SIAM J. MATH. ANAL.
Vol. 51, No. 5, pp. 4074–4104

ON THE EXISTENCE AND INSTABILITY OF SOLITARY WATER WAVES WITH A FINITE DIPOLE*

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Abstract. This paper considers the existence and stability properties of two-dimensional solitary waves traversing an infinitely deep body of water. We assume that above the water is air and that the waves are acted upon by gravity with surface tension effects on the air-water interface. In particular, we study the case where there is a finite dipole in the bulk of the fluid, that is, the vorticity is a sum of two weighted δ-functions. Using an implicit function theorem argument, we construct a family of solitary waves solutions for this system that is exhaustive in a neighborhood of 0. Our main result is that this family is conditionally orbitally unstable. This is proved using a modification of the Grillakis–Shatah–Strauss method recently introduced by Varholm, Wahlén, and Walsh.

Key words. finite dipole, point vortices, solitary water waves, existence, instability, spectrum

AMS subject classifications. 35Q35, 37K45, 76B25, 35B35

DOI. 10.1137/18M1231638

1. Introduction.

This paper is motivated by the following simple experiment. Imagine that a surface water wave passes over a thin submerged body. Boundary layer effects may then produce so-called *shed vortices*—highly localized vortical regions in the object's wake. A natural idealization for this phenomenon is a *finite dipole*, which is a solution of a weakened version of the Euler equations whose vorticity ω consists of a pair of Dirac δ-measures (called point vortices) of nearly opposite strength that are separated by a fixed distance.

It is well-known that, if the problem is posed in the plane, then there are exact (stable) solutions for which the pair of vortices translate in parallel at a fixed velocity. Here, we wish to study the far more complicated situation where the dipole lies inside a water wave. We prove that there exist traveling wave solutions to this system. However, our main result shows that they are conditionally orbitally unstable. Physically, this indicates that a pair of counterrotating shed vortices moving with a wave will not persist over long periods of time. For instance, they may approach and then breach the surface.

1.1. Main equations.

For each time $t \geq 0$, let $\Omega_t \subset \mathbb{R}^2$ be the fluid domain:

$$\Omega_t := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 < \eta(t, x_1)\right\},$$

where the a priori unknown function $\eta = \eta(t, x_1)$ describes the free surface between air and water. We define the water wave with a finite dipole problem as follows.

Let $v = v(t, \cdot) : \Omega_t \to \mathbb{R}^2$ be the fluid velocity. The vorticity $\omega = \omega(t, \cdot) : \Omega_t \to \mathbb{R}$ is the (scalar) curl of v. In the physics literature, a finite dipole consists of two counterrotating point vortices. Mathematically, this means that we look for solutions with

$$\omega := \partial_{x_2} v_1 - \partial_{x_1} v_2 = -\epsilon\gamma_1 \delta_{\bar{x}} + \epsilon\gamma_2 \delta_{\bar{y}}, \tag{1.1}$$

*Received by the editors December 10, 2018; accepted for publication (in revised form) September 9, 2019; published electronically October 17, 2019.
https://doi.org/10.1137/18M1231638
Funding: The work of the author was supported by National Science Foundation grants DMS-1549934 and DMS-1710989.
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in the sense of distributions. Here $\bar{x} = \bar{x}(t)$ and $\bar{y} = \bar{y}(t)$ are the vortex centers, and $\epsilon\gamma_1$ and $-\epsilon\gamma_2$ are the strengths, respectively. We require that v is a weak solution of the incompressible Euler equations away from the two point vortices:

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \nabla p + g e_2 = 0 \quad \text{in } \Omega_t \backslash \{\bar{x}, \bar{y}\}, \\ \nabla \cdot v = 0 \quad \text{in } \Omega_t. \end{cases} \tag{1.2}$$

We assume that there is finite excess kinetic energy, which corresponds to $v(t, \cdot) \in L^1_{\text{loc}}(\Omega_t) \cap L^2(\Omega_t \backslash N_t)$ for any open set N_t containing $\{\bar{x}, \bar{y}\}$.

On the free surface $S_t := \partial\Omega_t$, we have the kinematic and dynamic boundary condition:

$$\partial_t \eta = -\eta' v_1 + v_2, \qquad p = b\kappa \quad \text{on } S_t, \tag{1.3}$$

where primes indicate derivatives with respect to x_1, and $\kappa = \kappa(t, x_1)$ is the mean curvature of the surface

$$\kappa(t, x_1) = -\frac{\eta''(t, x_1)}{\langle \eta'(t, x_1) \rangle^3}.$$

Here we are using the bracket notation: $\langle \cdot \rangle := (1 + (\cdot)^2)^{\frac{1}{2}}$. The constant $b > 0$ in (1.3) is the coefficient of surface tension.

Finally, the motion of the vortices is governed by the Kirchhoff–Helmholtz model [13, 15]:

$$\begin{cases} \partial_t \bar{x} = \left(v - \frac{\gamma_1}{2\pi} \epsilon \nabla^\perp \log |x - \bar{x}| \right)\Big|_{\bar{x}}, \\ \partial_t \bar{y} = \left(v + \frac{\gamma_2}{2\pi} \epsilon \nabla^\perp \log |x - \bar{y}| \right)\Big|_{\bar{y}} \end{cases} \tag{1.4}$$

with $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. This system mandates that the point vortices are transported by the irrotational part of the fluid velocity field and also attract each other due to the opposite vortex strengths.

1.2. Statement of main results. We are interested in both showing the existence of solitary waves solutions to (1.1)–(1.4) and determining their stability. As long as the two point vortices are separated from each other and the surface, the fluid velocity v can be decomposed as

$$v = \nabla\Phi + \epsilon\nabla\Theta \tag{1.5}$$

in a neighborhood of S_t, where Φ is a harmonic function and Θ represents the influence of the dipole. Note that Θ can be written explicitly in terms of $\bar{x}$ and $\bar{y}$. To determine v, it is enough to know η and the restriction of Φ to the surface S_t:

$$\varphi = \varphi(t, x_1) := \Phi\left(t, x_1, \eta(t, x_1)\right). \tag{1.6}$$

For the steady problem, we look for solutions of the form

$$\eta = \eta^c(x_1 - ct), \quad \varphi = \varphi^c(x_1 - ct), \quad \bar{x} = cte_1 + (-a + \rho)e_2, \quad \bar{y} = cte_1 + (-a - \rho)e_2,$$

where (η^c, φ^c) are time-independent and spatially localized. Here, $(0, -a)$ is the center of the dipole, and 2ρ is the separation between the point vortices. Specifically, we work in the space

$$(\eta, \varphi, a, \rho) \in X = X_1 \times X_2 \times X_3 \times X_4 := H^k_e(\mathbb{R}) \times \left(\dot{H}^{k-1}_o(\mathbb{R}) \cap \dot{H}^{1/2}_o(\mathbb{R}) \right) \times \mathbb{R} \times \mathbb{R} \tag{1.7}$$

with real number $k > \frac{3}{2}$ that can be freely chosen,

$$H^k_e(\mathbb{R}) := \{f \in H^k(\mathbb{R}) : f \text{ is even in } x_1\}, \quad H^k_o(\mathbb{R}) := \{f \in H^k(\mathbb{R}) : f \text{ is odd in } x_1\},$$

and let $\dot{H}^k_o(\mathbb{R})$ be the corresponding homogeneous space. Then our first result is the existence of traveling capillary-gravity water waves with a finite dipole. This theorem is an analogue of the work of Varholm on the water wave problem with point vortices in finite depth [31].

THEOREM 1.1 (existence). *Fix a real number $k > \frac{3}{2}$ and integer $\ell \geq 1$. Let $a_0 \in (0, \infty)$, $\rho_0 \in (0, a_0)$, $\gamma^0_1 > 0$, and $\gamma^0_2 > 0$ be given subject to the compatibility condition*

$$\gamma^0_2 = \frac{a^3_0 + \rho^3_0}{a^3_0 - \rho^3_0}\gamma^0_1. \tag{1.8}$$

Then, there exists $\epsilon_1 > 0$, $\gamma^1_1 > 0$, $\gamma^1_2 > 0$, and a C^ℓ family of traveling water waves with a finite dipole:

$$\begin{aligned}\mathscr{C}_{\text{loc}} = \{(\eta(\epsilon, \gamma_1, \gamma_2), \varphi(\epsilon, \gamma_1, \gamma_2), a(\epsilon, \gamma_1, \gamma_2), \rho(\epsilon, \gamma_1, \gamma_2), c(\epsilon, \gamma_1, \gamma_2)) :\\ |\epsilon| < \epsilon_1, |\gamma_1 - \gamma^0_1| < \gamma^1_1, |\gamma_2 - \gamma^0_2| < \gamma^1_2\}\\ \subset X \times \mathbb{R}.\end{aligned}$$

In particular, at $(\epsilon, \gamma_1, \gamma_2) = (0, \gamma^0_1, \gamma^0_2)$, $(\eta, \varphi, c) = (0, 0, 0)$ and $(a, \rho) = (a_0, \rho_0)$.

Remark 1.2. More precise asymptotics for the family $\mathscr{C}_{\text{loc}}$ are given in (2.9) and (2.10). Also, for simplicity, we have suppressed one of the parameters; for further details, see the proof in section 2.3.

Because the vorticity is conserved for the time-dependent problem, when we analyze the stability of these waves it is more natural to fix ϵ, γ_1, γ_2. In the construction of $\mathscr{C}_{\text{loc}}$, we show that $c = \epsilon\tilde{c}_0 + O(\epsilon^2)$, where $\tilde{c}_0$ is explicitly determined by a_0, ρ_0, γ^0_1, and γ^0_2. Therefore, we can fix $0 < |\epsilon| \ll 1$ and reparameterize locally in terms of c. This results in a curve of solitary waves indexed by the wave speed:

$$U_c := (\eta(c), \varphi(c), \bar{x}(c), \bar{y}(c)).$$

The compatibility condition (1.8) implies that the lower vortex at $\bar{y}$ must have a greater strength than the upper vortex at $\bar{x}$, that is, $\gamma_2 > \gamma_1$ for $0 < \rho_0 < a_0$ sufficiently small. This is a consequence of the fact that $\bar{x}$ is closer to the free surface S_t and is therefore influenced by it more strongly. We emphasize that the compatibility condition is not artificial; rather, it is necessary for the existence of a steady solution to the classical problem of a dipole moving through the lower half-plane bounded above by a fixed rigid lid. Moreover, as the family $\mathscr{C}_{\text{loc}}$ is exhaustive in a neighborhood of 0 in the space X, (1.8) must hold for any sufficiently small-amplitude, slow moving waves with even symmetry and having the regularity in (1.7).

Returning to the time-dependent problem, we introduce two important spaces. Let

$$\mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \times \mathbb{X}_4 := H^1(\mathbb{R}) \times \dot{H}^{1/2}(\mathbb{R}) \times \mathbb{R}^4, \tag{1.9}$$

and set

$$\mathbb{W} := \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3 \times \mathbb{W}_4 := H^{3+}(\mathbb{R}) \times \left(\dot{H}^{5/2+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right) \times \mathbb{R}^4, \tag{1.10}$$

where H^{k+} means H^{k+s} for some fixed $0 < s \ll 1$. We think of $\mathbb{W}$ as the *well-posedness* space for (1.1)–(1.4). A local well-posedness result for irrotational capillary-gravity water waves with this degree of regularity was proved by Alazard, Burq, and Zuily [1]. On the other hand, $\mathbb{X}$ is the natural *energy space*. This is discussed in more detail in section 3.1. Finally, for the problem to be well-defined, the finite dipole must be away from the free surface, so we take

$$\mathcal{O} := \{u \in \mathbb{X} : \bar{x}_2 < \eta(\bar{x}_1) < -\bar{x}_2, \quad \bar{y}_2 < \eta(\bar{y}_1) < -\bar{y}_2, \quad \bar{x} \neq \bar{y}\}. \tag{1.11}$$

To simplify our computation, we will incorporate the reflection of each point vortex over the free surface, and hence the upper limits $-\bar{x}_2$ and $-\bar{y}_2$ ensure those reflections stay in the air region.

