Observer-based Synchronization of Multi-agent Systems Using Intermittent Output Measurements

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***Abstract*— The problem of synchronizing multiple continuous-time linear time-invariant systems connected over a complex network, with intermittently available measurements of their outputs, is considered. To solve this problem, we propose a distributed observer-based feedback controller that utilizes a local hybrid observer to estimate neighboring states only from output measurements at such potentially nonperiodic isolated event times. Due to the inherent continuous and discrete dynamics emerging from coupling the impulsive measurement updates and the interconnected networked systems, we use hybrid systems to model and analyze the resulting closed-loop system. The problem of synchronization and state estimation is then recast as a set stabilization problem, and, utilizing a Lyapunov-based analysis for hybrid systems, we provide sufficient conditions for global exponential stability of the synchronization and zero estimation error set. A numerical example is provided to illustrate the results.**

I. Introduction

Synchronization of multiagent networked systems is the natural tendency of distributed agents to self-organize to evolve together over time. Synchronization has a wide range of applications over a variety of modalities of science and engineering. In fact, synchronization is a natural phenomena seen in spiking neurons [1], [2], and in engineering applications such as formation control and flocking maneuvers [3], and satellite and aerial formation design [4].

In this paper, we are interested in the topic of synchronization of continuous-time linear time-invariant (LTI) systems interconnected over a general graph where each agent can only measure the output of its neighbors at some isolated time instant. We aim to design a distributed observer-based control algorithm to drive the agents to each other when output measurements between the agents are not continuously or even periodically available. The problem space of communicating networked systems comes with many challenges. One such challenge comes from the agents' output measurements arriving at isolated and non-periodic (namely, intermittent) times.

The wide applicability of synchronization in science and engineering has promoted a rich set of theoretical results for a variety of classes of dynamical systems. The study of convergence and stability of synchronization comes through the use of systems theory tools such as Lyapunov functions [5], [6], contraction theory [7], and incremental input-to-state stability [8]. Results for asymptotic synchronization with continuous coupling between agents exist in both the continuous-time domain and the discrete-time domain; see, e.g., [9], [10], where the latter reference is a detailed survey about coordination and consensus of integrator dynamics.

Synchronization in continuous-time systems where communication coupling occurs at discrete events, also called sampled-data systems, is an emergent area of study [11]. An observer-based event policy was developed in [12] for a network of linear time-invariant systems where communication events occur when the distance between the local state and its estimate is larger than a threshold. Using a sample-and-hold self-triggered controller policy, a practical synchronization result was established in [13] for the case of first-order integrator dynamics.

The algorithm designed in this paper achieves synchronization (in the limit, with stability) using only neighboring output measurements. Moreover, measurements occur at isolated, possibly nonperiodic, time instances. The main contribution of this work lies on the establishment of sufficient conditions for synchronization of multiagent systems with intermittent output measurements. Our solution integrates two components, namely, each agent contains both an observer to estimate the state of its neighbors and an observer-based control algorithm which utilizes this information to drive the agents to synchronization, while simultaneously accurately estimating the states of the agents. Each agent only has access to their neighboring agents' outputs at the measurement times. Due to the fact that each agent contains both continuous-time and discrete-time dynamics, we use the hybrid system framework in [14] to model the closed-loop multiagent system. We show that the proposed design conditions guarantee that the states of each agent synchronize and that the estimates converge at an exponential rate. Precisely, through an appropriate choice of coordinates and with a Lyapunov-based analysis, we provide sufficient conditions for global exponential stability of the synchronization and zero estimation error set.

This work builds on and combines our previous work in [15] and [16]. Namely, in [15], we consider a state estimation problem of a single agent through a distributed sensor network where information is exchanged between agents asynchronously. In [16], we consider the synchronization problem as in this work, however, each agent has full access to the state of its neighbors. The main contribution of this

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work is merging these algorithms since, as it turns out, each agent must have numerous local states to generate the estimates of the state of its neighbors. In this paper, we combine these strategies and develop a distributed observer-based feedback controller for each agent to generate an estimate of their neighboring states to drive themselves (in a distributed way) to synchronization.

The remainder of this work is organized as follows. In Section II, we introduce the main notation and some preliminaries on graph theory used in this work. In Section III, we formulate the problem under consideration and provide our proposed controller and observer design. Section IV models the closed loop system as a hybrid system and gives the main results. In Section V, we provide a numerical simulation showcasing our results.

