

Resilient Sensor Placement for Kalman Filtering in Networked Systems: Complexity and Algorithms

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Abstract—Given a linear dynamical system affected by noise, we study the problem of optimally placing sensors (at design-time) subject to a sensor placement budget constraint in order to minimize the trace of the steady-state error covariance of the corresponding Kalman filter. While this problem is NP-hard in general, we consider the underlying graph associated with the system dynamics matrix, and focus on the case when there is a single input at one of the nodes in the graph. We provide an optimal strategy (computed in polynomial-time) to place the sensors over the network. Next, we consider the problem of attacking (i.e., removing) the placed sensors under a sensor attack budget constraint in order to maximize the trace of the steady-state error covariance of the resulting Kalman filter. Using the insights obtained for the sensor placement problem, we provide an optimal strategy (computed in polynomial-time) to attack the placed sensors. Finally, we consider the scenario where a system designer places the sensors under a sensor placement budget constraint, and an adversary then attacks the placed sensors subject to a sensor attack budget constraint. The resilient sensor placement problem is to find a sensor placement strategy to minimize the trace of the steady-state error covariance of the Kalman filter corresponding to the sensors that survive the attack. We show that this problem is NP-hard, and provide a pseudo-polynomial-time algorithm to solve it.

I. INTRODUCTION

In large-scale control system design, one of the key problems is to place sensors or actuators on the system in order to achieve certain performance criteria (e.g., [1], [2]). In cases involving linear systems with process or measurement noise, researchers have studied how to place sensors (at design-time) in order to minimize certain metrics of the error covariance of the corresponding Kalman filter (e.g., [3]–[7]). The problem has been shown to be NP-hard and inapproximable within any constant factor in general [8]. This motivates us to consider special classes of this problem in this paper and seek polynomial-time algorithms for the optimal sensor placement problem. Specifically, we consider a discrete-time linear dynamical system whose states represent nodes in a directed network, and interact according to the topology of the network. The nodes of the network are possibly affected by stochastic inputs. Such networked systems with stochastic inputs have received much attention from researchers recently (e.g., [9]–[12]). These models encompass diffusion networks (e.g., [13]) that arise in many different areas, including information and influence diffusion over social networks [14], spreading of diseases in populations [15], and diffusion of chemicals in

certain environments [16]. In such applications, estimating the states of the entire network is an important objective.

In this paper, we focus on the case when there is a single node of the network that has a stochastic input. Specifically, we consider the scenario where a system designer can spend a limited budget on placing sensors (at design-time) over the network in order to minimize the trace of the steady-state error covariance of the Kalman filter corresponding to the placed sensors. A sensor placed at a certain node gives measurements of the state corresponding to the node. In addition, placing a sensor at a node incurs a placement cost (which could vary across the nodes). We refer to this problem as the *Graph-based Kalman Filtering Sensor Placement (GKFSP)* problem.

Additionally, the systems that we are interested in monitoring may be targeted by adversaries, where an adversary can attack a subset of placed sensors. Different types of attacks have been studied previously, including Denial of Service (DoS) attacks (e.g., [17], [18]) and false data injection attacks (e.g., [19], [20]). Here, we consider adversaries that perform DoS attacks on sensors by simply removing them (or equivalently, dropping all the measurement data). The goal of the adversary is to remove a subset of placed sensors under a budget constraint in order to maximize the trace of the steady-state error covariance of the Kalman filter corresponding to the surviving sensors. We assume that attacking a sensor placed at a node incurs an attack cost (which could also vary across the nodes). In contrast with existing work in the literature, we analyze the problem using the graph structure of the systems. We refer to this problem as the *Graph-based Kalman Filtering Sensor Attack (GKFSA)* problem.

Finally, combining the two problems that we considered above, we formulate and study a resilient sensor placement problem for the networked system. We assume that the system designer is aware of the potential attack from an adversary who chooses to optimally attack the sensors (subject to an attack budget constraint) deployed by the system designer. The system designer's goal is to place sensors (under a placement budget constraint) among a subset of nodes in order to minimize the trace of the steady-state error covariance of the Kalman filter corresponding to the surviving sensors after the attack. We refer to this problem as the *Resilient Graph-based Kalman Filtering Sensor Placement (RGKFSP)* problem.

Contributions

First, we provide an optimal sensor placement strategy for the GKFSP problem using the graph structure of the system. Second, leveraging the insights for the GKFSP problem, we give an optimal sensor attack strategy for the GKFSA problem. Third, we show that the RGKFSP problem is NP-hard; we then provide an algorithm based on dynamic programming

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that can return an optimal solution to general instances of the RGKFSP problem in pseudo-polynomial time. Although the results are derived under the assumption that the sensors give perfect measurements, we show that how to apply these results to analyze the case with sensor measurement noise and provide numerical examples. A preliminary version of the results in this paper was presented in [21], where only the GKFSP problem was studied for a more restrictive class of system dynamics matrices.

Related work

The (design-time) sensor placement problem has been widely studied in the literature. For example, in [6], [22], the authors considered the Kalman filtering sensor placement problem over a finite number of time steps. Here, we study the problem of optimizing steady-state error covariances of the corresponding Kalman filter. In [7], [8], the authors considered the same sensor placement problem as the one considered here, but for general system dynamics. In such cases, they showed that finding the optimal placement for the general problem is NP-hard. Thus, in this paper, we impose additional structure on the problem (by considering the graph representation of the dynamics) in order to seek optimal solutions. In [23], [24], the authors studied the sensor placement problem for estimating a static variable (parameter) that does not change over time. Here, we study the problem of placing sensors to estimate the states of a linear dynamical system affected by stochastic inputs. In contrast to the sensor placement problem where the set of placed sensors cannot change over time, the sensor scheduling problem for Kalman filtering has also received much attention (e.g., [25]–[28]), where different sets of sensors can be chosen at different time steps.

In networked system settings, the authors in [29] considered the sensor placement problem for continuous-time diffusion dynamics, and applied the Wiener filter to estimate the system states using sensor measurements. Here, we consider discrete-time networked system dynamics and apply the Kalman filter to estimate the system states. The authors in [10]–[12] studied the leader selection problem in consensus networks with stochastic inputs. The problem is to select a subset of nodes whose states are fixed over time in order to minimize the H_2 norm of the system states at steady state. In contrast, we consider the problem of placing sensors among the nodes of systems with more general dynamics in order to minimize the trace of the steady-state error covariance of the Kalman filter.

Although both of the sensor placement and the sensor attack problems have received much attention from researchers, the resilient sensor placement is less explored. The authors in [30] considered the problem of resilient maximization of monotone submodular set functions under a cardinality constraint on the sets. They proposed a polynomial-time approximation algorithm for the problem with performance bounds that depend on the curvature of the objective function. In [31], the authors considered a resilient observation selection problem. The problem is to resiliently select observations of a scalar Gaussian process given that some of the selected observations could be removed by an adversary. The authors showed that this problem is NP-hard and proposed a greedy algorithm

with a provable performance guarantee. Here, we consider the resilient sensor placement problem for Kalman filtering of (vector) linear dynamical systems subject to general knapsack constraints. While we show this problem is NP-hard, we give an algorithm based on dynamic programming to solve the problem optimally in pseudo-polynomial time [32].

Notation and terminology

The sets of integers and real numbers are denoted as $\mathbb{Z}$ and $\mathbb{R}$, respectively. For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x . For a matrix $P \in \mathbb{R}^{n \times n}$, let P^T denote its transpose, P_{ij} (or $(P)_{ij}$) denote the element in the i th row and j th column of P , and P_i denote the i th row of P . Let $\mathbf{0}_{m \times n}$ denote a zero matrix; the subscript is dropped if the dimension of the matrix is clear from the context. The identity matrix of dimension n is denoted as I_n . A positive semi-definite matrix P is denoted by $P \succeq \mathbf{0}$ and $P \succeq Q$ if $P - Q \succeq \mathbf{0}$. The set of n by n positive definite (resp., positive semi-definite) matrices is denoted by $\mathbb{S}_{++}^n$ (resp., $\mathbb{S}_+^n$). For a vector x , denote its i th element as x_i , and let $\text{supp}(x)$ be its support, where $\text{supp}(x) = \{i : x_i \neq 0\}$. Define $\mathbf{e}_i$ to be a column vector where the i th element is 1 and all the other elements are zero; the dimension of the vector can be inferred from the context. We use $\mathbb{E}[x]$ to denote the expectation of a random variable (vector) x . For a set $\mathcal{A}$, let $|\mathcal{A}|$ be its cardinality. Given two functions $\varphi_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ and $\varphi_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, $\varphi_1(n)$ is $O(\varphi_2(n))$ if there exist positive constants c and N such that $|\varphi_1(n)| \leq c|\varphi_2(n)|$ for all $n \geq N$.

II. PROBLEM FORMULATION

We begin with the following definitions from graph theory. Further details can be found in, for example, [33] and [34].

Definition 1: For any given matrix $A \in \mathbb{R}^{n \times n}$, the directed graph of A , denoted as $\mathcal{G}(A)$, is defined as the directed graph on n vertices (or nodes) $x_1, x_2, \dots, x_n$ such that for all $i, j \in \{1, 2, \dots, n\}$, there is a directed edge in $\mathcal{G}(A)$ from x_j to x_i , denoted as (x_j, x_i) , if and only if $A_{ij} \neq 0$. Denoting the set of vertices and the set of edges of $\mathcal{G}(A)$ as $\mathcal{X}(A) \triangleq \{x_1, x_2, \dots, x_n\}$ and $\mathcal{E}(A)$, respectively, the graph $\mathcal{G}(A)$ is also denoted as $\mathcal{G}(A) = \{\mathcal{X}(A), \mathcal{E}(A)\}$.

Definition 2: Consider a directed graph $\mathcal{G} = \{\mathcal{X}, \mathcal{E}\}$, where $\mathcal{X} \triangleq \{x_1, x_2, \dots, x_n\}$. A directed path from x_{i_0} to x_{i_t} is a sequence of directed edges $(x_{i_0}, x_{i_1}), (x_{i_1}, x_{i_2}), \dots, (x_{i_{t-1}}, x_{i_t})$ in $\mathcal{G}$. The ordered list of vertices in the directed path is $x_{i_0}, x_{i_1}, \dots, x_{i_t}$. The length of a directed path is the number of directed edges in the directed path. A cycle is a directed path that begins and ends at the same vertex which occurs exactly twice in the ordered list of vertices in the directed path, and no other vertices occur more than once in the list. A cycle of length 1 is a self-loop at the corresponding vertex.

