Optimal Selection of Basis Functions for Robust Tracking Control of Linear Systems using Filtered Basis Functions

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Abstract— **There is growing interest in the use of the filtered basis functions (FBF) approach for feedforward tracking control of linear systems. The FBF approach expresses the control input to the plant as a linear combination of basis functions. The basis functions are forward filtered through the plant dynamics and the coefficients of the linear combination are selected such that the tracking error is minimized. It has been demonstrated that the FBF approach is more versatile compared to other methods in the literature. However, the tracking accuracy of the FBF approach deteriorates in the presence of model uncertainty, much like it does with other feedforward control methods. But, unlike other methods, the FBF approach presents flexibility in terms of the choice of the basis functions, which can be used to improve its accuracy in the presence of model uncertainty. This paper analyzes the effect of choice of the basis functions on the tracking accuracy of FBF, in the presence of uncertainty, using the Frobenius norm of the lifted system representation of FBF's error dynamics. Based on the analysis, a methodology for optimal selection of basis functions is presented. The effectiveness of the proposed methodology is demonstrated using examples. Large improvements in robustness are observed using the proposed optimal set of basis functions compared to popular basis functions, viz., B-splines, discrete cosine transform and block pulse functions.**

I. Introduction

Tracking control is a fundamental problem encountered in a wide range of fields such as manufacturing, robotics and aeronautics. The objective of tracking control is to force the output trajectory of the controlled system to follow a desired trajectory as closely as possible. Tracking control could be achieved using feedforward and/or feedback controllers. This paper is written in the context of feedforward tracking control of discrete-time linear systems.

Feedforward tracking control of linear systems can be ideally achieved using perfect tracking control (PTC) i.e., pole-zero cancellation [1]. But in practice, ideal feedforward control cannot be realized due to (i) nonminimum phase (NMP) zeros and (ii) uncertainty in the plant dynamics [2]. When applied to NMP systems, PTC results in highly oscillatory or unstable control trajectories which are unacceptable. NMP zeros are quite prevalent in practice. For example, they occur in systems with fast sampling rates [3], as well as in systems with noncollocated placement of sensors and actuators [4]. Hence, a lot of research on feedforward tracking control of NMP systems can be found in the literature ([5]–[7] present detailed literature reviews on the subject). Recently, the filtered basis functions (FBF) approach has been gaining attention as an effective approach for feedforward tracking control of linear NMP systems [8]–[12]. The FBF approach expresses the control input as a linear combination of user-defined basis functions (of which there are a wide range of choices, e.g., B-splines [8], cosine signals [13], radial basis functions [10], etc.). The basis functions are forward filtered through the plant dynamics and the coefficients are selected, using an elegant least squares solution, such that the tracking error is minimized. The FBF approach finds its origins in iterative learning control (ILC) [13] but was not applied to feedforward tracking control of NMP systems until recently [8]–[12]. Unlike most of the methods in the literature, the FBF approach is effective for a wide range of desired trajectories and plants including nonhyperbolic systems (systems with zeros on the unit circle in the z-plane) [9], [12], square and non-square multi input multi output (MIMO) systems, linear time varying (LTV) systems, linear parameter varying (LPV) systems [10], etc. Also, the tracking accuracy of FBF does not change significantly with plant dynamics, as compared to other popular methods in the literature [14].

A key challenge of feedforward controllers, including FBF, is how to improve their robustness in the presence of uncertainty in plant dynamics [2], [15]–[20]. Recently, optimal selection of coefficients [21] and optimal filtering of basis functions [22] have been explored as avenues to improve the robustness of the FBF approach. However, optimal selection of basis functions presents an opportunity for improving the robustness of the FBF approach that is unavailable to other feedforward tracking control methods; it could be used as an alternative or complement to existing methods. Recently, [23] has proposed an optimal set of basis functions to achieve a desired level of tracking accuracy with minimum control effort. The optimization was realized using the Frobenius norm of the lifted system representation (LSR) of the FBF's error as well as controller dynamics. In a similar vein as [23], this paper explores optimal basis function selection for robust tracking control using the FBF approach. Specifically, it makes the following original contributions to the literature:

1. It analyzes the effect of basis functions on tracking accuracy of FBF, in the presence of uncertainty, using the Frobenius norm of the LSR of the error dynamics.
2. It proposes an optimal set of basis functions for tracking control using FBF, in the presence of uncertainty.

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The paper is structured as follows: Section II presents some background information on the FBF approach and the Frobenius norm metric. The contributions of the paper are presented in Section III. Section IV demonstrates the effectiveness of the proposed basis functions using a simulation example and Section V concludes the paper.

