UFIFT-QG-19-06, CP3-19-43

Breaking of scaling symmetry by massless scalar on de Sitter

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ABSTRACT

We study the response of a classical massless minimally coupled scalar to a static point scalar charge on de Sitter. By considering explicit solutions of the problem we conclude that – even though the dynamics formally admits dilatation (scaling) symmetry – the physical scalar field profile necessarily breaks the symmetry. This is an instance of symmetry breaking in classical physics due to large infrared effects. The gravitational backreaction, on the other hand, does respect dilatation symmetry, making this an example of symmetry non-inheritance phenomenon.

PACS numbers: 04.50.Kd, 95.35.+d, 98.62.-g

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Point particle and scaling solution. In this note we investigate the system of a massless minimally coupled scalar (MMCS) field Φ in de Sitter space coupled to a scalar point charge. The action for the MMCS in an arbitrary curved space is given by,

$$S_0[\Phi] = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} (\partial_\mu \Phi) (\partial_\nu \Phi) \right] , \tag{1}$$

where $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$, $g = \det[g_{\mu\nu}]$, the metric signature is $(-,+,+,+)$ and its coupling to the point particle $\chi^\mu = \chi^\mu(\tau)$ is modeled by the action,

$$S_{\rm int}[\chi, \Phi] = - \int d\tau \sqrt{-g_{\mu\nu} \dot{\chi}^\mu(\tau) \dot{\chi}^\nu(\tau)} \, \lambda \Phi(\chi(\tau)) , \tag{2}$$

where τ is an affine parameter, λ is a dimensionless coupling, and $\dot{\chi}^\mu(\tau) = d\chi^\mu(\tau)/d\tau$. We assume that the point particle is at rest, sitting at the origin of the coordinate system on flat spatial slices of the Poincaré patch, $\chi^\mu(\tau) = (\tau, 0, 0, 0)$. The equation of motion for the MMCS descends from variation of the action (1) and (2),

$$\Box \Phi(x) = -\frac{1}{a^2} \Big(\partial_0^2 + 2aH\partial_0 - \nabla^2 \Big) \Phi(x) = \lambda \frac{\delta^3(\vec{x})}{a^3} , \tag{3}$$

where $a(\eta) = -1/(H\eta)$ is the scale factor of de Sitter space with η confrormal time, $H = (\partial_0 a)/a^2$ the (constant) Hubble rate, $\partial_0 = \partial/\partial\eta$ and ∇^2 is the Laplacian. While the sourceless equation would respect all of the isometries of de Sitter, the point source (3) breaks spatial special conformal transformations and spatial translations, leaving us with only *four* isometries, namely spatial rotations and dilatations, $x^\mu \to e^\alpha x^\mu$ with $\alpha \in \mathbb{R}$.

It is most natural to assume that the solution of (3) satisfies the background isometries, and that it depends only on the rotation-invariant and dilatation-invariant combination of coordinates $X = aHr$, $r = \|\vec{x}\|$, *i.e.* $\Phi(\eta, \vec{x}) \to \Phi(X)$, also known as the *scaling solution*, upon which the equation of motion (3) away from the origin turns into an ordinary one,

$$\left[(1 - X^2) \frac{d}{dX} + \frac{2}{X} (1 - 2X^2) \right] \frac{d}{dX} \Phi(X) = 0 . \tag{4}$$

This equation is integrated straightforwardly, and its general solution is

$$\Phi(X) = -\frac{\lambda H}{4\pi X} - \frac{\lambda H}{8\pi} \ln\!\left(\frac{1 - X}{1 + X} \right) + \Phi_0 . \tag{5}$$

One integration constant is completely fixed by the δ-function source term by means of the Green's integral theorem, while the remaining trivial constant Φ_0 remains undetermined.

A closer examination of the solution in (5) reveals some worrisome features. Most notably, the solution exhibits a logarithmic singularity at the horizon! At a first glance there seems to be nothing wrong with our assumptions. Perhaps it is that strong infrared effects that are known to exist for MMCS in de Sitter conspire to create, in a manner of speaking, a classical wall of fire – a barrier at which the geodesic equation for a test particle becomes singular.