To state the main result, we introduce some terminology. First, observe that the entire system is invariant under the one-parameter affine symmetry group $T(s) : \mathbb{X} \to \mathbb{X}$ defined by

$$T(s)u := T(s)(\eta, \varphi, \bar{x}, \bar{y})^T = (\eta(\cdot - s), \varphi(\cdot - s), \bar{x} + se_1, \bar{y} + se_1)^T. \tag{1.12}$$

This suggests that stability or instability should be understood modulo $T(s)$. With that in mind, for each $\rho > 0$, we define the tubular neighborhood

$$\mathcal{U}_\rho := \left\{ u \in \mathcal{O} : \inf_{s \in \mathbb{R}} \|T(s)U_c - u\|_{\mathbb{W}} < \rho \right\}.$$

Definition 1.3. *We say U_c is* orbitally unstable *provided that there is a $\nu_0 > 0$ such that for every $0 < \nu < \nu_0$ there exists initial data in $\mathcal{U}_\nu$ whose corresponding solution exits $\mathcal{U}_{\nu_0}$ in finite time.*

Our main theorem is as follows.

Theorem 1.4 (instability). *Assuming that* (1.1)–(1.4) *is locally well-posed in $\mathbb{W}$ in the sense that there exists $\nu_0 > 0$ and $t_0 > 0$ such that for all initial data $u_0 \in \mathcal{U}_{\nu_0}$, there exists a unique solution to the abstract Hamiltonian system* (3.8) *on the interval $[0, t_0)$. Then for any $\epsilon \neq 0$ sufficiently small, the corresponding family of solitary capillary-gravity water waves with a finite dipole U_c furnished by Theorem* 1.1 *is conditionally orbitally unstable.*

One physical interpretation for this is that, while we can construct steady configurations of counterrotating vortices moving in parallel through a water wave, these will not tend to persist over long periods of time. Instead, we expect them to migrate to the surface of the water, fail to keep pace with the surface wave, or otherwise destabilize. Moreover, this result covers all sufficiently small amplitude, wave speed, and vortex strength waves with even symmetry because $\mathscr{C}_{\text{loc}}$ comprises also such waves near 0 in X.

The instability result in this paper can be understood roughly as follows. The family $\mathscr{C}_{\text{loc}}$ of water waves is constructed using the implicit function theorem at a trivial state where the free surface is completely flat $\eta = 0$ and $\Phi = 0$. At leading order, this problem is analogous to the motion of a dipole in the lower half-plane bounded above by a rigid lid. This is a well-studied finite-dimensional Hamiltonian system. The compatibility condition (1.8) guarantees the existence of a family of steady solutions to this ODE; these are effectively in one-to-one correspondence to $\mathscr{C}_{\text{loc}}$. We find that, in a neighborhood of the bifurcation point, the finite-dimensional dynamics dominates, and so the orbital instability of the water waves is inherited from the instability of the corresponding dipole configuration in the lower half-plane. See the discussion in sections 2.4 and 3.4

1.3. History of the problem. The study of point vortices was initiated by Helmholtz [13] and Kirchhoff [16], who independently developed the model (1.4). Since then, there has been extensive research on this subject. The majority of this work concerns vortices in fixed fluid domains. For instance, Love found a condition under which the motion of two pairs of vortices may be periodic [18] and investigated the stability of Kirchhoff's elliptic vortex [19]. Aref and Pomphrey [3] examined the chaotic behavior of a more general system of four point vortices. Marchioro and Pulvirenti [20] later justified the connection between the incompressible Euler equation (1.2) and the Kirchhoff–Helmholtz model (1.4). Aref and Newton gave a thorough review of the results for the N-vortex problem in the plane [2, 22] or on the surface of the sphere [22]. Recently, Smets and Van Schaftingen [26] and Cao, Liu, and Wei [5] studied the existence of solutions to the point vortex problem in a bounded domain using either a variational or Lyapunov–Schmidt reduction approach. Point vortex models can also be used in studies of atmosphere and oceans [4].

When a dipole is placed inside a water wave, which is the case in this paper, investigating the existence and stability of solutions is much more involved mathematically as it requires developing an understanding of the interaction between the motion of the vortices and the free surface. Nonetheless, there have been a sizable number of studies in this regime. The first rigorous existence theory for steady solutions was given by Filippov [10] and Ter-Krikorov [29], who investigated the finite-depth regime in the purely gravitational case. Moreover, Shatah, Walsh, and Zeng constructed a family of traveling capillary gravity waves in infinite depth water with a single point vortex [24]. Using a similar method, Varholm obtained analogues for capillary-gravity waves with one or more vortices in finite depth [31]. Our existence theory follows in large part from the techniques in these two papers.

We also mention that recently several authors have considered the related problem of steady rotational water waves with one or more stagnation points in the bulk. As in the case of waves with a point vortex or dipole, these may have closed streamlines, which requires some inventiveness to treat. For instance, Ehrnström and Villari [9], Wahlén [33], and Ehrnström, Escher, and Wahlén [8] find families of small-amplitude periodic water waves with vorticity that have one or more critical layers (that is, lines of stagnation points in the interior of the fluid domain).

In terms of stability theory, our main source of inspiration is the recent paper by Varholm, Wahlén, and Walsh [32] that proves the orbital stability of traveling capillary gravity waves with a single point vortex. As we explain below, we will adopt a similar methodology. However, the dipole turns out to be significantly more difficult to analyze at a technical level. One way to account for this is to note that, at leading order, the stability theory is dictated by the corresponding finite-dimensional problem where the free surface is replaced by a rigid lid. With a single point vortex, we can introduce a mirror vortex and consider the system of two vortices in the plane. On the other hand, with a dipole, this leads to a four body problem, which is far more subtle.

It is well-known in the physics literature that the governing equations for water waves with submerged point vortices have a Hamiltonian structure. Rouhi and Wright gave the formulation for the motion of vortices in the presence of a free surface in two and three dimensions [23]. A similar formulation was later given by Zakharov [35].

There have also been a number of numerical results about vortex pairs in a fluid. The closest to the current problem is the recent paper of Curtis, Carter, and Kalisch [7], who studied how constant vorticity shear profile affects the motion of the particles both at and beneath waves in infinitely deep water. Many authors have looked at the related scenario where a submerged dipole is sent moving toward the free surface rather than moving with the wave; see, for example, [28, 34, 30]. In all of these papers, the authors found cases where the vortices are able to breach the upper boundary. The exact opposite scenario was considered by Su [27], who found that if a dipole initially moves away from the surface, the solution will persist over a long time scale. This is in stark contrast to the present paper, where we ask the dipole to move with the wave.

1.4. Plan of the article. This paper contains two main sections. In section 2, we show the existence of traveling capillary-gravity waves with a finite dipole. This follows from an implicit function theorem argument in the spirit of Varholm [31] and Shatah, Walsh, and Zeng [24]. Then, in section 3, we prove that these waves are orbitally unstable.

We first establish that (1.1)–(1.4) can be formulated as an infinite-dimensional Hamiltonian system of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = J(u)DE(u)$$

with J being the Poisson map and E the energy functional. This is similar but distinct from the version due to Rouhi and Wright [23]. We offer a rigorous derivation in Theorem 3.3.

In two seminal papers [11, 12], Grillakis, Shatah, and Strauss (GSS) provided a fairly simple method for determining the stability or instability of traveling wave solutions to systems of this form that are invariant under a continuous symmetry group. Among the hypotheses of this theory are that the Poisson map J is invertible and that the initial value problem is globally well-posed in time. Unfortunately, our J is state-dependent and not surjective. Moreover, we do not expect the problem to be well-posed in the energy space.

In this paper, we will use a recent variant of the GSS method developed by Varholm, Wahlén, and Walsh [32]. Among other improvements, this machinery permits J to have merely a dense range and also allows for a mismatch between the space where the problem is well-posed and the natural energy space. In the present context, the latter point relates to the fact that $\mathbb{W} \nsubseteq \mathbb{X}$. As one of the hypotheses to apply the instability theory [32], we must compute the spectrum of the second variation of the augmented Hamiltonian defined in section 3.2.

For the convenience of the reader, steady and unsteady equations for the velocity potential and stream functions are provided in Appendix A. In Appendix B, we record the variations of the energy and momentum functional.

2. Existence theory. This section is devoted to proving the existence of traveling capillary-gravity water waves with a finite dipole. We will adopt a methodology introduced by Varholm [31] and Shatah, Walsh, and Zeng [24]. The first step is to reformulate (1.1)–(1.4) in the spirit of Zakharov [35] and Craig and Sulem [6]. This entails reducing the problem to a nonlocal system involving only surface variables.

2.1. Harmonic conjugates and splitting. Recall that we wish to split the velocity field v as in (1.5). For the rotational contribution, we take

$$\Theta = \Theta_1 + \Theta_2 + \Theta_1^* + \Theta_2^*$$

with

$$\Theta_1(x) = -\frac{\gamma_1}{2\pi} \arctan\left(\frac{x_1 - \bar{x}_1}{|x - \bar{x}| + x_2 - \bar{x}_2}\right),$$
$$\Theta_1^*(x) = \frac{\gamma_1}{2\pi} \arctan\left(\frac{x_1 - \bar{x}_1^*}{|x - \bar{x}^*| - x_2 - \bar{x}_2^*}\right),$$
$$\Theta_2(x) = \frac{\gamma_2}{2\pi} \arctan\left(\frac{x_1 - \bar{y}_1}{|x - \bar{y}| + x_2 - \bar{y}_2}\right),$$
$$\Theta_2^*(x) = -\frac{\gamma_2}{2\pi} \arctan\left(\frac{x_1 - \bar{y}_1^*}{|x - \bar{y}^*| - x_2 - \bar{y}_2^*}\right).$$

Here $\bar{x}^* = (\bar{x}_1, -\bar{x}_2)$ and $\bar{y}^* = (\bar{y}_1, -\bar{y}_2)$ are the reflection of the two point vortices over the x_1-axis. This corresponds to making a branch cut straight down from the vortex centers. It is easy to see that $\nabla\Phi$ and $\epsilon\nabla\Theta$ are both in L^2 on the complement of any neighborhood of $\bar{x}$ and $\bar{y}$.

It is often useful to work with the harmonic conjugates of these functions. In particular, let Γ be the stream function corresponding to Θ, that is, $\nabla\Theta = \nabla^\perp\Gamma$. Then we have

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_1^* + \Gamma_2^*,$$

where

$$\Gamma_1(x) = \frac{\gamma_1}{2\pi} \log|x - \bar{x}|, \qquad \Gamma_2(x) = -\frac{\gamma_2}{2\pi} \log|x - \bar{y}|,$$
$$\Gamma_1^*(x) = -\frac{\gamma_1}{2\pi} \log|x - \bar{x}^*|, \qquad \Gamma_2^*(x) = \frac{\gamma_2}{2\pi} \log|x - \bar{y}^*|.$$

Notice that $-\Delta\Gamma = -\gamma_1\delta_{\bar{x}} + \gamma_2\delta_{\bar{y}}$, and hence $-\epsilon\Delta\Gamma = \omega$. Let Ψ be the harmonic conjugate of Φ, so that

$$v = \nabla^\perp\Psi + \epsilon\nabla^\perp\Gamma.$$

We denote by ψ the restriction of Ψ to the free boundary:

$$\psi = \psi(t, x_1) := \Psi(t, x_1, \eta(t, x_1)). \tag{2.1}$$

Finally, define

$$\Xi_1 := \Theta_1 - \Theta_1^*, \quad \Xi_2 := \Theta_2 - \Theta_2^*, \quad \Upsilon_1 := \Theta_1 + \Theta_1^*, \quad \Upsilon_2 := \Theta_2 + \Theta_2^*,$$

so that $\Theta = \Upsilon_1 + \Upsilon_2$. This will be convenient for computing $\partial_{\bar{x}}\Theta$.