Definition 3: Consider a directed graph $\mathcal{G} = \{\mathcal{X}, \mathcal{E}\}$. For any pair of distinct vertices $x_i, x_j \in \mathcal{X}$ such that there exists a directed path from x_i to x_j , the distance from x_i to x_j , denoted as l_{ij} , is defined as the shortest length over all such paths. Define $l_{mm} = 0$ for all $x_m \in \mathcal{X}$.

Definition 4: A directed graph $\mathcal{G} = \{\mathcal{X}, \mathcal{E}\}$ is strongly connected if for all pairs of distinct vertices $x_i, x_j \in \mathcal{X}$, there is a directed path from x_j to x_i in $\mathcal{G}$.

We start with a general system model. Consider a matrix $A \in \mathbb{R}^{n \times n}$ with the associated graph $\mathcal{G}(A) = \{\mathcal{X}(A), \mathcal{E}(A)\}$ (given in Definition 1). Suppose that $\mathcal{I} \triangleq \{x_{i_0}, x_{i_1}, \dots, x_{i_{n_1-1}}\} \subseteq \mathcal{X}(A)$ is the set of nodes that have stochastic inputs, where $n_1 \in \mathbb{Z}_{\geq 1}$. We then consider the following discrete-time linear system:

$$x[k+1] = Ax[k] + Bw[k], \quad (1)$$

where $x[k] \in \mathbb{R}^n$ is the system state at time step k , and $B \triangleq [e_{i_0} \dots e_{i_{n_1-1}}] \in \mathbb{R}^{n \times n_1}$ is the input matrix. The stochastic input $w[k] \in \mathbb{R}^{n_1}$ is a zero-mean white noise process with $\mathbb{E}[w[k](w[k])^T] = W \in \mathbb{S}_+^{n_1}$. The initial state $x[0]$ is a random vector with mean $\bar{x}_0 \in \mathbb{R}^n$ and covariance $\Pi_0 \in \mathbb{S}_+^n$, and is assumed to be independent of $w[k]$ for all $k \in \mathbb{Z}_{\geq 0}$. Each state of the system, denoted as $x_i[k]$, is associated with node x_i in $\mathcal{G}(A)$. As we mentioned in the introduction, [8] showed that the Kalman filtering sensor placement problem cannot be approximated within any constant factor in polynomial time (if $P \neq NP$) for general system dynamics matrices *even* when the measurement noise is zero. Moreover, under the networked system setting, [21] showed that if there are multiple input nodes in the graph, the Kalman filtering sensor placement problem becomes NP-hard *even* when the graph only contains a set of disjoint paths of length three and each path has a single input node. Hence, in order to bypass these inherent complexity issues, we focus on networked systems with a single input node $x_{i_0} \in \mathcal{X}(A)$ (i.e., $B = e_{i_0}$ and $\mathbb{E}[(w[k])^2] = \sigma_w^2 \in \mathbb{R}_{\geq 0}$), and seek efficient algorithms to optimally solve the corresponding sensor placement, sensor attack, and resilient sensor placement problems. We assume throughout this paper that the pair $(A, B\sigma_w)$ is stabilizable. The generality of this assumption will be justified later.

A. The Sensor Placement Problem

First, suppose that there is a system designer who can choose a subset of the vertices of the graph $\mathcal{G}(A)$ at which to place sensors under a budget constraint. Specifically, a sensor placed at node $x_i \in \mathcal{X}(A)$ has a placement cost $h_i \in \mathbb{Z}_{\geq 0}$; define the sensor placement cost vector as $h \triangleq [h_1 \dots h_n]^T$. The designer has a sensor placement budget $H \in \mathbb{Z}_{\geq 0}$ that can be spent on placing sensors at the nodes of $\mathcal{G}(A)$. A sensor that is placed at node $x_i \in \mathcal{X}(A)$ gives a measurement

$$y_i[k] = C_i x[k] + v_i[k], \quad (2)$$

where $C_i = e_i^T$ and $v_i[k] \in \mathbb{R}$ is a zero-mean white noise process. We further define $y[k] \triangleq [y_1[k] \dots y_n[k]]^T$, $C \triangleq [C_1^T \dots C_n^T]^T$ and $v[k] \triangleq [v_1[k] \dots v_n[k]]^T$. Thus, the output provided by all sensors together is given by

$$y[k] = Cx[k] + v[k], \quad (3)$$

where $C = I_n$. We denote $\mathbb{E}[v[k](v[k])^T] = V \in \mathbb{S}_+^n$ and consider $\mathbb{E}[v[k](w[j])^T] = 0$, $\forall k, j \in \mathbb{Z}_{\geq 0}$. The initial state $x[0]$ is also assumed to be independent of $v[k]$ for all $k \in \mathbb{Z}_{\geq 0}$.

After the sensors are placed, the Kalman filter is then applied to provide an estimate of the states using the measurements from the installed sensors. We define a vector $\mu \in \{0, 1\}^n$ as the indicator vector indicating the vertices

where sensors are placed. Specifically, $\mu_i = 1$ if and only if a sensor is placed at node $x_i \in \mathcal{X}(A)$. Denote $C(\mu)$ as the measurement matrix of the installed sensors indicated by μ , i.e., $C(\mu) \triangleq [C_{i_1}^T \dots C_{i_p}^T]^T$, where $\text{supp}(\mu) = \{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$. Similarly, denote $V(\mu)$ as the measurement noise covariance matrix of the installed sensors, i.e., $V(\mu) = \mathbb{E}[\tilde{v}[k](\tilde{v}[k])^T]$, where $\tilde{v}[k] \triangleq [(v[k])_{i_1} \dots (v[k])_{i_p}]^T$. The *a priori* and the *a posteriori* error covariance matrices of the Kalman filter at time step k , when the sensors indicated by μ are placed, are denoted as $\Sigma_{k/k-1}(\mu)$ and $\Sigma_{k/k}(\mu)$, respectively. The initial *a priori* error covariance is set as $\Sigma_{0/-1}(\mu) = \Pi_0$. The limit $\Sigma(\mu) \triangleq \lim_{k \rightarrow \infty} \Sigma_{k+1/k}$ (also known as the steady-state *a priori* error covariance), if it exists, satisfies the *discrete algebraic Riccati equation (DARE)* [35]:

$$\begin{aligned} \Sigma(\mu) &= A\Sigma(\mu)A^T + \sigma_w^2 BB^T - \\ &A\Sigma(\mu)C(\mu)^T (C(\mu)\Sigma(\mu)C(\mu)^T + V(\mu))^{-1} C(\mu)\Sigma(\mu)A^T, \end{aligned} \quad (4)$$

where $\sigma_w^2 \in \mathbb{R}_{\geq 0}$ and $B = e_{i_0}$. The limit $\Sigma^*(\mu) \triangleq \lim_{k \rightarrow \infty} \Sigma_{k/k}(\mu)$ (also known as the steady-state *a posteriori* error covariance), if it exists, satisfies the following equations [36]:

$$\begin{aligned} \Sigma^*(\mu) &= \Sigma(\mu) - \\ &\Sigma(\mu)C(\mu)^T (C(\mu)\Sigma(\mu)C(\mu)^T + V(\mu))^{-1} C(\mu)\Sigma(\mu), \end{aligned} \quad (5)$$

and

$$\Sigma(\mu) = A\Sigma^*(\mu)A^T + \sigma_w^2 BB^T. \quad (6)$$

The inverses in Eq. (4) and (5) are interpreted as the Moore-Penrose pseudo-inverses (which we denote using the notation “ $\dagger$ ”) if the arguments are not invertible [35]. We will use the following result from [35].

Lemma 1: For a given indicator vector μ , $\Sigma_{k/k-1}(\mu)$ (resp., $\Sigma_{k/k}(\mu)$) will converge, as $k \rightarrow \infty$, to a finite limit $\Sigma(\mu)$ (resp., $\Sigma^*(\mu)$), regardless of the initial covariance $\Sigma_{0/-1}(\mu)$, if and only if the pair $(A, C(\mu))$ is detectable and the pair $(A, B\sigma_w)$ is stabilizable. Furthermore, if the limit $\Sigma(\mu)$ (resp., $\Sigma^*(\mu)$) exists, it is also the only positive semi-definite solution to Eq. (4) (resp., Eq. (5)).

When the pair $(A, C(\mu))$ is not detectable, we define the limits $\Sigma(\mu) = +\infty$ and $\Sigma^*(\mu) = +\infty$. The priori and posteriori Graph-based Kalman Filtering Sensor Placement (GKFSP) problems are defined as follows.

Problem 1: (Priori and Posteriori GKFSP) Consider a system dynamics matrix $A \in \mathbb{R}^{n \times n}$ with the associated graph $\mathcal{G}(A) = \{\mathcal{X}(A), \mathcal{E}(A)\}$, a single vertex $x_{i_0} \in \mathcal{X}(A)$ that has a stochastic input with variance $\sigma_w^2 \in \mathbb{R}_{\geq 0}$, the measurement matrix $C = I_n$ (containing all of the individual sensor measurement matrices), a sensor noise covariance matrix $V \in \mathbb{S}_+^n$, a sensor placement cost vector $h \in \mathbb{Z}_{\geq 0}^n$ and a sensor placement budget $H \in \mathbb{Z}_{\geq 0}$. The priori Graph-based Kalman Filtering Sensor Placement (GKFSP) problem is to find the sensor placement μ , i.e., the indicator vector μ of the vertices where sensors are placed, that solves

$$\begin{aligned} \min_{\mu \in \{0,1\}^n} \quad & \text{trace}(\Sigma(\mu)) \\ \text{s.t.} \quad & h^T \mu \leq H, \end{aligned}$$

where $\Sigma(\mu)$ is given by Eq. (4) if the pair $(A, C(\mu))$ is detectable, and $\Sigma(\mu) = +\infty$ otherwise. The posteriori GKFSP Problem is to find the sensor placement μ that solves

$$\begin{aligned} \min_{\mu \in \{0,1\}^n} \quad & \text{trace}(\Sigma^*(\mu)) \\ \text{s.t.} \quad & h^T \mu \leq H, \end{aligned}$$

where $\Sigma^*(\mu)$ is given by Eq. (5) if the pair $(A, C(\mu))$ is detectable, and $\Sigma^*(\mu) = +\infty$ otherwise.

