

Ricci Subtraction for Cosmological Coleman-Weinberg Potentials

S. P. Miao^{1*}, S. Park^{2*} and R. P. Woodard^{3†}

¹ *Department of Physics, National Cheng Kung University,
No. 1 University Road, Tainan City 70101, TAIWAN*

² *CEICO, Institute of Physics of the Czech Academy of Sciences,
Na Slovance 2, 18221 Prague 8 CZECH REPUBLIC*

³ *Department of Physics, University of Florida,
Gainesville, FL 32611, UNITED STATES*

ABSTRACT

We reconsider the fine-tuning problem of scalar-driven inflation arising from the need to couple the inflaton to ordinary matter in order to make reheating efficient. Quantum fluctuations of this matter induce Coleman-Weinberg corrections to the inflaton potential, depending (for de Sitter background) in a complex way on the ratio of the inflaton to the Hubble parameter. These corrections are not Planck-suppressed and cannot be completely subtracted because they are not even local for a general geometry. A previous study showed that it is not satisfactory to subtract a local function of just the inflaton and the *initial* Hubble parameter. This paper examines the other allowed possibility of subtracting a local function of the inflaton and the Ricci scalar. The problem in this case is that the new, scalar degree of freedom induced by the subtraction causes inflation to end almost instantly.

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* e-mail: spmiao5@mail.ncku.edu.tw

* email: park@fzu.cz

† e-mail: woodard@phys.ufl.edu

1 Introduction

Primordial inflation driven by the potential energy of a scalar inflaton,

$$\mathcal{L} = \frac{R\sqrt{-g}}{16\pi G} - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} - V(\varphi)\sqrt{-g}, \quad (1)$$

suffers from many fine tuning problems [1]. These include the need to make the potential very flat, the need to choose very special initial conditions to make inflation start, and the need to keep inflation predictive by avoiding the formation of a multiverse [2]. The implications of the increasingly stringent upper limits on the tensor-to-scalar ratio have caused some of the pioneers of inflation to question its testability [3, 4, 5].

This paper is aimed at a different sort of fine tuning problem which may prove equally serious: the Coleman-Weinberg corrections [6] to the inflaton potential that are generated when ordinary matter is coupled to the inflaton to facilitate re-heating. These corrections are too large to be ignored because they are not Planck-suppressed [7]. The usual assumption has been that Coleman-Weinberg corrections are local functions of the inflaton which could be completely subtracted off, however, it has recently been shown that cosmological Coleman-Weinberg corrections depend *nonlocally* on the metric [8], which precludes their complete subtraction.

There are two possible local subtraction schemes:

1. Subtract a local function of the inflaton which exactly cancels the cosmological Coleman-Weinberg potential at the beginning of inflation; or
2. Subtract a local function of the inflaton and the Ricci scalar which exactly cancels the cosmological Coleman-Weinberg potential when the first slow roll parameter vanishes.

A recent study of the first possibility concluded that it is not viable [9]. When the inflaton is coupled to fermions, inflaton never ends unless the coupling constant is chosen so small as to endanger re-heating, and then an initial reduction of the expansion rate still results in de Sitter expansion at a lower rate. When a charged inflaton is coupled to gauge bosons, inflation ends almost immediately, again unless the coupling constant is chosen so small as to endanger re-heating.

The purpose of this paper is to study the second possible subtraction scheme. Section 2 details the form of cosmological Coleman-Weinberg potentials for fermionic and for bosonic couplings. Section 3 gives the evolution equations associated with the second subtraction scheme. The effect on the simple $V(\varphi) = \frac{1}{2}m^2\varphi^2$ model is worked out for fermion and boson couplings in section 4. We discuss the results in section 5

2 Review of Past Work on the Problem

The purpose of this section is to explain the cosmological Coleman-Weinberg corrections from fermions and from gauge bosons, which differ profoundly from the simple $\mp\varphi^4 \ln(\varphi)$ form that pertains in flat space [6]. The section begins by reviewing explicit results from computations in de Sitter background. We then explain our assumption for how to generalize these de Sitter results to a general spatially flat, homogeneous and isotropic background geometry,

$$ds^2 = -dt^2 + a^2(t)d\vec{x}\cdot d\vec{x} \quad \implies \quad H(t) \equiv \frac{\dot{a}}{a} \quad , \quad \epsilon(t) \equiv -\frac{\dot{H}}{H^2} . \quad (2)$$

The section closes with a discussion of the Ricci subtraction scheme.

2.1 Fermion Corrections on de Sitter

Explicit results have so far only been obtained for de Sitter background, which corresponds to $\epsilon = 0$, with H exactly constant. Suppose the inflaton φ is Yukawa-coupled to a massless, Dirac fermion on this background via the interaction $\mathcal{L}_{\text{Yukawa}} = -\lambda\varphi\bar{\psi}\psi\sqrt{-g}$. The one loop correction to the inflaton effective potential on de Sitter was originally derived by Candelas and Raine in 1975 [10]. Of course their result depends slightly on conventions of regularization and renormalization. Our more recent computation [11] employed dimensional regularization in D spacetime dimensions with conformal and quartic counterterms,

$$\Delta\mathcal{L}^f = -\frac{1}{2}\delta\xi^f\varphi^2R\sqrt{-g} - \frac{1}{4!}\delta\lambda^f\varphi^4\sqrt{-g} . \quad (3)$$

To simplify the result we took the dimensional regularization scale μ to be proportional to the constant Hubble parameter of de Sitter,

$$\delta\xi_0^f = \frac{4\lambda^2 H^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(1-\frac{D}{2})}{D(D-1)} + \frac{(1-\gamma)}{6} + O(D-4) \right\} , \quad (4)$$

$$\delta\lambda_0^f = \frac{24\lambda^4 H^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \Gamma\left(1-\frac{D}{2}\right) + 2\zeta(3) - 2\gamma + O(D-4) \right\}, \quad (5)$$

where $\gamma = 0.577\dots$ is the Euler-Mascheroni constant. These choices result in a cosmological Coleman-Weinberg potential of the form $\Delta V_0^f(\varphi, H) = -\frac{H^4}{8\pi^2} \times f(z)$, where $z \equiv \frac{\lambda\varphi}{H}$, and the function $f(z)$ is,

$$f(z) = 2\gamma z^2 - [\zeta(3) - \gamma]z^4 + 2\int_0^z dx (x+x^3) [\psi(1+ix) + \psi(1-ix)], \quad (6)$$

and $\psi(x) \equiv \frac{d}{dx} \ln[\Gamma(x)]$ is the digamma function.