2.2. Nonlocal formulation. As mentioned above, rather than working with the Euler equations in the domain Ω_t, we will reformulate the problem in terms of the surface variables (η, φ) or (η, ψ), and the location of the vortex centers $(\bar{x}, \bar{y})$. The normal component of the velocity field on the free boundary can be recovered from normal derivatives of Φ or tangential derivatives of Ψ. To express these in terms of φ or ψ, we use the Dirichlet–Neumann operator $\mathcal{G}(\eta) : \dot{H}^{1/2}(\mathbb{R}) \cap \dot{H}^k(\mathbb{R}) \to \dot{H}^{k-1}(\mathbb{R})$ defined by

$$\mathcal{G}(\eta)\phi := (-\eta'\partial_{x_1}\phi_{\mathcal{H}} + \partial_{x_2}\phi_{\mathcal{H}})|_{S_t}, \tag{2.2}$$

where $\phi_{\mathcal{H}} := \langle \mathcal{H}(\eta), \phi\rangle \in \dot{H}^{k+1/2}(\Omega_t)$ is the harmonic extension of ϕ to Ω_t determined uniquely by

$$\Delta\phi_{\mathcal{H}} = 0 \text{ in } \Omega, \quad \phi_{\mathcal{H}} = \phi \text{ on } S_t,$$

and $k > \frac{1}{2}$. It is well-known that for any $\eta \in H^{k_0}(\mathbb{R})$, $k_0 > 3/2$, $\mathcal{G}(\eta)$ is a bounded, invertible, and self-adjoint operator between these spaces when $k \in [1-k_0, k_0]$. Moreover, the mapping $\eta \mapsto \mathcal{G}(\eta)$ is C^∞ and $\mathcal{G}(0) = |\partial_{x_1}|$ (see, for example, the book by Lannes [17] and the paper by Shatah and Zeng [25]).

Using these ideas, the water wave problem can be rewritten as the following system for the unknowns $(\eta, \varphi, \bar{x}, \bar{y})$:

$$\begin{cases} \partial_t \eta = \mathcal{G}(\eta)\varphi + \epsilon \nabla_\perp \Theta, \\ \partial_t \varphi = -\frac{1}{2\langle \eta' \rangle^2}\left((\varphi')^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi - (\mathcal{G}(\eta)\varphi)^2\right) - \epsilon\partial_t\Theta - \epsilon\varphi'\partial_{x_1}\Theta - \frac{\epsilon^2}{2}|\nabla\Theta|^2 \\ \qquad - g\eta + b\frac{\eta''}{\langle \eta' \rangle^3}, \\ \partial_t \bar{x} = \nabla\Phi(\bar{x}) + \epsilon\nabla\Theta_1^*(\bar{x}) + \epsilon\nabla\Theta_2(\bar{x}) + \epsilon\nabla\Theta_2^*(\bar{x}), \\ \partial_t \bar{y} = \nabla\Phi(\bar{y}) + \epsilon\nabla\Theta_1(\bar{y}) + \epsilon\nabla\Theta_1^*(\bar{y}) + \epsilon\nabla\Theta_2^*(\bar{y}). \end{cases} \tag{2.3}$$

The equations for $\partial_t \bar{x}$ and $\partial_t \bar{y}$ come from the Kirchhoff–Helmholtz model (1.4). Note that due to symmetry, many of the components are zero. Nonetheless, we choose to write it as above to clarify the meaning of each term. Recall that $\varphi = \Phi|_{S_t}$ as in (1.6) and describes the irrotational part of the velocity field. Here we have made use of the differential operators

$$\nabla_\perp := \left.(-\eta'\partial_{x_1} + \partial_{x_2})\right|_{S_t}, \quad \nabla_\top := \left.(\partial_{x_1} + \eta'\partial_{x_2})\right|_{S_t}, \tag{2.4}$$

which come naturally as we take derivatives of functions restricted to the free surface.

Note that in (2.3), the equation for $\partial_t \eta$ can be derived from the kinematic boundary condition, but now Θ appears as a forcing term. We can see that the evolution of φ is determined by the unsteady Bernoulli equation (A.3).

2.3. Existence of traveling waves. Now we are prepared to prove the existence theorem. As this is done in the steady frame, we will simply write $S := S_t$ and $\Omega := \Omega_t$. We also use subscripts x_1 and x_2 to denote partial derivatives.

Proof of Theorem 1.1. For convenience, we prove this result using ψ, which immediately gives the stated theorem in terms of φ. Traveling wave solutions of (2.3) have the ansatz

$$\eta = \eta(x_1 - ct), \quad \psi = \psi(x_1 - ct), \quad \partial_t \bar{x} = ce_1, \quad \partial_t \bar{y} = ce_1$$

for a wave speed $c \in \mathbb{R}$. Inserting this into (2.3) and writing it in terms of ψ, we arrive at the steady problem. Stated as an abstract operator equation, it has the form

$$\mathcal{F}(\epsilon, c, \gamma_1, \gamma_2; \eta, \psi, a, \rho) = 0, \tag{2.5}$$

where $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) : \mathbb{R}^4 \times X \to Y$ is defined by

$$\begin{aligned} \mathcal{F}_1 &:= \frac{c}{1+(\epsilon\eta')^2}\left(\psi' + \eta'\mathcal{G}(\eta)\psi\right) + \epsilon c \Gamma_{x_2}|_S + \frac{1}{2(1+(\eta')^2)}\Big((\psi')^2 + (\mathcal{G}(\eta)\psi)^2\Big) \\ &\qquad + \frac{\epsilon}{1+(\eta')^2}\left(\mathcal{G}(\eta)\psi\nabla_\perp\Gamma + \psi'\nabla_\top\Gamma\right) + \frac{\epsilon^2}{2}|(\nabla\Gamma)|_S|^2 + g\eta + b\kappa(\eta), \\ \mathcal{F}_2 &:= c\eta' + \psi' + \epsilon(1,\eta')^T \cdot \nabla\Gamma, \\ \mathcal{F}_3 &:= c + \left(\partial_{x_2}\psi_{\mathcal{H}}\right)(\bar{x}) + \epsilon\Gamma_{2_{x_2}}(\bar{x}) + \epsilon\Gamma^*_{1_{x_2}}(\bar{x}) + \epsilon\Gamma^*_{2_{x_2}}(\bar{x}), \\ \mathcal{F}_4 &:= c + \left(\partial_{x_2}\psi_{\mathcal{H}}\right)(\bar{y}) + \epsilon\Gamma_{1_{x_2}}(\bar{y}) + \epsilon\Gamma^*_{1_{x_2}}(\bar{y}) + \epsilon\Gamma^*_{2_{x_2}}(\bar{y}). \end{aligned} \tag{2.6}$$

Here, $\nabla\Gamma$ is evaluated at $x_2 = \eta(x_1)$, the domain X is given by (1.7), and the codomain

$$Y = Y_1 \times Y_2 \times Y_3 \times Y_4 := H_e^{k-2}(\mathbb{R}) \times \left(\dot{H}_o^{k-2}(\mathbb{R}) \cap \dot{H}_o^{-1/2}(\mathbb{R})\right) \times \mathbb{R} \times \mathbb{R}$$

for $k > \frac{3}{2}$ fixed.

It is easy to calculate that

$$D_{(\eta,\psi,a,\rho)}\mathcal{F}\,(0,0,\gamma_1,\gamma_2;0,0,a_0,\rho_0) = \begin{pmatrix} g - b\partial_{x_1}^2 & 0 & 0 & 0 \\ 0 & \partial_{x_1} & 0 & 0 \\ 0 & (\partial_{x_2}\langle \mathcal{H}(0),\cdot\rangle)\,|_{(0,-a_0+\rho_0)} & 0 & 0 \\ 0 & (\partial_{x_2}\langle \mathcal{H}(0),\cdot\rangle)\,|_{(0,-a_0-\rho_0)} & 0 & 0 \end{pmatrix}.$$

We see from the first two rows of the operator matrix that an implicit function theorem argument allows us to uniquely solve $(\mathcal{F}_1,\mathcal{F}_2) = (0,0)$ locally for η and ψ in terms of ϵ, c, γ_1, γ_2, a, and ρ. Moreover, this dependence is at least C^1. Notice, however, that when $\epsilon = 0$, setting $(\eta,\psi) = (0,0)$ gives a solution to $(\mathcal{F}_1,\mathcal{F}_2) = (0,0)$ for any $(c,\gamma_1,\gamma_2,a,\rho)$. By uniqueness, this implies that when $\epsilon = 0$, all small-amplitude waves must be trivial in that η and ψ vanish. In fact, looking at the equations $(\mathcal{F}_3,\mathcal{F}_4) = (0,0)$, we can further infer that $c = 0$ as well.

With that in mind, we introduce a rescaling:

$$\eta =: \epsilon\tilde{\eta}, \quad \psi =: \epsilon\widetilde{\psi}, \quad \Psi =: \epsilon\widetilde{\Psi}, \quad c =: \epsilon\widetilde{c},$$

which will counteract the degeneracy in the linearized problem for $\mathcal{F}_3$ and $\mathcal{F}_4$. While this in principle restricts the analysis to a cone in X, by the above discussion, we see that it does not incur any loss of generality.

In the rescaled variables, the traveling wave problem for (2.3) becomes

$$\widetilde{\mathcal{F}}(\epsilon,\widetilde{c},\gamma_1,\gamma_2;\tilde{\eta},\widetilde{\psi},a,\rho) = 0,$$

where $\widetilde{\mathcal{F}} : \mathbb{R}^4 \times X \to Y$ is given by

$$\mathcal{F}(\epsilon,c,\gamma_1,\gamma_2;\eta,\psi,a,\rho) = \epsilon\widetilde{\mathcal{F}}(\epsilon,\tilde{c},\gamma_1,\gamma_2;\tilde{\eta},\tilde{\psi},a,\rho).$$

First, we look for a trivial solution to the rescaled problem. Setting $\epsilon = 0$, we see that $\widetilde{\mathcal{F}}_1 = 0$ becomes

$$g\tilde{\eta} - b\tilde{\eta}'' = 0,$$

and hence $\tilde{\eta}$ must be at least $O(\epsilon)$. Likewise, looking at $\widetilde{\mathcal{F}}_2 = 0$ shows that $\tilde{\psi}$ must vanish when $\epsilon = 0$.