II. Background

A. Tracking Control Problem

Given a discrete-time linear time invariant (LTI) single input single output (SISO) system $G(q)$, as shown in Fig. 1, which may represent an open loop plant or a closed loop controlled system, we can write

$$y(k) = G(q)u(k) \tag{1}$$

where k is the time index, q is the forward shift operator, y and u are the output and control input, respectively. The objective of tracking control is to design the feedforward controller $C(q)$ or find the control input $u(k)$ given by

$$u(k) = C(q)y_d(k) \tag{2}$$

where $y_d(k)$ is the desired trajectory, such that the tracking error $e(k)$

$$\begin{aligned} e(k) &= y_d(k) - y(k) \\ &= (1 - \underbrace{G(q)C(q)}_{L(q)})y_d(k) = E_{ff}(q)y_d(k) \end{aligned} \tag{3}$$

is minimized, where $L(q)$ and $E_{ff}(q)$ are the overall dynamics and the error dynamics of the controlled system, respectively.

For finite time, $0 \le k \le M$ (M+1 is the number of discrete points in the trajectory), the desired trajectory, control input, tracking error and output trajectory can be expressed using vectors

$$\begin{aligned} \mathbf{y}_d &= \begin{bmatrix} y_d(0) & y_d(1) & \dots & y_d(M) \end{bmatrix}^{\mathrm{T}}, \\ \mathbf{u} &= \begin{bmatrix} u(0) & u(1) & \dots & u(M) \end{bmatrix}^{\mathrm{T}}, \\ \mathbf{e} &= \begin{bmatrix} e(0) & e(1) & \dots & e(M) \end{bmatrix}^{\mathrm{T}}, \\ \mathbf{y} &= \begin{bmatrix} y(0) & y(1) & \dots & y(M) \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{4}$$

Accordingly, Eqs. (1), (2) and (3) can be expressed as

$$\mathbf{y} = \mathbf{G}\mathbf{u}; \quad \mathbf{u} = \mathbf{C}\mathbf{y}_d; \quad \mathbf{e} = (\mathbf{I} - \mathbf{L})\mathbf{y}_d = \mathbf{E}_{ff}\mathbf{y}_d \tag{5}$$

where $\mathbf{G}$, $\mathbf{C}$, $\mathbf{L}$ and $\mathbf{E}_{ff}$ are the lifted system representations [24] of G, C, L and E_{ff}, respectively, and $\mathbf{I}$ is the identity matrix of appropriate size. The use of boldface symbols to represent LSR of systems is maintained hereinafter.

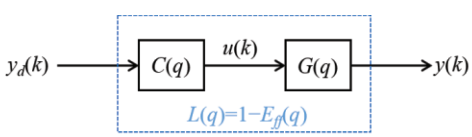

Fig. 1: Block diagram for feedforward tracking control

B. Filtered Basis Functions (FBF) Approach

The FBF approach relies on two assumptions:

- The desired trajectory is known a priori
- The control input $u(k)$ is expressed as a linear combination of n+1 user-defined basis functions $\varphi_i(k)$

$$u(k) = \sum_{i=0}^{n} \gamma_i \varphi_i(k) \tag{6}$$

where γ_i are unknown coefficients. Using vectors, Eq. (6) can be expressed as

$$\begin{aligned} &\mathbf{u} = \mathbf{\Phi}\boldsymbol{\gamma}; \\ &\mathbf{\Phi} = \begin{bmatrix} \boldsymbol{\varphi}_0 & \boldsymbol{\varphi}_1 & \dots & \boldsymbol{\varphi}_n \end{bmatrix}, \\ &\boldsymbol{\varphi}_i = \begin{bmatrix} \varphi_i(0) & \varphi_i(1) & \dots & \varphi_i(M) \end{bmatrix}^{\mathrm{T}}, \\ &\boldsymbol{\gamma} = \begin{bmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_n \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{7}$$

For a linear system $G(q)$ (with lifted system representation $\mathbf{G}$), $\mathbf{y}$ can be expressed as

$$\begin{aligned} &\mathbf{y} = \tilde{\mathbf{\Phi}}\boldsymbol{\gamma}; \\ &\tilde{\mathbf{\Phi}} = \mathbf{G}\mathbf{\Phi}; \quad \tilde{\boldsymbol{\varphi}}_i = \mathbf{G}\boldsymbol{\varphi}_i; \\ &\tilde{\mathbf{\Phi}} = \begin{bmatrix} \tilde{\boldsymbol{\varphi}}_0 & \tilde{\boldsymbol{\varphi}}_1 & \dots & \tilde{\boldsymbol{\varphi}}_n \end{bmatrix} \end{aligned} \tag{8}$$

where $\tilde{\mathbf{\Phi}}$ represents the filtered basis functions matrix. The control objective is to find the optimal coefficient vector $\boldsymbol{\gamma}$ such that the 2-norm of the tracking error

$$\mathbf{e}^{\mathrm{T}}\mathbf{e} = \left(\mathbf{y}_d - \tilde{\mathbf{\Phi}}\boldsymbol{\gamma}\right)^{\mathrm{T}}\left(\mathbf{y}_d - \tilde{\mathbf{\Phi}}\boldsymbol{\gamma}\right) \tag{9}$$

is minimized; the optimal solution is given by

$$\boldsymbol{\gamma}^* = \left(\tilde{\mathbf{\Phi}}^{\mathrm{T}}\tilde{\mathbf{\Phi}}\right)^{-1}\tilde{\mathbf{\Phi}}^{\mathrm{T}}\mathbf{y}_d \tag{10}$$