We assume that de Sitter results can be extended to general homogeneous and isotropic geometries (2) by simply replacing the constant de Sitter Hubble parameter with the time dependent $H(t)$ for a general background. However, we must be careful to keep the dimensional regularization scale constant, which amounts to a small change of the counterterms (4-5),

$$\delta\xi_1^f = \frac{4\lambda^2 \mu^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(1-\frac{D}{2})}{D(D-1)} + \frac{(1-\gamma)}{6} + O(D-4) \right\}, \quad (7)$$

$$\delta\lambda_1^f = \frac{24\lambda^4 \mu^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \Gamma\left(1-\frac{D}{2}\right) + 2\zeta(3) - 2\gamma + O(D-4) \right\}. \quad (8)$$

The net effect is to change $\Delta V_0^f(\varphi, H)$ to,

$$\Delta V_1^f(\varphi, H) = -\frac{H^4}{8\pi^2} \left\{ f(z) + z^2 \ln\left(\frac{H^2}{\mu^2}\right) + \frac{1}{2} z^4 \ln\left(\frac{H^2}{\mu^2}\right) \right\}. \quad (9)$$

2.2 Gauge Boson Corrections on de Sitter

The contribution of a gauge boson to a charged inflaton,

$$\mathcal{L}_{\text{vector}} = -(\partial_\mu + ieA_\mu)\varphi^*(\partial_\nu - ieA_\nu)\varphi g^{\mu\nu} \sqrt{-g}, \quad (10)$$

was originally computed on de Sitter background using mode sums by Allen in 1983 [12]. As always, the precise result depends on conventions of regularization and renormalization. Our more recent, dimensionally regulated computation [13] employed the massive photon propagator [14] with conformal and quartic counterterms,

$$\Delta\mathcal{L}^b = -\delta\xi^b \varphi^* \varphi R \sqrt{-g} - \frac{1}{4} \delta\lambda^b (\varphi^* \varphi)^2 \sqrt{-g}. \quad (11)$$

We again chose the dimensional regularization mass scale μ to be proportional to the constant de Sitter Hubble parameter,

$$\delta\xi_0^b = \frac{e^2 H^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{1}{4-D} + \frac{1}{2}\gamma + O(D-4) \right\}, \quad (12)$$

$$\delta\lambda_0^b = \frac{D(D-1)e^4 H^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{2}{4-D} + \gamma - \frac{3}{2} + O(D-4) \right\}, \quad (13)$$

These choices result in a cosmological Coleman-Weinberg potential of the form $\Delta V_0^b(\varphi^* \varphi, H^2) = +\frac{3H^4}{8\pi^2} \times b(z)$, where $z \equiv \frac{e^2 \varphi^* \varphi}{H^2}$, and $b(z)$ is,

$$b(z) = (-1+2\gamma)z + \left(-\frac{3}{2}+\gamma\right)z^2 + \int_0^z dx (1+x) \left[\psi\left(\frac{3}{2}+\frac{1}{2}\sqrt{1-8x}\right) + \psi\left(\frac{3}{2}-\frac{1}{2}\sqrt{1-8x}\right) \right]. \quad (14)$$

When generalizing the constant de Sitter Hubble parameter to the time-dependent one of a general homogeneous and isotropic geometry we must revise the counterterms (12-13) to keep the mass scale of dimensional regularization strictly constant,

$$\delta\xi_1^b = \frac{e^2 \mu^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{1}{4-D} + \frac{1}{2}\gamma + O(D-4) \right\}, \quad (15)$$

$$\delta\lambda_1^b = \frac{D(D-1)e^4 \mu^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{2}{4-D} + \gamma - \frac{3}{2} + O(D-4) \right\}. \quad (16)$$

The net effect is to change $\Delta V_0^b(\varphi^* \varphi, H^2)$ to,

$$\Delta V_1^b(\varphi^* \varphi, H^2) = +\frac{3H^4}{8\pi^2} \left\{ b(z) + z \ln\left(\frac{H^2}{\mu^2}\right) + \frac{1}{2}z^2 \ln\left(\frac{H^2}{\mu^2}\right) \right\}. \quad (17)$$

2.3 Ricci Subtraction

Ricci subtraction amounts to subtracting the primitive contribution with the replacement $H^2(t) \rightarrow \frac{1}{12}R(t)$. In a homogeneous and isotropic geometry this can be thought of as an ϵ -dependent Hubble parameter $\overline{H}(t)$,

$$R(t) = 12H^2(t) + 6\dot{H}(t) = 12\left[1 - \frac{1}{2}\epsilon(t)\right]H^2(t) \equiv 12\overline{H}^2(t). \quad (18)$$