Now, taking $\tilde{\eta}_0 = 0$ and $\widetilde{\psi}_0 = 0$ and fixing γ_1^0, γ_2^0, a_0, $\rho_0 \in \mathbb{R}$, it is clear that

$$\widetilde{\mathcal{F}}(0,\tilde{c}_0,\gamma_1^0,\gamma_2^0;\tilde{\eta}_0,\widetilde{\psi}_0,a_0,\rho_0) = 0$$

if and only if

$$\begin{aligned} \tilde{c}_0 &= -\Gamma_{2_{x_2}}(0,-a_0+\rho_0) - \Gamma^*_{1_{x_2}}(0,-a_0+\rho_0) - \Gamma^*_{2_{x_2}}(0,-a_0+\rho_0) \\ &= -\frac{\gamma_1^0}{4\pi(a_0-\rho_0)} + \frac{\gamma_2^0}{4\pi}\left(\frac{1}{a_0} + \frac{1}{\rho_0}\right) \end{aligned} \tag{2.7}$$

and the compatibility condition (1.8) holds. A simple computation shows that

$$\mathscr{L} := D_{(\widetilde{\eta},\widetilde{\psi},a,\rho)}\widetilde{\mathcal{F}}\left(0,\tilde{c}_0,\gamma_1^0,\gamma_2^0;0,0,a_0,\rho_0\right) = \begin{pmatrix} g-b\partial_{x_1}^2 & 0 & 0 & 0 \\ 0 & \partial_{x_1} & 0 & 0 \\ 0 & \left(\partial_{x_2}\langle\mathcal{H}(0),\cdot\rangle\right)|_{(0,-a_0+\rho_0)} & -\frac{\gamma_1^0}{4\pi(a_0-\rho_0)^2}+\frac{\gamma_2^0}{4\pi a_0^2} & \frac{\gamma_1^0}{4\pi(a_0-\rho_0)^2}+\frac{\gamma_2^0}{4\pi\rho_0^2} \\ 0 & \left(\partial_{x_2}\langle\mathcal{H}(0),\cdot\rangle\right)|_{(0,-a_0-\rho_0)} & -\frac{\gamma_1^0}{4\pi a_0^2}+\frac{\gamma_2^0}{4\pi(a_0+\rho_0)^2} & \frac{\gamma_2^0}{4\pi(a_0+\rho_0)^2}+\frac{\gamma_1^0}{4\pi\rho_0^2} \end{pmatrix}.$$

As before, inspecting the first two rows of $\mathscr{L}$, we see that $\widetilde{\eta}$ and $\widetilde{\psi}$ can be solved explicitly. This follows because $g-b\partial_{x_1}^2$ and ∂_{x_1} are bounded and invertible operators between the corresponding spaces. On the other hand, the operators in the column below ∂_{x_1} are bounded. Thus, the invertibility of $\mathscr{L}$ is equivalent to the invertibility of the 2×2 real submatrix:

$$\mathscr{T} := \begin{pmatrix} -\dfrac{\gamma_1^0}{4\pi(a_0-\rho_0)^2}+\dfrac{\gamma_2^0}{4\pi a_0^2} & \dfrac{\gamma_1^0}{4\pi(a_0-\rho_0)^2}+\dfrac{\gamma_2^0}{4\pi\rho_0^2} \\ -\dfrac{\gamma_1^0}{4\pi a_0^2}+\dfrac{\gamma_2^0}{4\pi(a_0+\rho_0)^2} & \dfrac{\gamma_2^0}{4\pi(a_0+\rho_0)^2}+\dfrac{\gamma_1^0}{4\pi\rho_0^2} \end{pmatrix}. \tag{2.8}$$

By the compatibility condition (1.8), we have

$$\det\mathscr{T} = -\frac{(\gamma_1^0)^2}{16\pi^2}\frac{6(a_0^4-a_0^2\rho_0^2+\rho_0^4)}{(a_0+\rho_0)(a_0-\rho_0)^3(a_0^2+a_0\rho_0+\rho_0^2)^2} < 0.$$

Thus, $\mathscr{L}$ is an isomorphism.

The implicit function theorem then tells us that, for any $\ell \geq 1$, there exists a C^ℓ family $\mathcal{C}_{\mathrm{loc}}$ of solutions of the form

$$\widetilde{\mathcal{F}}(\epsilon,\tilde{c},\gamma_1,\gamma_2;\tilde{\eta}(\epsilon,\tilde{c},\gamma_1,\gamma_2),\tilde{\psi}(\epsilon,\tilde{c},\gamma_1,\gamma_2),a(\epsilon,\tilde{c},\gamma_1,\gamma_2),\rho(\epsilon,\tilde{c},\gamma_1,\gamma_2)) = 0$$

for all $|\epsilon| \ll 1$, $|\tilde{c}-\tilde{c}_0| \ll 1$, $|\gamma_1-\gamma_1^0| \ll 1$, and $|\gamma_2-\gamma_2^0| \ll 1$.

Finally, we translate this result to the statement in Theorem 1.1. Recalling the scalings, we see that the wave speed can be viewed as a function $c = c(\epsilon,\tilde{c}) := \epsilon\tilde{c}$. Suppressing the dependence of (η,ψ,a,ρ,c) on $\tilde{c}$ gives $\mathscr{C}_{\mathrm{loc}}$. From the argument above, it is clear that all traveling wave solutions in a neighborhood of 0 in X are captured by this family. ∎

For the stability analysis, we rely on asymptotic information about the traveling waves constructed above. Using implicit differentiation, one can readily compute that

$$\begin{aligned} \eta(\epsilon,\tilde{c},\gamma_1,\gamma_2) &= -\epsilon^2(g-b\partial_{x_1}^2)^{-1}\left[\tilde{c}_0\Gamma_{x_2}(x_1,0)\right] \\ &\qquad + O(|\epsilon|^3+|\epsilon||c-c_0|^2+|\epsilon||\gamma_1-\gamma_1^0|^2+|\epsilon||\gamma_2-\gamma_2^0|^2), \\ \psi(\epsilon,\tilde{c},\gamma_1,\gamma_2) &= O(|\epsilon|^3+|\epsilon||c-c_0|^2+|\epsilon||\gamma_1-\gamma_1^0|^2+|\epsilon||\gamma_2-\gamma_2^0|^2), \end{aligned} \tag{2.9a}$$

in $C^1(U;X_1)$ and $C^1(U,X_2)$, respectively, where U is a neighborhood of $(0,c_0,\gamma_1^0,\gamma_2^0)$ in $\mathbb{R}^4$. Likewise,

$$\begin{aligned} a(\epsilon,\tilde{c},\gamma_1,\gamma_2) &= a_0+\epsilon\mathbf{a}_{\tilde{c}}(c-c_0)+\mathbf{a}_{\gamma_1}(\gamma_1-\gamma_1^0)+\mathbf{a}_{\gamma_2}(\gamma_2-\gamma_2^0) \\ &\qquad + O(|\epsilon|^2+|c-c_0|^2+|\gamma_1-\gamma_1^0|^2+|\gamma_2-\gamma_2^0|^2), \\ \rho(\epsilon,\tilde{c},\gamma_1,\gamma_2) &= \rho_0+\epsilon\boldsymbol{\rho}_{\tilde{c}}(c-c_0)+\boldsymbol{\rho}_{\gamma_1}(\gamma_1-\gamma_1^0)+\boldsymbol{\rho}_{\gamma_2}(\gamma_2-\gamma_2^0) \\ &\qquad + O(|\epsilon|^2+|c-c_0|^2+|\gamma_1-\gamma_1^0|^2+|\gamma_2-\gamma_2^0|^2), \end{aligned} \tag{2.9b}$$

in $C^1(U;\mathbb{R})$. Here, $\widetilde{c}_0$, $\mathscr{T}$ are given by (2.7)–(2.8). We note that the expression (2.9) contains both c and $\widetilde{c} = c/\epsilon$. The coefficients $\mathbf{a}_{\widetilde{c}}$, $\mathbf{a}_{\gamma_1}$, $\mathbf{a}_{\gamma_2}$, $\boldsymbol{\rho}_{\widetilde{c}}$, $\boldsymbol{\rho}_{\gamma_1}$, and $\boldsymbol{\rho}_{\gamma_2}$ are variations at the point $(0, \tilde{c}_0, \gamma_1^0, \gamma_2^0)$ with respect to $\tilde{c}$, γ_1, and γ_2, respectively. In particular,

$$\begin{aligned} \boldsymbol{a}_{\widetilde{c}} &= \frac{1}{\det \mathscr{T}} \left(-\frac{\gamma_2^0}{4\pi(a_0+\rho_0)^2} - \frac{\gamma_1^0}{4\pi\rho_0^2} + \frac{\gamma_1^0}{4\pi(a_0-\rho_0)^2} + \frac{\gamma_2^0}{4\pi\rho_0^2} \right), \\ \boldsymbol{\rho}_{\widetilde{c}} &= \frac{1}{\det \mathscr{T}} \left(\frac{\gamma_2^0}{4\pi(a_0+\rho_0)^2} - \frac{\gamma_1^0}{4\pi a_0^2} + \frac{\gamma_1^0}{4\pi(a_0-\rho_0)^2} - \frac{\gamma_2^0}{4\pi a_0^2} \right). \end{aligned} \tag{2.10}$$

2.4. The compatibility condition and dipoles in the half-plane. It is instructive to compare the submerged dipoles constructed in Theorem 1.1 to the classical problem of dipoles in the lower half-plane. Suppose for a moment that the x_1-axis represents a rigid lid and $\bar{x}(t)$ and $\bar{y}(t)$ represent point vortices in a fluid confined to the lower half-plane $\{x_2 < 0\}$. The Kirchhoff–Helmholtz model for this system is the (finite-dimensional) point vortex system:

$$\begin{cases} \partial_t \bar{x} = \epsilon \nabla^\perp \Gamma_1^*(\bar{x}) + \epsilon \nabla^\perp \Gamma_2(\bar{x}) + \epsilon \nabla^\perp \Gamma_2^*(\bar{x}), \\ \partial_t \bar{y} = \epsilon \nabla^\perp \Gamma_1(\bar{y}) + \epsilon \nabla^\perp \Gamma_1^*(\bar{y}) + \epsilon \nabla^\perp \Gamma_2^*(\bar{y}). \end{cases} \tag{2.11}$$

Note that the phantom vortices here ensure that the fluid velocity is tangential on the lid.

An elementary calculation confirms that

$$\bar{x}(t) = cte_1 + (-a+\rho)e_2, \qquad \bar{y}(t) = cte_1 + (-a-\rho)e_2$$

is a steady solution of (2.11) if and only if the compatibility condition (1.8) holds and the wave speed satisfies

$$c = -\frac{\gamma_1}{4\pi(a-\rho)} + \frac{\gamma_2}{4\pi}\left(\frac{1}{a} + \frac{1}{\rho}\right)$$

which is equivalent to (2.7). From this point of view, the family $\mathcal{C}_{\text{loc}}$ represents a perturbation of the above steady dipole configuration. Indeed, the free surface η is $O(|\epsilon|^2)$ and ψ is even higher order (in the appropriate norms), so it is reasonable to expect that the stability of the waves is determined by the stability of the corresponding solution to the half-plane problem.

3. Instability theory. In this section, we show that the traveling waves constructed in Theorem 1.1 are orbitally unstable. Heuristically, this can be attributed to the instability of dipoles in a half-plane as discussed above. To make this rigorous for the water wave problem, we follow the general strategy of Varholm–Wahlén–Walsh [32], which is an adaptation of the classical GSS method [11, 12]. In section 3.1, we rewrite the equations of capillary-gravity waves with a finite dipole (1.1)–(1.4) as a Hamiltonian system and give an explicit form for its energy functional E and momentum functional P. We also establish that the traveling waves are critical points of the so-called augmented Hamiltonian $E - cP$. In section 3.2, we analyze the spectrum of the second variation of the augmented Hamiltonian at a small-amplitude wave in $\mathscr{C}_{\text{loc}}$. It is shown that it has the required configuration for [11, 12]. This calculation is done in the spirit of Mielke's work on irrotational capillary-gravity waves [21]. Finally, in section 3.3, we complete the proof of our main result by computing the second derivative of the moment of instability for small waves in this family. This shows, via [32, Theorem 2.6], such waves are *not* local minima of the energy on the fixed momentum manifold and are consequently (conditionally) orbitally unstable.