II. Notation and Preliminaries on Graph Theory

A. Notation

Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$ and notation $[u^\top \;\; v^\top]^\top$ is equivalent to (u, v). The set $\mathbb{Z}_{\geq 1}$ denotes the set of positive integers, i.e., $\mathbb{Z}_{\geq 1} := \{1, 2, 3, \dots\}$. $\mathbb{N}$ denotes the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. Given a symmetric matrix P, $\overline{\lambda}(P) := \max\{\lambda : \; \lambda \in \text{eig}(P)\}$ and $\underline{\lambda}(P) := \min\{\lambda : \; \lambda \in \text{eig}(P)\}$. Given matrices A, B with proper dimensions, we define the operator $\text{He}(A, B) := A^\top B + B^\top A$; $A \otimes B$ defines the Kronecker product; $\text{diag}(A, B)$ denotes a 2×2 block matrix with A and B being the diagonal entries. Given $N \in \mathbb{Z}_{\geq 1}$, $I_N \in \mathbb{R}^{N \times N}$ defines the identity matrix, $\mathbf{1}_N$ is the vertical vector of N ones, and $0_N \in \mathbb{R}^{N \times N}$ is the zero matrix. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. The graph of a set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined as gph $G = \{(x, y) : x \in \mathbb{R}^n, y \in G(x)\}$.

B. Preliminaries on Graph Theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathbb{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \dots, N\}$, and the edges are the pairs in the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two nodes, i.e., an edge from i to k, denoted by (i, k), implies that agent i can receive information from agent k. The adjacency matrix of the digraph Γ is denoted by $\mathbb{G} \in \mathbb{R}^{N \times N}$, where its (i, k)-th entry g_{ik} is equal to one if $(i, k) \in \mathcal{E}$ and zero otherwise. A digraph is undirected if $g_{ik} = g_{ki}$ for all $i, k \in \mathcal{V}$. Without loss of generality, we assume that $g_{ii} = 0$ for all $i \in \mathcal{V}$. The in-degree and out-degree of agent i are defined by $d_i^{in} = \sum_{k=1}^N g_{ik}$ and $d_i^{out} = \sum_{k=1}^N g_{ki}$. The in-degree matrix $\mathbb{D}$ is the diagonal matrix with the i-th diagonal entry equal to d_i^{in} for each $i \in \mathcal{V}$. The Laplacian matrix of the graph Γ, denoted by $\mathcal{L} \in \mathbb{R}^{N \times N}$, is defined as $\mathcal{L} = \mathbb{D} - \mathbb{G}$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (i, k) \in \mathcal{E}\}$. A digraph is *undirected* if communication between every distinct node is bidirectional, namely, for each edge (i, k) in the edge set $\mathcal{E}$, the edge (k, i) is also in $\mathcal{E}$. Let the digraph be strongly connected and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ be the eigenvalues of $\mathcal{L}$.[1] Then, $\lambda_1 = 0$ is a simple eigenvalue of $\mathcal{L}$ associated with the eigenvector 1_N; $\mathcal{L}$ is positive semi-definite and, therefore, there exists an orthonormal matrix $\Psi \in \mathbb{R}^{N \times N}$ such that $\Psi \mathcal{L} \Psi^\top = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$. The digraph is undirected if and only if the Laplacian is symmetric. According to [18], if the Laplacian is symmetric then we have the following properties. We define $\widetilde{\Psi} = (\psi_2, \psi_3, \dots, \psi_N) \in \mathbb{R}^{N \times N-1}$ with $\psi_i = (\psi_{i1}, \psi_{i2}, \dots, \psi_{iN})$ being the orthonormal eigenvector corresponding to the nonzero eigenvalue λ_i, $i \in \{2, 3, \dots, N\}$, which satisfies $\sum_{k=1}^N \psi_{ik} = 0$. Moreover, $\widetilde{\Psi}$ satisfies the following:

$$\widetilde{\Psi}\widetilde{\Psi}^\top = \frac{1}{N} \begin{bmatrix} N-1 & -1 & \dots & -1 \\ -1 & N-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & N-1 \end{bmatrix} =: U \qquad (1)$$

$\widetilde{\Psi}^\top \widetilde{\Psi} = I$, $U^2 = U$, $\Lambda := \widetilde{\Psi}^\top \mathcal{L} \widetilde{\Psi} = \text{diag}(\lambda_2, \lambda_3, \dots, \lambda_N)$. Note that $\widetilde{\Psi}$ does not contain the eigenvector associated to the zero eigenvalue of the Laplacian. We denote $\bar{\mathcal{L}}$ as a fat matrix which takes the rows of the Laplacian ℓ_i for each $i \in \mathcal{V}$ as blocks and builds it into a diagonal block matrix, namely, $\bar{\mathcal{L}} := \text{diag}(\ell_1, \ell_2, \dots, \ell_N)$.