Because the quartic terms cancel between the primitive potential and the Ricci subtraction, the full fermionic result takes the form $\Delta V_2^f(\varphi, H) - \Delta V_2^f(\varphi, \overline{H})$ where,

$$\Delta V_2^f(\varphi, H) = -\frac{H^4}{8\pi^2} \left\{ \Delta f(z) + z^2 \ln\left(\frac{H^2}{H_{\text{inf}}^2}\right) \right\} \quad , \quad z \equiv \frac{\lambda\varphi}{H} \quad , \quad (19)$$

$$\Delta f(z) = 2\gamma z^2 - \frac{1}{2}z^4 \ln(z^2) + 2\int_0^z dx (x+x^3) [\psi(1+ix) + \psi(1-ix)] \quad . \quad (20)$$

Note that we have chosen the constant mass scale to be the Hubble parameter at the beginning of inflation, $\mu = H_{\text{inf}}$. The full bosonic result takes the form $\Delta V_2^b(\varphi^*\varphi, H^2) - \Delta V_2^b(\varphi^*\varphi, \overline{H}^2)$ where,

$$\Delta V_2^b(\varphi^*\varphi, H^2) = +\frac{3H^4}{8\pi^2} \left\{ \Delta b(z) + z \ln\left(\frac{H^2}{H_{\text{inf}}^2}\right) \right\} \quad , \quad z \equiv \frac{e^2\varphi^*\varphi}{H^2} \quad , \quad (21)$$

$$\begin{aligned} \Delta b(z) = & \left(-1 + 2\gamma \right) z - \frac{1}{2} z^2 \ln(2z) \\ & + \int_0^z dx (1+x) \left[\psi\left(\frac{3}{2} + \frac{1}{2}\sqrt{1-8x}\right) + \psi\left(\frac{3}{2} - \frac{1}{2}\sqrt{1-8x}\right) \right] \quad . \quad (22) \end{aligned}$$

3 The Modified Evolution Equations

The purpose of this section is to work out the two Friedman equations for the case in which the cosmological Coleman-Weinberg potential depends on the Hubble parameter, and it is subtracted by a function which depends on the Ricci scalar. We also change to dimensionless dependent and independent variables.

It is useful to change the evolution variable from co-moving time t to the number of e-foldings from the *beginning* of inflation,

$$n \equiv \ln\left(\frac{a(t)}{a(t_i)}\right) \quad \implies \quad \frac{d}{dt} = H \frac{d}{dn} \quad , \quad \frac{d^2}{dt^2} = H^2 \left[\frac{d^2}{dn^2} - \epsilon \frac{d}{dn} \right] \quad . \quad (23)$$

It is also useful to make the dependent variables dimensionless,

$$\phi(n) \equiv \sqrt{8\pi G} \times \varphi(t) \quad , \quad \chi(n) \equiv \sqrt{8\pi G} \times H(t) \quad . \quad (24)$$

With these variables the slow roll approximation to the (already dimensionless) scalar power spectrum becomes,

$$\Delta_{\mathcal{R}}^2 \simeq \frac{GH^2}{\pi\epsilon} = \frac{1}{8\pi^2} \frac{\chi^2}{\epsilon} \quad . \quad (25)$$

Finally, it is natural to use a dimensionless potential and mass parameter,

$$U \equiv (8\pi G)^2 \times V \quad , \quad k^2 \equiv 8\pi G \times m^2 . \quad (26)$$

The simplest way of expressing the modified field equations is to imagine that the dimensionless form of the classical potential plus the primitive Coleman-Weinberg takes the form $U(\phi, \chi)$. The Ricci-subtraction takes the similar form $U_{\text{sub}}(\phi, \bar{\chi})$, where we define,

$$\bar{\chi} \equiv \sqrt{1 - \frac{1}{2}\epsilon} \chi = \sqrt{\chi^2 + \frac{1}{2}\chi\chi'} . \quad (27)$$

The two potentials enter the scalar evolution equation the same way,

$$\chi^2 [\phi'' + (3-\epsilon)\phi'] + \frac{\partial U}{\partial \phi} + \frac{\partial U_{\text{sub}}}{\partial \phi} = 0 . \quad (28)$$

However, the fact that the subtracted potential U_{sub} depends upon ϵ , in addition to χ , makes the form of its contributions to the gravitational field equations very different. The 1st Friedmann equation becomes,

$$3\chi^2 = \frac{1}{2}\chi^2\phi'^2 + U - \chi \frac{\partial U}{\partial \chi} + U_{\text{sub}} - \frac{1}{2}(1-\epsilon)\chi^2 \frac{\partial U_{\text{sub}}}{\partial \bar{\chi}^2} + \frac{1}{2}\chi^2 \frac{d}{dn} \frac{\partial U_{\text{sub}}}{\partial \bar{\chi}^2} . \quad (29)$$

The 2nd Friedmann equation is,

$$\begin{aligned} -(3-2\epsilon)\chi^2 &= \frac{1}{2}\chi^2\phi'^2 - U + \chi \frac{\partial U}{\partial \chi} + \frac{1}{3}\chi \frac{d}{dn} \frac{\partial U}{\partial \chi} \\ &\quad - U_{\text{sub}} + \frac{1}{2}\left(1-\frac{1}{3}\epsilon\right)\chi^2 \frac{\partial U_{\text{sub}}}{\partial \bar{\chi}^2} - \frac{1}{6}\chi^2 \left[\frac{d}{dn} + 2 - \epsilon\right] \frac{d}{dn} \frac{\partial U_{\text{sub}}}{\partial \bar{\chi}^2} . \end{aligned} \quad (30)$$

One consequence of the final term in equation (29) is that the first Friedmann equation involves second derivatives of $\chi(n)$. To see this, use the chain rule to exhibit the implicit higher derivatives,

$$\frac{1}{2}\chi^2 \frac{d}{dn} \frac{\partial U_{\text{sub}}}{\partial \bar{\chi}^2} = \frac{1}{2}\chi^2 \left\{ \phi' \frac{\partial^2 U_{\text{sub}}}{\partial \phi \partial \bar{\chi}^2} + \left[2\chi\chi' + \frac{1}{2}\chi'^2 + \frac{1}{2}\chi\chi'' \right] \frac{\partial^2 U_{\text{sub}}}{\partial \bar{\chi}^4} \right\} . \quad (31)$$