That (5) cannot be a physical solution can be seen by considering the energy-momentum tensor, $T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi$, accompanying the solution (5), which in spherical coordinates reads,

$$T^\mu{}_\nu = H^2 \begin{pmatrix} -\frac{1}{2}(1+X^2) & -X & 0 & 0 \\ X & \frac{1}{2}(1+X^2) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(1-X^2) & 0 \\ 0 & 0 & 0 & -\frac{1}{2}(1-X^2) \end{pmatrix} \left(\frac{\partial\Phi}{\partial X}\right)^2 . \tag{6}$$

Near the horizon it diverges quadratically, as can be easily seen from,

$$\left(\frac{\partial\Phi}{\partial X}\right)^2 \overset{X\to1}{\sim} \frac{\lambda^2 H^2}{64\pi^2}\frac{1}{(1-X)^2} . \tag{7}$$

This divergence of the diagonal terms would generate a large classical backreaction onto the background space-time. In particular, there is a positive radial energy density flux $T^r{}_0$, which also diverges quadratically at the horizon. While the divergence at the origin $\propto 1/(ar)^4$ is the usual divergence generated by a point charge that is dealt with in the usual way, the divergence at the Hubble horizon cannot be a part of the physical solution. In order to shed light on the origin of the problem, in the next section we consider the equivalent problem for a massive scalar and construct a solution that is regular everywhere except at the origin.

Massive scalar on de Sitter. A massive scalar field satisfies the equation of motion,

$$\Big(\Box - m^2\Big)\Phi(x) = \lambda\frac{\delta^3(\vec{x})}{a^3} . \tag{8}$$

This equation still possesses dilatation symmetry, and thus admits a scaling solution that away from the origin satisfies a homogeneous equation,

$$\left[(1-X^2)\frac{d^2}{dX^2}+\frac{2}{X}(1-2X^2)\frac{d}{dX}-\frac{m^2}{H^2}\right]\Phi(X)=0\,. \tag{9}$$

The general solution can be written in terms of two hypergeometric functions,

$$\begin{aligned}\Phi(X)=&-\frac{\lambda H}{4\pi X}\times{}_2F_1\biggl(\Bigl\{\frac{1}{4}+\frac{\nu}{2},\frac{1}{4}-\frac{\nu}{2}\Bigr\},\Bigl\{\frac{1}{2}\Bigr\},X^2\biggr)\\&+\frac{\lambda H}{2\pi}\times\frac{\Gamma\bigl(\frac{3}{4}+\frac{\nu}{2}\bigr)\,\Gamma\bigl(\frac{3}{4}-\frac{\nu}{2}\bigr)}{\Gamma\bigl(\frac{1}{4}+\frac{\nu}{2}\bigr)\,\Gamma\bigl(\frac{1}{4}-\frac{\nu}{2}\bigr)}\times{}_2F_1\biggl(\Bigl\{\frac{3}{4}+\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2}\Bigr\},\Bigl\{\frac{3}{2}\Bigr\},X^2\biggr)\,,\end{aligned} \tag{10}$$

where,

$$\nu=\sqrt{\frac{9}{4}-\frac{m^2}{H^2}}\,. \tag{11}$$

The constant in front of the first hypergeometric function is fixed by the source in (8), while the second one is fixed by the requirement of regularity at the horizon. Moreover, the behaviour of the solution for $X\to\infty$ is regular. One can add to (10) a homogeneous solution that breaks scaling symmetry, but such contributions tend to be subdominant at late times.

Examining the result (10) in the small mass limit is instructive for understanding the issues involved in the massless scaling solution (5),

$$\begin{aligned}\Phi(X)\overset{m\to0}{\sim}&-\frac{\lambda H}{4\pi X}-\frac{\lambda H}{8\pi}\ln\biggl(\frac{1-X}{1+X}\biggr)\\&-\frac{\lambda H}{2\pi}\biggl[\frac{3H^2}{2m^2}+\ln(2)-\frac{7}{6}\biggr]+\frac{\lambda H}{8\pi}\ln\bigl(1-X^2\bigr)\,,\end{aligned} \tag{12}$$

The first line in this expansion comes from the first line of the full solution (10), and reproduces the massless solution (5) up to a constant. The second line above comes from the small mass expansion of the second line in (10). It is clear there is no singularity at the horizon even in this limit. However, it is also clear that this limit is singular due to the constant term $\sim 1/m^2$. One might try to employ the observation that the massless solution (5) is defined up to a constant in order to remove the divergent term above. This though does not work, as (12) with the divergent constant removed simply does not satisfy the massless equation of motion (4). The proper conclusion is that the scaling solution of our problem is singular in the massless limit, and (5) does not represent a valid physical solution. In other words, there is no scaling solution for the massless case that is regular away from the origin.

The physical interpretation of this behavior is clear: the point source generates a large amount of classical infrared scalar modes such that it breaks scaling (dilatation) symmetry in the limit of small mass. This is the reason behind why the naîve scaling solution (5) we found in the massless case has a pathological behavior at the horizon. The small mass behavior in (12) is reminiscent of the well understood massless limit of the MMCS propagator in de Sitter space, which we briefly recap in the following.