B. The Sensor Attack Problem

Suppose that the sensors indicated by the sensor placement $\mu \in \{0,1\}^n$ are placed and installed by the system designer, and there is an adversary who aims to attack (i.e., remove) a subset of the installed sensors. To attack a sensor placed at node $x_i \in \mathcal{X}(A)$, the adversary needs to pay a cost $f_i \in \mathbb{Z}_{\geq 0}$. Define the sensor attack cost vector as $f \triangleq [f_1 \ \cdots \ f_n]^T$. The adversary has a total sensor attack budget $F \in \mathbb{Z}_{\geq 0}$ for attacking the installed sensors. We define a vector $\nu \in \{0,1\}^n$ as the indicator vector indicating the subset of sensors that are attacked, where $\nu_i = 1$ if and only if the sensor at $x_i \in \mathcal{X}(A)$ is attacked. Note that $\text{supp}(\nu) \subseteq \text{supp}(\mu)$ is always assumed implicitly in the sequel. Denote the matrix $C(\mu \setminus \nu)$ as the measurement matrix of the surviving sensors corresponding to μ and ν , i.e., $C(\mu \setminus \nu) \triangleq [C_{j_1}^T \ \cdots \ C_{j_q}^T]^T$, where $\{j_1, \dots, j_q\} = \text{supp}(\mu) \setminus \text{supp}(\nu)$. Denote $\text{supp}(\mu) \setminus \text{supp}(\nu) \triangleq \text{supp}(\mu \setminus \nu)$. Similarly, define $V(\mu \setminus \nu)$ as the measurement noise covariance of the surviving sensors. The Kalman filter is then applied based on the measurements of the surviving sensors. The resulting *a priori* and *a posteriori* error covariances of the Kalman filter at time step k are denoted as $\Sigma_{k/k-1}(\mu \setminus \nu)$ and $\Sigma_{k/k}(\mu \setminus \nu)$, respectively, whose limits as $k \rightarrow \infty$ are denoted as $\Sigma(\mu \setminus \nu)$ and $\Sigma^*(\mu \setminus \nu)$, respectively.

The priori and posteriori Graph-based Kalman Filtering Sensor Attack (GKFSA) problems are then defined as follows.

Problem 2: (Priori and Posteriori GKFSA) Consider a system dynamics matrix $A \in \mathbb{R}^{n \times n}$ with the associated graph $\mathcal{G}(A) = \{\mathcal{X}(A), \mathcal{E}(A)\}$, a single vertex $x_{i_0} \in \mathcal{X}(A)$ that has a stochastic input with variance $\sigma_w^2 \in \mathbb{R}_{\geq 0}$, the measurement matrix $C = I_n$ (containing all of the individual sensor measurement matrices), a sensor noise covariance matrix $V \in \mathbb{S}_{+}^n$, a sensor attack cost vector $f \in \mathbb{Z}_{\geq 0}^n$, a sensor attack budget $F \in \mathbb{Z}_{\geq 0}$, and a sensor placement vector $\mu \in \{0,1\}^n$. The priori Graph-based Kalman Filtering Sensor Attack (GKFSA) problem is to find the sensor attack ν , i.e., the indicator vector ν of the vertices where the installed sensors (indicated by μ) are attacked, that solves

$$\begin{aligned} \max_{\nu \in \{0,1\}^n} \quad & \text{trace}(\Sigma(\mu \setminus \nu)) \\ \text{s.t.} \quad & f^T \nu \leq F, \end{aligned}$$

where $\Sigma(\mu \setminus \nu)$ is given by Eq. (4) if the pair $(A, C(\mu \setminus \nu))$ is detectable, and $\Sigma(\mu \setminus \nu) = +\infty$ otherwise. The posteriori GKFSA problem is to find the sensor attack ν that solves

$$\begin{aligned} \max_{\nu \in \{0,1\}^n} \quad & \text{trace}(\Sigma^*(\mu \setminus \nu)) \\ \text{s.t.} \quad & f^T \nu \leq F, \end{aligned}$$

where $\Sigma^*(\mu \setminus \nu)$ is given by Eq. (5) if the pair $(A, C(\mu \setminus \nu))$ is detectable, and $\Sigma^*(\mu \setminus \nu) = +\infty$ otherwise.

C. The Resilient Sensor Placement Problem

We next consider the scenario where the system designer is aware of the potential attack from a strategic adversary (who can perform optimal sensor attacks under budget constraints), and aims to choose a resilient sensor placement. We first define feasible sensor placements for the system designer as follows.

Definition 5: A sensor placement $\mu \in \{0,1\}^n$ is said to be feasible if $h^T \mu \leq H$ (i.e., the sensor placement budget constraint is satisfied), and for all $\nu \in \{0,1\}^n$ such that $f^T \nu \leq F$, $\text{supp}(\mu \setminus \nu) \neq \emptyset$ (i.e., for all sensor attacks that satisfy the sensor attack budget constraint, at least one sensor indicated by μ is left over by the adversary).

Remark 1: Note that if a sensor placement μ is not feasible, there is an attack (satisfying the attacker's budget constraint) such that the pair $(A, C(\mu \setminus \nu))$ is not detectable if the system dynamics matrix A is not stable.

The priori and posteriori Resilient Graph-based Kalman Filtering Sensor Placement (RGKFSP) problems are then given by the following.

Problem 3: (Priori and Posteriori RGKFSP) Consider a system dynamics matrix $A \in \mathbb{R}^{n \times n}$ with the associated graph $\mathcal{G}(A) = \{\mathcal{X}(A), \mathcal{E}(A)\}$, a single vertex $x_{i_0} \in \mathcal{X}(A)$ that has a stochastic input with variance $\sigma_w^2 \in \mathbb{R}_{\geq 0}$, the measurement matrix $C = I_n$ (containing all of the individual sensor measurement matrices), a sensor noise covariance matrix $V \in \mathbb{S}_{+}^n$, a sensor placement cost vector $h \in \mathbb{Z}_{\geq 0}^n$, a sensor placement budget $H \in \mathbb{Z}_{\geq 0}$, a sensor attack cost vector $f \in \mathbb{Z}_{\geq 0}^n$, and a sensor attack budget $F \in \mathbb{Z}_{\geq 0}$. The priori Resilient Graph-based Kalman Filtering Sensor Placement (RGKFSP) problem is to find the sensor placement μ that solves

$$\begin{aligned} \min_{\mu \in \{0,1\}^n} \quad & \max_{\nu \in \{0,1\}^n} \text{trace}(\Sigma(\mu \setminus \nu)) \\ \text{s.t.} \quad & h^T \mu \leq H, \text{ and } f^T \nu \leq F, \end{aligned}$$

where $\Sigma(\mu \setminus \nu)$ is given by Eq. (4) if the pair $(A, C(\mu \setminus \nu))$ is detectable, and $\Sigma(\mu \setminus \nu) = +\infty$ otherwise. The posteriori RGKFSP problem is to find the sensor placement μ that solves

$$\begin{aligned} \min_{\mu \in \{0,1\}^n} \quad & \max_{\nu \in \{0,1\}^n} \text{trace}(\Sigma^*(\mu \setminus \nu)) \\ \text{s.t.} \quad & h^T \mu \leq H, \text{ and } f^T \nu \leq F, \end{aligned}$$

where $\Sigma^*(\mu \setminus \nu)$ is given by Eq. (5) if the pair $(A, C(\mu \setminus \nu))$ is detectable, and $\Sigma^*(\mu \setminus \nu) = +\infty$ otherwise.

III. SOLVING THE GKFSP AND GKFSA PROBLEMS

In this section, we provide algorithms to optimally solve the GKFSP and GKFSA problems, respectively, when the sensor noise covariance is $V = \mathbf{0}_{n \times n}$. We will make the following assumptions on the instances of the GKFSP and GKFSA problems in the sequel.

Assumption 1: The pair $(A, B\sigma_w)$ is assumed to be stabilizable. The pair $(A, C(\mu))$ is assumed to be detectable for all sensor placements $\mu \in \{0,1\}^n$ with $\text{supp}(\mu) \neq \emptyset$.

Assumption 2: The graph $\mathcal{G}(A) = \{\mathcal{X}(A), \mathcal{E}(A)\}$ (associated with the system dynamics matrix $A \in \mathbb{R}^{n \times n}$) is assumed

to satisfy the property that for all $x_j \in \mathcal{X}(A)$ and $x_j \neq x_{i_0}$, there exists a directed path from x_{i_0} to x_j . The system dynamics matrix A is assumed to satisfy $(A^m)_{ji_0} \neq 0$ if $l_{i_0j} = m$, where l_{i_0j} is the distance from x_{i_0} to x_j .

Remark 2: Note that Assumptions 1-2 are satisfied by large classes of systems. For example, it was shown in [37] that Assumption 1 holds if the system dynamics matrix A is row-stochastic and irreducible.¹ Assumption 2 holds if the system dynamics matrix A is nonnegative and irreducible [34]. Since any row-stochastic matrix is also nonnegative, Assumptions 1-2 hold for any system dynamics matrix A that is row-stochastic and irreducible. Furthermore, using techniques in control theory pertaining to linear structured systems (e.g., [38], [39]), one can show that Assumption 1 holds for almost any system dynamics matrix A such that the graph $\mathcal{G}(A)$ is strongly connected, using approaches from [40], [41]. Specifically, one can consider the system dynamics matrix A to be structured, i.e., each entry of the system dynamics matrix A is either a fixed zero or an independent free parameter (which can attain any real value including zero), where the graph $\mathcal{G}(A)$ is defined according to the free parameters of the structured matrix A . One can then show that the set of parameters for which Assumption 1 does not hold has Lebesgue measure zero. Moreover, using similar techniques to those above and the result from [34] that shows that Assumption 2 holds for all nonnegative irreducible matrices A , one can show that Assumption 2 holds for almost any choice of free parameters in the structured matrix A such that the graph $\mathcal{G}(A)$ is strongly connected. Note that the systems where Assumptions 1-2 hold are not limited to the cases described above.

Remark 3: We can generalize our analysis to system dynamics matrices A where $\mathcal{G}(A)$ has multiple strongly connected components [33]. Suppose that the input node can only reach (via directed paths in $\mathcal{G}(A)$) nodes that are in the same strongly connected component. Then, under Assumption 1, we only need to consider the strongly connected component of $\mathcal{G}(A)$ that contains the input node, since one can show that the mean square estimation error of the Kalman filter remains zero for the states corresponding to nodes that are not in the strongly connected component containing the input node.

The first main result of this section is as follows.