Recalling that $\epsilon = -\chi'/\chi$ allows us to express the first Friedmann equation (29) as,

$$\begin{aligned} \epsilon' &= -4\epsilon + 2\epsilon^2 + \frac{4}{\chi^4 \frac{\partial^2 U_{\text{sub}}}{\partial \bar{\chi}^4}} \left\{ -\chi^2 \left[3 - \frac{1}{2}\phi'^2 \right] + U + U_{\text{sub}} \right. \\ &\quad \left. - \chi^2 \left[2 \frac{\partial U}{\partial \chi^2} + \frac{1}{2}(1-\epsilon) \frac{\partial U_{\text{sub}}}{\partial \bar{\chi}^2} \right] + \frac{1}{2}\chi^2 \phi' \frac{\partial^2 U_{\text{sub}}}{\partial \phi \partial \bar{\chi}^2} \right\} . \end{aligned} \quad (32)$$

The natural initial conditions derive from the slow roll solutions for the purely classical model ($U = \frac{1}{2}k^2\phi^2$ and $U_{\text{sub}} = 0$),

$$\phi^2(n) \simeq \phi^2(0) - 4n \quad , \quad \chi^2(n) \simeq \frac{1}{6}k^2[\phi^2(0) - 4n] \quad , \quad \epsilon(n) \simeq \frac{2}{\phi^2(0) - 4n} . \quad (33)$$

Hence we obtain a 2-parameter family of initial conditions based on $\phi(0) \equiv \phi_0$ and k ,

$$\phi(0) = \phi_0 \quad , \quad \phi'(0) = -\frac{2}{\phi_0} , \quad (34)$$

$$\chi(0) = \frac{k\phi_0}{\sqrt{6 - (\frac{2}{\phi_0})^2}} \quad , \quad \chi'(0) = -\frac{2\chi_0}{\phi_0^2} . \quad (35)$$

Note that these initial conditions (34-35) exactly satisfy the classical Friedmann equation $3\chi_0^2 = \frac{1}{2}\chi_0^2\phi_0'^2 + \frac{1}{2}k^2\phi_0^2$ and also make the first slow roll parameter $\epsilon_0 = 2/\phi_0^2$ agree with the slow roll approximation (33).

Using the slow roll approximations (33) we see that $\phi_0 = 20$ will give about 100 total e-foldings of inflation. We can also express the power spectrum and the scalar spectral index in terms of the evolving first slow roll parameter $\epsilon(n)$,

$$\Delta_{\mathcal{R}}^2 \simeq \frac{1}{8\pi^2} \frac{\chi^2}{\epsilon} \longrightarrow \frac{1}{8\pi^2} \frac{k^2}{3\epsilon^2} , \quad (36)$$

$$1 - n_s \simeq 2\epsilon + \frac{\epsilon'}{\epsilon} \longrightarrow 4\epsilon . \quad (37)$$

Of course relations (36) and (37) allow us to determine the constant k in terms of the measured scalar amplitude A_s and spectral index n_s [15],

$$k \simeq \pi(1 - n_s) \sqrt{\frac{3}{2}A_s} \simeq 6.13 \times 10^{-6} . \quad (38)$$

4 The Fate of the $m^2\varphi^2$ Model

The purpose of this section is to numerically simulate the effect of primitive Coleman-Weinberg potentials with Ricci subtraction in the context of the classical $V_{\text{class}} = \frac{1}{2}m^2\varphi^2$ model. We begin with the case of fermionic corrections, and then discuss bosonic corrections. The generic problem in each case is that the $\chi''(n)$ terms in the first Friedmann equation (29) excite a new scalar degree of freedom that causes inflation to end almost immediately when starting from the classical initial conditions (34-35).

4.1 Fermionic Corrections

The Ricci subtraction scheme for fermionic corrections is defined by the potentials,

$$U^f(\phi, \chi) = \frac{1}{2}k^2\phi^2 - \frac{\chi^4}{8\pi^2} \left[\Delta f\left(\frac{\lambda\phi}{\chi}\right) + \left(\frac{\lambda\phi}{\chi}\right)^2 \ln\left(\frac{\chi^2}{\chi^2(0)}\right) \right], \quad (39)$$

$$U_{\text{sub}}^f(\phi, \bar{\chi}) = +\frac{\bar{\chi}^4}{8\pi^2} \left[\Delta f\left(\frac{\lambda\phi}{\bar{\chi}}\right) + \left(\frac{\lambda\phi}{\bar{\chi}}\right)^2 \ln\left(\frac{\bar{\chi}^2}{\chi^2(0)}\right) \right], \quad (40)$$

where $\Delta f(z)$ was defined in (20) and $\bar{\chi} \equiv \sqrt{1 - \frac{1}{2}\epsilon} \chi$. Figure 1 displays the classical evolution (in long yellow dashes) versus the quantum-corrected model (in blue dots) for a moderate coupling of $\lambda = 5.5 \times 10^{-4}$.