III. Problem Formulation and Approach

We are interested in the problem of synchronizing the states of N identical LTI systems (referred to as agents) connected over a intermittently available network, where agent can measure their neighbors' output measurements. Each agent in the network satisfies the following dynamics:

$$\dot{x}_i = A x_i + B u_i \qquad (2)$$

for each $i \in \mathcal{V} := \{1, 2, \dots, N\}$, where $x_i \in \mathbb{R}^n$ is the state and $u_i \in \mathbb{R}^m$ is the input for the i-th agent. The agents are able to intermittently measure the output of their connected neighbors y_{ik} at isolated time events where $k \in \mathcal{N}(i)$. More specifically, each agent measures itself and receives the output measurement from its connected neighbors at time instances given by the sequence of increasing times $\{t_s\}_{s=1}^\infty$, where $s \in \mathbb{N} \setminus \{0\}$ is the measurement time instance index; i.e., at each such s, each i-th agent measures its output and the output of neighboring agents given by

$$y_{ik}(t_s) = H x_k(t_s)$$

where $k \in \mathcal{N}(i)$. The sequence of times $\{t_s\}_{s=1}^\infty$ are constrained to satisfy

$$\begin{aligned} & T_1 \leq t_{s+1} - t_s \leq T_2 && \forall s \in \{1, 2, \dots, \}, \\ & t_1 \leq T_2 \end{aligned} \qquad (3)$$

where the positive scalars $T_2 \geq T_1$ are the time parameters that define the lower and upper bounds, respectively, of the

[1] See [17] for more information on algebraic graph theory.

time allowed to elapse between consecutive measurement event times[2]. Note that this formulation considers the case when no information is known a priori for each agent and the first measurement is received at time t_1 for all agents.

Our goal is to design an observer-based feedback controller, that, using local measurements, drives each agent to synchronization, asymptotically with stability; namely, for each $i, k \in \mathcal{V}$,

$$\lim_{t\to\infty} |x_i(t) - x_k(t)| = 0 \tag{4}$$

while also rendering the set of points $x_i = x_k$ stable.

In the following sections, we introduce the observer-based feedback controller in two parts, namely, the distributed hybrid observer and the synchronizing controller. The distributed hybrid observer uses neighboring information to update a dynamic state which drives the estimate towards the true state value, and the synchronizing controller leverages the accuracy of the estimates to drive the states of the controllers towards synchronization asymptotically.

A. Distributed Hybrid Observer

Due to the fact that we do not have perfect and continuously available knowledge of the state, we propose a distributed hybrid observer to estimate the state of the neighboring agents. Each hybrid observer runs locally, at the i-th agent, to generate an estimate of the state x_k of the k-th agent x_k. To achieve this, the observer utilizes two dynamic states, the state estimate $\hat{x}_{ik} \in \mathbb{R}^n$ and an auxiliary state η_{ik} to capture the updated information at event times. Inspired by [15], each observer features an auxiliary state η_{ik} that captures the local output estimation error. In between update times, the observer states are each continuously updated as

$$\begin{aligned} \dot{\hat{x}}_{ik} &= A\hat{x}_{ik} + \eta_{ik}, \\ \dot{\eta}_{ik} &= \Pi \eta_{ik} \end{aligned} \tag{5}$$

for each $i \in \mathcal{V}$, where Π is a real matrix of appropriate dimensions to be designed. Note that, in between events, the observer operates open loop as there is no external information affecting the dynamics of the pair $(\hat{x}_{ik}, \eta_{ik})$. At measurement event times $\{t_s\}_{s=1}^{\infty}$, each agent receives measurements from their neighbors that can be used to update the state of their local observer. This leads to the following discrete update for the pair $(\hat{x}_{ik}, \eta_{ik})$:

$$\begin{aligned} \hat{x}_{ik}^+ &= \hat{x}_{ik}, \\ \eta_{ik}^+ &= L(H\hat{x}_{ik} - y_{ik}) \end{aligned} \tag{6}$$

where L is a real matrix that is to be designed. Note that the construction of the observer is such that the state of the local agent is not used; only a function of the state is used (i.e., the output).