Theorem 1: Consider a system dynamics matrix $A \in \mathbb{R}^{n \times n}$ with the associated graph $\mathcal{G}(A) = \{\mathcal{X}(A), \mathcal{E}(A)\}$, a single vertex $x_{i_0} \in \mathcal{X}(A)$ that has a stochastic input with variance $\sigma_w^2 \in \mathbb{R}_{\geq 0}$, the measurement matrix $C = I_n$ (containing all of the individual sensor measurement matrices), and the sensor noise covariance matrix $V = \mathbf{0}_{n \times n}$. Suppose that Assumptions 1-2 hold. For any sensor placement $\mu \in \{0, 1\}^n$ such that $\text{supp}(\mu) \neq \emptyset$, denote $\zeta = \min_{j \in \text{supp}(\mu)} l_{i_0j} \geq 0$, where l_{i_0j} is the distance from vertex x_{i_0} to vertex x_j . The following expressions hold:

$$\Sigma(\mu) = \sigma_w^2 \sum_{m=0}^{\zeta} A^m B B^T (A^T)^m, \quad (7)$$

¹Note that the matrix A is irreducible if and only if the graph $\mathcal{G}(A)$ is strongly connected [34].

and

$$\Sigma^*(\mu) = \begin{cases} \sigma_w^2 \sum_{m=0}^{\zeta-1} A^m B B^T (A^T)^m & \text{if } \zeta \geq 1, \\ \mathbf{0} & \text{if } \zeta = 0, \end{cases} \quad (8)$$

where $\Sigma(\mu)$ (resp., $\Sigma^*(\mu)$) is the steady-state *a priori* (resp., *a posteriori*) error covariance of the corresponding Kalman filter, and $B = \mathbf{e}_{i_0}$.

Proof: The existence of $\Sigma(\mu)$ and $\Sigma^*(\mu)$ follows directly from Lemma 1 and Assumption 1. Considering any sensor placement μ such that $\zeta \geq 1$, i.e., sensors are not placed at the input vertex x_{i_0} , we first prove Eq. (7) by verifying that Eq. (7) satisfies Eq. (4). Note that $C_i = \mathbf{e}_i^T$ for all $x_i \in \mathcal{X}(A)$. Denote $\mathcal{X}_\mu \subseteq \mathcal{X}(A)$ as the set of vertices indicated by μ where sensors are placed and $\mathcal{X}_\zeta \subseteq \mathcal{X}(A)$ as the set of vertices that have distance ζ from the input vertex x_{i_0} . Since performing elementary row operations on $C(\mu)$ does not change $\Sigma(\mu)$, we assume without loss of generality that $\mu = [\mu_1^T \ \mu_2^T]^T$ such that $\mu_1 = \mathbf{1}_{|\mathcal{X}_\zeta \cap \mathcal{X}_\mu|}$ and $\mu_2 \in \{0, 1\}^{n-|\mathcal{X}_\zeta \cap \mathcal{X}_\mu|}$. In other words, μ_1 contains all sensors placed at vertices that have distance ζ from the input vertex x_{i_0} , and $l_{i_0j} > \zeta$ for all $j \in \text{supp}(\mu_2)$. The corresponding measurement matrix is given by $C(\mu) = \begin{bmatrix} C(\mu_1) \\ C(\mu_2) \end{bmatrix}$, where $C(\mu_1) \in \mathbb{R}^{|\text{supp}(\mu_1)| \times n}$ and $C(\mu_2) \in \mathbb{R}^{|\text{supp}(\mu_2)| \times n}$. Substituting Eq. (7) into the right hand side (RHS) of Eq. (4), we obtain:

RHS of Eq. (4)

$$\begin{aligned} &= \sigma_w^2 \sum_{m=1}^{\zeta+1} A^m B B^T (A^T)^m + \sigma_w^2 B B^T - \sigma_w^2 A^{\zeta+1} B B^T (A^T)^\zeta \\ &\quad \times (C(\mu))^T (C(\mu) A^\zeta B B^T (A^T)^\zeta (C(\mu))^T)^\dagger \\ &\quad \times C(\mu) A^\zeta B B^T (A^T)^{\zeta+1} \quad (9) \end{aligned}$$

$$\begin{aligned} &= \sigma_w^2 \sum_{m=0}^{\zeta+1} A^m B B^T (A^T)^m - \sigma_w^2 A^{\zeta+1} B B^T (A^T)^\zeta \\ &\quad \times [(C(\mu_1))^T \ (C(\mu_2))^T] \left(\begin{bmatrix} C(\mu_1) \\ C(\mu_2) \end{bmatrix} A^\zeta B B^T (A^T)^\zeta \right. \\ &\quad \times [(C(\mu_1))^T \ (C(\mu_2))^T]^\dagger \left. \begin{bmatrix} C(\mu_1) \\ C(\mu_2) \end{bmatrix} A^\zeta B B^T (A^T)^{\zeta+1} \right) \\ &= \sigma_w^2 \sum_{m=0}^{\zeta+1} A^m B B^T (A^T)^m - \sigma_w^2 A^{\zeta+1} B \\ &\quad \times \begin{bmatrix} B^T (A^T)^\zeta (C(\mu_1))^T \ \mathbf{0}_{1 \times |\text{supp}(\mu_2)|} \\ \left[(C(\mu_1) A^\zeta B B^T (A^T)^\zeta (C(\mu_1))^T)^\dagger \ \mathbf{0} \right] \\ \mathbf{0} \end{bmatrix} \\ &\quad \times \begin{bmatrix} C(\mu_1) A^\zeta B \\ \mathbf{0}_{|\text{supp}(\mu_2)| \times 1} \end{bmatrix} B^T (A^T)^{\zeta+1} \quad (10) \end{aligned}$$

$$\begin{aligned} &= \sigma_w^2 \sum_{m=0}^{\zeta+1} A^m B B^T (A^T)^m - \sigma_w^2 A^{\zeta+1} B B^T (A^T)^\zeta (C(\mu_1))^T \\ &\quad \times (C(\mu_1) A^\zeta B B^T (A^T)^\zeta (C(\mu_1))^T)^\dagger C(\mu_1) A^\zeta B B^T (A^T)^{\zeta+1}, \quad (11) \end{aligned}$$

where Eq. (9) uses the fact that $(A^m)_{ji_0} = 0$ for all $j \in \text{supp}(\mu)$ whenever $m \in \{0, 1, \dots, \zeta - 1\}$, which implies

that $C(\mu)A^m B = \mathbf{0}$ for all $m \in \{0, 1, \dots, \zeta - 1\}$. Similarly, Eq. (10) follows from the fact that $C(\mu_2)A^m B = \mathbf{0}$ for all $m \in \{0, 1, \dots, \zeta\}$. Denoting $\psi \triangleq C(\mu_1)A^\zeta B \in \mathbb{R}^{|\text{supp}(\mu_1)|}$ and noting that $\psi \neq \mathbf{0}$ from Assumption 2, one can show that $\psi^T(\psi\psi^T)^\dagger\psi = 1$. We then have from Eq. (11):

RHS of Eq. (4)

$$\begin{aligned} &= \sigma_w^2 \sum_{m=0}^{\zeta+1} A^m B B^T (A^T)^m - \sigma_w^2 A^{\zeta+1} B B^T (A^T)^{\zeta+1} \\ &= \sigma_w^2 \sum_{m=0}^{\zeta} A^m B B^T (A^T)^m. \end{aligned}$$

Since $\sigma_w^2 \sum_{m=0}^{\zeta} A^m B B^T (A^T)^m \succeq \mathbf{0}$, we know from Lemma 1 that the limit $\Sigma(\mu)$ is given by Eq. (7). We then obtain from Eq. (6) that the limit $\Sigma^*(\mu)$ is given by Eq. (8) (when $\zeta \geq 1$).

Next, we consider any sensor placement μ such that $\zeta = 0$, i.e., a sensor is placed at the input vertex x_{i_0} . Using similar arguments to those above, we can also show that Eq. (7)-(8) hold when $\zeta = 0$. This completes the proof of the theorem. ■

To verify the results in Theorem 1, let us consider the following example.

Example 1: Consider the graph in Fig. 1, where x_2 is the input node (i.e., $B = \mathbf{e}_2$) with variance $\sigma_w^2 = 1$. Suppose $A = \begin{bmatrix} 0.5 & 2.1 & 0 & 0 \\ 0.3 & 0 & 1.5 & 0 \\ 0 & 0.6 & 0 & 0.5 \\ 0 & 0 & -0.8 & 1 \end{bmatrix}$, $C = I_4$ and $V = \mathbf{0}_{4 \times 4}$. Denote $\mu_2 = [0 \ 1 \ 0 \ 0]^T$ and $\mu_4 = [0 \ 0 \ 0 \ 1]^T$. It can be verified that $\Sigma(\mu_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B B^T$, $\Sigma^*(\mu_2) = \mathbf{0}_{4 \times 4}$, $\Sigma(\mu_4) = \begin{bmatrix} 5.5125 & 1.6065 & 1.26 & -0.504 \\ 1.6065 & 3.3409 & 0 & -0.7344 \\ 1.26 & 0 & 0.36 & 0 \\ -0.504 & -0.7344 & 0 & 0.2304 \end{bmatrix} = \sum_{m=0}^2 A^m B B^T (A^T)^m$ and $\Sigma^*(\mu_4) = \begin{bmatrix} 4.41 & 0 & 1.26 & 0 \\ 0 & 1 & 0 & 0 \\ 1.26 & 0 & 0.36 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sum_{m=0}^1 A^m B B^T (A^T)^m$, as provided by Theorem 1.



Fig. 1: Graph for Example 1

A. An Optimal Solution to GKFSP

Using the above discussions, we give the following result that characterizes an optimal solution to GKFSP (Problem 1).

Theorem 2: Supposing that Assumptions 1-2 hold, an optimal solution, denoted as μ^* , to the priori (resp., posteriori) GKFSP problem is to place a single sensor at a vertex x_j in order to minimize $l_{i_0 j}$, i.e., the distance from the input vertex x_{i_0} to x_j , while satisfying the budget constraint.

Proof: Under Assumptions 1-2, we first note from Eq. (7)-(8) that the *a priori* and the *a posteriori* error covariance matrices only depend on ζ , i.e., the shortest distance from the input node to the sensor nodes. Hence, it is sufficient to consider sensor placements $\mu \in \{0, 1\}^n$ such that $|\text{supp}(\mu)| = 1$ in terms of minimizing the trace of the *a priori* (resp., *a posteriori*) steady-state error covariance of the Kalman filter. Moreover, we know from Eq. (7) in Theorem 1 that $\Sigma(\mu) = \sigma_w^2 \sum_{m=0}^{\zeta} A^m B B^T (A^T)^m$, where $\zeta = \min_{j \in \text{supp}(\mu)} l_{i_0 j}$. Noting that the matrix $A^m B B^T (A^T)^m$ is positive semi-definite for all $m \in \mathbb{Z}_{\geq 0}$, it follows that $\text{trace}(A^m B B^T (A^T)^m) \geq 0$

for all $m \in \mathbb{Z}_{\geq 0}$. Hence, $\text{trace}(\Sigma(\mu))$ is minimized by finding a sensor placement μ^* with $|\text{supp}(\mu^*)| = 1$ such that ζ is minimized while satisfying the budget constraint. Using similar arguments, we can show that μ^* is also an optimal solution to the posteriori GKFSP problem. ■

Based on Theorem 2, we can find the optimal solution μ^* to the priori (resp., posteriori) GKFSP problem using polynomial-time algorithms such as the Breadth-First Search (BFS) algorithm which runs in time $O(n + |\mathcal{E}(A)|)$ [42].