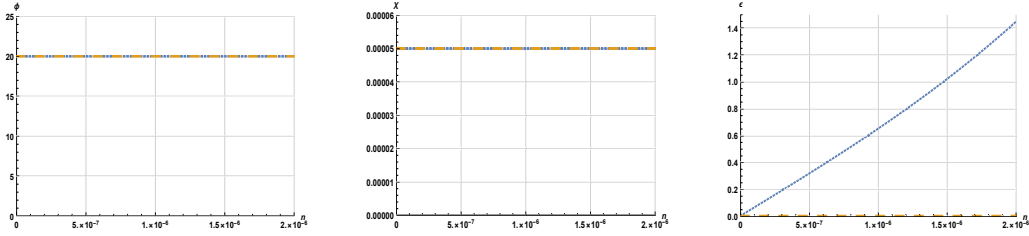


Figure 1: Plots of the dimensionless scalar $\phi(n)$ (on the left), the dimensionless Hubble parameter $\chi(n)$ (middle) and the first slow roll parameter $\epsilon(n)$ (on the right) for classical model (in long yellow dashes) and the quantum-corrected model (in blue dots) with Yukawa coupling $\lambda = 5.5 \times 10^{-4}$.

While the initial evolution of the scalar and the Hubble parameter is not visibly affected by the quantum correction, the first slow roll parameter rises above the inflationary threshold of $\epsilon = 1$ almost immediately.

To understand why Ricci subtraction engenders immediate deviations for $\lambda = 5.5 \times 10^{-4}$, first note that the initial conditions of the classical model (34-35) force the initial value of the parameter $z \equiv \frac{\lambda\phi}{\chi}$ to be much larger than one,

$$z_0 = \frac{\lambda\phi_0}{\chi_0} = \frac{\lambda}{k} \sqrt{6 - \left(\frac{2}{\phi_0}\right)^2} \simeq 220. \quad (41)$$

This means it is valid to use the large z expansion of (20) [8],

$$\Delta f(z) = -\frac{1}{4}z^4 + z^2 \ln(z^2) - \left(\frac{5}{6} - 2\gamma\right)z^2 + \frac{11}{60} \ln(z^2) + O(1). \quad (42)$$

Substituting (42) in expressions (39-40) implies,

$$U^f(\phi, \chi) = \frac{1}{2}k^2\phi^2 - \frac{\chi^4}{8\pi^2} \left\{ -\frac{1}{4}\left(\frac{\lambda\phi}{\chi}\right)^4 + \left[\ln\left(\frac{\lambda^2\phi^2}{\chi_0^2}\right) - \frac{5}{6} + 2\gamma\right]\left(\frac{\lambda\phi}{\chi}\right)^2 + \frac{11}{60}\ln\left(\frac{\lambda^2\phi^2}{\chi^2}\right) + \dots \right\}, \quad (43)$$

$$U_{\text{sub}}^f(\phi, \bar{\chi}) = \frac{\bar{\chi}^4}{8\pi^2} \left\{ -\frac{1}{4}\left(\frac{\lambda\phi}{\bar{\chi}}\right)^4 + \left[\ln\left(\frac{\lambda^2\phi^2}{\chi_0^2}\right) - \frac{5}{6} + 2\gamma\right]\left(\frac{\lambda\phi}{\bar{\chi}}\right)^2 + \frac{11}{60}\ln\left(\frac{\lambda^2\phi^2}{\bar{\chi}^2}\right) + \dots \right\}. \quad (44)$$

The $(\lambda\phi)^4$ terms cancel out between (43) and (44) so that the leading contribution usually comes from the $(\lambda\phi)^2$ term,

$$\begin{aligned} U^f + U_{\text{sub}}^f - \chi^2 \left[2\frac{\partial U^f}{\partial \chi^2} + \frac{1}{2}(1-\epsilon)\frac{\partial U_{\text{sub}}^f}{\partial \bar{\chi}^2} \right] \\ = \frac{1}{2}k^2\phi^2 + \frac{\chi^4}{8\pi^2} \times \frac{3}{2}z^2 \left[\ln\left(\frac{\lambda^2\phi^2}{\chi_0^2}\right) - \frac{5}{6} + 2\gamma \right] + \dots \end{aligned} \quad (45)$$

$$\frac{1}{2}\chi^2\phi' \frac{\partial^2 U_{\text{sub}}^f}{\partial \phi \partial \bar{\chi}^2} = \frac{\chi^4}{8\pi^2} \times \frac{\phi'}{\phi} z^2 \left[\ln\left(\frac{\lambda^2\phi^2}{\chi_0^2}\right) + \frac{1}{6} + 2\gamma \right] + \dots \quad (46)$$

However, the $(\lambda\phi)^2$ term makes no contribution to the denominator, so one must go one order higher,

$$\chi^4 \frac{\partial^2 U_{\text{sub}}^f}{\partial \bar{\chi}^4} = \frac{\chi^4}{8\pi^2} \times \frac{11}{60} \left[2\ln\left(\frac{\lambda^2\phi^2}{\bar{\chi}^2}\right) - 3 \right] + \dots \quad (47)$$

Taking account of the fact that the classical terms initially cancel, and that $\phi'_0/\phi_0 = -\epsilon_0$, the large z_0 form of (32) is,

$$\epsilon'_0 \simeq -4\epsilon_0 + 2\epsilon_0^2 + \frac{6z_0^2[\ln(z_0^2) - \frac{5}{6} + 2\gamma] - 4\epsilon_0 z_0^2[\ln(z_0^2) + \frac{1}{6} + 2\gamma]}{\frac{11}{60}[2\ln(z_0^2) - 2\ln(1 - \frac{1}{2}\epsilon_0) - 3]} \simeq 9.4 \times 10^5. \quad (48)$$

This compares with the slow roll result of $\epsilon'_0 = 2\epsilon_0^2 = 5 \times 10^{-5}$, and explains why the subtraction term brings inflation to such an abrupt end.

Because the large fraction in (48) scales like λ^2 one might expect that decreasing λ reduces $\epsilon'(0)$. This is indeed true for as long as the large z

regime pertains, but $\epsilon'(0)$ approaches a constant value of about 6 in the small z regime, as can be seen from Figure 2.