Remark 3.1: Note that the estimate of x_k generated at agent i, namely, $\hat{x}_{ik}$ does not reset when new measurements are available, but instead only the auxiliary state η_{ik} associated to this agent is updated. In between updates, η_{ik} is injected in the continuous dynamics of $\hat{x}_{ik}$ to drive the estimate $\hat{x}_{ik}$ to the true value of x_k.

B. Distributed Synchronizing Controller

Since the actual states are not available to the agents, but rather linear functions of them, we propose a feedback controller that utilizes estimates generated from the hybrid observer to achieve synchronization. At each agent x_i, we assign the input u_i to a function of the estimates $\hat{x}_{ik}$ of the states of the agents' neighbors. This feedback law is given by

$$u_i = K \sum_{k \in \mathcal{N}(i)} (\hat{x}_{ii} - \hat{x}_{ik}) \tag{7}$$

where K is a real matrix that is to be designed.

IV. Hybrid Modeling and Main Results

A. Hybrid Modeling

Due to the impulsive and non-periodic nature of the measurement events over the network, we employ the time triggering model proposed in [19]. More precisely, we use a decreasing timer τ to capture the sequence of events times $\{t_s\}_{s=1}^{\infty}$. The timer decreases with ordinary time and, upon reaching zero, is impulsively reset to a point within the interval $[T_1, T_2]$. To model this mechanism and the closed-loop system, we employ the hybrid systems framework[3] in [14], where a hybrid system with state $\xi \in \mathbb{R}^n$ is denoted by $\mathcal{H} = (C, f, D, G)$ and is written in the compact form

$$\mathcal{H}: \quad \xi \in \mathbb{R}^n \quad \begin{cases} \dot{\xi} = f(\xi) & \xi \in C \\ \xi^+ \in G(\xi) & \xi \in D \end{cases} . \tag{8}$$

Using this framework, the evolution of the timer τ is given by the following dynamics:

$$\begin{aligned} \dot{\tau} &= -1 & \tau &\in [0, T_2], \\ \tau^+ &\in [T_1, T_2] & \tau &= 0. \end{aligned} \tag{9}$$

Note that any sequence of times $\{t_s\}_{s=1}^{\infty}$ that satisfies (3) is captured by the timer model in (9).

Inspired by [19], for each $i, k \in \mathcal{V}$, consider the change in coordinates

$$\begin{aligned} e_{ik} &= \hat{x}_{ik} - x_k, \\ \theta_{ik} &= L(H\hat{x}_{ik} - y_{ik}) - \eta_{ik}. \end{aligned} \tag{10}$$

The quantity e_{ik} defines the estimation error between the local estimate of agent k held at agent i. The quantity θ_{ik} is the difference between the output estimation error multiplied by L and the auxiliary state η_{ik}. Let $e = (e_1, e_2, e_3, \ldots, e_N)$, $e_i = (e_{i1}, e_{i2}, \ldots, e_{iN})$, $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_N)$, $\theta_i = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{iN})$, $x = (x_1, x_2, \ldots, x_N)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$, and $\eta_i = (\eta_{i1}, \eta_{i2}, \ldots, \eta_{iN})$. Then, with the change of variables in (10), the closed-loop hybrid system

[2]If $T_2 = T_1$, then the outputs are measured at periodic times.

[3]A hybrid system is given by four objects (C, f, D, G) defining its data: the *flow set* is a set $C \subset \mathbb{R}^n$ specifying the points where the continuous evolution (or flows) is possible; the *flow map* is a single-valued map $f : \mathbb{R}^n \to \mathbb{R}^n$ defining the flows; the *jump set* is a set $D \subset \mathbb{R}^n$ specifying the points where the discrete evolution (or jumps) is possible; and the *jump map* is a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the value of the state after jumps.