Based on Eqs. (5), (7), (8) and (10), the LSRs of the controller and error dynamics can be expressed as

$$\begin{aligned} \mathbf{C}_{FBF} &= \mathbf{\Phi}\left(\tilde{\mathbf{\Phi}}^{\mathrm{T}}\tilde{\mathbf{\Phi}}\right)^{-1}\tilde{\mathbf{\Phi}}^{\mathrm{T}} \\ \mathbf{E}_{ff,FBF} &= \mathbf{I} - \underbrace{\tilde{\mathbf{\Phi}}\left(\tilde{\mathbf{\Phi}}^{\mathrm{T}}\tilde{\mathbf{\Phi}}\right)^{-1}\tilde{\mathbf{\Phi}}^{\mathrm{T}}}_{\mathbf{L}_{FBF}} \end{aligned} \tag{11}$$

Remark 1: $\mathbf{C}_{FBF}$ and $\mathbf{E}_{ff,FBF}$ both depend on the system as well as the selected basis functions. Both matrices are, in general, non-Toeplitz and non-triangular implying that the FBF controller is, in general, LTV and non-causal [9].

Remark 2: Although this paper describes the FBF approach in the context of LTI SISO systems, it is applicable to other types of linear systems such as LTV and MIMO systems.

C. Frobenius Norm Metric

As a tracking performance evaluation metric, [14], [23] proposed the following metric, J_e, based on the Frobenius norm of $\mathbf{E}_{ff}$

$$\begin{aligned} &J_e = \frac{\left\|\mathbf{E}_{ff}\right\|_F}{\sqrt{M+1}}; \\ &\left\|\mathbf{E}_{ff}\right\|_F = \sqrt{trace(\mathbf{E}_{ff}^{\mathrm{T}}\mathbf{E}_{ff})} = \sqrt{\sum_{\forall i}\left\{\sigma_i(\mathbf{E}_{ff})\right\}^2} \end{aligned} \tag{12}$$

The Frobenius norm is selected because it takes into account all singular values/gains (σ_i) of $\mathbf{E}_{ff}$, as opposed to $\|\mathbf{E}_{ff}\|_2$, which considers only the maximum singular value/gain.

Note that for a normalized desired trajectory ($\|\mathbf{y}_d\|_2 = 1$),

$$\mathbf{e}_{RMS} = \frac{\|\mathbf{e}\|_2}{\sqrt{M+1}} \le \frac{\left\|\mathbf{E}_{ff}\right\|_F}{\sqrt{M+1}} = J_e \tag{13}$$

The implication is that J_e is an upper bound on the RMS tracking error ($\mathbf{e}_{RMS}$). Moreover, it is shown in [23], that for an LTI system

$$\frac{\left\|\mathbf{E}_{ff}\right\|_F}{\sqrt{M+1}} \rightarrow \left\|E_{ff}\left(q\right)\right\|_2 \quad \text{as } M \rightarrow \infty \tag{14}$$

In other words, J_e approaches the system error 2-norm criterion (sometimes used in the design and analysis of tracking controllers [25]).

III. Optimal Basis Functions for Robust Tracking

This section analyzes the robustness of the FBF approach and proposes an optimal set of basis functions for robust tracking control by: (i) providing a metric expression for FBF including plant uncertainty, and (ii) formulating basis functions selection as an optimization problem and solving the optimization problem to find the best set of basis functions for robust tracking.