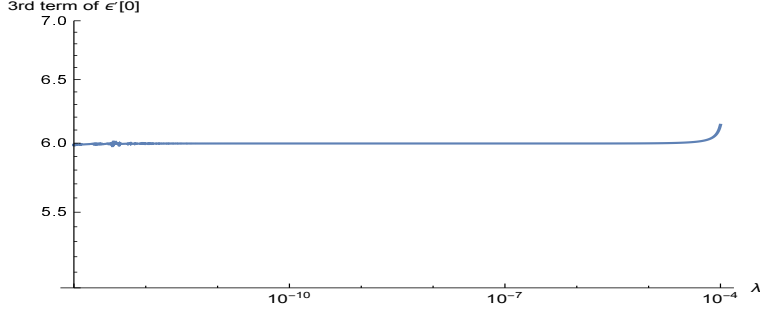


Figure 2: Log-log plot of the final term in relation (32) for $\epsilon'(0)$ as a function of λ in the range $10^{-13} < \lambda < 10^{-4}$.

The asymptotic limit of $\epsilon'(0) \simeq 6$ is still much too large, corresponding to only about one e-folding of inflation.

To understand the small z limit of $\epsilon'(0)$, note first that the small z expansion of $\Delta f(z)$ is [8],

$$\Delta f(z) = -\frac{1}{2}z^4 \ln(z^2) + [\zeta(3) - \gamma]z^4 + \frac{2}{3}[\zeta(3) - \zeta(5)]z^6 + O(z^8) . \quad (49)$$

Comparison with expressions (39) and (40) implies that the small λ limiting forms derive from the conformal renormalization,

$$U^f(\phi, \chi) - \frac{1}{2}k^2\phi^2 \longrightarrow -\frac{\lambda^2\phi^2\chi^2}{8\pi^2} \ln\left(\frac{\chi^2}{\chi_0^2}\right) , \quad (50)$$

$$U_{\text{sub}}^f(\phi, \bar{\chi}) \longrightarrow +\frac{\lambda^2\phi^2\bar{\chi}^2}{8\pi^2} \ln\left(\frac{\bar{\chi}^2}{\chi_0^2}\right) . \quad (51)$$

Substituting (50-51) in the final term of (32) gives,

$$\epsilon'_0 \longrightarrow -4\epsilon_0 + 2\epsilon_0^2 + \frac{6-2\epsilon_0+2(1-2\epsilon_0)\ln(1-\frac{1}{2}\epsilon_0)}{(1-\frac{1}{2}\epsilon_0)^{-1}} . \quad (52)$$

A striking feature of Figure 2 and expression (52) is that the limit $\lambda \rightarrow 0$ fails to agree with the case of $\lambda = 0$ for which there is no change to classical inflation. This seems contradictory but is in fact the standard signature of

a perturbation that changes the number of derivatives. A simple example is the higher derivative extension of the simple harmonic oscillator considered in section 2.2 of [16]. The oscillator's position is $x(t)$ and its Lagrangian is,

$$L = -\frac{\epsilon m}{2\omega^2}\ddot{x}^2 + \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2. \quad (53)$$

When $\epsilon = 0$ this system reduces to the simple harmonic oscillator which has two pieces of initial value data and whose energy is bounded below. However, for any nonzero value of ϵ the system has *four* pieces of initial value data, and its energy is unbounded below.

4.2 Corrections from Gauge Bosons

Making the inflaton complex causes a few small changes in the key equations of section 3. Because the two potentials $U^b(\phi^*\phi, \chi^2)$ and $U_{\text{sub}}^b(\phi^*\phi, \bar{\chi}^2)$ depend on the norm-squared of the scalar, the evolution equation for the inflaton becomes,

$$\chi^2[\phi'' + (3-\epsilon)\phi'] + \phi\left[\frac{\partial U^b(\phi^*\phi, \chi^2)}{\partial \phi^*\phi} + \frac{\partial U_{\text{sub}}^b(\phi^*\phi, \bar{\chi}^2)}{\partial \phi^*\phi}\right] = 0. \quad (54)$$

The first Friedmann equation takes the form,

$$3\chi^2 = \chi^2\phi'^*\phi' + U^b + U_{\text{sub}}^b - \chi^2\left[2\frac{\partial U^b}{\partial \chi^2} + \frac{1}{2}(1-\epsilon)\frac{\partial U_{\text{sub}}^b}{\partial \bar{\chi}^2}\right] + \frac{1}{2}\chi^2\frac{d}{dn}\frac{\partial U_{\text{sub}}^b}{\partial \bar{\chi}^2}. \quad (55)$$

This gives an evolution equation for the first slow roll parameter analogous to (32),

$$\begin{aligned} \epsilon' = -4\epsilon + 2\epsilon^2 + \frac{4}{\chi^4\frac{\partial^2 U_{\text{sub}}^b}{\partial \bar{\chi}^4}} \Bigg\{ & -\chi^2[3-\phi'^*\phi'] + U^b + U_{\text{sub}}^b \\ & -\chi^2\left[2\frac{\partial U^b}{\partial \chi^2} + \frac{1}{2}(1-\epsilon)\frac{\partial U_{\text{sub}}^b}{\partial \bar{\chi}^2}\right] + \frac{1}{2}\chi^2(\phi^*\phi)'\frac{\partial^2 U_{\text{sub}}^b}{\partial \phi^*\phi\partial \bar{\chi}^2} \Bigg\}. \quad (56) \end{aligned}$$