Scalar propagator in de Sitter. The small mass behavior in (12) is reminiscent of the better known example in linear quantum physics in de Sitter space. There exists a de Sitter invariant two-point Wightman function for a massive scalar in de Sitter [1],

$$\big\langle \hat{\phi}(x)\hat{\phi}(x')\big\rangle = \frac{H^2\,\Gamma\big(\frac{3}{2}+\nu\big)\,\Gamma\big(\frac{3}{2}-\nu\big)}{(4\pi)^2} \times {}_2F_1\bigg(\bigg\{\frac{3}{2}+\nu,\frac{3}{2}-\nu\bigg\},\bigg\{2\bigg\},1-\frac{y}{4}\bigg)\,, \tag{13}$$

where ν is again the one from (11), and y is the de Sitter invariant function of the coordinates,

$$y(x;x') = a(\eta)a(\eta')H^2\Big[\|\vec{x}-\vec{x}\,'\|^2 - \big(\eta-\eta'-i\varepsilon\big)^2\Big]\,. \tag{14}$$

The small mass expansion of this expression is,

$$\big\langle \hat{\phi}(x)\hat{\phi}(x')\big\rangle \overset{m\to 0}{\sim} \frac{H^2}{(2\pi)^2}\bigg[\frac{1}{y} - \frac{1}{2}\ln(y) + \frac{3H^2}{2m^2} + \ln(2) - \frac{11}{12}\bigg]\,, \tag{15}$$

which tells us there is no physical and finite de Sitter invariant solution for the massless scalar field due to strong infrared effects. However, demanding that the state respects only spatial homogeneity and isotropy yields a perfectly physical behavior [2, 3, 4, 5, 6, 7, 8, 9],

$$\big\langle \hat{\phi}(x)\hat{\phi}(x')\big\rangle = \frac{H^2}{(2\pi)^2}\bigg[\frac{1}{y} - \frac{1}{2}\ln(y) + \frac{1}{2}\ln(aa') + 1 - \gamma_E\bigg]\,, \tag{16}$$

where the (non-universal) constant is fixed by taking the $D=4$ limit of the massless scalar propagator from [10]. This lesson prompts us to look for a physical solution in the case at hand which does not respect the background isometries to resolve the conundrum.

Breaking of dilatation symmetry. Here we derive the solution of (3) by using the Green's function method. Let us assume that the scalar point charge starts acting on the scalar field at some initial moment of time η_0. We use the method of Green's function to determine the reaction of the scalar field to this charge. The retarded Green's functions for a massless scalar field on de Sitter space can be straightforwardly obtained from (16),

$$G_R(x;x') = -\frac{\theta(\Delta\eta)}{2\pi}\left[\frac{\delta\big(\Delta\eta^2-\|\Delta\vec{x}\|^2\big)}{a(\eta)\,a(\eta')} + \frac{H^2}{2}\theta\big(\Delta\eta-\|\Delta\vec{x}\|\big)\right], \tag{17}$$

where $\Delta\eta=\eta-\eta'$, and $\Delta\vec{x}=\vec{x}-\vec{x}'$. The scalar potential that solves (3) is now obtained by integrating the retarded Green's function against the point source,

$$\Phi(\eta,r) = \int_{\eta_0}^{0} d\eta' \int d^3x'\, a^4(\eta')\, G_R(x;x') \;\times \lambda\frac{\delta^3(\vec{x}')}{a^3(\eta')}\,, \tag{18}$$

which evaluates to,

$$\Phi(\eta,r) = \theta\big(\eta-\eta_0-r\big)\left[-\frac{\lambda H}{4\pi X} - \frac{\lambda H}{4\pi}\ln\!\left(\frac{a}{1+X}\right)\right], \tag{19}$$

where $\eta_0 = -1/H$ such that $a(\eta_0) = 1$. The step function in front of the solution accounts for causality, restricting the effect of the interaction to within the forward light cone of the source. Of course, Green's second identity includes surface integrations of the Green's function (and its derivative) times the solution (and its derivative) on the initial value surface. Eq. (19) has implicitly assumed that the solution and its first time derivative vanish at $\eta = \eta_0$. It is more natural to take the initial values from the term inside the square brackets, in which case the solution becomes,

$$\Phi(\eta,r) = -\frac{\lambda H}{4\pi X} - \frac{\lambda H}{4\pi}\ln\!\left(\frac{a}{1+X}\right). \tag{20}$$

