Asymptotic Stability of Limit Cycles in Hybrid Systems with Explicit Logic States

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Abstract— **This work pertains to the study of stability of limit cycles for hybrid systems with explicit logic states within a hybrid systems framework. We first focus on constructing the hybrid systems with explicit logic states and revealing basic properties of limit cycles. Application to model switched systems under dwell-time switching as such a hybrid system is provided. In addition, we establish sufficient and necessary conditions for stability of the limit cycles relying on Poincaré maps. Examples illustrate the results. A discussion about the case of systems with nonunique solutions is also included.**

I. Introduction

Hybrid systems are ubiquitous in realistic systems due to their ability to capture models having state variables that can evolve continuously (flows) and/or discretely (jumps). In recent years, the study of limit cycles in hybrid systems has received substantial attention. This is mainly due to the existence of limit cycles with jumps in many engineering applications, such as robotics [1], phase-locked loop [2], gene networks [3], etc. In this paper, we consider a subset of hybrid systems with great utility: hybrid systems with explicit logic states; that is, hybrid systems containing discrete modes or logic variables. Such systems can be modeled using hybrid automata, in particular, and arise in a variety of applications including the modeling of the dynamics of genetic networks with binary hysteresis [3], the modeling of heating control systems with desired temperature bands [4], and the modeling of the DC-DC boost converter under different switching modes [5].

Particular motivation for the study of hybrid systems with explicit logic states comes from limit cycles in switched systems with a sequence of modes. The problems of ensuring the stability of limit cycles have been studied for specific classes of switched systems. In particular, for a class of switched linear systems in [6], Olsder applied a generalized implicit function theorem to show characteristics of the periodic solutions around operating points for sufficiently small period. In [7], a sufficient condition given in terms of a set of linear matrix inequalities for exponential stability of limit cycles in a class of switched affine linear systems was proposed using a discrete-time state description of the system. In [2], Flieller et al. utilized the sensitivity analysis method to determine limit cycles of switched systems and analyzed their local stability through the computation of the Jacobian, which relies on the knowledge of the switching sequence. In [8], the trajectory sensitivity approach was employed to develop sufficient conditions for stability of limit cycles in switched differential-algebraic systems. In [9], Li studied the maximum number of limit cycles in a class of discontinuous quadratic polynomial differential systems with ε-order terms. Recently, Benmiloud et al. in [10] provided a new methodology to guarantee local asymptotic stability of the desired limit cycle in planar switched systems by designing one switching surface. Existing works in this area mostly focus on deriving stability conditions for switched linear systems or planar switched systems. We believe that conditions for stability of limit cycles in hybrid systems with explicit logic states should play a more prominent role in analysis and control of limit cycles with jumps. To the best of our knowledge, tools for the analysis of asymptotic stability of limit cycles in such hybrid systems are not available in the literature.

In this paper, we exploit the main idea proposed in [11], formulating the stability problem of limit cycles for hybrid systems with explicit logic states in a hybrid dynamical systems framework [12]. The main contributions of this paper can be summarized as follows:

1) We employ the hybrid systems framework [12] to model the hybrid systems with explicit logic states and the mechanisms generating switching modes in a way that is amenable to the tools in [11] for the study of attractivity, stability, and robustness of limit cycles.
2) As an application, we model the switched systems under dwell-time switching as a hybrid system with explicit logic states. A notion of limit cycle for such hybrid systems is introduced and some of its properties are presented including compactness and transversality.
3) We establish sufficient and necessary conditions for guaranteeing global asymptotic stability of limit cycles for hybrid systems with explicit logic states, which can further be extended for characterizing robustness of asymptotic stability under perturbations of such systems using the recent results in [12].

The structure of the paper is as follows. We start with a motivational example in Section II. The formulation of the hybrid model and an application to switched systems with dwell-time are given in Section III. Section IV introduces the definition of limit cycle and gives some of its basic

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properties. Section V presents the stability notions as well as sufficient and necessary conditions for stability of limit cycles. Section VI discusses the case of nonuniqueness of solutions in switched systems with dwell-time.

Notation: Specifically, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}_{\geqslant 0}$ denotes the set of nonnegative real numbers, i.e., $\mathbb{R}_{\geqslant 0} := [0, +\infty)$, and $\mathbb{N}$ denotes the set of natural numbers including 0, i.e., $\mathbb{N} := \{0, 1, 2, \cdots\}$. The equivalent notation $[x^\top\ y^\top]^\top$ and (x, y) are used for the same vector. Given a set $\mathcal{A} \subset \mathbb{R}^n$, $\overline{\mathcal{A}}$ denotes its closure. Given a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, the Lie derivative of h at x in the direction of f is denoted by $L_f h(x) := \langle \nabla h(x), f(x) \rangle$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}$.

II. Motivational Example

The following example motivates the study of limit cycles for hybrid systems with explicit logic states in this paper. The state-triggered switched system features a limit cycle.