A. *Effect of Uncertainty and Basis Functions on Tracking Accuracy of FBF*

Assume that the actual plant dynamics belongs to the set $\{G_{aj}\}$, j =1, 2, …, l. The set could represent a plant with additive uncertainty, multiplicative uncertainty, parametric uncertainty, etc. Without loss of generality, this paper assumes that the set is discrete. Sampling of uncertainty has been used in literature [20], [26] for robust controller design. If the controller C (see Fig. 1) is designed based on nominal plant dynamics G_{nom} and the error dynamics corresponding to G_{aj} is given by E_{ffj}, then to analyze the robustness of tracking controllers, the Frobenius norm metric $J_{e,r}$ can be expressed as

$$\begin{aligned} J_{e,r}^2 &= \sum_{j=1}^{l} \lambda_j J_{ej}^2; \\ J_{ej} &= \frac{\left\|\mathbf{E}_{ffj}\right\|_F}{\sqrt{M+1}} = \frac{\left\|\mathbf{I}-\mathbf{G}_{aj}\mathbf{C}\right\|_F}{\sqrt{M+1}}; \quad \sum_{j=1}^{l}\lambda_j = 1 \end{aligned} \tag{15}$$

where $\{\lambda_j\}$ denotes weights associated with the distribution of the uncertainty. Note that the nominal plant dynamics G_{nom} may or may not belong to the set $\{G_{aj}\}$.

Remark 3: This paper focuses on FBF, hence, the modified metric will only be explored in the context of FBF in the remainder of the paper. However, the metric can be used to analyze robustness of other tracking controllers in the literature.

If the FBF controller C is designed using the nominal plant dynamics G_{nom}, then its LSR $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{\Phi}\left(\tilde{\mathbf{\Phi}}_{nom}^{\mathrm{T}}\tilde{\mathbf{\Phi}}_{nom}\right)^{-1}\tilde{\mathbf{\Phi}}_{nom}^{\mathrm{T}} \tag{16}$$

Analysis using the pseudoinverse is quite cumbersome and hence, the filtered basis functions matrix $\tilde{\mathbf{\Phi}}_{nom}=\mathbf{G}_{nom}\mathbf{\Phi}$ is transformed into the decoupled filtered basis functions matrix $\tilde{\mathbf{\Psi}}_{nom}=\mathbf{G}_{nom}\mathbf{\Psi}_{nom}$ (for more details see [9]). After transformation, the LSRs $\mathbf{C}$ and $\mathbf{E}_{ffj}$ can be expressed as

$$\begin{aligned} &\mathbf{C} = \mathbf{\Psi}_{nom}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}}; \quad \tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}}\tilde{\mathbf{\Psi}}_{nom} = \mathbf{I}_{n+1}; \\ &\mathbf{E}_{ffj} = \mathbf{I}_{M+1} - \tilde{\mathbf{\Psi}}_{aj,nom}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}} \end{aligned} \tag{17}$$

where $\tilde{\mathbf{\Psi}}_{aj,nom}$ is obtained by filtering $\mathbf{\Psi}_{nom}$ using LSR of the actual plant dynamics $\mathbf{G}_{aj}$.

Proposition 1: For the set of possible actual plant dynamics $\{G_{aj}\}$ and associated weights $\{\lambda_j\}$, j =1, 2, …, l, the metric $J_{e,r}$ can be expressed in terms of the uncertainty and basis functions as

$$J_{e,r}^2 = 1 - \frac{n+1}{M+1} + \sum_{j=1}^{l}\lambda_j \frac{\left\|\left(\mathbf{G}_{aj}-\mathbf{G}_{nom}\right)\mathbf{\Psi}_{nom}\right\|_F^2}{M+1} \tag{18}$$

Proof: This proof first finds the metric J_{ej} and then finds $J_{e,r}$ using Eq. (15). Based on Eqs. (15) and (17),

$$\begin{aligned} \left\|\mathbf{E}_{ffj}\right\|_F^2 &= trace\left(\mathbf{E}_{ffj}^{\mathrm{T}}\mathbf{E}_{ffj}\right) \\ &= trace\begin{pmatrix} (\mathbf{I}_{M+1} - \tilde{\mathbf{\Psi}}_{aj,nom}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}})^{\mathrm{T}} \\ (\mathbf{I}_{M+1} - \tilde{\mathbf{\Psi}}_{aj,nom}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}}) \end{pmatrix} \\ &= trace\begin{pmatrix} \mathbf{I}_{M+1} - \tilde{\mathbf{\Psi}}_{aj,nom}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}} \\ -\tilde{\mathbf{\Psi}}_{nom}\tilde{\mathbf{\Psi}}_{aj,nom}^{\mathrm{T}} \\ +\tilde{\mathbf{\Psi}}_{nom}\tilde{\mathbf{\Psi}}_{aj,nom}^{\mathrm{T}}\tilde{\mathbf{\Psi}}_{aj,nom}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}} \end{pmatrix} \\ &= trace\begin{pmatrix} \mathbf{I}_{M+1} \\ -\left\{\tilde{\mathbf{\Psi}}_{nom} + \tilde{\mathbf{\Psi}}_{aj-nom,nom}\right\}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}} \\ -\tilde{\mathbf{\Psi}}_{nom}\left\{\tilde{\mathbf{\Psi}}_{nom} + \tilde{\mathbf{\Psi}}_{aj-nom,nom}\right\}^{\mathrm{T}} \\ +\tilde{\mathbf{\Psi}}_{nom}\left\{\tilde{\mathbf{\Psi}}_{nom} + \tilde{\mathbf{\Psi}}_{aj-nom,nom}\right\}^{\mathrm{T}} \\ \left\{\tilde{\mathbf{\Psi}}_{nom} + \tilde{\mathbf{\Psi}}_{aj-nom,nom}\right\}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}} \end{pmatrix}; \\ \tilde{\mathbf{\Psi}}_{aj-nom,nom} &\triangleq \tilde{\mathbf{\Psi}}_{aj,nom} - \tilde{\mathbf{\Psi}}_{nom} \end{aligned} \tag{19}$$