$\mathcal{H}$ comprises the agents' dynamics (2), the observer design in (5) and (6), the synchronizing controller in (7), and the timer dynamics in (9) triggering measurement events. Using the error coordinates, the state of $\mathcal{H}$ is defined as $\xi = (z, \tau) \in (\mathbb{R}^{nN} \times \mathbb{R}^{nN^2} \times \mathbb{R}^{nN^2}) \times [0, T_2] =: \mathcal{X}$ where $z = (x, e, \theta)$. The continuous dynamics of x are

$$\dot{x} = \widetilde{A}x + \widetilde{B}\widetilde{K}x + \widetilde{B}\bar{K}e \tag{11}$$

where $\widetilde{A} = I_N \otimes A$, $\widetilde{B} = I_N \otimes B$, $\widetilde{K} = \mathcal{L} \otimes K$, and $\bar{K} = \bar{\mathcal{L}} \otimes K$ with $\mathcal{L}$ and $\bar{\mathcal{L}}$ defined in Section II. From the observer design in (5) and (6) and the dynamics of the agents in (2), it follows that the error state e_{ik} satisfies the following dynamics

$$\begin{aligned}\dot{e}_{ik} = Ae_{ik} - BK \sum_{r \in \mathcal{N}(k)} (e_{kk} - e_{kr}) + \eta_{ik} \\ - BK \sum_{r \in \mathcal{N}(k)} (x_k - x_r)\end{aligned}$$

Then, we have that the error dynamics e can be written compactly as

$$\dot{e} = I_N \otimes \widetilde{A}e + \eta - \mathbf{1}_N \otimes \widetilde{B}\bar{K}e - \mathbf{1}_N \otimes \widetilde{B}\widetilde{K}x.$$

From the definition of θ_{ik} in (10), it follows that combining θ_{ik} to θ leads to

$$\theta = \widetilde{L}\widetilde{H}e - \eta \tag{12}$$

where $\widetilde{L} = I_{n^2} \otimes L$ and $\widetilde{H} = I_{n^2} \otimes H$ which leads to

$$\dot{e} = I_N \otimes \widetilde{A}e + \widetilde{L}\widetilde{H}e - \theta - \mathbf{1}_N \otimes \widetilde{B}\bar{K}e - \mathbf{1}_N \otimes \widetilde{B}\widetilde{K}x. \tag{13}$$

The continuous dynamics of θ are given by

$$\dot{\theta} = \widetilde{L}\widetilde{H}\dot{e} - \dot{\eta}.$$

From (5), (12) and (13), we have that

$$\begin{aligned}\dot{\theta} = -\widetilde{L}\widetilde{H}(\mathbf{1}_N \otimes \widetilde{B}\widetilde{K})x - (\widetilde{L}\widetilde{H} + \widetilde{\Pi})\theta \\ + (\widetilde{L}\widetilde{H}(I_N \otimes \widetilde{A} + \widetilde{L}\widetilde{H} - \mathbf{1}_N \otimes \widetilde{B}\bar{K}) - \widetilde{\Pi}\widetilde{L}\widetilde{H})e\end{aligned} \tag{14}$$

where $\widetilde{\Pi} = I_{N^2} \otimes \Pi$. Note that the dynamics of x_i and $\hat{x}_{ik}$ for each $i, k \in \mathcal{V}$ are continuous and do not change at measurement event times (e.g., when $\tau = 0$). More specifically, $x_i^+ = x_i$ and $\hat{x}_{ik}^+ = \hat{x}_{ik}$, which leads to the update in the error state to also be continuous, i.e., $e^+ = e$. Lastly, due to the definition of θ_{ik} in (10) it follows that

$$\theta_{ik}^+ = L(H\hat{x}_{ik}^+ - x_k^+) - \eta_{ik}^+ = 0$$

due to the update of η_{ik} to $L(H\hat{x}_{ik} - x_k)$ for each $i, k \in \mathcal{V}$. This leads to an autonomous closed-loop hybrid system $\mathcal{H}$ as in (8) with state ξ and dynamics given by

$$\begin{aligned} \dot{\xi} &= (\mathcal{F}z, -1) & \tau \in [0, T_2] \\ \xi^+ &= (\mathcal{G}z, [T_1, T_2]) & \tau = 0 \end{aligned} \tag{15}$$

where $\mathcal{F}$ and $\mathcal{G}$ are given by

$$\mathcal{F} = \begin{bmatrix} \widetilde{A} + \widetilde{B}\widetilde{K} & \widetilde{B}\bar{K} & 0 \\ \mathcal{F}_{21} & \mathcal{F}_{22} & -I_{nN^2} \\ \widetilde{L}\widetilde{H}\mathcal{F}_{21} & \widetilde{L}\widetilde{H}\mathcal{F}_{22} - \widetilde{\Pi}\widetilde{L}\widetilde{H} & -(\widetilde{L}\widetilde{H} + \widetilde{\Pi}) \end{bmatrix} \tag{16}$$