Consider a thermostat that controls the temperature ξ of a heating system [4]. The heater in the system is used to maintain the temperature of the thermostat within a desired temperature band. The proposed model does not consider the influence of the internal temperature of the heater. As we will show later, the system can be modeled as a hybrid system with a logic variable $q \in \mathcal{Q} := \{1, 2\}$. When $q = 1$ and the temperature is at the higher end b_2 (or above) of the desired temperature band, reset q to 2 and the heater turns off until the temperature reaches the lower end b_1 of the desired band. When $q = 2$ and the temperature is less than or equal to b_1, reset q to 1 and the heater turns on again. Fig. 1 shows a hybrid automaton modeling this system [4]. The parameter a is the natural cooling coefficient of the thermostat and the parameter c characterizes the effectiveness of the heater. The desired temperature band for the thermostat is chosen as the interval $[b_1, b_2]$. If the parameters a, b_1, b_2, and c satisfy $a > 0$ and $c > ab_2 > ab_1 > 0$, the system exhibits periodic behavior for any initial condition, as shown in Fig. 2.

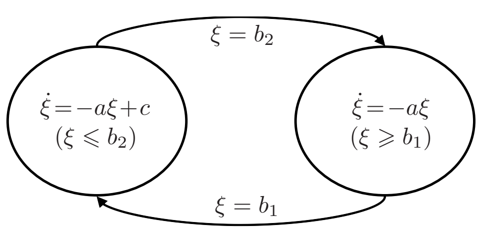


Fig. 1. The hybrid automaton of a thermostat

In this paper, our interest is in developing analysis methods that can be applied to such a system with explicit logic states as well as to switched systems under dwell-time so as to guarantee asymptotic stability of limit cycles. While the models considered in [9], [10] are specific cases of switched systems with logic modes, they only apply to planar switched systems. On the other hand, the results presented here are suitable for general hybrid systems with explicit logic states.

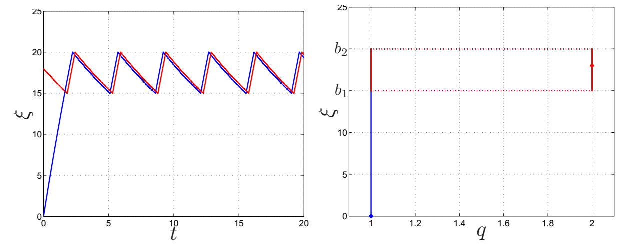


Fig. 2. State trajectories of the thermostat system with initial values 0 for ξ and 1 for q (blue) or 18 for ξ and 2 for q (red), where $a = 0.1$, $b_1 = 15$, $b_2 = 20$ and $c = 10$. Left: Transient response of ξ; Right: Phase plot of state variables q and ξ.

III. Problem Statement and Mathematical Modeling

A. System Description

Following [12, Chapter 1.4], consider the hybrid system with explicit logic states given by

$$\mathcal{H} \begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} f_q(\xi) \\ 0 \end{bmatrix} & \xi \in C_q, q \in \mathcal{Q} \\ \begin{bmatrix} \xi^+ \\ q^+ \end{bmatrix} = g_q(\xi) & \xi \in D_q, q \in \mathcal{Q} \end{cases} \tag{1}$$

where $\dot{\xi}$ and $\dot{q}$ denote the derivatives with respect to time, respectively, and ξ^+ and q^+ denote the values of the state ξ and the logic variable q after a jump, respectively. The state vector of $\mathcal{H}$ is given by $x := (\xi, q) \in \mathbb{R}^{n+1}$, where $\xi \in \mathbb{R}^n$ is the continuous state component, $q \in \mathcal{Q}$ is the discrete state component associated with the explicit logic mode, and $\mathcal{Q}$ is a finite index set of $q_{\max}$ elements, namely, $\mathcal{Q} := \{1, 2, \cdots, q_{\max}\}$. The explicit logic state q, by its nature, can change only via a jump. For each $q \in \mathcal{Q}$, the function $f_q : \mathbb{R}^n \to \mathbb{R}^n$ defines the evolution of ξ during flows and the function $g_q : \mathbb{R}^n \times \mathcal{Q} \to \mathbb{R}^{n+1}$ determines the value of the state after jumps.

System (1) with state $x = (\xi, q)$ can be written within the hybrid systems framework in [12] as follows:

$$\mathcal{H} \begin{cases} \dot{x} = f(x) := \begin{bmatrix} f_q(\xi) \\ 0 \end{bmatrix} & x \in C \\ x^+ = g(x) := g_q(\xi) & x \in D \end{cases} \tag{2}$$

where $C := \bigcup_{q \in \mathcal{Q}} (C_q \times \{q\})$ denotes the flow set and $D := \bigcup_{q \in \mathcal{Q}} (D_q \times \{q\})$ denotes the jump set. The data of the hybrid system $\mathcal{H}$ is given by (C, f, D, g). The elements of data that represent $\mathcal{H}$ on the state space $\mathbb{R}^{n+1}$ have the following properties:

- Given functions f_q for each $q \in \mathcal{Q}$, the flow map $f : \mathbb{R}^n \times \mathcal{Q} \to \mathbb{R}^{n+1}$ defines the continuous evolution of x.
- The jump map g is given by, for each $q \in \mathcal{Q}$, function $g_q : \mathbb{R}^n \to \mathbb{R}^n \times \mathcal{Q}$ that defines the changes of x at jumps.
- For each $q \in \mathcal{Q}$, the sets C_q and D_q are subsets of $\mathbb{R}^n$, on which the flows are effective and jumps (with such value of q) can occur, respectively.

