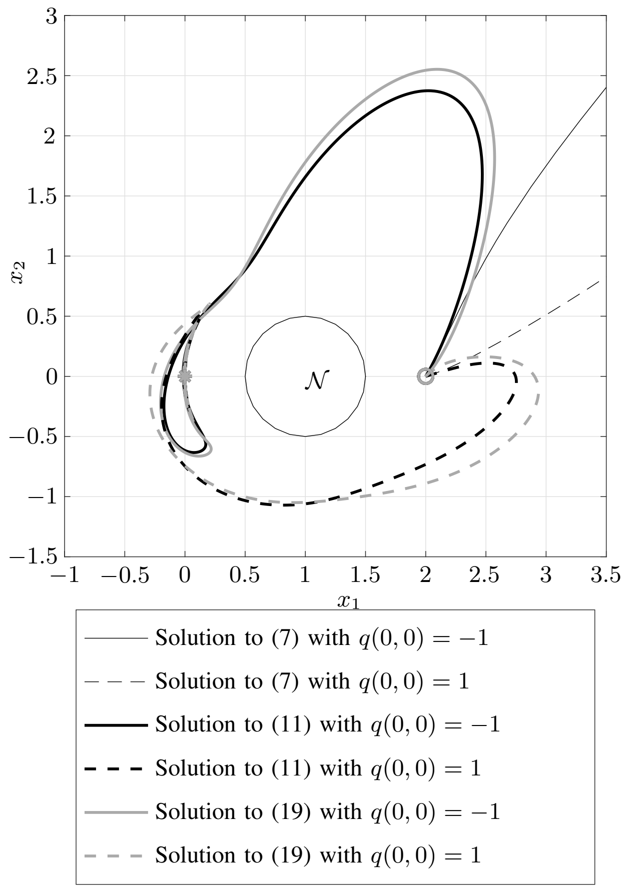


Fig. 1. Representation of the trajectories of the $x$ component of the solutions to (7), (11) and (19) under the influence of nonzero disturbance $\theta = (1,1)$, with $x(0,0) = (2,0)$ and distinct initial values of the logic variable $q \in \mathcal{Q} := \{-1, 1\}$. It is possible to verify that the state trajectories for (11) and (19) converge to the origin, but the solution to (7) is not able to compensate the effect of nonzero disturbances.

Figure 1 represents the component $x := (x_1, x_2)$ of two solutions for each of the systems (7), (11) and (19) with $f$ given by (24) and $\theta = (1,1)$. All the solutions share the same initial condition for the state variable $x$, given by $x(0,0) = (2,0)$, but distinct initial values of the logic variable $q \in \mathcal{Q} := \{-1, 1\}$. We verify that, although $\mathcal{A}$ is globally asymptotically stable for (7) when $\theta = 0$ (c.f. Theorem 1), Figure 1 shows that this nominal controller is not able to compensate for the presence of nonzero disturbances $\theta = (1,1)$, since the state of the system leaves the reasonably large boundaries of the boxed region we chose to represent. On the other hand, the solutions to (11) and (19) with initial conditions $\hat{\theta}(0,0) = 0$, $u(0,0) = 0$ and controller parameters $\Gamma_0 = \Gamma_1 = I_2$, $k_u = 10$, are able to overcome the presence of nonzero disturbances. In fact, in both of these cases, the state of the estimator $\hat{\theta}$ converges to the value of the disturbance $\theta$. Finally, notice that the way in which the system circumvents the obstacle depends on the initial value of the logic variable $q$, but the objective of global asymptotic stabilization of the origin is achieved regardless of the initial conditions.

## VII. Conclusions

In this paper, we modified the hybrid controller that is induced by a Synergistic Lyapunov Function and Feedback pair to achieve global asymptotic stabilization of a compact set in the presence of matched uncertainties. We also showed that the synergism property of the proposed adaptive feedback can be preserved through backstepping without compromising the global asymptotic stabilization objective. We applied the proposed controllers to the problem of obstacle avoidance and we demonstrated this application by means of simulation results. Future work will focus on the development of a controller for global trajectory tracking in the presence of exogenous disturbances and multiple obstacles.