B. An Optimal Solution to GKFSa

Given a sensor placement μ , we know from the insights obtained above for GKFSP that the steady-state *a priori* and the *a posteriori* error covariances of the Kalman filter (after an attack that removes some of those sensors) only depend on the surviving sensors that have the shortest distance from the input vertex x_{i_0} . We then have the following result whose proof is similar to that of Theorem 2 and is thus omitted.

Theorem 3: Suppose that Assumptions 1-2 hold. Given a sensor placement μ , an optimal solution, denoted as ν^* , to the priori (resp., posteriori) GKFSa problem can be found by maximizing the shortest distance from the input vertex x_{i_0} to the surviving sensors, i.e., solving the following optimization problem

$$\begin{aligned} &\max_{\nu \in \{0, 1\}^n} \min_{j \in \text{supp}(\mu \setminus \nu)} l_{i_0 j} \\ &\text{s.t. } f^T \nu \leq F, \end{aligned} \quad (12)$$

where $l_{i_0 j}$ is the distance from vertex x_{i_0} to vertex x_j , and $l_{i_0 j} = +\infty$ if $\text{supp}(\mu \setminus \nu) = \emptyset$.

The optimal solution ν^* to the priori (resp., posteriori) GKFSa problem described by Theorem 3 can be found as follows. Given a sensor placement μ , the adversary starts by inspecting the placed sensors (indicated by μ) that have the shortest distance from the input vertex x_{i_0} . The adversary will remove all of these sensors if the sum of the corresponding sensor attack costs is less than or equal to the budget constraint F , and terminate the process if otherwise. The above process is then repeated for the placed sensors that have the second shortest distance from the input vertex x_{i_0} , based on the remaining budget. This process continues with the placed sensors that have the third shortest distance from the input vertex x_{i_0} , etc. Hence, polynomial-time algorithms such as the BFS algorithm can be used to find the optimal sensor attack ν^* for the adversary in time $O(n + |\mathcal{E}(A)|)$.

IV. SOLVING THE RGKFSP PROBLEM

We now turn to the RGKFSP problem (Problem 3). Recall that Theorem 2 showed that it is enough to consider only sensor placements μ with $|\text{supp}(\mu)| = 1$ for the GKFSP problem (i.e., the system designer does not necessarily need to utilize all of the sensor placement budget H). However, an optimal sensor placement μ^* for the RGKFSP problem does not necessarily satisfy $|\text{supp}(\mu^*)| = 1$, since the adversary could have enough budget to remove the single sensor placed by the system designer, which causes the trace of the *a priori* (resp., *a posteriori*) error covariance of the Kalman filter to

be infinite (if the system dynamics matrix A is not stable). Note that the steady-state *a priori* and the *a posteriori* error covariance matrices of the Kalman filter (after the attack) only depend on the surviving sensors that have the shortest distance from the input vertex x_{i_0} . Using similar arguments to those for Theorems 2-3, we have that an optimal solution to the RGKFSP problem can be found by minimizing the shortest distance from the input vertex x_{i_0} to the sensors after the corresponding optimal sensor attack, and a sensor placement μ^* is optimal for the priori RGKFSP problem if and only if it is optimal for the posteriori RGKFSP problem.

We thus focus on the priori RGKFSP problem in this section. Although we provided polynomial-time algorithms to solve the GKFSP and GKFSa problems, we will show that the RGKFSP problem is NP-hard, i.e., there exist classes of the RGKFSP problem that cannot be solved by any polynomial-time algorithm if $P \neq NP$. To do this, we first recall from Remark 2 that Assumptions 1-2 hold for any system dynamics matrix A that is row-stochastic and irreducible. Therefore, Eq. (7) and Eq. (8) in Theorem 1 also hold for such A matrices.

To show that the RGKFSP problem is NP-hard, we reduce the subset sum problem [32] to RGKFSP.

Definition 6: An instance of the subset sum problem is given by a finite set U and a positive integer K , where each $s \in U$ has a size $\kappa(s) \in \mathbb{Z}_{>0}$.

We use the following result from [32].

Lemma 2: Given an instance of the subset sum problem as described in Definition 6, the problem of determining whether there is a subset $U' \subseteq U$ such that $\sum_{s \in U'} \kappa(s) = K$ is NP-complete.

We are now in place to prove the following result.

Theorem 4: The RGKFSP problem is NP-hard even when both of the following two conditions are satisfied: (1) the sensor placement cost and the sensor attack cost satisfy $h_i = f_i$ for all $i \in \{1, 2, \dots, n\}$, and (2) there is a feasible sensor placement for the system designer.

Proof: We prove the result by giving a polynomial-time reduction from the subset sum problem. Consider any instance of the subset sum problem defined in Definition 6. Denote $U = \{s_1, s_2, \dots, s_{|U|}\}$. Denote the number of bits of the binary representation of the positive integer K as $b(K)$, i.e., $b(K) \triangleq \lfloor \log_2(K) \rfloor + 1$. We then construct an instance of the priori RGKFSP problem as follows. The system dynamics matrix $A \in \mathbb{R}^{(|U|+b(K)) \times (|U|+b(K))}$ is chosen such that the graph $\mathcal{G}(A)$ is an undirected path of length $|U| + b(K) - 1$. Specifically, we set $A_{ij} = A_{ji} = \frac{1}{3}$ for all $i \in \{1, 2, \dots, |U| + b(K) - 1\}$ and $j = i + 1$, $A_{mm} = \frac{1}{3}$ for all $m \in \{2, 3, \dots, |U| + b(K) - 1\}$, $A_{mm} = \frac{2}{3}$ for all $m \in \{1, |U| + b(K)\}$, and all the other entries in A are zero. The vertex x_1 is set as the only vertex that has the stochastic input with variance $\sigma_w^2 = 1$. The sensor placement cost vector is set as $h_i = \kappa(s_i)$ for all $i \in \{1, 2, \dots, |U|\}$, and $h_i = 2^{i-|U|-1}$ for all $i \in \{|U|+1, |U|+2, \dots, |U|+b(K)\}$. The sensor attack cost is set as $f_i = h_i$ for all $i \in \{1, 2, \dots, |U| + b(K)\}$. Note that the sensor placement vector and the sensor attack vector are given by $\mu \in \{0, 1\}^{|U|+b(K)}$ and $\nu \in \{0, 1\}^{|U|+b(K)}$, respectively. The sensor placement budget of the system designer is set as $H = K$, and the sensor attack budget of

the adversary is set as $F = K - 1$. We also note that the matrix A that we constructed is row-stochastic and irreducible. Therefore, Eq. (7) in Theorem 1 holds for the A matrix that we constructed. We claim that the answer to the given subset sum instance is “yes” if and only if the optimal solution to the constructed instance of the priori RGKFSP problem, denoted as μ^* , satisfies $\text{trace}(\Sigma(\mu^* \setminus \nu^*)) \leq \text{trace}(\sum_{i=0}^{|U|-1} A^i B B^T A^i)$, where ν^* is the optimal sensor attack given μ^* .

Suppose that the answer to the given subset sum instance is “yes”, i.e., there exists $U' \subseteq U$ such that $\sum_{s \in U'} \kappa(s) = K$. It follows that for the instance of the priori RGKFSP problem as constructed above, there exists a sensor placement vector $\tilde{\mu}$ such that $\sum_{i=1}^{|U|} h_i \tilde{\mu}_i = K \leq H$. Therefore, for any sensor attack $\tilde{\nu}$ that satisfies the sensor attack budget constraint, i.e., $\sum_{i=1}^{|U|} f_i \tilde{\nu}_i \leq F = K - 1$, we have $\text{supp}(\tilde{\mu} \setminus \tilde{\nu}) \cap \{1, \dots, |U|\} \neq \emptyset$, which implies that there exists $j \in \{1, \dots, |U|\}$ such that $j \in \text{supp}(\tilde{\mu} \setminus \tilde{\nu})$. Noting that $A^m B B^T A^m \succeq \mathbf{0}$ for all $m \in \mathbb{Z}_{\geq 0}$ and $l_{1j} = j - 1 \leq l_{1|U|} = |U| - 1$, it then follows from Eq. (7) that $\text{trace}(\Sigma(\tilde{\mu} \setminus \tilde{\nu})) \leq \text{trace}(\sum_{i=0}^{j-1} A^i B B^T A^i) \leq \text{trace}(\sum_{i=0}^{|U|-1} A^i B B^T A^i)$, for any sensor attack $\tilde{\nu}$ such that $\sum_{i=1}^{|U|} f_i \tilde{\nu}_i \leq F$. Since $\text{trace}(\Sigma(\mu^* \setminus \nu^*)) \leq \text{trace}(\Sigma(\tilde{\mu} \setminus \tilde{\nu}))$, we have $\text{trace}(\Sigma(\mu^* \setminus \nu^*)) \leq \text{trace}(\sum_{i=0}^{|U|-1} A^i B B^T A^i)$.