Compared to our previous results in [11], [13], the explicit logic state in the hybrid system $\mathcal{H}$ allows us to capture certain families of logic mode patterns or switching signals explicitly. We extend and specialize, as needed, our previous results to fit such a model. When modeling a system with explicit logic states as a well-posed hybrid system, the tools in [12] for robust stability of limit cycles apply.

A solution to $\mathcal{H}$ is parameterized by ordinary time t and a counter j for jumps. It is given by a hybrid arc[1] $\phi : \text{dom}\,\phi \to \mathbb{R}^{n+1}$ that satisfies the dynamics of $\mathcal{H}$; see [12] for more details. A solution ϕ to $\mathcal{H}$ is said to be complete if $\text{dom}\,\phi$ is unbounded. It is precompact if it is complete and bounded. It is said to be maximal if it is not a truncated version of another solution. The set of maximal solutions to $\mathcal{H}$ from the set K is denoted as

$$\mathcal{S}_{\mathcal{H}}(K) := \{\phi : \phi \text{ is a maximal solution to } \mathcal{H} \text{ with } \phi(0,0) \in K\}.$$

We define $t \mapsto \phi^f(t, x_0)$ as a solution of the flow dynamics

$$\dot{x} = f(x) \quad x \in C$$

from $x_0 \in \overline{C}$. Note that by construction, the q component of the solutions to $\mathcal{H}$ remains constant during flows.

B. Special Case: Switched Systems with Dwell-Time

The continuous state ξ of $\mathcal{H}$ in (2) may contain an auxiliary state component χ that is useful in modeling switched systems under dwell-time switching. A dwell-time switching signal has switching times t_i satisfying $t_{i+1} - t_i \geqslant T_\chi$ for $i = 1, 2, \cdots$, where $T_\chi > 0$ denotes the dwell-time parameter. In fact, a hybrid system modeling switched systems with dwell-time switching signals is given by[2]

$$\left.\begin{aligned} \dot{z} &= \widetilde{f}_q(z) \\ \dot{\chi} &= 1 \\ \dot{q} &= 0 \end{aligned}\right\} \quad z \in \widetilde{C}_q, \chi \geqslant 0$$
$$\left.\begin{aligned} z^+ &= z \\ \chi^+ &= 0 \\ q^+ &= \widetilde{g}_q(z) \end{aligned}\right\} \quad z \in \widetilde{D}_q, \chi \in [T_\chi, \infty) \tag{3}$$

with state $x = (\xi, q) = ((z, \chi), q) \in \mathbb{R}^{n+1} \times \mathcal{Q}$ and dwell time parameter $T_\chi > 0$. System (3) fits the framework of hybrid systems in (2) with $\xi := (z, \chi) \in \mathbb{R}^{n+1}$, $f_q(\xi) := (\widetilde{f}_q(z), 1)$, $g_q(\xi) := (z, 0, \widetilde{g}_q(z))$, $C_q := \widetilde{C}_q \times \mathbb{R}_{\geqslant 0}$, and $D_q := \widetilde{D}_q \times [T_\chi, \infty)$. A model with switches occurring every T_χ seconds (after the first switch) is given as in (3) but with flow and jump sets $C_q := \widetilde{C}_q \times [0, T_\chi]$ and $D_q := \widetilde{D}_q \times \{T_\chi\}$, respectively.

The following example illustrates the latter model in a system that features a limit cycle.

Example 3.1: (A boost converter) Consider the simplified boost DC-DC converter shown in Fig. 3. The boost circuit consists of a DC voltage source E, an inductor L, an ideal diode d, a capacitor c, a resistor R, and an ideal switch S. The controlled switch S can be either on or off, defining two main steady-state modes of operation (cf. [14]).

mode 1: S is on   mode 2: S is off

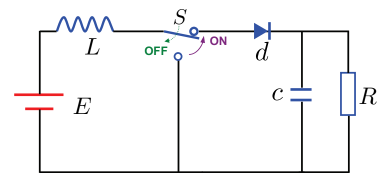


Fig. 3. The boost converter

The circuit associated to each mode is shown in Fig. 4. In mode 1, in which the switch is on, the inductor offloads power to the resistor. In mode 2, in which the switch is off, the input source charges the inductor and the capacitor feeds the load. Assuming no parasitic effects, the dynamics of the boost DC-DC converter on such modes are modeled by

$$\dot{z} = A_q z + b_q \tag{4}$$

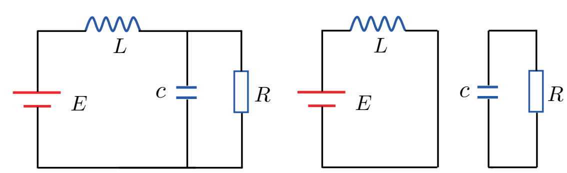
where $z = (z_1, z_2) \in \mathbb{R}^2_{\geqslant 0}$ is the state vector with z_1 the inductor current, z_2 the voltage over the capacitor, and $q \in \{1, 2\}$ a logic variable used to indicate whether the switch S is on or off. When the logic variable q is equal to 1 (i.e., the switch S is on), the dynamics of the system are governed by

$$A_1 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{c} & -\frac{1}{Rc} \end{bmatrix}, \; b_1 = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}.$$

When the logic variable q is equal to 2 (i.e., the switch S is off), the dynamics of the system are governed by

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{Rc} \end{bmatrix}, \; b_2 = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}.$$

As suggested in [6], one would expect different behaviors



(a) mode 1   (b) mode 2

Fig. 4. Two different modes for the boost converter

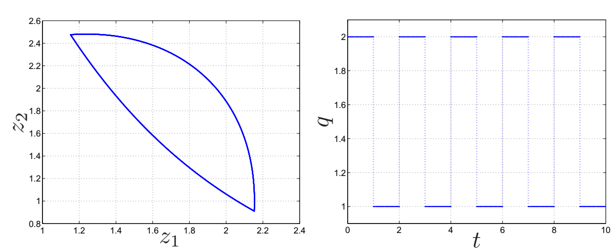


Fig. 5. A limit cycle of the boost converter system in (5) with initial condition $(1.153, 2.476)$ from mode 2. Left: Phase plot of state variables z_1 and z_2; Right: Transient response of q.