3.1. Hamiltonian formulation.

Hamiltonian structure and functional analytic setting. We first show that the system of equations (2.3) has a Hamiltonian structure in terms of the state variable $u = (\eta, \varphi, \bar{x}, \bar{y})^T$. Define the energy $E = E(u)$ to be

$$E(u) := K(u) + V(u), \tag{3.1}$$

where K is the (excess) kinetic energy and V is the potential energy. The submerged dipole does not affect the latter, and so we expect

$$V(u) := \int_{\mathbb{R}} \left(\frac{1}{2} g\eta^2 + b(\langle \eta' \rangle - 1) \right) \mathrm{d}x_1. \tag{3.2}$$

However, some care is needed in deriving the correct expression for K. Formally, we take the classical kinetic energy $\frac{1}{2}\int_\Omega |v|^2 \,\mathrm{d}x$, split v according to (1.5), and then integrate by parts. We will end up with terms on the boundary plus terms at the vortex centers. The Newtonian potentials in Γ will naturally lead to some of these being singular, and those we discard. This process is equivalent to removing the self-advection of the point vortices as in the Helmholtz–Kirchhoff model (see, for example, [23]). What results is the following:

$$\begin{aligned} K(u) &:= K_0(u) + \epsilon K_1(u) + \epsilon^2 K_2(u) \\ &= \frac{1}{2}\int_{\mathbb{R}} \varphi \mathcal{G}(\eta)\varphi \,\mathrm{d}x_1 + \epsilon \int_{\mathbb{R}} \varphi \nabla_\perp \Theta \,\mathrm{d}x_1 + \epsilon^2 \left(\frac{1}{2}\int_{\mathbb{R}} \Theta|_{S_t} \nabla_\perp \Theta \,\mathrm{d}x_1 + \Gamma^* \right), \end{aligned}$$

where

$$\Gamma^* := \frac{\gamma_1}{2}\Big(\Gamma_1^*(\bar{x}) + \Gamma_2(\bar{x}) + \Gamma_2^*(\bar{x})\Big) - \frac{\gamma_2}{2}\Big(\Gamma_1(\bar{y}) + \Gamma_1^*(\bar{y}) + \Gamma_2^*(\bar{y})\Big).$$

Note that $K_0 = \frac{1}{2}\int_\Omega |\nabla\Phi|^2 \,\mathrm{d}x$ and hence represents the kinetic energy contributed by the purely irrotational part of the velocity. On the other hand, K_1 is the interaction between the irrotational and rotational parts, and K_2 is the kinetic energy attributed to the rotational part. Finally, $\epsilon^2\Gamma^*$ is the kinetic energy for dipoles in the lower half-plane.

Recall that the energy space $\mathbb{X}$ was defined by (1.9) and the well-posedness space $\mathbb{W}$ was defined by (1.10). As $\mathbb{X}$ is a Hilbert space, it is isomorphic to its continuous dual $\mathbb{X}^*$, and the isomorphism $I : \mathbb{X} \to \mathbb{X}^*$ takes the form

$$I = (1 - \partial_{x_1}^2, |\partial_{x_1}|, \mathrm{Id}_{\mathbb{R}^2}, \mathrm{Id}_{\mathbb{R}^2}),$$

where $\mathrm{Id}_{\mathbb{R}^2}$ is the 2×2 identity matrix. For the Dirichlet–Neumann operator in E to be well-defined, we must have that η is at least Lipschitz continuous. This forces us to work in a smoother space than $\mathbb{X}$:

$$\mathbb{V} := \mathbb{V}_1 \times \mathbb{V}_2 \times \mathbb{V}_3 \times \mathbb{V}_4 := H^{3/2+}(\mathbb{R}) \times \left(\dot{H}^{1+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right) \times \mathbb{R}^4. \tag{3.3}$$

As before, we are using the standard shorthand H^{s+} for $H^{s+\varepsilon}$ with fixed $\varepsilon > 0$ taken sufficiently small and then suppressed.

From the definition of the energy in (3.1), we see that $E \in C^\infty(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$. In short, our Hamiltonian structure is formulated in $\mathbb{X}$, and that is where we will conduct our spectral analysis. On the other hand, the conserved quantities are only smooth in

$\mathbb{V}$. One of the main ideas in [32] is that one can bridge this gap to control quantities in $\mathbb{V}$ using the assumption of boundedness in the local well-posedness space $\mathbb{W}$ and interpolation. For that, we use the following lemma, which is an immediate consequence of the Gagliardo–Nirenberg interpolation inequality.

LEMMA 3.1 (spaces). *There exist constants $\theta \in (0,1]$ and $C > 0$ such that*

$$\|u\|_{\mathbb{V}}^3 \le C\|u\|_{\mathbb{X}}^{2+\theta}\|u\|_{\mathbb{W}}^{1-\theta}$$

for all $u \in \mathbb{W}$.

Next, in order to give the symplectic structure of the Hamiltonian system, we need to describe the Poisson map $J(u)$. We first encode the structures for water waves and point vortices by themselves by considering the closed operator $\widehat{J} : \mathcal{D}(\widehat{J}) \subset \mathbb{X}^* \to \mathbb{X}$ defined by

$$\widehat{J} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & (\epsilon\gamma_1)^{-1}\mathcal{J} & 0 \\ 0 & 0 & 0 & -(\epsilon\gamma_2)^{-1}\mathcal{J} \end{pmatrix}, \tag{3.4}$$

where $\mathcal{J}$ is a 2×2 real matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathcal{D}(\widehat{J}) = (H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})) \times (H^1(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R})) \times \mathbb{R}^2 \times \mathbb{R}^2.$$

It is well-known that the irrotational water waves problem possesses a canonical Hamiltonian structure [6, 35]; this is represented by the upper-left 2×2 submatrix in (3.4). Likewise, the motion of a finite dipole in the plane is Hamiltonian with the Poisson map given by the lower-right 4×4 submatrix (see, for example, [22]).

We must now account for wave-vortex interaction. As we saw in the proof of Theorem 1.1, this should intuitively be a lower order term. Let $B \in C^1(\mathcal{O}; \mathrm{Lin}(\mathbb{X})) \cap C^1(\mathcal{O} \cap \mathbb{W}; \mathrm{Lin}(\mathbb{W}))$ be defined by

$$B(u) := \mathrm{Id}_{\mathbb{X}} + Z(u), \tag{3.5}$$

where

$$Z(u)\dot{w} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\epsilon(\gamma_1)^{-1}(\mathcal{J}\xi|_{S_t})^T & \epsilon(\gamma_2)^{-1}(\mathcal{J}\zeta|_{S_t})^T & \epsilon\xi^T|_{S_t} & \epsilon\zeta^T|_{S_t} \\ \gamma_1^{-1}\mathcal{J} & 0 & 0 & 0 \\ 0 & -(\gamma_2)^{-1}\mathcal{J} & 0 & 0 \end{pmatrix} \begin{bmatrix} \langle \xi|_{S_t}, \dot{\eta}\rangle \\ \langle \zeta|_{S_t}, \dot{\eta}\rangle \\ \dot{\bar{x}} \\ \dot{\bar{y}} \end{bmatrix}$$

for all $\dot{w} = (\dot{\eta}, \dot{\varphi}, \dot{\bar{x}}, \dot{\bar{y}})^T \in \mathcal{O}$ with

$$\xi := -\nabla_{\bar{x}}\Theta = (\Upsilon_{1_{x_1}}, \Xi_{1_{x_2}})^T, \qquad \zeta := -\nabla_{\bar{y}}\Theta = (\Upsilon_{2_{x_1}}, \Xi_{2_{x_2}})^T.$$

Writing Z this way, while hard to motivate physically, is convenient in that it is clearly finite-rank. Here we are following a similar idea from [32].

Finally, for each $u \in \mathcal{O}$, let the Poisson map $J(u) : \mathcal{D}(\widehat{J}) \subset \mathbb{X}^* \to \mathbb{X}$ be defined by

$$J(u) := B(u)\widehat{J}.$$

We see that B is state-*dependent*, but it is a C^1 compact perturbation of identity and bijective (see parts (iii) and (iv) of Lemma 3.2). On the other hand, $\widehat{J}$ is state-*independent*, but it is not surjective. The decomposition of J into $\widehat{J}$ and B allows us to treat each of the difficulties one at a time. In particular, this makes it easy to confirm that J depends smoothly on u in an appropriate way.

It is useful to write down explicitly the form of $J(u)$, which can be computed directly from the definition of B and $\widehat{J}$:

$$J(u) := B(u)\widehat{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & J_{22} & J_{23} & J_{24} \\ 0 & J_{32} & (\epsilon\gamma_1)^{-1}\mathcal{J} & 0 \\ 0 & J_{42} & 0 & -(\epsilon\gamma_2)^{-1}\mathcal{J} \end{pmatrix}, \tag{3.6}$$

where

$$\begin{aligned} J_{22} := {} & \epsilon\Upsilon_{1_{x_1}}|_{S_t}\ \langle\cdot, -\gamma_1^{-1}\Xi_{1_{x_2}}|_{S_t}\rangle + \epsilon\Xi_{1_{x_2}}|_{S_t}\ \langle\cdot, \gamma_1^{-1}\Upsilon_{1_{x_1}}|_{S_t}\rangle \\ & + \epsilon\Upsilon_{2_{x_1}}|_{S_t}\ \langle\cdot, \gamma_2^{-1}\Xi_{2_{x_2}}|_{S_t}\rangle + \epsilon\Xi_{2_{x_2}}|_{S_t}\ \langle\cdot, -\gamma_2^{-1}\Upsilon_{2_{x_1}}|_{S_t}\rangle, \end{aligned}$$

$$J_{23} := \left(\gamma_1^{-1}\Xi_{1_{x_2}}|_{S_t},\ -\gamma_1^{-1}\Upsilon_{1_{x_1}}|_{S_t}\right), \qquad J_{24} := \left(-\gamma_2^{-1}\Xi_{2_{x_2}}|_{S_t},\ \gamma_2^{-1}\Upsilon_{2_{x_1}}|_{S_t}\right),$$

$$J_{32} := \begin{pmatrix} \langle\cdot, -\gamma_1^{-1}\Xi_{1_{x_2}}|_{S_t}\rangle \\ \langle\cdot, \gamma_1^{-1}\Upsilon_{1_{x_1}}|_{S_t}\rangle \end{pmatrix}, \qquad J_{42} := \begin{pmatrix} \langle\cdot, \gamma_2^{-1}\Xi_{2_{x_2}}|_{S_t}\rangle \\ \langle\cdot, -\gamma_2^{-1}\Upsilon_{2_{x_1}}|_{S_t}\rangle \end{pmatrix}.$$

We also note that the terms J_{22}, J_{23}, J_{24}, J_{32}, and J_{42} are from the operator B. In particular, J_{23}, J_{32}, J_{24}, and J_{42} represent interactions between point vortices and water waves.

The next lemma verifies that the Poisson map J has the properties required by the general theory in [32, section 2]. The proof of can be obtained similarly as in [32, Lemma 5.2].

Lemma 3.2 (Poisson map). *The Poisson map $J(u)$ satisfies the following:*

(i) *The domain $\mathcal{D}(\widehat{J})$ is dense in $\mathbb{X}^*$.*

(ii) *$\widehat{J}$ is injective.*

(iii) *For each $u \in \mathcal{O}\cap\mathbb{V}$, the operator $B(u)$ is bijective.*

(iv) *The map $u \mapsto B(u)$ is of class $C^1(\mathcal{O}\cap\mathbb{V}; \mathrm{Lin}(\mathbb{X})) \cap C^1(\mathcal{O}\cap\mathbb{W}; \mathrm{Lin}(\mathbb{W}))$.*

(v) *For each $u \in \mathcal{O}\cap\mathbb{V}$, $J(u)$ is skew-adjoint in the sense that*

$$\langle J(u)v, w\rangle = -\langle v, J(u)w\rangle$$

for all $v, w \in \mathcal{D}(\widehat{J})$.

Part (i) is a technical fact needed for the general theory. Many water waves problems, including our system (2.3), do not have a Poisson map that is bijective and hence do not satisfy the hypotheses for the stability theory in [11, 12]. However, the work [32] allows J to be only injective (as in part (ii)) with dense range. The injectivity of J excludes the existence of Casimir invariants, which simplifies the arguments. Part (v) is the characteristic of a Poisson map in a Hamiltonian system.