Conversely, suppose that the answer to the subset sum instance is “no”, i.e., for any $U' \subseteq U$, we have $\sum_{s \in U'} \kappa(s) \neq K$. Considering the instance of the priori RGKFSP problem we constructed, for any sensor placement vector μ such that $\sum_{i=1}^{|U|+b(K)} h_i \mu_i \leq H = K$, we have $\sum_{i=1}^{|U|} h_i \mu_i \neq K$, which implies $\sum_{i=1}^{|U|} h_i \mu_i \leq K - 1$. Denote $\sum_{i=1}^{|U|} h_i \mu_i \triangleq K|U|$. Therefore, for any sensor placement vector μ with $\sum_{i=1}^{|U|+b(K)} h_i \mu_i \leq H$, there exists an attack $\hat{\nu}$ such that $\sum_{i=1}^{|U|} f_i \hat{\nu}_i = K|U| \leq K - 1$, which implies $\text{supp}(\mu \setminus \hat{\nu}) \cap \{1, 2, \dots, |U|\} = \emptyset$. Moreover, note that $K = H \geq H - K|U| > F - K|U|$. Since we set the sensor placement cost vector and the sensor attack cost vector to satisfy $h_i = f_i = 2^{i-|U|-1}$ for all $i \in \{|U|+1, |U|+2, \dots, |U|+b(K)\}$, where $b(K)$ is the number of bits for the binary representation of K , we have that for any $U' \subseteq U$, there exists $\bar{U}' \subseteq \{|U|+1, |U|+2, \dots, |U|+b(K)\}$ such that $\sum_{s \in U'} \kappa(s) + \sum_{i \in \bar{U}'} h_i = H$. Therefore, the system designer can always use all the sensor placement budget by placing sensors at an appropriate subset of the vertices in the vertex set $\{x_{|U|+1}, x_{|U|+2}, \dots, x_{|U|+b(K)}\}$ and guarantee to have at least one sensor left after any attack that satisfies the sensor attack budget constraint. Formally, we have that for any sensor placement μ with $\sum_{i=1}^{|U|+b(K)} h_i \mu_i = H$, there exists $j' \in \{|U|+1, \dots, |U|+b(K)\}$ such that $j' \in \text{supp}(\mu \setminus \nu)$, where ν is any sensor attack satisfying the sensor attack budget constraint. Meanwhile, any sensor placement μ such that $\sum_{i=1}^{|U|+b(K)} h_i \mu_i < H$ is not a feasible sensor placement. Therefore, there is always a feasible sensor placement for the system designer under the constructed instance of the priori RGKFSP problem when the answer to the subset sum instance is “no”. Note that the matrix $A^m B B^T A^m \succeq \mathbf{0}$ for all $m \in \mathbb{Z}_{\geq 0}$ and $l_{1j'} = j' - 1 \geq l_{1|U|} + 1 = |U|$. Combining the arguments above together, it then follows from Eq. (7) that for any μ such that $\sum_{i=1}^{|U|+b(K)} h_i \mu_i = H$, we have $\text{trace}(\Sigma(\mu \setminus$

$\nu)) \geq \text{trace}(\sum_{i=0}^{j'-1} A^i B B^T A^i) \geq \text{trace}(\sum_{i=0}^{|U|} A^i B B^T A^i)$, where ν is any sensor attack satisfying the sensor attack budget constraint. Since $(A^m)_{11} > 0$ for all $m \in \mathbb{Z}_{\geq 0}$, we have $\text{trace}(A^{|U|} B B^T A^{|U|}) > 0$ and thus $\text{trace}(\Sigma(\mu \setminus \nu)) > \text{trace}(\sum_{i=0}^{|U|-1} A^i B B^T A^i)$. Since the above arguments hold for any μ with $\sum_{i=1}^{|U|+b(K)} h_i \mu_i = H$, they also hold for the optimal solution μ^* to the constructed priori RGKFSP instance, i.e., $(\Sigma(\mu^* \setminus \nu^*)) > \text{trace}(\sum_{i=0}^{|U|-1} A^i B B^T A^i)$, where ν^* is the optimal sensor attack given μ^* . This completes the proof of the claim above.

Since the subset sum problem is NP-complete and RGKFSP $\notin$ NP, we conclude that RGKFSP is NP-hard even under the additional conditions as stated. ■

A. An Algorithm for RGKFSP

It follows directly from Theorem 4 that there is no polynomial-time algorithm that would solve all instances of RGKFSP if $P \neq NP$. However, we now provide a pseudo-polynomial-time algorithm² (Algorithm 1) for RGKFSP by relating it to the knapsack problem defined as follows.

Definition 7: Given a finite set $U \triangleq \{s_1, s_2, \dots, s_{|U|}\}$, a size $\kappa(s_i) \in \mathbb{Z}_{>0}$ and a value $\phi(s_i) \in \mathbb{Z}_{>0}$ for each $i \in \{1, 2, \dots, |U|\}$, and a positive integer K , the knapsack problem is to find an indicator vector $\pi \in \{0, 1\}^{|U|}$ that solves

$$\begin{aligned} & \max_{\pi \in \{0, 1\}^{|U|}} \sum_{i=1}^{|U|} \phi(s_i) \pi_i \\ & \text{s.t. } \sum_{i=1}^{|U|} \kappa(s_i) \pi_i \leq K. \end{aligned} \quad (13)$$

Denoting the given instance of knapsack as a tuple $\{\phi, \kappa, K\}$, where $\phi \triangleq (\phi(s_1), \phi(s_2), \dots, \phi(s_{|U|}))$ and $\kappa \triangleq (\kappa(s_1), \kappa(s_2), \dots, \kappa(s_{|U|}))$,³ the corresponding optimal indicator vector for (13) is denoted as $\pi^*(\phi, \kappa, K)$, and the corresponding optimal value of the objective function in (13) is denoted as $z(\phi, \kappa, K)$.

The steps of Algorithm 1 for RGKFSP are as follows. Algorithm 1 starts by relabeling the input vertex as vertex x_1 and relabeling the other vertices in terms of a non-decreasing order of the distances from the vertex x_1 (Lines 1-2). Denoting $l_{\max} \triangleq \max_{x_j \in \mathcal{X}(A)} l_{i_0 j}$, Algorithm 1 then finds the smallest $m \in \{0, 1, \dots, l_{\max}\}$ such that by placing sensors (under the budget constraint) solely at nodes that have distances less than or equal to m from x_1 (after the relabeling), the sum of the sensor attack costs of the placed sensors is greater than the sensor attack budget, i.e., there is at least one sensor that survives the corresponding optimal sensor attack. This is done by iteratively solving a knapsack problem at increasingly longer distances from the input node, where at each distance, the goal is to find a set of sensor locations that fits within the sensor placement budget constraint H but maximizes the

sum of the sensor attack costs. Algorithm 1 returns $\mu = \mathbf{0}_{n \times 1}$ if there is no feasible sensor placement. We now prove that Algorithm 1 returns an optimal solution to RGKFSP.

Algorithm 1 Algorithm for RGKFSP

Input: An instance of the RGKFSP problem.

Output: A sensor placement $\mu \in \{0, 1\}^n$.

- 1: Find the distance $l_{i_0 j}$ for all $x_j \in \mathcal{X}(A) \setminus \{x_{i_0}\}$ via BFS and denote $l_{\max} \triangleq \max_{x_j \in \mathcal{X}(A)} l_{i_0 j}$.
- 2: Relabel the vertices of $\mathcal{G}(A)$ such that x_1 is the input vertex and $l_{1j} \leq l_{1t}$ for all $x_j, x_t \in \mathcal{X}(A) \setminus \{x_1\}$ with $j \leq t$.
- 3: $\mu = \mathbf{0}_{n \times 1}$
- 4: **for** $m = 0$ to $l_{\max}$ **do**
- 5: Find $j_m \triangleq \max\{j : l_{1j} = m, x_j \in \mathcal{X}(A)\}$.
- 6: Find $\pi^*((f_1, \dots, f_{j_m}), (h_1, \dots, h_{j_m}), H)$
- 7: **if** $z((f_1, \dots, f_{j_m}), (h_1, \dots, h_{j_m}), H) > F$ **then**
- 8: $[\mu_1 \dots \mu_{j_m}]^T = \pi^*((f_1, \dots, f_{j_m}), (h_1, \dots, h_{j_m}), H)$
- 9: **return** μ

Theorem 5: Under Assumptions 1-2, Algorithm 1 returns an optimal solution to the RGKFSP problem.

Proof: Denote an optimal solution to the RGKFSP problem as μ^* and denote the solution returned by Algorithm 1 as μ' . Suppose that μ' is a feasible sensor placement. Suppose that the vertices in $\mathcal{G}(A)$ are relabeled as indicated by Lines 1-2 in Algorithm 1, i.e., vertex x_1 is labeled as the input vertex and the other vertices are labeled in terms of a non-decreasing order of the distances from vertex x_1 (note that the relabeling of the vertices does not change the optimal solution to the RGKFSP problem other than permuting it). Assume for the sake of contradiction that $\text{trace}(\Sigma(\mu^* \setminus \nu^*)) < \text{trace}(\Sigma(\mu' \setminus \nu'))$, where ν^* and ν' are the optimal sensor attacks given μ^* and μ' , respectively. Denote $j^* \triangleq \max J$ and $j' \triangleq \max J'$, where $J \triangleq \arg \min_{m \in \text{supp}(\mu^* \setminus \nu^*)} l_{1m}$ and $J' \triangleq \arg \min_{m \in \text{supp}(\mu' \setminus \nu')} l_{1m}$. In other words, among those sensors that are closest to the input vertex in $\text{supp}(\mu^* \setminus \nu^*)$ (resp., $\text{supp}(\mu' \setminus \nu')$), j^* (resp., j') is the largest index. Noting that $\sum_{m=1}^{j^*} f_m \mu_m^* > F$ (otherwise the optimal sensor attack ν^* given μ^* would remove the sensor placed at vertex x_{j^*} as argued previously in Section III-B), it follows from Definition 7 that $z((f_1, \dots, f_{j^*}), (h_1, \dots, h_{j^*}), H) > F$, which implies that $z((f_1, \dots, f_{j_m}), (h_1, \dots, h_{j_m}), H) > F$, where j_m is defined in Line 5 of Algorithm 1 with $m = l_{1j^*}$. We then know from the definition of Algorithm 1 that the sensor placement μ' returned by Algorithm 1 would satisfy $j' \leq j_m$, which implies that $l_{1j'} \leq l_{1j^*}$ (by the way that Algorithm 1 relabels the vertices). Moreover, we have from Theorem 1 that

$$\Sigma(\mu^* \setminus \nu^*) = \sigma_w^2 \sum_{m=0}^{l_{1j^*}} A^m B B^T (A^T)^m, \quad (14)$$

and

$$\Sigma(\mu' \setminus \nu') = \sigma_w^2 \sum_{m=0}^{l_{1j'}} A^m B B^T (A^T)^m, \quad (15)$$

²A pseudo-polynomial-time algorithm is an algorithm that runs in time that is bounded by a polynomial in the largest integer in its input [32].

³Note that the elements in ϕ and κ are ordered, and the i th element of ϕ (resp., κ) corresponds to the value (resp., weight) of $s_i \in U$ for all $i \in \{1, \dots, |U|\}$. The dependency of $\{\phi, \kappa, K\}$ on U is dropped since each element of ϕ (resp., κ) represents an element of U .

hold under Assumptions 1-2. Since the matrix $A^m B B^T (A^T)^m \succeq \mathbf{0}$ for all $m \in \mathbb{Z}_{\geq 0}$, we have from the assumption $\text{trace}(\Sigma(\mu^* \setminus \nu^*)) < \text{trace}(\Sigma(\mu' \setminus \nu'))$ and Eq. (14)-(15) that $l_{1j^*} < l_{1j'}$. Thus, we get a contradiction.