[1] A hybrid arc is a function ϕ defined on a hybrid time domain and for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous. A *compact hybrid time domain* is a set $\mathcal{E} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ of the form $\mathcal{E} = \bigcup_{j=0}^{J}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{J+1}$. The set $\mathcal{E}$ is a *hybrid time domain* if for all $(T, J) \in \mathcal{E}$, $\mathcal{E} \cap ([0, T] \times \{0, 1, \cdots, J\})$ is a compact hybrid time domain.

[2] For a more general switched system with dwell-time signals, see [12, Example 2.13].

by toggling the switch S. In this work, we are interested in periodic behavior. For instance, when we change the position of the switch S every second and the parameters are chosen as $c = 0.5, L = 1, R = 2$, and $E = 1$, the boost converter exhibits a limit cycle for any initial condition. A limit cycle, denoted $\mathcal{O}$, of the system in (5) is depicted in Fig. 5.[3]

In a similar manner as in [12, Chapter 2.4], we model the switching frequency/events using a timer variable χ that increases during flow, and then triggers a jump once it reaches a given threshold T_χ. Following (3), when $q = 1$, we make χ to increase as ordinary time t, and whenever $\chi = T_\chi$, reset χ to 0 and the logic variable q to 2. When $q = 2$, we make χ to increase as ordinary time t, and whenever $\chi = T_\chi$ again, reset χ to 0 and the logic variable q to 1. Hence, the whole system composed by the circuit states z_1 and z_2, timer variable χ, and the logic variable q can be represented by the following hybrid system $\mathcal{H}_\mathrm{B} = (C_\mathrm{B}, f_\mathrm{B}, D_\mathrm{B}, g_\mathrm{B})$:

$$\mathcal{H}_\mathrm{B}: \begin{cases} \dot{x} = \begin{bmatrix} \dot{z} \\ \dot{\chi} \\ \dot{q} \end{bmatrix} = f_\mathrm{B}(x) := \begin{bmatrix} A_q z + b_q \\ 1 \\ 0 \end{bmatrix} & x \in C_\mathrm{B} \\ x^+ = \begin{bmatrix} z^+ \\ \chi^+ \\ q^+ \end{bmatrix} = g_\mathrm{B}(x) := \begin{bmatrix} z \\ 0 \\ 3-q \end{bmatrix} & x \in D_\mathrm{B} \end{cases} \tag{5}$$

where $x = (\xi, q) = (z, \chi, q)$, $z = (z_1, z_2) := (i_L, u_C)$, $C_\mathrm{B} = \mathbb{R}^2_{\geqslant 0} \times [0, T_\chi] \times \mathcal{Q}$, $D_\mathrm{B} = \mathbb{R}^2_{\geqslant 0} \times \{T_\chi\} \times \mathcal{Q}$, and $\mathcal{Q} = \{1, 2\}$. △

IV. Limit Cycles and Basic Properties

A. Definitions

In this section, we introduce the notion of limit cycles for systems as in (2) and, in the next section, reveal their basic properties. As in [11], we consider a class of periodic solutions defined as follows.

Definition 4.1: (periodic solution) Consider a hybrid system $\mathcal{H}$ with explicit logic states in (2). Let ϕ^* be a complete solution to $\mathcal{H}$. Then ϕ^* is periodic with period T^* and J^* jumps in each period if $T^* \in (0, \infty)$ and $J^* \in \mathbb{N} \setminus \{0\}$ are the smallest numbers such that $\phi^*(t+T^*, j+J^*) = \phi^*(t, j)$ for all $(t, j) \in \operatorname{dom} \phi^*$.

The definition of a periodic solution ϕ^* with period T^* and J^* jumps implies that there exist J^* jumps and a switching sequence $\{q_1, q_2, \cdots, q_{J^*}\}$ with elements in $\mathcal{Q}$ in each period. Moreover, it implies that if $(t, j) \in \operatorname{dom} \phi^*$, then $(t+T^*, j+J^*) \in \operatorname{dom} \phi^*$. A periodic solution to $\mathcal{H}$ generates a limit cycle.

Definition 4.2: (limit cycle) A periodic solution ϕ^* to $\mathcal{H}$ with period $T^* \in (0, \infty)$ and $J^* \in \mathbb{N} \setminus \{0\}$ jumps in each period defines a limit cycle[4] $\mathcal{O} = \{x \in \mathbb{R}^n \times \mathcal{Q} : x = \phi^*(t, j), (t, j) \in \operatorname{dom} \phi^*\} = \bigcup_{q \in \mathcal{Q}} (\mathcal{O}_q \times \{q\})$, where $\mathcal{O}_q$ is the range of ϕ^* with logic variable component equal to q.