Even though we do not use it, the second Friedmann equation is,

$$\begin{aligned} -(3-2\epsilon)\chi^2 = \chi^2\phi'^*\phi' - U^b - U_{\text{sub}}^b + \chi^2\left[2\frac{\partial U^b}{\partial \chi^2} + \frac{1}{2}\left(1-\frac{1}{3}\epsilon\right)\frac{\partial U_{\text{sub}}^b}{\partial \bar{\chi}^2}\right] \\ + \frac{1}{3}\chi^2\left[\frac{d}{dn}-\epsilon\right]\frac{\partial U^b}{\partial \chi^2} - \frac{1}{6}\chi^2\left[\frac{d}{dn}+2-\epsilon\right]\frac{d}{dn}\frac{\partial U_{\text{sub}}^b}{\partial \bar{\chi}^2}. \quad (57) \end{aligned}$$

And the initial values (assuming ϕ_0 is real) become,

$$\phi(0) = \phi_0 \quad , \quad \phi'(0) = -\frac{2}{\phi_0} \quad , \quad (58)$$

$$\chi(0) = \frac{k\phi_0}{\sqrt{3 - (\frac{2}{\phi_0})^2}} \quad , \quad \chi'(0) = -\frac{2\chi_0}{\phi_0^2} \quad . \quad (59)$$

We continue to use $\phi_0 = 20$, with the value of k given in (38).

The Ricci subtraction scheme for bosons is defined by these potentials,

$$U^b(\phi^*\phi, \chi^2) = k^2\phi^*\phi + \frac{3\chi^4}{8\pi^2} \left[\Delta b\left(\frac{e^2\phi^*\phi}{\chi^2}\right) + \frac{e^2\phi^*\phi}{\chi^2} \ln\left(\frac{\chi^2}{\chi^2(0)}\right) \right] \quad , \quad (60)$$

$$U_{\text{sub}}^b(\phi^*\phi, \bar{\chi}^2) = -\frac{3\bar{\chi}^4}{8\pi^2} \left[\Delta b\left(\frac{e^2\phi^*\phi}{\bar{\chi}^2}\right) + \frac{e^2\phi^*\phi}{\bar{\chi}^2} \ln\left(\frac{\bar{\chi}^2}{\chi^2(0)}\right) \right] \quad , \quad (61)$$

where $\Delta b(z)$ was defined in (22) and $\bar{\chi} \equiv \sqrt{1 - \frac{1}{2}\epsilon} \chi$. Figure 3 compares the classical evolution (in blue) with the quantum-corrected (in red) for a charge $e^2 \simeq 2.9 \times 10^{-10}$ which is three hundred million times weaker than electromagnetism.

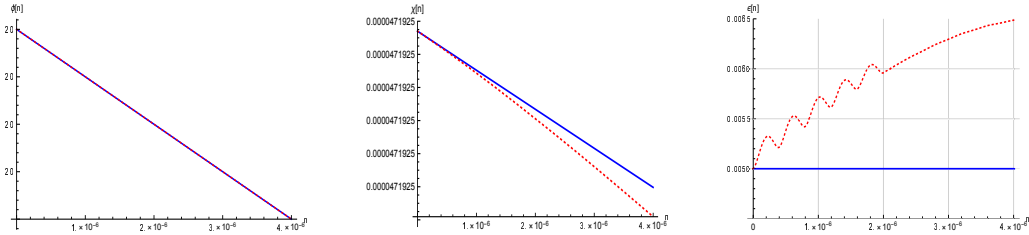


Figure 3: Plots of the dimensionless scalar $\phi(n)$ (on the left), the dimensionless Hubble parameter $\chi(n)$ (middle) and the first slow roll parameter $\epsilon(n)$ (on the right) for classical model (in blue) and the quantum-corrected model (in red) with the charge-squared $e^2 = \frac{4\pi}{137} \times 10^{-8.5} \simeq 2.9 \times 10^{-10}$.

The rapid onset of deviations from classical evolution evident in Figure 3 has the same explanation for bosons as for fermions. Even for the small coupling $e^2 \simeq 2.9 \times 10^{-10}$ the initial value of $z = \frac{e^2\phi^*\phi}{\chi^2}$ is larger than one,

$$z_0 \equiv \frac{e^2\phi_0^2}{\chi_0^2} = \frac{e^2}{k^2} \left[3 - \frac{4}{\phi_0^2} \right] \simeq 23.1 \quad . \quad (62)$$

Just as for fermions, this means we can simplify relation (56) using the large z expansion of $\Delta b(z)$ [8],

$$\Delta b(z) = -\frac{1}{4}z^2 + z \ln(2z) - \left(\frac{5}{3} - 2\gamma\right)z + \frac{19}{60} \ln(z) + O(1) . \quad (63)$$

The corresponding large argument expansions of the potentials are,

$$U^b(\phi^*\phi, \chi^2) = k^2\phi^*\phi + \frac{3\chi^4}{8\pi^2} \left\{ -\frac{1}{4} \left(\frac{e^2\phi^*\phi}{\chi^2} \right)^2 + \left[\ln\left(\frac{2e^2\phi^*\phi}{\chi_0^2} \right) - \frac{5}{3} + 2\gamma \right] \frac{e^2\phi^*\phi}{\chi^2} + \frac{19}{60} \ln\left(\frac{e^2\phi^*\phi}{\chi^2} \right) + \dots \right\}, \quad (64)$$

$$U_{\text{sub}}^b(\phi^*\phi, \bar{\chi}^2) = -\frac{3\bar{\chi}^4}{8\pi^2} \left\{ -\frac{1}{4} \left(\frac{e^2\phi^*\phi}{\bar{\chi}^2} \right)^2 + \left[\ln\left(\frac{2e^2\phi^*\phi}{\chi_0^2} \right) - \frac{5}{3} + 2\gamma \right] \frac{e^2\phi^*\phi}{\bar{\chi}^2} + \frac{19}{60} \ln\left(\frac{e^2\phi^*\phi}{\bar{\chi}^2} \right) + \dots \right\}. \quad (65)$$