We then suppose that the solution μ' returned by Algorithm 1 is not feasible, i.e., $\text{supp}(\mu' \setminus \nu') = \emptyset$. Again, we assume that $\text{trace}(\Sigma(\mu^* \setminus \nu^*)) < \text{trace}(\Sigma(\mu' \setminus \nu'))$, i.e., $\text{supp}(\mu^* \setminus \nu^*) \neq \emptyset$. Via similar arguments to those above, we have that there exists $j^{*'} \in \{1, \dots, n\}$ such that $z((f_1, \dots, f_{j^{*'}}, (h_1, \dots, h_{j^{*'}}, H) > F$, which implies that $z((f_1, \dots, f_{j_m}, (h_1, \dots, h_{j_m}, H) > F$, where j_m is defined in Line 5 of Algorithm 1 with $m = l_{1j^{*'}}$. Therefore, Algorithm 1 would also return a solution μ' such that $\text{supp}(\mu' \setminus \nu') \neq \emptyset$, which is a contradiction. We then conclude that $\text{trace}(\Sigma(\mu^* \setminus \nu^*)) = \text{trace}(\Sigma(\mu' \setminus \nu'))$, i.e., Algorithm 1 returns an optimal solution to the RGKFSP problem. ■

Since the knapsack problem is NP-hard, there is no polynomial-time algorithm to solve it optimally (if $P \neq NP$) [32]. Various algorithms exist to approximate or optimally solve it, including greedy algorithms, linear programming relaxation and dynamic programming [43]. When implementing Algorithm 1, we can use existing algorithms for knapsack to find $\pi^*((f_1, \dots, f_{j_m}, (h_1, \dots, h_{j_m}, H)$ in Line 6 and $z((f_1, \dots, f_{j_m}, (h_1, \dots, h_{j_m}, H)$ in Line 7 when we range m from 0 to $l_{\max}$. Specifically, we call a pseudo-polynomial-time algorithm for knapsack (that solves it optimally) at most $l_{\max} + 1$ times to achieve this. For example, a typical dynamic programming approach for knapsack finds $\pi^*((f_1, \dots, f_{j_m}, (h_1, \dots, h_{j_m}, H)$ and $z((f_1, \dots, f_{j_m}, (h_1, \dots, h_{j_m}, H)$ in time $O(j_m H)$ for each $m \in \{0, \dots, l_{\max}\}$ [43]. Since BFS runs in time $O(n + |\mathcal{E}(A)|)$, Algorithm 1 runs in time $O(l_{\max} n H + n + |\mathcal{E}(A)|)$.

V. NOISY SENSOR MEASUREMENT CASE

The results we obtained so far hold under the assumption that $V = \mathbf{0}_{n \times n}$. In this section, we provide a bound on the suboptimality of the proposed strategies when there is sensor measurement noise. We will use the following result whose proof is in the appendix.

Lemma 3: Consider a system dynamics matrix $A \in \mathbb{R}^{n \times n}$, an input matrix $B \in \mathbb{R}^{n \times n_1}$, a sensor measurement matrix $C \in \mathbb{R}^{n_2 \times n}$, an input covariance matrix $W \in \mathbb{S}_+^{n_1}$, and a sensor measurement noise covariance matrix $\tilde{V} \in \mathbb{S}_+^{n_2}$. Suppose that the pair $(A, BW^{1/2})$ is stabilizable and the pair (A, C) is detectable. Denote $\tilde{\Sigma}$ (resp., $\tilde{\Sigma}^*$) as the steady-state *a priori* (resp., *a posteriori*) error covariance of the Kalman filter corresponding to the measurement noise covariance $\tilde{V}$, and denote Σ (resp., Σ^*) as the steady-state *a priori* (resp., *a posteriori*) error covariance of the corresponding Kalman filter when $V = \mathbf{0}_{n_2 \times n_2}$. Then, $\tilde{\Sigma} \preceq \Sigma + E$ and $\tilde{\Sigma}^* \preceq \Sigma^* + (I_n - LC)E$, where E is given by

$$E \triangleq \sum_{m=0}^{\infty} (A - KC)^m K \tilde{V} K^T ((A - KC)^T)^m, \quad (16)$$

with $K \triangleq A \Sigma C^T (C \Sigma C^T)^{-1}$ and $L \triangleq \Sigma C^T (C \Sigma C^T)^{-1}$.⁴

⁴The inverses are interpreted as the Moore-Penrose pseudo-inverses if the arguments are not invertible [35].

Note that E exists and is finite since the matrix $A - KC$ is stable. See the proof in the appendix for more details. We have the following result for the GKFSP problem.

Theorem 6: Suppose that Assumptions 1-2 hold. Let $\tilde{\Sigma}(\mu)$ (resp., $\tilde{\Sigma}^*(\mu)$) be the steady-state *a priori* (resp., *a posteriori*) error covariance matrix of the Kalman filter associated with μ when $V = \tilde{V} \in \mathbb{S}_+^{n_2}$. Denote $\tilde{\mu}_1^*$ (resp., $\tilde{\mu}_2^*$) as the optimal solution to the priori (resp., posteriori) GKFSP problem when $V = \tilde{V}$, and denote μ^* as the optimal solution to the priori (resp., posteriori) GKFSP problem when $V = \mathbf{0}_{n \times n}$. Then, $\text{trace}(\tilde{\Sigma}(\mu^*)) \leq \text{trace}(\tilde{\Sigma}(\tilde{\mu}_1^*)) + \text{trace}(E(\mu^*))$ and $\text{trace}(\tilde{\Sigma}^*(\mu^*)) \leq \text{trace}(\tilde{\Sigma}^*(\tilde{\mu}_2^*)) + \text{trace}(E^*(\mu^*))$, where $E(\mu^*)$ and $E^*(\mu^*)$ are defined in Lemma 3 with $C = C(\mu^*)$, and $E^*(\mu^*) \triangleq (I_n - L(\mu^*)C(\mu^*))E(\mu^*)$.

Proof: First, we know from Lemma 3 that $\tilde{\Sigma}(\mu^*) \preceq \Sigma(\mu^*) + E(\mu^*)$, where $\Sigma(\mu^*)$ is the steady-state *a priori* error covariance of the Kalman filter corresponding to μ^* when $V = \mathbf{0}$. This implies $\text{trace}(\tilde{\Sigma}(\mu^*)) \leq \text{trace}(\Sigma(\mu^*)) + \text{trace}(E(\mu^*))$. Since μ^* is the optimal solution to the priori GKFSP problem when $V = \mathbf{0}$, we have $\text{trace}(\Sigma(\mu^*)) \leq \text{trace}(\Sigma(\tilde{\mu}_1^*))$. Moreover, one can show that the error covariance of the Kalman filter is always lower bounded (in the positive semi-definite sense) by the error covariance of the Kalman filter with zero measurement noise covariance (with the other system matrices fixed). We obtain $\text{trace}(\Sigma(\tilde{\mu}_1^*)) \leq \text{trace}(\tilde{\Sigma}(\tilde{\mu}_1^*))$. It then follows from the above arguments that $\text{trace}(\tilde{\Sigma}(\mu^*)) \leq \text{trace}(\tilde{\Sigma}(\tilde{\mu}_1^*)) + \text{trace}(E(\mu^*))$. Similarly, we can show that $\text{trace}(\tilde{\Sigma}^*(\mu^*)) \leq \text{trace}(\tilde{\Sigma}^*(\tilde{\mu}_2^*)) + \text{trace}(E^*(\mu^*))$. ■

The above result has the following interpretation. Consider an instance of the priori (resp., posteriori) GKFSP problem with $V = \tilde{V} \in \mathbb{S}_+^{n_2}$. If we simply take $V = \mathbf{0}$ and apply the algorithm described in Section III-A, we will obtain an optimal solution, denoted as μ^* , to the corresponding instance of the priori (resp., posteriori) GKFSP problem (with $V = \mathbf{0}$). Theorem 6 shows that the performance (i.e., suboptimality) of this sensor placement (i.e., μ^*) for the original priori (resp., posteriori) GKFSP instance with $V = \tilde{V}$ can be bounded by $\text{trace}(\tilde{\Sigma}(\mu^*)) \leq \text{trace}(\tilde{\Sigma}(\tilde{\mu}_1^*)) + \text{trace}(E(\mu^*))$ (resp., $\text{trace}(\tilde{\Sigma}^*(\mu^*)) \leq \text{trace}(\tilde{\Sigma}^*(\tilde{\mu}_2^*)) + \text{trace}(E^*(\mu^*))$), where $\tilde{\mu}_1^*$ (resp., $\tilde{\mu}_2^*$) is the optimal solution to the instance of the priori (resp., posteriori) GKFSP problem when $V = \tilde{V}$. Moreover, we have from Eq. (16) that as $\tilde{V}$ goes to zero, $\text{trace}(E(\mu^*))$ (resp., $\text{trace}(E^*(\mu^*))$) will go to zero, which implies that $\text{trace}(\tilde{\Sigma}(\mu^*))$ (resp., $\text{trace}(\tilde{\Sigma}^*(\mu^*))$) will go to $\text{trace}(\tilde{\Sigma}(\tilde{\mu}_1^*))$ (resp., $\text{trace}(\tilde{\Sigma}^*(\tilde{\mu}_2^*))$). Similar performance bounds can be obtained for the GKFSA and RGKFSP problems, respectively.