[3]The code is available online: `https://github.com/HybridSystemsLab/BoostConverterLimitCycle`.

[4]Alternatively, the limit cycle $\mathcal{O}$ can be written as $\{(\xi, q) \in \mathbb{R}^n \times \mathcal{Q} : (\xi, q) = (\phi^*_\xi(t, j), \phi^*_q(t, j)), t \in [t_s, t_s + T^*], (t, j) \in \operatorname{dom} \phi^*\}$ for some $t_s \in \mathbb{R}_{\geqslant 0}$.

B. Basic Properties of Limit Cycles

In what follows, we focus on a class of hybrid systems with explicit logic states in (2) that satisfies the following assumption. In particular, the systems in the motivational example (Section II) and Example 3.1 satisfy them.

Assumption 4.3: For a hybrid system $\mathcal{H} = (C, f, D, g)$ in (2) with state $x := (\xi, q)$ on $\mathbb{R}^n \times \mathcal{Q}$, there exist compact sets $M_q \subset \mathbb{R}^n$ and continuously differentiable functions $h_q : \mathbb{R}^n \to \mathbb{R}$ for each $q \in \mathcal{Q}$ such that, for each $q \in \mathcal{Q}$,

1) the set C_q can be written as $C_q = \{\xi \in \mathbb{R}^n : h_q(\xi) \geqslant 0\}$ and the set D_q can be written as $D_q = \{\xi \in \mathbb{R}^n : h_q(\xi) = 0, L_{f_q} h_q(\xi) \leqslant 0\}$;
2) the function f_q is continuously differentiable on an open neighborhood of $M_q \cap C_q$, and the jump map g_q is continuous on $M_q \cap D_q$;
3) $L_{f_q} h_q(\xi) < 0$ for all $\xi \in M_q \cap D_q$ and $g((M_q \cap D_q) \times \{q\}) \cap (M_\mathcal{Q} \cap D) = \emptyset$, where $M_\mathcal{Q} := \bigcup_{q \in \mathcal{Q}} (M_q \times \{q\})$;
4) $\mathcal{H}_M := (M_\mathcal{Q} \cap C, f, M_\mathcal{Q} \cap D, g)$ has a periodic solution ϕ^* with period $T^* \in (0, \infty)$ and $J^* \in \mathbb{N} \setminus \{0\}$ jumps per period that defines a limit cycle $\mathcal{O} \subset M_\mathcal{Q} \cap (C \cup D)$.

Remark 4.4: Item 1) in Assumption 4.3 implies that flows occur when every h_q is nonnegative while jumps only occur at points in zero level sets of h_q. The continuity property of f_q in item 2) of Assumption 4.3 is further required for the existence of solutions to $\dot{x} = f(x)$ according to [12, Proposition 2.10]. Moreover, item 2) also guarantees that solutions to $\dot{x} = f(x)$ continuously depend on initial conditions. The first condition in item 3) is necessary to establish a transversality property of limit cycles. The second condition in item 3) implies that, for each $q, p \in \mathcal{Q}$ such that $q \neq p$, we have $((M_q \cap D_q) \times \{q\}) \cap ((M_p \cap D_p) \times \{p\}) = \emptyset$. The set $M_\mathcal{Q}$ restricts the analysis of the hybrid system $\mathcal{H}$ to a particular region of the state space, leading to the restriction of $\mathcal{H}$ given by $\mathcal{H}_M$ in item 4) of Assumption 4.3.

We revisit the motivational example in Section II to illustrate the properties of a hybrid system $\mathcal{H}_M$ satisfying Assumption 4.3.

Example 4.5: (Thermostat, revisited) Consider the thermostat system in the motivational example. Using a logic variable $q \in \mathcal{Q} := \{1, 2\}$, the system can be modeled as a hybrid system in (2), in which switches are triggered by conditions involving the temperature state ξ and the logic variable q. The resulting hybrid system $\mathcal{H}_\mathrm{Tem} = (C_\mathrm{Tem}, f_\mathrm{Tem}, D_\mathrm{Tem}, g_\mathrm{Tem})$ has state $x = (\xi, q)$ and dynamics

$$\mathcal{H}_\mathrm{Tem} \begin{cases} \dot{x} = f_\mathrm{Tem}(x) := \begin{bmatrix} -a\xi + c(2-q) \\ 0 \end{bmatrix} & x \in C_\mathrm{Tem} \\ x^+ = g_\mathrm{Tem}(x) := \begin{bmatrix} \xi \\ 3-q \end{bmatrix} & x \in D_\mathrm{Tem} \end{cases} \tag{6}$$

where $C_\mathrm{Tem} := \{x \in [0, \bar{b}] \times \mathcal{Q} : (q = 1, \xi \leqslant b_2) \text{ or } (q = 2, \xi \geqslant b_1)\}$ and $D_\mathrm{Tem} := \{x \in [0, \bar{b}] \times \mathcal{Q} : (q = 1, \xi = b_2) \text{ or } (q = 2, \xi = b_1)\}$. The parameters a, b_1, b_2, $\bar{b}$ and c satisfy $a > 0$, $c > ab_2 > ab_1 > 0$, and $\bar{b} > b_2$. Define compact sets $M_q \subset \mathbb{R}$, $q \in \{1, 2\}$, as $M_1 := [0, b_2]$