where $\mathcal{F}_{21} = -\mathbf{1}_N \otimes \widetilde{B}\widetilde{K}$ and $\mathcal{F}_{22} = I_N \otimes \widetilde{A} + \widetilde{L}\widetilde{H} - (\mathbf{1}_N \otimes \widetilde{B}\bar{K})$ and

$$\mathcal{G} = \text{diag}(I_{nN}, I_{nN^2}, 0_{nN^2}). \tag{17}$$

The matrix (16) captures the (linear) continuous-time evolution of the hybrid system in the error coordinates between event times and (17) captures the update at event times, i.e., when $\tau = 0$.

In general, a solution ϕ to a hybrid system $\mathcal{H}$ in (8) is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain dom $\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in$ dom ϕ, the set dom $\phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\bigcup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The time instances t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction.

Remark 4.1: Note that the jump set of the hybrid system $\mathcal{H}$ in (15) is completely dependent on the timer state τ and the jump map resets the value of τ to a point in the interval $[T_1, T_2]$. Therefore, if we consider a maximal solution ϕ to $\mathcal{H}$, then the time instances t_j satisfy the event times given by the constraints in (3), namely,

$$\begin{aligned} T_1 \leq t_{j+1} - t_j \leq T_2 \\ t_1 \leq T_2 \end{aligned}$$

for each $(t_j, j), t_{j+1}, j) \in \operatorname{dom} \phi$.

With the definition of solutions to hybrid systems above, we have the following result which implicitly removes the possibility of Zeno solutions.

Lemma 4.2: Let $0 \leq T_1 \leq T_2$ be given. Every maximal solution ϕ to the hybrid system in (15) *is complete and each $(t, j) \in \operatorname{dom} \phi$ satisfies $T_1(j - 1) \leq t \leq (j + 1)T_2$.*

B. Synchronization and Estimation as a Set Stabilization Problem

As mentioned in Section III, (global) synchronization is characterized as every solution, starting from arbitrary initial conditions, converges to the set of points $x = (x_1, x_2, \ldots, x_N)$ such that $x_1 = x_2 = \cdots = x_N$, with stability. In this section, we recast the synchronization problem as a set stabilization problem. Namely, our goal is to stabilize the set of points $\xi = ((x, e, \theta), \tau)$ such that each component of x is synchronized, and the estimation error e and the state θ are driven to zero. The state τ will continue to evolve within $[0, T_2]$ indefinitely; therefore, the set to stabilize will allow τ to belong to $[0, T_2]$. In particular, given a (maximal) complete solution $\phi = (\phi_x, \phi_e, \phi_\theta, \phi_\tau)$ to the hybrid system $\mathcal{H}$ in (15), the goal is to ensure that $\lim_{t+j\to\infty} |\phi_{x_i}(t,j) - \phi_{x_k}(t,j)| = 0$, $\lim_{t+j\to\infty} |\phi_{\hat{x}_{ik}}(t,j) - \phi_{x_k}(t,j)| = 0$ and $\lim_{t+j\to\infty} \phi_\theta(t,j) = 0$ for each $i, k \in \mathcal{V}$. To determine such a property, we recast our problem as a stabilization of the hybrid system $\mathcal{H}$ in (15) to the following set of points:

$$\mathcal{A} := \{((x, e, \theta), \tau) \in \mathcal{X} : x_i = x_k, e_{ik} = 0, \theta_{ik} = 0, \\ \forall i, k \in \mathcal{V}\}. \quad (18)$$

In the next section, we determine sufficient conditions that yield this set globally exponentially stable for $\mathcal{H}$.

Definition 4.3: Let a hybrid system $\mathcal{H}$ as in (8) be defined on $\mathbb{R}^n$. Let $\mathcal{A}$ be closed. The set $\mathcal{A}$ is said to be globally exponentially stable (GES) for $\mathcal{H}$ if there exists $k, \alpha > 0$ such that every maximal solution ϕ to $\mathcal{H}$ is complete and satisfies

$$|\phi(t,j)|_\mathcal{A} = k \exp(-\alpha(t+j))|\phi(0,0)|_\mathcal{A}$$

for each $(t,j) \in \operatorname{dom} \phi$.