In the next subsection, we will use spectral theory to analyze the Hamiltonian system. Since there is a mismatch between space $\mathbb{V}$, where the energy is differentiable, and the energy space $\mathbb{X}$ itself, we must therefore show that DE can be realized as an element of $\mathbb{X}^*$. With that in mind, we define the extension $\nabla E \in C^0(\mathcal{O}\cap\mathbb{V}; \mathbb{X}^*)$ by

$$\begin{aligned} \langle \nabla E(u), v\rangle_{\mathbb{X}^*\times\mathbb{X}} := {} & \langle E'_\eta(u), v_1\rangle_{H^{-1}\times H^1} + \langle E'_\varphi(u), v_2\rangle_{\dot{H}^{-1/2}\times\dot{H}^{1/2}} \\ & + (E'_{\bar{x}}(u), v_3)_{\mathbb{R}^2} + (E'_{\bar{y}}(u), v_4)_{\mathbb{R}^2}, \end{aligned} \tag{3.7}$$

where

$$E'_\varphi(u) := \mathcal{G}(\eta)\varphi + \epsilon\nabla_\perp\Theta,$$

$$E'_\eta(u) := \frac{1}{2}\int_{\mathbb{R}} \varphi\langle D_\eta\mathcal{G}(\eta)\cdot,\varphi\rangle\,\mathrm{d}x_1 + \epsilon\varphi'\Theta_{x_1}|_{S_t} + \frac{\epsilon^2}{2}|(\nabla\Theta)|_{S_t}|^2 + g\eta - b\left(\frac{\eta'}{\langle\eta'\rangle}\right)',$$

and

$$E'_{\bar{x}}(u) := -\epsilon\int_{\mathbb{R}}\varphi\nabla_\perp\xi\,\mathrm{d}x_1 - \frac{\epsilon^2}{2}\int_{\mathbb{R}}\Big(\xi\nabla_\perp\Theta + \Theta|_{S_t}\nabla_\perp\xi\Big)\,\mathrm{d}x_1 + \nabla_{\bar{x}}\Gamma^*,$$

$$E'_{\bar{y}}(u) := -\epsilon\int_{\mathbb{R}}\varphi\nabla_\perp\zeta\,\mathrm{d}x_1 - \frac{\epsilon^2}{2}\int_{\mathbb{R}}\Big(\zeta\nabla_\perp\Theta + \Theta|_{S_t}\nabla_\perp\zeta\Big)\,\mathrm{d}x_1 + \nabla_{\bar{y}}\Gamma^*.$$

See Appendix B for details.

THEOREM 3.3 (Hamiltonian formulation). *A function* $u := (\eta,\varphi,\bar{x},\bar{y})^T \in C^1([0,t_0);\mathcal{O}\cap\mathbb{W})$ *is a solution of* (2.3) *if and only if it is a solution to the abstract Hamiltonian system*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = J(u)\nabla E(u), \tag{3.8}$$

where E *is the energy functional defined in* (3.1) *and* J *is the skew-symmetric operator defined by* (3.6).

Proof. Throughout the proof, we make repeated use of the identities

$$\nabla_\perp f_{x_1} = \nabla_\top f_{x_2} = \Big(f_{x_2}|_{S_t}\Big)', \quad \nabla_\perp f_{x_2} = -\nabla_\top f_{x_1} = -\Big(f_{x_1}|_{S_t}\Big)', \tag{3.9}$$

where f is any function harmonic in a neighborhood of S_t, and recall $\nabla_\perp$ and $\nabla_\top$ are defined in (2.4).

Suppose we have a solution u of (3.8). From the expressions for J in (3.6) and the differential equation (3.8), we see that

$$\partial_t\eta = E'_\varphi(u) = \mathcal{G}(\eta)\varphi + \epsilon\nabla_\perp\Theta,$$

which is the kinematic condition (1.3).

Next, we verify that

$$\partial_t\bar{x} = J_{32}\varphi + (\epsilon\gamma_1)^{-1}\mathcal{J}\nabla_{\bar{x}}E(u)$$

is equivalent to the ODE for $\bar{x}$ in (1.4). Explicitly, the first component of the equation is

$$\partial_t\bar{x}_1 = -(\epsilon\gamma_1)^{-1}\partial_{\bar{x}_2}E(u) + \left\langle E'_\varphi(u), -\gamma_1^{-1}\Xi_{1_{x_2}}|_{S_t}\right\rangle. \tag{3.10}$$

Using the fact that

$$\nabla_\perp\Phi = \nabla_\top\Psi, \quad \nabla_\top\Phi = -\nabla_\perp\Psi, \quad \nabla_\perp\Theta = \nabla_\top\Gamma, \quad \nabla_\top\Theta = -\nabla_\perp\Gamma$$

and the identities (3.9), (3.10) becomes

$$\begin{aligned}\partial_t\bar{x} &= \frac{1}{\gamma_1}\int_{\mathbb{R}}\left(-\Xi_{1_{x_1}}|_{S_t}\nabla_\perp\Psi + \Psi|_{S_t}\nabla_\perp\Xi_{1_{x_1}}\right)\,\mathrm{d}x_1 \\ &\quad + \frac{\epsilon}{2\gamma_1}\int_{\mathbb{R}}\left(-\Xi_{1_{x_1}}|_{S_t}\nabla_\perp\Gamma - \Xi_{1_{x_2}}|_{S_t}\nabla_\perp\Theta\right)\,\mathrm{d}x_1 - \epsilon\Gamma^*_{1_{x_2}}(\bar{x}) - \epsilon\Gamma_{2_{x_2}}(\bar{x}) - \epsilon\Gamma^*_{2_{x_2}}(\bar{x}) \\ &=: \frac{1}{\gamma_1}\mathscr{A} + \frac{\epsilon}{2\gamma_1}\mathscr{B} - \epsilon\Gamma^*_{1_{x_2}}(\bar{x}) - \epsilon\Gamma_{2_{x_2}}(\bar{x}) - \epsilon\Gamma^*_{2_{x_2}}(\bar{x}).\end{aligned}$$

Since Ψ and Θ_1^* are harmonic in Ω_t, for any $0 < r \ll 1$ we have

$$\begin{aligned}\mathscr{A} &= -\int_{\partial B_r(\bar{x})} \Big(-\Theta_{1_{x_1}} N \cdot \nabla\Psi + \Psi N \cdot \nabla\Theta_{1_{x_1}} \Big) \,\mathrm{d}S_t \\ &= -\int_{\partial B_r(\bar{x})} \left(\frac{\gamma_1}{2\pi} \frac{x_2 - \bar{x}_2}{|x-\bar{x}|^2} N \cdot \nabla\Psi + \Psi \frac{\gamma_1}{2\pi} \frac{x_2 - \bar{x}_2}{|x-\bar{x}|^3} \right) \mathrm{d}S_t \\ &= -\frac{\gamma_1}{2\pi} \int_0^{2\pi} \left(\frac{r\sin\theta}{r^2} \partial_r(\Psi) + \Psi \frac{r\sin\theta}{r^3} \right) r \,\mathrm{d}\theta.\end{aligned}$$

Expanding Ψ around $r = 0$ gives

$$\begin{aligned}\mathscr{A} = -\frac{\gamma_1}{2\pi} \int_0^{2\pi} \bigg[&\sin\theta(\Psi_{x_1}\cos\theta + \Psi_{x_2}\sin\theta) + \frac{\sin\theta}{r}(\Psi(\bar{x}) + \Psi_{x_1}(\bar{x}) r\cos\theta \\ &+ \Psi_{x_2}(\bar{x}) r \sin\theta) \bigg] \mathrm{d}\theta + o(r) = -\gamma_1 \Psi_{x_2}(\bar{x}) + o(r)\end{aligned}$$

as $r \to 0$. A direct computation along the same lines shows $\mathscr{B} = 0$. Thus, (3.10) is equivalent to

$$\partial_t \bar{x}_1 = -\Psi_{x_2}(\bar{x}) - \epsilon\Gamma^*_{1_{x_2}}(\bar{x}) - \epsilon\Gamma_{2_{x_2}}(\bar{x}) - \epsilon\Gamma^*_{2_{x_2}}(\bar{x}),$$

which agrees with the Kirchhoff–Helmholtz model (1.4). By nearly identical arguments, we likewise confirm that the same holds for $\partial_t \bar{x}_2$ and then $\partial_t \bar{y}$.

Finally, we claim that

$$\begin{aligned}\partial_t \varphi = &-E'_\eta(u) + \xi|_{S_t} \cdot (\gamma_1^{-1}\mathcal{J})\nabla_{\bar{x}} E(u) + \zeta|_{S_t} \cdot (\gamma_2^{-1}\mathcal{J})\nabla_{\bar{y}} E(u) \\ &+ \epsilon\xi|_{S_t} \left\langle E'_\varphi(u), (-\gamma_1^{-1}\mathcal{J})\xi \right\rangle + \epsilon\zeta|_{S_t} \left\langle E'_\varphi(u), (-\gamma_2^{-1}\mathcal{J})\zeta \right\rangle\end{aligned} \tag{3.11}$$

is equivalent to the unsteady Bernoulli condition in (2.3). By a well-known formula for the derivative of $\mathcal{G}(\eta)$ (see, for example, [21, Proposition 2.1]), we know that

$$\int_{\mathbb{R}} \varphi \langle D_\eta \mathcal{G}(\eta)\dot{\eta}, \varphi\rangle \,\mathrm{d}x_1 = \int_{\mathbb{R}} \frac{1}{\langle \eta' \rangle^2} \Big((\varphi')^2 - (\mathcal{G}(\eta)\varphi)^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi \Big) \dot{\eta} \,\mathrm{d}x_1.$$

Then

$$\begin{aligned}\partial_t \varphi = &-\frac{1}{\langle \eta' \rangle^2} \Big((\varphi')^2 - (\mathcal{G}(\eta)\varphi)^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi \Big) + \epsilon\varphi'\Gamma_{x_2}|_{S_t} - \frac{\epsilon^2}{2} |(\nabla\Theta)|_{S_t}|^2 - V'_\eta(u) \\ &+ \epsilon\Theta_{1_{x_1}}|_{S_t}\, \partial_t\bar{x}_1 + \epsilon\Theta_{1_{x_2}}|_{S_t}\, \partial_t\bar{x}_2 + \epsilon\Theta_{2_{x_1}}|_{S_t}\, \partial_t\bar{y}_1 + \epsilon\Theta_{2_{x_2}}|_{S_t}\, \partial_t\bar{y}_2.\end{aligned}$$

Here we have used the fact that for $\Theta = (\Theta_1 + \Theta_1^* + \Theta_2 + \Theta_2^*)(x_1, x_2, \bar{x}, \bar{y})$,

$$\begin{aligned}(\partial_t\Theta)|_S = &(-\Theta_{1_{x_1}} - \Theta^*_{1_{x_1}})|_{S_t}\, \partial_t\bar{x}_1 + (-\Theta_{1_{x_2}} + \Theta^*_{1_{x_2}})|_{S_t}\, \partial_t\bar{x}_2 + (-\Theta_{2_{x_1}} - \Theta^*_{2_{x_1}})|_{S_t}\, \partial_t\bar{y}_1 \\ &+ (-\Theta_{2_{x_2}} + \Theta^*_{2_{x_2}})|_{S_t}\, \partial_t\bar{y}_2 \\ = &-\Upsilon_{1_{x_1}}|_{S_t}\, \partial_t\bar{x}_1 - \Xi_{1_{x_2}}|_{S_t}\, \partial_t\bar{x}_2 - \Upsilon_{2_{x_1}}|_{S_t}\, \partial_t\bar{y}_1 - \Xi_{2_{x_2}}|_{S_t}\, \partial_t\bar{y}_2.\end{aligned}$$

Thus, comparing this to the equations for φ in (2.3), the claim has been proved. ☐