and $M_2 := [b_1, \bar{b}]$, and define continuously differentiable functions $h_q : M_q \to \mathbb{R}$, $q = \{1, 2\}$, as $h_1(\xi) := b_2 - \xi$ and $h_2(\xi) := \xi - b_1$. Then, C_{Tem} and D_{Tem} can be rewritten as $C_{\text{Tem}} = \bigcup_{q=1}^{2}(C_{\text{Tem}_q} \times \{q\})$ with $C_{\text{Tem}_q} = \{\xi \in M_q : h_q(\xi) \geqslant 0\}$ and $D_{\text{Tem}} = \bigcup_{q=1}^{2}(D_{\text{Tem}_q} \times \{q\})$ with $D_{\text{Tem}_q} = \{\xi \in M_q : h_q(\xi) = 0, L_{f_{\text{Tem}_q}} h_q(\xi) \leqslant 0\}$, where $f_{\text{Tem}_q}(\xi) = -a\xi + c(2-q)$ and we used the properties $L_{f_{\text{Tem}_1}} h_1(\xi) = -(-a\xi + c(2-q)) = ab_2 - c < 0$ for each $\xi \in M_1 \cap D_{\text{Tem}_1}$ and $L_{f_{\text{Tem}_2}} h_2(\xi) = -a\xi + c(2-q) = -ab_1 < 0$ for each $\xi \in M_2 \cap D_{\text{Tem}_2}$. By design, the sets $M_{\mathcal{Q}}$, C_{Tem} and D_{Tem} are compact, where $M_{\mathcal{Q}} := \bigcup_{q \in \mathcal{Q}}(M_q \times \{q\})$. In addition, since the functions f_{Tem} and g_{Tem} are continuously differentiable, item 2) in Assumption 4.3 holds. Furthermore, it can be verified that for each $q \in \{1, 2\}$, $g_{\text{Tem}}((M_q \cap D_{\text{Tem}_q}) \times \{q\}) \cap (M_{\mathcal{Q}} \cap D_{\text{Tem}}) = \emptyset$ since $g_{\text{Tem}}(x) = (\xi, 3-q)$ as defined in (6). Therefore, for the hybrid system $\mathcal{H}_{\text{Tem}_M} = (M_{\mathcal{Q}} \cap C_{\text{Tem}}, f_{\text{Tem}}, M_{\mathcal{Q}} \cap D_{\text{Tem}}, g_{\text{Tem}})$, Assumption 4.3 holds. △

The following properties hold for $\mathcal{H}_M$ defined in item 4) of Assumption 4.3.

Lemma 4.6: Let Assumption 4.3 hold. Then, the data of the hybrid system $\mathcal{H}_M = (M_{\mathcal{Q}} \cap C, F, M_{\mathcal{Q}} \cap D, G)$ satisfies the hybrid basic conditions given by (A1)-(A3) in [12, Proposition 6.5].

The following result shows that a limit cycle generated by periodic solutions as in Definition 4.2 is closed and bounded.

Lemma 4.7: Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ with explicit logic states in (2) and compact sets $M_q \subset \mathbb{R}^n$ for each $q \in \mathcal{Q}$ satisfying Assumption 4.3. Then, any limit cycle $\mathcal{O}$ for $\mathcal{H}_M$ is compact.

The following result establishes a transversality[5] property of limit cycles for $\mathcal{H}_M$.

Lemma 4.8: Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ with explicit logic states in (2) and compact sets $M_q \subset \mathbb{R}^n$ for each $q \in \mathcal{Q}$ satisfying Assumption 4.3. Any limit cycle $\mathcal{O}$ for $\mathcal{H}_M$ is transversal to $M_{\mathcal{Q}} \cap D$ at every jump, where $M_{\mathcal{Q}} := \bigcup_{q \in \mathcal{Q}}(M_q \times \{q\})$.

Following the construction in [1], for the hybrid system $\mathcal{H}$ in (2), for each $q \in \mathcal{Q}$, the *time-to-impact function with respect to D_q* is defined by $T_{D_q} : C \cup D \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, where[6]

$$T_{D_q}(x) := \inf\{t \geqslant 0 : \phi(t, j) \in D_q \times \{q\}, \phi \in \mathcal{S}_{\mathcal{H}}(x)\}$$

for each $x = (\xi, q) \in C \cup D$.

Next, let us introduce the Poincaré map for a hybrid system $\mathcal{H} = (C, f, D, g)$ with explicit logic states in (2). For each $q \in \mathcal{Q}$, the hybrid Poincaré map $P_q : (M_q \cap D_q) \times \{q\} \to (M_q \cap D_q) \times \{q\}$ is given by

$$P_q(x) := \{\phi(T_{D_q}(g(x)), j) : \phi \in \mathcal{S}_{\mathcal{H}_M}(g(x)), (T_{D_q}(g(x)), j) \in \operatorname{dom}\phi\} \quad (7)$$

for all $x \in (M_q \cap D_q) \times \{q\}$.

V. Stability of Limit Cycles

In this section, we present stability properties of limit cycles for hybrid systems with explicit logic states. First, we define asymptotic stability using the hybrid Poincaré map P_q in (7). Below, for each $q \in \mathcal{Q}$, P_q^k denotes k compositions of the hybrid Poincaré map P_q with itself.