## References


[1] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[2] C. G. Mayhew and A. R. Teel. Global asymptotic stabilization of the inverted equilibrium manifold of the 3-D pendulum by hybrid feedback. In *Proceedings of the 49th IEEE Conference on Decision and Control*, pages 679–684. IEEE, 2010.

[3] S. Berkane and A. Tayebi. Construction of Synergistic Potential Functions on SO(3) with Application to Velocity-Free Hybrid Attitude Stabilization. *IEEE Transactions on Automatic Control*, 62(1), 2017.

[4] P. Casau, R.G. Sanfelice, R. Cunha, and C. Silvestre. A globally asymptotically stabilizing trajectory tracking controller for fully actuated rigid bodies using landmark-based information. *International Journal of Robust and Nonlinear Control*, 25(18), 2015.

[5] P. Casau, R.G. Sanfelice, R. Cunha, D. Cabecinhas, and C. Silvestre. Robust global trajectory tracking for a class of underactuated vehicles. *Automatica*, 58:90–98, 2015.

[6] C.G. Mayhew, R.G. Sanfelice, and A.R. Teel. Quaternion-Based Hybrid Control for Robust Global Attitude Tracking. *IEEE Transactions on Automatic Control*, 56(11):2555–2566, 2011.

[7] Christopher G. Mayhew and Andrew R. Teel. Synergistic hybrid feedback for global rigid-body attitude tracking on SO(3). *IEEE Transactions on Automatic Control*, 58(11):2730–2742, nov 2013.

[8] T. Lee. Global Exponential Attitude Tracking Controls on SO(3). *IEEE Transactions on Automatic Control*, 60(10):2837–2842, 2015.

[9] S. P. Bhat and D. S. Bernstein. A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems & Control Letters*, 39(1):63–70, 2000.

[10] T. Rybus. Obstacle avoidance in space robotics: Review of major challenges and proposed solutions. *Progress in Aerospace Sciences*, 101:31–48, 2018.

[11] N. Malone, H.-T. Chiang, K. Lesser, M. Oishi, and L. Tapia. Hybrid Dynamic Moving Obstacle Avoidance Using a Stochastic Reachable Set-Based Potential Field. *IEEE Transactions on Robotics*, 33(5), 2017.

[12] A. Bloch, M. Camarinha, and L. Colombo. Dynamic interpolation for obstacle avoidance on Riemannian manifolds. *ArXiV*, pages 1–22, 2018.

[13] D. E. Koditschek and E. Rimon. Robot navigation functions on manifolds with boundary. *Advances in Applied Mathematics*, 11(4):412–442, 1990.

[14] R.G. Sanfelice, M.J. Messina, S. Emre Tuna, and A.R. Teel. Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points. In *Proceedings of the 2006 American Control Conference*, pages 3352–3357, 2006.

[15] C.G. Mayhew, R.G. Sanfelice, and A.R. Teel. Synergistic Lyapunov functions and backstepping hybrid feedbacks. In *Proceedings of the 2011 American Control Conference*, pages 3203–3208, 2011.

[16] R.K. Sundaram. *A First Course in Optimization Theory*. Cambridge University Press, New York, USA, 20th edition, 1996.

[17] Jing Zhou and Changyun Wen. *Adaptive Backstepping Control of Uncertain Systems: Nonsmooth Nonlinearities, Interactions or Time-Variations*. Springer, 2008.

[18] Z. Cai, M. S. Queiroz, and D. M. Dawson. A sufficiently smooth projection operator. *IEEE Transactions on Automatic Control*, 51(1):135–139, 2006.

[19] I. J. Cox and G. T. Wilfong. *Autonomous robot vehicles*. Springer, 1990.