C. Main Results

In this section, we establish a sufficient condition that guarantees the synchronization and estimation properties via stability analysis of the set $\mathcal{A}$ in (18) for the hybrid system in (15). We establish such a result by using a Lyapunov function candidate $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$. An appropriate choice of V must satisfy $V(\xi) = 0$ for each $\xi \in \mathcal{A}$, and for any $\xi \in \mathcal{X} \setminus \mathcal{A}$, $V(\xi) > 0$. We first define the Lyapunov function candidate

$$V(\xi) = x^\top \Psi P_1 \Psi^\top x + e^\top P_2 e + \exp(\sigma\tau)\theta^\top P_3 \theta \quad (19)$$

where P_1, P_2, and P_3 are positive definite symmetric matrices, and $\Psi = \widetilde{\Psi} \otimes I_n$, where $\widetilde{\Psi}$ is defined in Section II-B. The Lyapunov function V in (19) satisfies [20, Definition 3.16] which makes it a suitable Lyapunov function candidate for stability analysis of $\mathcal{A}$ in (18). The following result exploits the fact that, under certain conditions, V decreases during flows and, at jumps, it does not increase. To guarantee exponential stability, we leverage the proof of Proposition 3.24 in [14] which uses the fact that, since every solution to $\mathcal{H}$ persistently flows and does not increase during jumps then solutions must converge to the set $\mathcal{A}$. The Lyapunov function V in (19) is inspired by [16] where we focus on synchronization with state feedback; in that case, the Lyapunov function V also decreases during flows and does not increase at jumps.

Theorem 4.4: Let the hybrid system $\mathcal{H}$ in (8) *and positive scalars $T_1 \leq T_2$ be given. Let the graph Γ be undirected. If there exist a scalar $\sigma > 0$, matrices K, h, L, and positive definite matrices symmetric P_1, P_2 and P_3 satisfying*

$$P(\nu)\bar{\Psi}^\top \mathcal{F}\bar{\Psi} + \bar{\Psi}^\top \mathcal{F}\bar{\Psi}P(\nu) - \sigma\bar{P}_3(\nu) < 0 \quad (20)$$

for each $\nu \in [0, T_2]$ where $P(\tau) = diag(P_1, P_2, P_3 \exp(\sigma\tau))$, $\bar{P}_3(\tau) = diag(0, 0, P_3 \exp(\sigma\tau))$, $\bar{\Psi} = diag(\Psi, I_{nN^2}, I_{nN^2})$ and $\mathcal{F}$ is given in (16)*, then the set $\mathcal{A}$ in* (18) *is globally exponentially stable for the hybrid system in* (15).

Note that the matrix inequality in (20) must be satisfied for an infinite number of points, i.e., for each $\nu \in [0, T_2]$. To alleviate this issue, we have the following result.

Proposition 4.5: Let the positive scalars $T_1 \leq T_2$ be given. The inequality in (20) *holds if there exist, matrices K, h, L, and positive definite symmetric matrices P_1, P_2, and P_3, and a scalar $\sigma > 0$ satisfying*

$$P(0)\bar{\Psi}^\top \mathcal{F}\bar{\Psi} + \bar{\Psi}^\top \mathcal{F}\bar{\Psi}P(0) - \sigma\bar{P}_3(0) < 0 \quad (21)$$

$$P(T_2)\bar{\Psi}^\top \mathcal{F}\bar{\Psi} + \bar{\Psi}^\top \mathcal{F}\bar{\Psi}P(T_2) - \sigma\bar{P}_3(T_2) < 0 \quad (22)$$

where $P(\tau) = diag(P_1, P_2, P_3 \exp(\sigma\tau))$, $\bar{P}_3(\tau) = diag(0, 0, P_3 \exp(\sigma\tau))$, $\bar{\Psi} = diag(\widetilde{\Psi} \otimes I_n, I_{nN^2}, I_{nN^2})$ and $\mathcal{F}$ is given in (16)*.*

Remark 4.6: Note that the matrices in (21) and (22) involve nonlinear terms involving K, Π, and L. The presence of these terms in (16) makes the problem nonlinear and difficult to solve numerically. However, it can be shown that LMI conditions can be derived following ideas in [21].