Next, a relationship between stability of fixed points of Poincaré maps[7] and the stability of the corresponding limit cycles is established.[8]

Theorem 5.1: Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ with explicit logic states in (2) and compact sets $M_q \subset \mathbb{R}^n$ for each $q \in \mathcal{Q}$ satisfying Assumption 4.3. Suppose every maximal solution to $\mathcal{H}_M = (M_{\mathcal{Q}} \cap C, f, M_{\mathcal{Q}} \cap D, g)$ is complete. Then, the following statements hold:

1) *for each $q \in \mathcal{Q}$, $x_q^* := (\xi_q^*, q) \in (M_q \cap D_q) \times \{q\}$ is a stable fixed point of the Poincaré map P_q in (7) if and only if the limit cycle $\mathcal{O}$ to $\mathcal{H}_M$ from $\phi^*(0, 0) = g(x_q^*)$ for each $q \in \mathcal{Q}$ is stable for $\mathcal{H}_M$;*
2) *for each $q \in \mathcal{Q}$, $x_q^* := (\xi_q^*, q) \in (M_q \cap D_q) \times \{q\}$ is a globally asymptotically stable[9] fixed point of the Poincaré map P_q if and only if the unique limit cycle $\mathcal{O}$ to $\mathcal{H}_M$ from $\phi^*(0, 0) = g(x_q^*)$ for each $q \in \mathcal{Q}$ is asymptotically stable for $\mathcal{H}_M$ with basin of attraction containing every point in[10] $M_{\mathcal{Q}} \cap (C \cup D)$.*

The following example illustrates the stability of the corresponding limit cycle using Theorem 5.1.

Example 5.2: (Thermostat, revisited) Consider the temperature control system $\mathcal{H}_{\text{Tem}_M}$ in Example 4.5. Assumption 4.3 is verified in Example 4.5. Moreover, it can be shown that every maximal solution to $\mathcal{H}_{\text{Tem}_M}$ is complete. For each $q \in \mathcal{Q} = \{1, 2\}$, let the Poincaré maps for $\mathcal{H}_{\text{Tem}_M}$ be given by P_q with its associated fixed point (ξ_q^*, q). Using Theorem 5.1, to show that the limit cycle $\mathcal{O}$ of $\mathcal{H}_{\text{Tem}_M}$ is asymptotically stable with basin of attraction containing every point in $M_{\mathcal{Q}} \cap (C_{\text{Tem}} \cup D_{\text{Tem}})$, it suffices to check, for each $q \in \mathcal{Q}$, the eigenvalues of the Jacobian matrices associated to the Poincaré maps P_q at its fixed point (ξ_q^*, q).

Due to the linear form of the flow map of $\mathcal{H}_{\text{Tem}_M}$, the Jacobian matrices of the Poincaré maps have explicit analytic forms. Since q keeps constant during flow, the flow solution ϕ^f to the flow dynamics $\dot{\xi} = -a\xi + c(2-q)$ from ξ_0 are given by $\phi^f(t, \xi_0) = e^{-at}(\xi_0 - \frac{c}{a}) + \frac{c}{a}$ when $q = 1$ and

[5] A limit cycle $\mathcal{O}$ with J^* jumps in each period is transversal to $M_{\mathcal{Q}} \cap D$ at every jump (where $J^* \in \mathbb{N} \backslash \{0\}$ and D is the union of J^* jump sets, i.e., $D = \bigcup_{q \in \mathcal{Q}}(D_q \times \{q\})$), if it intersects each jump set $(M_q \cap D_q) \times \{q\}$ at exactly one point $(\bar{\xi}, q) := \mathcal{O} \cap M_{\mathcal{Q}} \cap (D_q \times \{q\})$ with the property $L_{f_q} h_q(\bar{\xi}) \neq 0$, where $q \in \mathcal{Q}$.

[6] In particular, when there does not exist $t \geqslant 0$ such that $\phi^f(t, x) \in D_q \times \{q\}$, we have $\{t \geqslant 0 : \phi^f(t, x) \in D_q \times \{q\}\} = \emptyset$ for each $q \in \mathcal{Q}$, which gives $T_{D_q}(x) = \infty$.

[7] A point x^* is a fixed point of a Poincaré map $P : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ if $x^* = P(x^*)$.

[8] For stability notions of fixed points of Poincaré maps and limit cycles, we refer the reader to [11].

[9] In this paper, our results employ the term "global" as in [12] and related references, which requires careful treatment.

[10] A "global" property for $\mathcal{H}_M$ implies a "global" property of the original system $\mathcal{H}$ only when M is equal to $C \cup D$. For tools to establish the asymptotic stability property, see [12].