Using the fact that *trace* is a linear mapping and is invariant under cyclic permutations

$$\left\|\mathbf{E}_{ffj}\right\|_F^2 = \left(M+1\right) - \left(n+1\right) + \left\|\tilde{\mathbf{\Psi}}_{aj-nom,nom}\right\|_F^2 \tag{20}$$

Substituting Eq. (20) in Eq. (15) gives

$$\begin{aligned} J_{e,r}^2 &= \sum_{j=1}^{l}\lambda_j J_{ej}^2 \\ &= 1 - \frac{n+1}{M+1} + \sum_{j=1}^{l}\lambda_j \frac{\left\|\left(\mathbf{G}_{aj}-\mathbf{G}_{nom}\right)\mathbf{\Psi}_{nom}\right\|_F^2}{M+1} \end{aligned} \tag{21}$$

(End of Proof)

Remark 4: The metric can be expressed as

$$\begin{aligned} J_{e,r}^2 &= J_{e,nom}^2 + J_{e,unc}^2 \\ J_{e,nom}^2 &\triangleq 1 - \frac{n+1}{M+1} \\ J_{e,unc}^2 &\triangleq \sum_{j=1}^{l}\lambda_j \frac{\left\|\left(\mathbf{G}_{aj}-\mathbf{G}_{nom}\right)\mathbf{\Psi}_{nom}\right\|_F^2}{M+1} \end{aligned} \tag{22}$$

The implication is that the metric is the summation of two components – nominal and uncertainty-related. The nominal component is identical to the value of the metric in the absence of the uncertainty (discussed in prior work [14]); it only depends on the number of basis functions and is independent of the plant dynamics and the choice of basis functions. However, the uncertainty-related component depends on the uncertainty, choice of nominal model and the type and number of basis functions.

B. *Optimal Selection of Basis Functions for Robust Tracking Control*

This section finds an optimal set of basis functions that minimize the uncertainty-related component of the metric $J_{e,unc}$, for a given value of the nominal component of the metric $J_{e,nom}$. The procedure for optimal basis function selection is outlined in Proposition 2.

Proposition 2: For the set of possible actual plant dynamics $\{G_{aj}\}$, associated weights $\{\lambda_j\}$, j =1, 2, …, l, and nominal model G_{nom}, the n+1 basis functions $\mathbf{\Psi}_{nom}$ that minimize $J_{e,unc}$ are given by

$$\begin{aligned}
&\mathbf{\Psi}_{nom} = \mathbf{W}_{nom}\mathbf{\Sigma}_{nom}^{-1}\mathbf{V}_{nom}^{\mathrm{T}}\mathbf{W}_{\Delta}\begin{bmatrix}\mathbf{0}_{(M-n)\times(n+1)}\\ \mathbf{I}_{n+1}\end{bmatrix};\\
&\mathbf{G}_{nom} = \mathbf{V}_{nom}\mathbf{\Sigma}_{nom}\mathbf{W}_{nom}^{\mathrm{T}};\\
&\mathbf{\Delta} = \mathbf{V}_{\Delta}\mathbf{\Sigma}_{\Delta}\mathbf{W}_{\Delta}^{\mathrm{T}};\\
&\mathbf{\Delta}^{\mathrm{T}}\mathbf{\Delta}\\
&= \sum_{j=1}^{l}\lambda_j\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)^{\mathrm{T}}\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)
\end{aligned} \tag{23}$$

where $\mathbf{V}_{nom}$, $\mathbf{\Sigma}_{nom}$ and $\mathbf{W}_{nom}$ denote the left singular vector matrix, singular value matrix and right singular vector matrix of $\mathbf{G}_{nom}$, respectively. Similarly, $\mathbf{V}_{\Delta}$, $\mathbf{\Sigma}_{\Delta}$ and $\mathbf{W}_{\Delta}$ denote the left singular vector matrix, singular value matrix and right singular vector matrix of $\mathbf{\Delta}$, respectively.