Just as for fermions, the order $e^2\phi^*\phi$ terms in (64-65) make the dominant contributions to the numerator of expression (56), but the denominator is a crucial order weaker,

$$\epsilon'_0 \simeq -4\epsilon_0 + 2\epsilon_0^2 + \frac{6z_0[\ln(2z_0) - \frac{5}{3} + 2\gamma] - 4\epsilon_0 z_0[\ln(2z_0) - \frac{2}{3} + 2\gamma]}{\frac{19}{60}[2\ln(z_0) - 2\ln(1 - \frac{1}{2}\epsilon_0) - 3]} \simeq 441 . \quad (66)$$

Just as we found for fermions, $\epsilon'(0)$ can be decreased by reducing the coupling constant, but it eventually approaches a value that is still much too large. This can be seen from Figure 4.

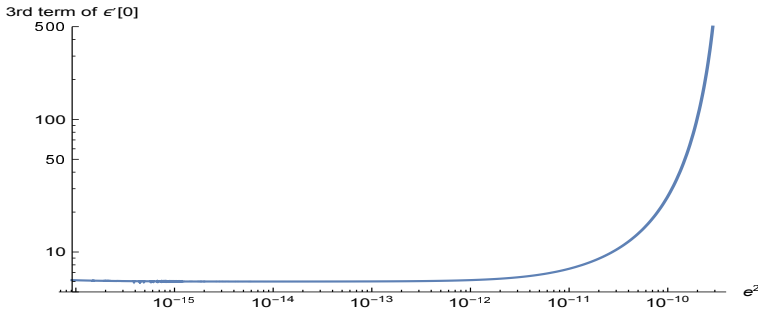


Figure 4: Plot of the 3rd term in expression (56) for $\epsilon'(0)$ as a function of e^2 .

The analytic derivation follows from the small e^2 limiting forms of the quantum part of U^b and U_{sub}^b ,

$$U^b(\phi^*\phi, \chi^2) - k^2\phi^*\phi \longrightarrow \frac{3e^2\phi^*\phi\chi^2}{8\pi^2} \ln\left(\frac{\chi^2}{\chi_0^2}\right), \quad (67)$$

$$U_{\text{sub}}^b(\phi^*\phi, \bar{\chi}^2) \longrightarrow -\frac{3e^2\phi^*\phi\bar{\chi}^2}{8\pi^2} \ln\left(\frac{\bar{\chi}^2}{\chi_0^2}\right). \quad (68)$$

The analysis, and even the result, is the same as for fermions. Note that the limit $e^2 \rightarrow 0$, for which there is always an instability, again fails to agree with $e^2 = 0$ model, which is classical inflation.

5 Discussion

Cosmological Coleman-Weinberg potentials are induced when the inflaton is coupled to ordinary matter, typically to facilitate re-heating. Without subtraction, these potentials are disastrous to inflation because they are far too steep and not Planck-suppressed. If they depended only on the inflaton it would be straightforward to subtract them but they also involve the metric in a deep and profound way. Explicit computations on de Sitter background, for fermions [10, 11] and for vector bosons [12, 13], reveal complicated functions of the dimensionless ratio of the coupling constant times the inflaton, all divided by the Hubble parameter. Indirect arguments show that the constant Hubble parameter of de Sitter in this ratio cannot be constant for a general geometry, nor can it even be local [8]. That poses a major obstacle to subtracting away cosmological Coleman-Weinberg potentials because only local functions of the inflaton and the Ricci scalar can be employed [17], and neither can completely subtract the potentials.

A previous study explored the possibility of subtracting a function of the just inflaton, chosen to completely cancel the cosmological Coleman-Weinberg potential at the onset of inflation [9]. What we found for moderate coupling constants is that inflation never ends for the corrections due to fermions, and it ends too soon for the corrections due to vector bosons. Making the Yukawa coupling very small results in a nearly classical evolution until late times, at which point the universe approaches de Sitter with a much smaller Hubble parameter. An acceptable evolution can only be obtained by making the vector boson coupling very small, and this degrades the efficiency of re-heating.

This paper studied the other possibility: subtracting a function of the inflaton and the Ricci scalar. One might think (as we originally hoped) that corrections for this type of subtraction would be suppressed by the smallness of the first slow roll parameter. However, the higher time derivatives in the subtraction change the first Friedmann equation (29) from being algebraic in the Hubble parameter to containing second derivatives of it, and the particular way (32) this change manifests is fatal for inflation. We were able to construct an analytic proof (52) — supported by explicit numerical analysis in Figures 2 and 4 — that the initial value of ϵ' can never be less than about 6. That compares with its initial value of 5×10^{-5} in the classical model, and it means that inflation cannot last more than a single e-folding. So the Ricci subtraction scheme is much worse than the initial time subtraction, but neither method is satisfactory.

Before closing we should make a few comments. First, the problem with $\epsilon'(0)$ is almost completely independent of the classical model of inflation. So one should not expect that a different model would lead to a different result. Second, we need better control of the ϵ dependence of cosmological Coleman-Weinberg potentials. The present study was carried out by assuming that the constant Hubble parameter of de Sitter background becomes the instantaneous Hubble parameter of an evolving geometry. In reality, the cosmological Coleman-Weinberg potential should depend as well on $\epsilon(n)$ [18]. Accounting for this dependence will tighten the argument, and we expect it to extend the $\epsilon'(0)$ problem even to the initial time subtraction. Finally, it should be possible to extend these studies to the case in which derivatives of the inflaton are coupled to matter. Such a coupling would not induce a cosmological Coleman-Weinberg potential but would change the kinetic term. It would be very interesting to work out the consequences for evolution and the generation of perturbations.

Acknowledgements

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