$\phi^f(t,\xi_0) = e^{-at}\xi_0$ when $q=2$. From the definition of the Poincaré map and the solution of the flow dynamics from $x=(\xi,q)$ with $q=1$ and $\xi=b_2\in M_1\cap D_{\mathrm{Tem}_1}$, it follows that $P_1(x)=(b_2,1)$. Similarly, from the solution of the flow dynamics from $x=(\xi,q)$ with $q=2$ and $\xi=b_1\in M_2\cap D_{\mathrm{Tem}_2}$, it follows that $P_2(x)=(b_1,2)$. Then, the fixed points for P_1 and P_2 are $x_1^*=(\xi_1^*,1)=(b_2,1)$ and $x_2^*=(\xi_2^*,2)=(b_1,2)$, respectively. The Jacobian matrices of P_q at the fixed points x_q^* for each $q\in\{1,2\}$ are both the zero matrices. According to Theorem 5.1, the hybrid limit cycle $\mathcal{O}$ of the hybrid system $\mathcal{H}_{\mathrm{Tem}_M}$ is asymptotically stable with basin of attraction containing every point in $M_{\mathcal{Q}}\cap(C_{\mathrm{Tem}}\cup D_{\mathrm{Tem}})$. In fact, the (unique) limit cycle $\mathcal{O}$ is defined by a periodic solution ϕ^* to $\mathcal{H}_{\mathrm{Tem}_M}$ from $\phi^*(0,0)=(b_1,1)$ with $T^*=\frac{1}{a}\ln\frac{ab_1-c}{ab_2-c}+\frac{1}{a}\ln\frac{b_2}{b_1}$ and two jumps per period. △

VI. Remarks on Systems with Nonunique Limit Cycles

Compared to the switched system in (3), where χ flows at a constant rate of 1, general switched systems under dwell-time switching, where χ flows at variable rates, allow for nonuniqueness of solutions; see, e.g., (3) and the first set of definitions for C_q and D_q below it. The following example presents a switched system with multiple limit cycles under dwell-time switching, which is modeled using one of the models given in Section III-B.

Consider the hybrid inclusion $\mathcal{H}_{\mathrm{A}}=(C_{\mathrm{A}},F_{\mathrm{A}},D_{\mathrm{A}},G_{\mathrm{A}})$ with state $x=(z,\chi,q)\in\mathbb{R}^4$ modeling a switched system under dwell-time switching

$$\mathcal{H}_{\mathrm{A}}\begin{cases}\dot{x} & \in & F_{\mathrm{A}}(x) & x\in C_{\mathrm{A}}\\ x^+ & = & G_{\mathrm{A}}(x) & x\in D_{\mathrm{A}}\end{cases}\tag{8}$$

where $z=(z_1,z_2)\in\mathbb{R}^2$, $C_{\mathrm{A}}=\mathbb{R}^2\times[0,T_\chi]\times\mathcal{Q}$, $D_{\mathrm{A}}=\mathbb{R}^2\times\{T_\chi\}\times\mathcal{Q}$, $\mathcal{Q}=\{1,2\}$, $F_{\mathrm{A}}(x)=(f_q(z),[\epsilon,1],0)$, [11] $\epsilon>0$, $G_{\mathrm{A}}(x)=(z,0,3-q)$, $f_1(z)=A_1z+b_1$, $f_2(z)=A_2z+b_2$, with $A_1=\begin{bmatrix}0&-1\\5&-1\end{bmatrix}$, $b_1=\begin{bmatrix}1\\0\end{bmatrix}$, $A_2=\begin{bmatrix}0&0\\0&-1\end{bmatrix}$ and $b_2=\begin{bmatrix}0\\0\end{bmatrix}$. The hybrid system $\mathcal{H}_{\mathrm{A}}$ can exhibit periodic behaviors. Note that the system $\mathcal{H}_{\mathrm{A}}$ allows for nonuniqueness of solutions due to the set-valued flow map. Due to this, multiple limit cycles within a period are possible. In fact, $\mathcal{H}_{\mathrm{A}}=(C_{\mathrm{A}},F_{\mathrm{A}},D_{\mathrm{A}},G_{\mathrm{A}})$ at least has a periodic solution ϕ^* with period $T^*=4T_\chi$ and $J^*=2$ jumps per period for $\epsilon\in(0,0.5]$ and a periodic solution ϕ^* with period $T^*=2T_\chi$ and $J^*=2$ jumps per period.

Note that the system $\mathcal{H}_{\mathrm{A}}$ may have no limit cycles when $\epsilon=0$. In fact, $\mathcal{H}_{\mathrm{A}}$ has a solution that flows for all time in such a case. For a hybrid system with explicit logic states and nonunique solutions, such as the above example, to guarantee (asymptotic) stability of limit cycles using the proposed method in this paper, a proper definition of set-valued Poincaré maps will be required due to the existence of multiple solutions from the same initial condition.

[11] Here, the differential inclusion $\dot{\chi}\in[\epsilon,1]$ leads to switching instants t_j satisfying $0\leqslant t_1\leqslant T_\chi/\epsilon$ and $t_{j+1}-t_j\leqslant T_\chi/\epsilon$ for each $j\in\mathbb{N}\setminus\{0\}$.

VII. Conclusion

This paper introduced a Poincaré map to analyze asymptotic stability of limit cycles in a class of hybrid systems with explicit logic states, which can model switched systems under dwell-time switching. The compactness and transversality properties of limit cycles have been presented. Via the constructions of time-to-impact functions and the Poincaré map, sufficient and necessary conditions for asymptotic stability of limit cycles have been established. An example for a switched system with dwell-time, which allows for nonuniqueness of solutions, suggests that multiple limit cycles within a period are possible. Due to existence of set-valued maps in such a switched system, ways to establish conditions for asymptotic stability of limit cycles remain a challenging problem. For future work, we will also investigate the robust stability of limit cycles in hybrid systems with explicit logic states.

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