From the point of view of a local observer on de Sitter, Eq. (20) is valid on the entire manifold. The solution (20) is the *principal result* of this letter. It can be obtained by adding to (5) a homogeneous solution, $\Phi_h = \frac{\lambda H}{8\pi}\ln[(1–X^2)/a^2]$, resulting in a solution that is *regular everywhere* except at the origin. However, the scaling symmetry is broken by the term $\propto \ln(a)$. It should be noted that at late times and at large radial separations the dominant contribution is time-independent and grows logarithmically with the comoving distance,

$$\Phi(\eta, r) \overset{r\to\infty}{\sim} \frac{\lambda H}{4\pi} \ln\bigl(Hr\bigr) \,. \tag{21}$$

The energy-momentum tensor for (20) reads,

$$T^{\mu}{}_{\nu} = \frac{\lambda^2 H^4}{32\pi^2} \begin{pmatrix} -\Theta^2 - \Psi^2 & -2\Theta\Psi & 0 & 0 \\ 2\Theta\Psi & \Theta^2 + \Psi^2 & 0 & 0 \\ 0 & 0 & \Theta^2 - \Psi^2 & 0 \\ 0 & 0 & 0 & \Theta^2 - \Psi^2 \end{pmatrix} , \tag{22}$$

with $\Psi = \frac{1}{X^2} + \frac{1}{1+X}$, $\Theta = X\Psi - 1$. It is regular everywhere away from the origin and decays as $\sim 1/X^2$ for large radial distances. Remarkably, the energy-momentum tensor in (22) respects dilatation symmetry, even though the field profile in (20) does not. This is a cosmological example of the phenomenon of (perturbative) *symmetry non-inheritance*, which has attracted significant attention in recent literature [11, 12, 13, 14, 15].

Summary and discussion. We investigate the classical response of a massless scalar field to a static point-like scalar charge on de Sitter. The point charge breaks spatial special conformal isometries, as well as spatial translations of de Sitter space. The resulting equation (3) possesses only four isometries, namely spatial rotations and dilatations, also known as global scaling transformations. We show that any solution that respects all four isometries exhibits a logarithmic singularity at the Hubble horizon, making this naîve solution (5) unphysical. Inspired by the quantum case of a massless scalar propagator on de Sitter, we then show that the classical physical solution (20) necessarily breaks scaling symmetry and it is regular everywhere except at the point charge location. Remarkably, the energy-momentum tensor associated with this solution *does* respect dilatation symmetry. Therefore, our solution provides an example of the phenomenon of symmetry non-inheritance in gravitational systems [11, 12, 13, 14, 15]. Our analysis can be generalized to D space-time dimensions, in which case the naîve scaling solution also exhibits a logarithmic singularity at the horizon, [1] and therefore the physical solution must break scaling symmetry in arbitrary number of dimensions.

[1] The scaling solution which generalizes (5) to D dimensions is,

$$\Phi(X) = -\frac{\lambda H^{D-3}\Gamma\left(\frac{D-3}{2}\right)}{4\pi^{\frac{D-1}{2}} X^{D-3}} \times {}_2F_1\left(\left\{1, \frac{3-D}{2}\right\}, \left\{\frac{5-D}{2}\right\}, X^2\right) + \Phi_0 \,,$$

It would be of interest to study physical consequences of such a classical breaking of scaling symmetry, and in particular whether there are observable late time effects of this symmetry breaking. For example, our solution can be helpful for improving our understanding of how point charges in inflation affect temperature fluctuations in the cosmic microwave background radiation [16, 17].

After the completion of this work it came to our attention that some of our results, including equation (20), but not the breakdown of the dilatation invariant solution (5), were previously obtained by Akhmedov, Roura and Sadofyev [18].

Acknowledgements. We are grateful to Ivica Smolić for a critical reading of the manuscript and for drawing our attention to the symmetry noninheritance phenomena. This work was partially supported by the Fonds de la Recherche Scientifique – FNRS under Grant IISN 4.4517.08 – Theory of fundamental interactions, by Taiwan MOST grants 103-2112-M-006-001-MY3 and 107-2119-M-006-014; by the D-ITP consortium, a program of the Netherlands Organization for Scientific Research (NWO) that is funded by the Dutch Ministry of Education, Culture and Science (OCW); by NSF grants PHY-1806218 and PHY-1912484; and by the Institute for Fundamental Theory at the University of Florida.

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which exhibits a logarithmic singularity at the horizon,

$$\Phi(X) \overset{X\to 1}{\sim} -\frac{\lambda H^{D-3}\,\Gamma\big(\frac{D-1}{2}\big)}{4\pi^{\frac{D-1}{2}}}\ln\big(1-X\big)\,.$$

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