Proof: Using Eqs. (17) and (22), the optimization problem corresponding to Proposition 2 can be expressed as

$$\begin{aligned}
&\min_{\mathbf{\Psi}_{nom}}\left[J_{e,unc}^2 = \sum_{j=1}^{l}\lambda_j\frac{\left\|\left(\mathbf{G}_{aj}-\mathbf{G}_{nom}\right)\mathbf{\Psi}_{nom}\right\|_F^2}{M+1}\right]\\
&\text{s.t. } \tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}}\tilde{\mathbf{\Psi}}_{nom} = \mathbf{I}_{n+1}
\end{aligned} \tag{24}$$

Note that the constraint is a result of the decoupling process (Eq. (17)). The objective can be expressed in terms of $\tilde{\mathbf{\Psi}}_{nom}$ as follows

$$\begin{aligned}
J_{e,unc}^2 &= \sum_{j=1}^{l}\lambda_j\frac{\left\|\left(\mathbf{G}_{aj}-\mathbf{G}_{nom}\right)\mathbf{G}_{nom}^{-1}\tilde{\mathbf{\Psi}}_{nom}\right\|_F^2}{M+1}\\
&= \sum_{j=1}^{l}\lambda_j\frac{\left\|\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)\tilde{\mathbf{\Psi}}_{nom}\right\|_F^2}{M+1}
\end{aligned} \tag{25}$$

Hence, the optimization problem given by Eq. (24) can be re-written as

$$\begin{aligned}
&\min_{\mathbf{\Psi}_{nom}}\left[J_{e,unc}^2 = \sum_{j=1}^{l}\lambda_j\frac{\left\|\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)\tilde{\mathbf{\Psi}}_{nom}\right\|_F^2}{M+1}\right]\\
&\text{s.t. } \tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}}\tilde{\mathbf{\Psi}}_{nom} = \mathbf{I}_{n+1}
\end{aligned} \tag{26}$$

Using definition of Frobenius norm,

$$\begin{aligned}
&\sum_{j=1}^{l}\lambda_j\left\|\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)\tilde{\mathbf{\Psi}}_{nom}\right\|_F^2\\
&= \sum_{j=1}^{l} trace\begin{pmatrix}\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}}\lambda_j\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)^{\mathrm{T}}\\ \left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)\tilde{\mathbf{\Psi}}_{nom}\end{pmatrix}
\end{aligned} \tag{27}$$

Using the fact that *trace* is a linear mapping

$$\begin{aligned}
&\sum_{j=1}^{l}\lambda_j\left\|\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)\tilde{\mathbf{\Psi}}_{nom}\right\|_F^2\\
&= trace\left(\tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}}\mathbf{\Delta}^{\mathrm{T}}\mathbf{\Delta}\tilde{\mathbf{\Psi}}_{nom}\right) = \left\|\mathbf{\Delta}\tilde{\mathbf{\Psi}}_{nom}\right\|_F^2;\\
&\mathbf{\Delta}^{\mathrm{T}}\mathbf{\Delta}\\
&\triangleq \sum_{j=1}^{l}\lambda_j\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)^{\mathrm{T}}\left(\mathbf{I}_{M+1}-\mathbf{G}_{aj}\mathbf{G}_{nom}^{-1}\right)
\end{aligned} \tag{28}$$

Hence, the optimization problem given by Eq. (26) can be re-written as

$$\begin{aligned}
&\min_{\tilde{\mathbf{\Psi}}_{nom}}\left[J_{e,unc}^2 = \frac{\left\|\mathbf{\Delta}\tilde{\mathbf{\Psi}}_{nom}\right\|_F^2}{M+1}\right]\\
&\text{s.t. } \tilde{\mathbf{\Psi}}_{nom}^{\mathrm{T}}\tilde{\mathbf{\Psi}}_{nom} = \mathbf{I}_{n+1}
\end{aligned} \tag{29}$$

The solution to the optimization problem is the set of right singular vectors of the matrix $\mathbf{\Delta}$, corresponding to its n+1 smallest singular values [23]

$$\tilde{\mathbf{\Psi}}_{nom} = \mathbf{W}_{\Delta}\begin{bmatrix}\mathbf{0}_{(M-n)\times(n+1)}\\ \mathbf{I}_{n+1}\end{bmatrix};\quad \mathbf{\Delta} = \mathbf{V}_{\Delta}\mathbf{\Sigma}_{\Delta}\mathbf{W}_{\Delta}^{\mathrm{T}} \tag{30}$$

and the corresponding basis functions are

$$\begin{aligned}
&\mathbf{\Psi}_{nom} = \mathbf{W}_{nom}\mathbf{\Sigma}_{nom}^{-1}\mathbf{V}_{nom}^{\mathrm{T}}\mathbf{W}_{\Delta}\begin{bmatrix}\mathbf{0}_{(M-n)\times(n+1)}\\ \mathbf{I}_{n+1}\end{bmatrix};\\
&\mathbf{G}_{nom} = \mathbf{V}_{nom}\mathbf{\Sigma}_{nom}\mathbf{W}_{nom}^{\mathrm{T}}
\end{aligned} \tag{31}$$

The optimal value of the metric is given by

$$J_{e,unc}^2 = \frac{\sum_{i=M-n+1}^{M+1}\sigma_{\Delta i}^2}{M+1} \tag{32}$$

where $\{\sigma_{\Delta i}\}$, i = 1, 2, …, M+1, denote the singular values of the matrix $\mathbf{\Delta}$ in the descending order.