The momentum associated to a solution of the system (3.8) is given by

$$P = P(u) = -\epsilon\gamma_1\bar{x}_2 + \epsilon\gamma_2\bar{y}_2 - \int_{\mathbb{R}} \eta'(\varphi + \epsilon\Theta|_{S_t}) \,\mathrm{d}x_1. \tag{3.12}$$

It is clear that $P \in C^\infty(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$. Similarly to the Fréchet derivatives of the energy, DP can be extended to $\nabla P \in C^0(\mathcal{O} \cap \mathbb{V}; \mathbb{X}^*)$:

$$\begin{aligned}\langle \nabla P(u), v\rangle_{\mathbb{X}^*\times\mathbb{X}} := \langle P'_\eta(u), v_1\rangle_{H^{-1}\times H^1} + \langle P'_\varphi(u), v_2\rangle_{\dot{H}^{-1/2}\times\dot{H}^{1/2}} \\ + (P'_{\bar{x}}(u), v_3)_{\mathbb{R}^2} + (P'_{\bar{y}}(u), v_4)_{\mathbb{R}^2}\end{aligned} \tag{3.13}$$

with

$$P'_\eta(u) := \varphi' + \epsilon\Theta_{x_1}|_{S_t}, \qquad P'_\varphi(u) := -\eta',$$
$$P'_{\bar{x}}(u) := -\epsilon\gamma_1 e_2 + \epsilon\int_{\mathbb{R}} \eta'\xi|_{S_t}\,\mathrm{d}x_1, \qquad P'_{\bar{y}}(u) := \epsilon\gamma_2 e_2 + \epsilon\int_{\mathbb{R}} \eta'\zeta|_{S_t}\,\mathrm{d}x_1.$$

Observe also that ∇P is in $\mathcal{D}(\widehat{J})$ and

$$J(u)\nabla P(u) = (-\eta', -\varphi', 1, 0, 1, 0)^T. \tag{3.14}$$

The next lemma records the fact that the momentum and the energy are conserved.

LEMMA 3.4 (conservation). *Suppose that $u \in C^0([0,t_0); \mathcal{O}\cap\mathbb{W})$ is a distributional solution to the Cauchy problem* (3.8) *with initial data $u_0 \in \mathcal{O}\cap\mathbb{W}$. Then*

$$E(u(t)) = E(u_0) \quad \textit{and} \quad P(u(t)) = P(u_0) \quad \textit{for all } t \in [0,t_0).$$

Proof. The fact that the energy is conserved is a consequence of the well-posedness definition. For the conservation of momentum, let $u \in C^0([0,t_0); \mathcal{O}\cap\mathbb{W})$. Then using the chain rule for distributional solutions as in [11, Lemma 4.6], we compute

$$\begin{aligned}\partial_t P(u) &= -\langle J(u)\nabla P(u), \nabla E(u)\rangle \\ &= \langle \eta', E'_\eta\rangle_{H^{-1}\times H^1} + \langle \varphi', E'_\varphi\rangle_{\dot{H}^{-1/2}\times\dot{H}^{1/2}} - \partial_{\bar{x}_1}E(u) - \partial_{\bar{y}_1}E(u) \\ &= \int_{\mathbb{R}} \frac{1}{2\langle\eta'\rangle^2}\Big((\varphi')^2 - (\mathcal{G}(\eta)\varphi)^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi\Big)\eta'\,\mathrm{d}x_1 + \epsilon\int_{\mathbb{R}} \eta'\varphi'\Theta_{x_1}|_S\,\mathrm{d}x_1 \\ &\quad + \frac{\epsilon^2}{2}\int_{\mathbb{R}} \eta'|(\nabla\Theta)|_S|^2\,\mathrm{d}x_1 + \int_{\mathbb{R}} \eta'\Big(\eta - b\frac{\eta''}{\langle\eta'\rangle^3}\Big)\,\mathrm{d}x_1 + \int_{\mathbb{R}} (\mathcal{G}(\eta)\varphi + \epsilon\nabla_\perp\Theta)\varphi'\,\mathrm{d}x_1 \\ &\quad + \epsilon\int_{\mathbb{R}} \varphi\nabla_\perp\Theta_{x_1}\,\mathrm{d}x_1 + \frac{\epsilon^2}{2}\int_{\mathbb{R}} \Big(\Theta_{x_1}|_S\nabla\Theta + \Theta|_S\nabla_\perp\Theta_{x_1}\Big)\,\mathrm{d}x_1 \\ &= P_0 + \epsilon P_1 + \epsilon^2 P_2.\end{aligned}$$

For terms without ϵ, using the divergence theorem, we obtain

$$\begin{aligned}P_0 &= \int_{\mathbb{R}} \Bigg(\nabla_\perp\Phi\nabla_\top\Phi + \frac{1}{2\langle\eta'\rangle^2}\Big((\nabla_\top\Phi)^2 - (\nabla_\perp\Phi)^2 - 2\eta'\nabla_\perp\Phi\nabla_\top\Phi\Big)\eta' \\ &\qquad + \eta'\left(\eta - b\frac{\eta''}{\langle\eta'\rangle^3}\right)\Bigg)\,\mathrm{d}x_1 \\ &= \int_{\mathbb{R}} \left(\frac{1}{2\langle\eta'\rangle^2}\Big(\eta'(\nabla_\top\Phi)^2 - \eta'(\nabla_\perp\Phi)^2 + 2\nabla_\perp\Phi\nabla_\top\Phi\Big) + \partial_{x_1}\Big(\frac{1}{2}(\eta)^2 + b\langle\eta'\rangle^{-1}\Big)\right)\mathrm{d}x_1 \\ &= \frac{1}{2}\int_{\mathbb{R}} \Big(\Phi_{x_1}\nabla_\perp\Phi - \Psi_{x_1}\nabla_\perp\Psi\Big)\,\mathrm{d}x_1 = \frac{1}{2}\int_S \Big(\Phi_{x_1}N\cdot\nabla\Phi - \Psi_{x_1}N\cdot\nabla\Psi\Big)\,\mathrm{d}S \\ &= \frac{1}{2}\int_\Omega \Big(\nabla\Phi_{x_1}\cdot\nabla\Phi - \nabla\Psi_{x_1}\cdot\nabla\Psi\Big)\,\mathrm{d}x = \frac{1}{4}\int_\Omega \partial_{x_1}\Big(|\nabla\Phi|^2 - |\nabla\Psi|^2\Big)\,\mathrm{d}x = 0.\end{aligned}$$

Through direct computation, we can show that $P_1 = P_2 = 0$. Thus, $\partial_t P(u) = 0$, which means the momentum is conserved. ∎

Properties of the symmetry group. Next, we consider the symmetry group T defined by (1.12) and confirm that it indeed satisfies Assumption 4 (symmetry group) in [32]. As this follows essentially from the same argument as in [32, Lemma 5.4], which is itself largely straightforward, we will omit the details of the proofs.

The linear part of T is

$$dT(s)u := T(s)u - T(s)0 = (\eta(\cdot - s), \varphi(\cdot - s), \bar{x}, \bar{y})^T \quad \text{for all } s \in \mathbb{R}, u \in \mathbb{X}, \tag{3.15}$$

and the infinitesimal generator of T is the unbounded affine operator

$$T'(0)u := (-\eta', -\varphi', e_1, e_1)^T \quad \text{for all } u \in \mathcal{D}(T'(0))$$

with domain

$$\mathcal{D}(T'(0)) := H^2(\mathbb{R}) \times \left(\dot{H}^{3/2}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right) \times \mathbb{R}^2 \times \mathbb{R}^2.$$

From the definition of T, the neighborhood $\mathcal{O}$ and the subspaces $\mathbb{V}$ and $\mathbb{W}$ are all invariant under the symmetry group. T is strongly continuous on both $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$. Moreover, $I^{-1}\mathcal{D}(\widehat{J})$ is invariant under the *linear* symmetry group, or equivalently, $\mathcal{D}(\widehat{J})$ is invariant under the adjoint $dT^*(s) : \mathbb{X}^* \to \mathbb{X}^*$. We also have $T(0) = dT(0) = \mathrm{Id}_{\mathbb{X}}$, and for all $s, r \in \mathbb{R}$,

$$T(s+r) = T(s)T(r), \quad \text{and hence} \quad dT(s+r) = dT(s)dT(r).$$

From the definition (3.15), the linear part $dT(s)$ is a unitary operator on $\mathbb{X}$ for each $s \in \mathbb{R}$ and an isometry on the spaces $\mathbb{V}$ and $\mathbb{W}$.

The affine part of T affects only the location of the point vortices and therefore behaves the same way on $\mathbb{X}$, $\mathbb{W}$, and $\mathbb{V}$. In particular, we have the bound

$$\|T(s)0\|_{\mathbb{W}}, \|T(s)0\|_{\mathbb{X}} \sim |s|.$$

We also have that T commutes with the symplectic structure in the following way: for all $s \in \mathbb{R}$,

$$\widehat{J}I\, dT(s) = dT(s)\widehat{J}I,$$
$$dT(s)B(u) = B(T(s)u)dT(s) \quad \text{for all } u \in \mathcal{O} \cap \mathbb{V}.$$

By the previous discussion about the momentum P, we see that $\nabla P(u) \in \mathcal{D}(\widehat{J})$ for every $u \in \mathcal{D}(T'(0)|_{\mathbb{V}}) \cap \mathcal{O}$ and that

$$T'(0)u = J(u)\nabla P(u) \qquad \text{and} \qquad \widehat{J}I\, dT'(0) = dT'(0)\widehat{J}I.$$

This property and Lemma 3.2 ensure that the traveling waves $U_c \in \mathcal{D}(T'(0))$, and U_c is a critical point of the linearized Hamiltonian defined later in section 3.2.

We also observe that

$$\operatorname{Rng} \widehat{J} = \left(H^1(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R})\right) \times \left(H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right) \times \mathbb{R}^4,$$
$$\mathcal{D}(T'(0)|_{\mathbb{W}}) = H^4(\mathbb{R}) \times \left(\dot{H}^{7/2}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right) \times \mathbb{R}^4,$$

and hence,

$$\operatorname{Rng} \widehat{J} \cap \mathcal{D}(T'(0)|_{\mathbb{W}}) = \left(H^4(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R})\right) \times \left(H^{-1}(\mathbb{R}) \cap \dot{H}^{7/2}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right) \times \mathbb{R}^4.$$

Then using [32, Lemma A.1], we conclude that

$$\mathcal{D}(T'(0)|_{\mathbb{W}}) \cap \operatorname{Rng} \widehat{J} \text{ is dense in } \mathbb{X}.$$

This property is one of the most important ingredients in proving the instability result for [32].

Finally, it clear from their definitions in (3.1) and (3.12) that the energy and momentum are conserved by flow of the symmetry group. In particular, for all $u \in \mathcal{O} \cap \mathbb{V}$,

$$E(u) = E(T(s)u) \quad \text{and} \quad P(u) = P(T(s)u) \tag{3.16}$$

hold for all $s \in \mathbb{R}$.

Bound states. We finish this section by recasting the family $\mathscr{C}_{\mathrm{loc}}$ of solitary water waves from Theorem 1.1 as solutions of the above Hamiltonian system. We say that a solution u is a *bound state* of the Hamiltonian system (3.8) if it is in the form

$$u(t) = T(ct)U_c$$

for some time-independent $U_c \in \mathbb{W}$. Since $T(s)$ corresponds to horizontal translation, clearly the bound states correspond to traveling waves with wave speed c. Notice also that, by conservation of energy and momentum under the group (3.16), the energy and momentum of $T(s)U_c$ are independent of s.