In the next section, we give an example showcasing the results in this section. Namely, we consider the synchronization of four impulsively coupled ideal mass-spring systems (such systems can also be considered to be linear oscillators with unitary spring and mass coefficients). Such systems are known to have cyclic behaviors if initialized away from the origin. As we will show, we can use Proposition 4.5 to design matrices K, Π, and L that yield global exponential stability of the synchronization set $\mathcal{A}$ in (18).

V. Simulations

In this section, we present a simulation which illustrates the main results. We consider the case of four agents connected over a network defined by the graph

$$\mathbb{G} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad (23)$$

The dynamics of each agent are governed by a continuous-time linear oscillator with state $x \in \mathbb{R}^2$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \quad (24)$$

Note that the eigenvalues of the system matrix in (24) are $\pm i$ implying strictly oscillatory behavior in open loop. The agents can measure the output given by

$$y(t_s) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t_s)$$

and that of their neighbors at events governed by the sequence of times $\{t_s\}_{s=1}^\infty$ satisfying (3) with

$$T_1 = 0.001 \qquad T_2 = 0.15.$$

In Figure 1, we show a particular numerical solution to

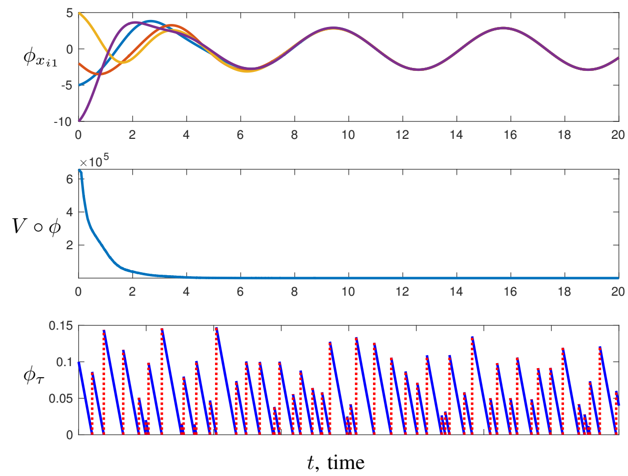

Fig. 1. A numerical simulation of the closed-loop network system. (top) The trajectory of the first component of the state of each agent. (middle) The Lyapunov function V defined in (19) evaluated over the solution. (bottom) The measurement times governed by the timer state τ occur when $\tau = 0$.

this example. Namely, we found that for the gain matrices

$$K = \begin{bmatrix} -0.5 & -0.5 \end{bmatrix}$$
$$L = \begin{bmatrix} -1 & -2 \end{bmatrix}^\top$$
$$\Pi = -0.5$$

it is possible to find scalars $\sigma > 0$ and positive definite matrices P_1, P_2, and P_3 that satisfy the matrix inequalities (21) and (22) implying global exponential stability. The initial conditions of the states x are given by

$$x_1 = \begin{bmatrix} -5 & 1 \end{bmatrix}^\top \qquad x_2 = \begin{bmatrix} -2 & -3 \end{bmatrix}^\top$$
$$x_3 = \begin{bmatrix} 5 & -3 \end{bmatrix}^\top \qquad x_4 = \begin{bmatrix} -10 & 4 \end{bmatrix}^\top.$$

The initial conditions of both the estimation states $\hat{x}_{ik}$ and the local auxiliary state η_{ik} were nonzero and, in fact, randomly chosen in the interval $[-5, 5]$. From Figure 1, we can see that the solutions converge to synchronization through the convergence of the first component of the solution to x_{i1} to each other. The middle plot of Figure 1 shows the Lyapunov function in (19), converging exponentially to zero. Lastly, the bottom plot in Figure 1 shows the points in time where measurements occur, namely, when $\tau = 0$ (indicated by the dashed red lines which reset the timer randomly point inside the bounds $[0.1, 0.15]$).

VI. Conclusion

In this paper, we provided a solution to the synchronization problem where the measurements of neighboring agents are not continuously available. We proposed a distributed local observer-based feedback controller that impulsively updates an auxiliary state at measurement event times to drive the estimate of the observer to the true value of the state. The static controller assigned to the input is based on the local estimated states. Through modeling the closed-loop system as a hybrid system, we recasted the synchronization problem to a set synchronization problem and utilized Lyapunov stability tools. The main result was given in terms of sufficient conditions for exponential stability of the synchronization set in terms of matrix inequalities.

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