(End of Proof)

Remark 5: If the nominal plant dynamics has NMP zeros or more poles than zeros, then the LSR of the nominal plant $\mathbf{G}_{nom}$ has very small singular values, which might not be ideal for inversion of $\mathbf{\Sigma}_{nom}$ and $\mathbf{G}_{nom}$. This undesirable inversion might result in very high control inputs, as discussed in prior work [23]. This issue can be resolved by imposing additional constraints, to avoid the small singular values, in the optimization problem. This situation will be investigated further in future work.

Remark 6: Since, the choice of basis functions only affects the uncertainty-related component and does not affect the nominal component (Remark 4), the proposed optimal basis functions are selected such that robust tracking is realized without significantly affecting the nominal tracking accuracy of FBF (Proposition 2). This is unlike many other robust tracking controllers in the literature [19], [22], whose improved robustness in tracking is achieved at the cost of deterioration in nominal tracking accuracy.

IV. Example

To demonstrate the effectiveness of the proposed optimal set of basis functions, this section uses a damped oscillator with parametric uncertainty:

$$\begin{aligned} &G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}; \\ &\omega_{n,nom} = 200\text{Hz}, \ \zeta_{nom} = 0.01, \\ &\omega_n \in [180, \quad 220]\text{Hz}, \ \zeta \in [0.001, \quad 0.1] \end{aligned} \tag{33}$$

where ω_n and ζ denote the natural frequency and damping ratio, and the subscript '*nom*' denotes the nominal value. The system in Eq. (33) is representative of the dynamics of a vibration-prone 3D printer with uncertainty [11], [22]. The system is sampled at 1 kHz. The set of actual plant dynamics $\{G_{aj}\}$, is generated by selecting l = 410 evenly distributed realizations of the plant defined by Eq. (33) such that $\lambda_j = 1/l$. The nominal values of the plant parameters are used to generate the filter G_{nom}. The desired signal $\mathbf{y}_d$ is a white noise signal with zero mean and unit variance (M = 1000). The choice of white noise for $\mathbf{y}_d$ is because its broadband nature eliminates biased results based on arbitrarily selected $\mathbf{y}_d$. There is a wide choice in terms of the basis functions that can be used with the FBF approach. This work uses discrete cosine transform (DCT) [27], block pulse functions (BPF) [28] and B-splines [29] for comparison with the proposed optimal basis functions (the Appendix gives more details about the basis functions).

Figure 2 shows the normalized RMS tracking error $\mathbf{e}_{RMS}/\mathbf{y}_{d,RMS}$ for DCT, BPF and B-splines and the proposed optimal basis functions (obtained using Proposition 2), for various numbers of basis functions (n = 10 to 990), using the 410 realizations of the actual plant dynamics G, described above. The metrics used for comparison are the mean and standard deviation of $\mathbf{e}_{RMS}/\mathbf{y}_{d,RMS}$. It is observed that for all values of n, the optimal basis functions result in minimum values of mean and standard deviation as compared to DCT, BPF and B-splines. For example, at n = 500, compared to DCT, BPF and B-splines, the optimal basis functions result in improvements in mean and standard deviations of $\mathbf{e}_{RMS}/\mathbf{y}_{d,RMS}$ by up to 1.5 times and 77 times, respectively. The nominal values of $\mathbf{e}_{RMS}/\mathbf{y}_{d,RMS}$ for DCT, BPF, B-spline and optimal basis functions are 0.6686, 0.6708, 0.6662 and 0.6794, respectively. This demonstrates that the significant improvement in mean and standard deviations of $\mathbf{e}_{RMS}/\mathbf{y}_{d,RMS}$ is achieved without significantly affecting nominal tracking accuracy of the FBF approach. It is also observed that the tracking accuracy of the FBF approach, in the presence of uncertainty, does not necessarily improve with increase in the number of basis functions. This is unlike prior work [9], [23], which showed that in the absence of uncertainty, increasing the number of basis functions generally improves the tracking accuracy of the FBF approach.