We provide simulations to show the performance of the algorithms in Section III-A, Section III-B, and Section IV, when applied to solve the GKFSP, GKFSA, and RGKFSP problems with measurement noise, respectively. Specifically, consider a strongly connected graph $\mathcal{G}(A)$ with $\mathcal{X}(A) = \{x_1, \dots, x_{10}\}$ and $|\mathcal{E}(A)| = 15$, where node x_1 has the stochastic input with variance $\sigma_w^2 = 0.1$. Set the measurement matrix $C = I_{10}$ and the sensor noise covariance $V = \sigma_v^2 I_{10}$, where $\sigma_v^2 \in \mathbb{R}_{\geq 0}$. Under a fixed cost $h_i \in \mathbb{Z}_{\geq 0}$ to place sensor at x_i , a budget $H \in \mathbb{Z}_{\geq 0}$, a fixed cost $f_i \in \mathbb{Z}_{\geq 0}$ to attack sensor at x_i , and an attack budget $F \in \mathbb{Z}_{\geq 0}$, we randomly generate the corresponding system dynamics matrix $A \in \mathbb{R}^{10 \times 10}$ by selecting each

nonzero element of A from a standard normal distribution. Fig. 2(a) and Fig. 2(b) show the performance of the algorithm described in Section III-A, when applied to solve the (priori) GKFSF instances with $V = \sigma_v^2 I_{10}$. Specifically, Fig. 2(a) is obtained for a single realization of A , which compares the gap (i.e., difference) between the optimal solution to the GKFSF problem (found by brute force and denoted as OPT) and the solution returned by the algorithm (denoted as ALG), with the bound (on the difference) provided in Theorem 6, when σ_v^2 ranges from 0.01 to 0.5. Fig. 2(b) shows a histogram of the suboptimality of the algorithm, computed as $\frac{ALG - OPT}{OPT}$, over 1000 realizations of A , when $\sigma_v^2 = 5$. Similarly, Fig. 2(c)-(d) and Fig. 2(e)-(f) show the performance of the algorithm described in Section III-B for GKFSF and Algorithm 1 for RGKFSF, respectively. Note that we fix a sensor placement μ when solving the GKFSF instances. Moreover, the objective function of RGKFSF associated with the solution returned by Algorithm 1 is computed against the corresponding optimal sensor attack when $V = \sigma_v^2 I_{10}$. The simulations show that the bounds in Theorem 6 are conservative and that the algorithms (for zero sensor noise) give solutions that are close to optimal for the noisy measurement instances, particularly for RGKFSF, even when σ_w^2/σ_v^2 becomes small.

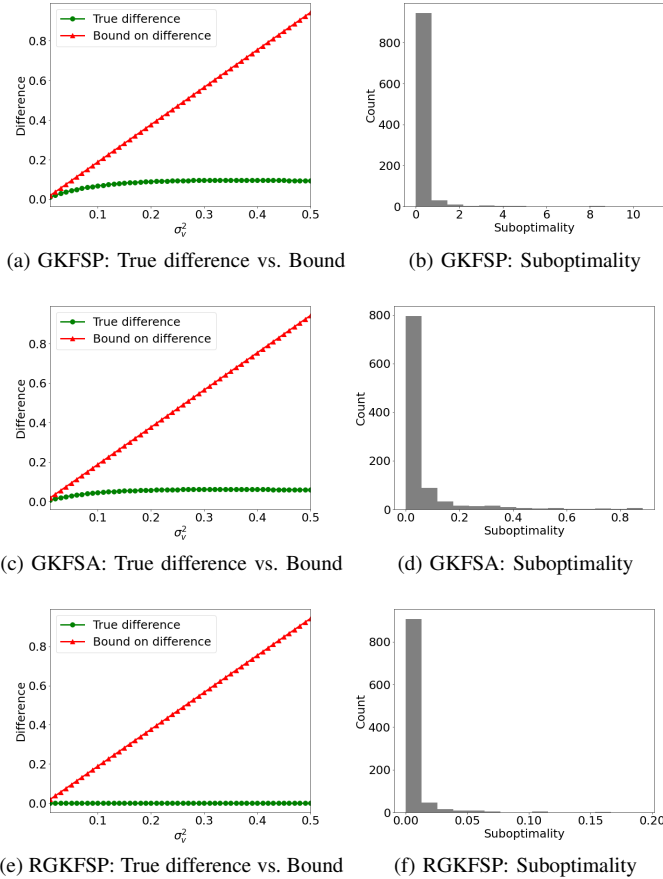


Fig. 2: Performance of the algorithms

VI. CONCLUSION

We considered networked dynamical systems affected by a stochastic input. Under this setting, we first studied the problem for a system designer to optimally place sensors

over the network subject to a budget constraint in order to minimize the trace of the steady-state error covariance of the corresponding Kalman filter. We then studied the optimal sensor attack problem where an adversary can attack the placed sensors under an attack budget constraint in order to maximize the trace of the steady-state error covariance of the Kalman filter corresponding to the surviving sensors. Using the graph structure of the networked system, we provided polynomial-time algorithms to solve these two problems. Furthermore, we studied the resilient sensor placement for the system designer when faced with an adversary. We showed that this problem is NP-hard, and provided a pseudo-polynomial-time algorithm to solve it. Although these results are obtained when there is no sensor noise, we provided bounds on the suboptimality of the proposed strategies in the presence of sensor measurement noise. Future work on characterizing optimal solutions when there is sensor noise and providing algorithms for systems with multiple stochastic inputs are of interest.

APPENDIX

Proof of Lemma 3:

Denote $\Sigma_{k/k-1}$ (resp., $\Sigma_{k/k}$) as the *a priori* (resp., *a posteriori*) error covariance of the Kalman filter at time step k when $V = 0$, and denote $\tilde{\Sigma}_{k/k-1}$ (resp., $\tilde{\Sigma}_{k/k}$) as the *a priori* (resp., *a posteriori*) error covariance of the Kalman filter at time step k when $V = \tilde{V}$. Denoting $\bar{W} \triangleq BWB^T$, we have (from [35]):

$$\begin{aligned} \tilde{\Sigma}_{k+1/k} &= (A - \tilde{K}_k C) \tilde{\Sigma}_{k/k-1} (A - \tilde{K}_k C)^T + \bar{W} + \tilde{K}_k \tilde{V} \tilde{K}_k^T, \\ \text{where } k \geq 0 \text{ and } \tilde{K}_k &\triangleq A \tilde{\Sigma}_{k/k-1} C^T (C \tilde{\Sigma}_{k/k-1} C^T + \tilde{V})^{-1} \text{ is the corresponding Kalman gain at time step } k. \text{ For any time step } k, \text{ the Kalman gain } \tilde{K}_k \text{ satisfies} \\ \tilde{K}_k &= \arg \min_J \{ (A - JC) \tilde{\Sigma}_{k/k-1} (A - JC)^T + \bar{W} + J \tilde{V} J^T \}, \end{aligned} \quad (17)$$

where the minimization is in the positive semi-definite sense [35]. Since the pair $(A, BW^{1/2})$ (resp., (A, C)) is stabilizable (resp., detectable), we know from a more general version of Lemma 1 for general system matrices in [35] that the limit $\tilde{\Sigma} = \lim_{k \rightarrow \infty} \tilde{\Sigma}_{k+1/k}$ exists, and satisfies

$$\tilde{\Sigma} = (A - \tilde{K} C) \tilde{\Sigma} (A - \tilde{K} C)^T + \bar{W} + \tilde{K} \tilde{V} \tilde{K}^T,$$

where $\tilde{K} \triangleq A \tilde{\Sigma} C^T (C \tilde{\Sigma} C^T + \tilde{V})^{-1}$ is the corresponding (steady-state) Kalman gain. Similarly, we have

$$\Sigma = (A - KC) \Sigma (A - KC)^T + \bar{W}, \quad (18)$$

where $K \triangleq A \Sigma C^T (C \Sigma C^T)^{-1}$. Noting the optimality of the Kalman gains from Eq. (17), there exists, as argued in [35], a suboptimal filter (when $V = \tilde{V}$) with a (time-invariant) suboptimal gain given by K such that the corresponding *a priori* error covariance at time step $k+1$, denoted as $\hat{\Sigma}_{k+1/k}$, satisfies

$$\hat{\Sigma}_{k+1/k} = (A - KC) \hat{\Sigma}_{k/k-1} (A - KC)^T + \bar{W} + K \tilde{V} K^T. \quad (19)$$

Furthermore, the limit $\hat{\Sigma} \triangleq \lim_{k \rightarrow \infty} \hat{\Sigma}_{k+1/k}$ exists and satisfies $\hat{\Sigma} \succeq \tilde{\Sigma}$ [35]. We then obtain from Eq. (18) and (the steady-state version of) Eq. (19) the following:

$$E = (A - KC) E (A - KC)^T + K \tilde{V} K^T, \quad (20)$$

where $E = \hat{\Sigma} - \Sigma$. Since the matrix $A - KC$ is stable [35], we have that there exists a unique finite positive semi-definite matrix E that satisfies Eq. (20) and can be written as $E = \sum_{m=0}^{\infty} (A - KC)^m K \tilde{V} K^T ((A - KC)^T)^m$ (e.g., [35]). It then follows from the arguments above that $\hat{\Sigma} = E + \Sigma \succeq \tilde{\Sigma}$.

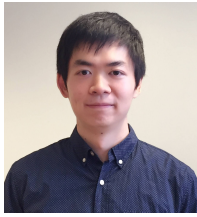
Similarly, we have from [35] that $\tilde{\Sigma}_{k/k}$ satisfies $\tilde{\Sigma}_{k/k} = (I_n - \tilde{L}_k C) \tilde{\Sigma}_{k/k-1}$, where $\tilde{L}_k \triangleq \tilde{\Sigma}_{k/k-1} C^T (C \tilde{\Sigma}_{k/k-1} C^T + \tilde{V})^{-1}$. Moreover, the limits $\tilde{\Sigma}^* \triangleq \lim_{k \rightarrow \infty} \tilde{\Sigma}_{k/k}$ and $\Sigma^* \triangleq \lim_{k \rightarrow \infty} \Sigma_{k/k}$ exist and satisfy $\tilde{\Sigma}^* = (I_n - \tilde{L} C) \tilde{\Sigma}$ and $\Sigma^* = (I_n - L C) \Sigma$, respectively, where $\tilde{L} \triangleq \tilde{\Sigma} C^T (C \tilde{\Sigma} C^T + \tilde{V})^{-1}$ and $L \triangleq \Sigma C^T (C \Sigma C^T + V)^{-1}$. Similarly, the *a posteriori* error covariance at time step k of the suboptimal filter (when $V = \tilde{V}$) as described above, denoted as $\hat{\Sigma}_{k/k}$, is given by

$$\hat{\Sigma}_{k/k} = (I_n - LC) \hat{\Sigma}_{k/k-1}. \quad (21)$$

Since the limit $\hat{\Sigma} = \lim_{k \rightarrow \infty} \hat{\Sigma}_{k+1/k}$ exists, we know from Eq. (21) that the limit $\hat{\Sigma}^* \triangleq \lim_{k \rightarrow \infty} \hat{\Sigma}_{k/k}$ also exists. Using similar arguments to those in [35], one can show that $\hat{\Sigma}^* \succeq \tilde{\Sigma}^*$. Thus, we have $\hat{\Sigma}^* - \Sigma^* = (I_n - LC)(\hat{\Sigma} - \Sigma) = (I_n - LC)E$, which implies $\hat{\Sigma}^* \succeq \Sigma^* + (I_n - LC)E$. ■

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