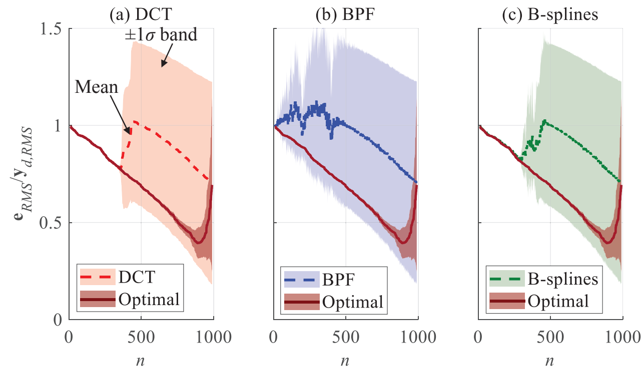

Fig. 2: Comparison of normalized RMS tracking error for optimal basis functions, DCT, BPF and B-splines, in the presence of uncertainty, for various values of number basis functions, n

V. Conclusion

This paper has proposed an optimal set of basis functions for feedforward tracking control of uncertain plants, using the FBF approach. By defining a robustness metric based on the Frobenius norm of the LSR of error dynamics, the effect of the uncertainty, basis functions and nominal plant model (filter) on tracking accuracy is quantified. It is demonstrated that, unlike the nominal tracking case studied in prior work, the tracking accuracy of FBF in the presence of uncertainty depends on the plant dynamics as well as basis functions.

An optimal set of basis functions that minimize the uncertainty-related component, while maintaining the desired level of nominal tracking is selected. The proposed basis functions try to ensure that the deviation between the tracking error due to the uncertainty and the nominal tracking error is minimal. In many applications, this property of the proposed basis functions could ensure that the uncertainty does not affect tracking accuracy significantly, resulting in near consistent tracking even in the presence of uncertainty. For example, Fig. 2 shows that the standard deviation of the normalized tracking error for the proposed optimal basis functions is very small for a larger range of number of basis functions as compared to popular basis functions such as DCT, BPF and B-splines, for the damped oscillator example.

The results of this paper (e.g., Fig. 2) reveal that there is an opportunity to maximize robustness using an optimal number of basis functions along with the proposed optimal set of basis functions. This opportunity will be explored in future work. There is also an opportunity to explore the optimal selection of filter using the Frobenius norm metric. Unlike prior work on optimal filtering [22] which was restricted to LTI systems, the use of the Frobenius norm metric will help in formulation of a robust feedforward controller for other types of linear systems, for example, LPV and LTV systems.

Appendix

The discrete cosine transform (DCT) is a frequency-based transform that is widely used in signal processing; its basis functions are real-valued cosines defined as [22]

$$\varphi_i(k) = \beta_i \cos\left(\frac{\pi(2k+1)i}{2(M+1)}\right); \ \beta_i = \begin{cases} \frac{1}{\sqrt{M+1}} & i = 0 \\ \sqrt{\frac{2}{M+1}} & i > 0 \end{cases} \tag{34}$$

The block pulse function (BPF) basis are given by

$$\varphi_i(k) = \begin{cases} 1 & \begin{matrix} k \in \left[i\frac{M}{n+1}, (i+1)\frac{M}{n+1} \right), 0 \le i < n \\ \& \ k \in \left[i\frac{M}{n+1}, (i+1)\frac{M}{n+1} \right], i = n \end{matrix} \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

The BPF expressed in Eq. (35) seeks to divide the time interval from 0 to M among n+1 basis functions in a quasi-uniform manner.

For a B-spline of degree m, having $n+1 \le M+1$ control points (coefficients), γ_0, γ_1, ..., γ_n, and knot vector $[\eta_0 \ \eta_1 \ ... \ \eta_{m+n+1}]^T$, its real-valued basis functions, $\varphi_{i,m}$, are given by [29]

$$\begin{aligned} \varphi_i(k) &:= \varphi_{i,m}(\xi_k) \\ &= \frac{\xi_k - \eta_i}{\eta_{i+m} - \eta_i}\varphi_{i,m-1}(\xi) + \frac{\eta_{i+m+1} - \xi_k}{\eta_{i+m+1} - \eta_{i+1}}\varphi_{i+1,m-1}(\xi) \\ \varphi_{i,0}(\xi_k) &= \begin{cases} 1 & \eta_i \le \xi_k \le \eta_{i+1} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{36}$$

where i = 0, 1, ..., n with $\xi_k \in [0,1]$, representing normalized time, discretized into M+1 points, ξ_0, ξ_1 …ξ_M, and η_j is a uniform knot vector, selected such that

$$\eta_j = \begin{cases} 0 & 0 \le j \le m \\ \frac{j-m}{n-m+1} & m+1 \le j \le n \\ 1 & n+1 \le j \le m+n+1 \end{cases} \tag{37}$$

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