Now, recall that in Theorem 1.1 we constructed a family of solitary capillary-gravity waves $\{(\eta, \varphi, \bar{x}, \bar{y}, c)\}$ parameterized smoothly by $(\epsilon, \tilde{c}, \gamma_1, \gamma_2)$ lying in a neighborhood of $(0, \tilde{c}_0, \gamma_1^0, \gamma_2^0)$. However, $-\epsilon\gamma_1$ and $\epsilon\gamma_2$ describe the strengths of the two point vortices, which are conserved quantities. For the stability analysis, it is therefore more natural to set them to a fixed value and imagine varying only the scaled wave speed $\tilde{c}$. This choice is further motivated by a variational characterization of the bound states that will be encountered in the next section.

In view of the above discussion, let us fix $(\epsilon, \gamma_1, \gamma_2)$ with $0 < |\epsilon| \ll 1$, $|\gamma_1 - \gamma_1^0| \ll 1$, $|\gamma_2 - \gamma_2^0| \ll 1$. We may then freely exchange $\tilde{c}$ with c, which will lie in an open interval $\mathcal{I}$ containing

$$c_0 = \epsilon\widetilde{c}_0 = -\frac{\epsilon\gamma_1^0}{4\pi(a_0 - \rho_0)} + \frac{\epsilon\gamma_2^0}{4\pi}\left(\frac{1}{a_0} + \frac{1}{\rho_0}\right).$$

The resulting one-parameter family of traveling waves we denote by

$$\{U_c = (\eta(c), \varphi(c), \bar{x}(c), \bar{y}(c)) : c \in \mathcal{I}\}. \tag{3.17}$$

The next lemma records some basic facts about these bound states that are needed for the general theory. In particular, these play an important role in showing that one can smoothly quotient out the action of the symmetry group; see [32, Lemma 3.1].

LEMMA 3.5 (bound states). *Fix any choice of* $0 < \rho_0 < a_0$, *and consider the corresponding surface of solutions* $\mathscr{C}_{\mathrm{loc}}$ *furnished by Theorem* 1.1. *Then the corresponding one-parameter family of bound states* $\{U_c\}_{c\in\mathcal{I}}$ *satisfies Assumption* 5 *(bound states) in* [32]:

(i) *The mapping* $c \in \mathcal{I} \mapsto U_c \in \mathcal{O} \cap \mathbb{W}$ *is* C^1.
(ii) *The nondegeneracy condition* $T'(0)U_c \neq 0$ *holds for every* $c \in \mathcal{I}$. *Equivalently,* U_c *is never a critical point of the momentum.*

(iii) *For all* $c \in \mathcal{I}$,

$$U_c \in \mathcal{D}(T'''(0)) \cap \mathcal{D}(\widehat{J}IT'(0))$$

and

$$\widehat{J}IT'(0)U_c \in \mathcal{D}(T'(0)|_{\mathbb{W}}).$$

(iv) $\liminf_{|s|\to\infty} \|T(s)U_c - U_c\|_{\mathbb{X}} > 0.$

Parts (i) and (ii) are automatically given by the implicit function theorem arguments in the proof of Theorem 1.1. Because one can take arbitrarily high regularity in the existence theory, it is always possible to ensure that part (iii) holds. Part (iv) is a technical requirement for the theory [32], but essentially always holds for solitary waves like $\{U_c\}$. Indeed,

$$\|T(s)U_c - U_c\|_{\mathbb{X}} \geq 2|s|.$$

3.2. Spectrum of the augmented potential. It is well-known that traveling water waves can frequently be characterized as constrained extrema of the energy with fixed momentum. With that in mind, we define the *augmented Hamiltonian* to be the functional

$$E_c(u) := E(u) - cP(u).$$

By (3.8) and (3.14), we have $JDE(U_c) - cJDP(U_c) = 0$, and hence

$$DE_c(U_c) = DE(U_c) - cDP(U_c) = 0. \tag{3.18}$$

Thus, each traveling wave U_c is indeed a critical point of the augmented Hamiltonian. Formally, at least, this is suggestive of a constrained minimization problem, with c being the Lagrange multiplier. We might hope, then, that the stability or instability of the bound state will relate to whether it locally minimizes the energy on a fixed momentum manifold.

This idea, though quite elegant, is far from straightforward to carry out. As a first step, in this section we compute the second variation of the augmented Hamiltonian and determine its spectrum. Our main result is that, as in the setting of [11, 12], $D^2E_c(U_c)$ can be associated to a bounded self-adjoint operator on $\mathbb{X}$ whose spectrum takes the form $\{-\mu_c^2\} \cup \{0\} \cup \Sigma_c$, where $\Sigma_c \subset \mathbb{R}_+$ is uniformly bounded away from 0 and $-\mu_c^2 < 0$. This also corresponds to Assumption 6 (spectrum) in [32]. It is important to note that the presence of an unstable eigenvalue does not immediately imply instability.

We first note that 0 is in the spectrum. Indeed, for all $s \in \mathbb{R}$, $T(s)U_c$ is also a traveling wave solution. Therefore,

$$DE_c(T(s)U_c) = 0$$

for all s. Differentiating with respect to s gives

$$\left\langle D^2E_c(T(s)U_c), T'(0)U_c \right\rangle = 0,$$

and hence $T'(0)U_c$ is an eigenfunction for eigenvalue 0.

Following Mielke's approach [21], we will determine the remaining spectrum by first considering the *augmented potential*

$$V_c = V_c(\eta, \bar{x}, \bar{y}) := \min_{\varphi \in \mathbb{V}_2} E_c(\eta, \varphi, \bar{x}, \bar{y}) =: E_c(\eta, \varphi_m, \bar{x}, \bar{y}) \tag{3.19}$$

for $(\eta, \bar{x}, \bar{y}) \in \mathbb{V}_1 \times \mathbb{V}_3 \times \mathbb{V}_4$. Thus,

$$D_\varphi E_c(\eta, \varphi_m, \bar{x}, \bar{y}) = 0. \tag{3.20}$$

Because φ occurs quadratically in the energy, it is easy to see that this minimum is attained exactly when

$$\varphi_m(\eta, \bar{x}, \bar{y}) = \mathcal{G}(\eta)^{-1} \left(-c\eta' - \epsilon \nabla_\perp \Theta\right). \tag{3.21}$$

Since we will be doing many calculations where φ is fixed, we adopt the notational convention that for $u = (u_1, u_2, u_3, u_4)$,

$$v := (u_1, u_3, u_4)$$

and write a variation in the direction v as $\dot{v}$. We also use the short hand $\mathbb{V}_{1,3,4} := \mathbb{V}_1 \times \mathbb{V}_3 \times \mathbb{V}_4$, and for convenience, define

$$u_m(v) := (\eta, \varphi_m, \bar{x}, \bar{y}) \in \mathbb{V}.$$

Also, when we evaluate derivatives of the Dirichlet–Neumann operator, we will encounter the quantities

$$\mathfrak{a} := \left(\nabla \langle \mathcal{H}(\eta), \varphi_m \rangle\right)|_S, \quad \mathfrak{b} := \mathfrak{a} + \epsilon (\nabla \Theta)|_S - c e_1.$$

See Appendix A for an explicit formula giving $\mathfrak{a}$ in terms of φ and η. Physically, $\mathfrak{b}$ is the restriction of the full relative velocity to the interface. Therefore, $\mathfrak{b}_2 = \eta' \mathfrak{b}_1$ due to (3.21).

While it is not completely obvious, we will see that the spectral properties of $D^2 E_c(U_c)$ can be inferred from those of $D^2 V_c(v)$. With that in mind, the first step is to derive a formula for the second variation of the augmented potential.

LEMMA 3.6. *For all $v \in \mathbb{V}_{1,3,4} \cap \mathcal{O}_{1,3,4}$ and all variations $\dot{v} \in \mathbb{V}_{1,3,4}$, we have*

$$\begin{aligned} \left\langle D^2 V_c(v)\dot{v}, \dot{v} \right\rangle_{\mathbb{V}_{1,3,4}^* \times \mathbb{V}_{1,3,4}} &= -\left\langle \mathcal{L}(v)\dot{v}, \mathcal{G}(\eta)^{-1} \mathcal{L}(v)\dot{v} \right\rangle_{\mathbb{X}_2^* \times \mathbb{X}_2} \\ &\quad + \left\langle D_v^2 E_c(u_m(v))\dot{v}, \dot{v} \right\rangle_{\mathbb{V}_{1,3,4}^* \times \mathbb{V}_{1,3,4}}, \end{aligned} \tag{3.22}$$

where $\mathcal{L}(v) \in \operatorname{Lin}(\mathbb{X}_{1,3,4}; \mathbb{X}_2^*)$ defined by

$$\mathcal{L}(v)\dot{v} := \mathcal{G}(\eta)(\mathfrak{a}_2 \dot{\eta}) + (\mathfrak{b}_1 \dot{\eta})' + \epsilon \nabla_\perp \xi \cdot \dot{\bar{x}} + \epsilon \nabla_\perp \zeta \cdot \dot{\bar{y}}. \tag{3.23}$$

The proof follows by a straightforward adaptation of [32, Lemma 6.2], and we therefore omit it. In the next lemma, we refine expression (3.22) to derive a quadratic form representation of $D^2 V_c$.

LEMMA 3.7 (quadratic form). *For all $v \in \mathbb{V}_{1,3,4} \cap \mathcal{O}_{1,3,4}$, there is a self-adjoint linear operator $A(v) \in \operatorname{Lin}(\mathbb{X}_{1,3,4}; \mathbb{X}_{1,3,4}^*)$ such that*

$$\left\langle D^2 V_c(v)\dot{v}, \dot{w} \right\rangle_{\mathbb{V}_{1,3,4}^* \times \mathbb{V}_{1,3,4}} = \langle A\dot{v}, \dot{w} \rangle_{\mathbb{X}_{1,3,4}^* \times \mathbb{X}_{1,3,4}}$$

for all $\dot{v}, \dot{w} \in \mathbb{V}_{1,3,4}$. The form of A is given in (3.25).

Proof. From [21, Proposition 2.1], we have

$$\int_{\mathbb{R}} \hat{\varphi} \langle D_\eta \mathcal{G}(\eta)\dot{\eta}, \varphi \rangle \, dx_1 = \int_{\mathbb{R}} \dot{\eta} (\mathfrak{a}_1 \hat{\varphi}' - \mathfrak{a}_2 \mathcal{G}(\eta)\hat{\varphi}) \, dx_1$$

and

$$\int_{\mathbb{R}} \varphi \langle \langle D_\eta^2 \mathcal{G}(\eta)\dot{\eta}, \dot{\eta}\rangle, \varphi\rangle \, \mathrm{d}x_1 = 2\int_{\mathbb{R}} \left(\dot{\eta}^2 \mathfrak{a}_1' \mathfrak{a}_2 + \mathfrak{a}_2 \dot{\eta} \mathcal{G}(\eta)(\mathfrak{a}_2\dot{\eta})\right) \mathrm{d}x_1.$$

Letting the self-adjoint operator $\mathcal{M}$ be defined by

$$\mathcal{M}\dot{\eta} := -\mathfrak{b}_1(\mathcal{G}(\eta)^{-1}(\mathfrak{b}_1\dot{\eta})')'$$

and using the fact that $\mathcal{G}(\eta)^{-1}$ is a self-adjoint operator, we can compute

$$\begin{aligned}
&\int_{\mathbb{R}} \mathcal{L}(v)\dot{v}\mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v} \, \mathrm{d}x_1 \\
&\quad = \int_{\mathbb{R}} \mathfrak{a}_2\dot{\eta}\mathcal{G}(\eta)(\mathfrak{a}_2\eta)\, \mathrm{d}x_1 + \int_{\mathbb{R}} \dot{\eta}\mathcal{M}\dot{\eta}\, \mathrm{d}x_1 + \int_{\mathbb{R}} (\mathfrak{a}_2\mathfrak{b}_1' - \mathfrak{a}_2'\mathfrak{b}_1)\dot{\eta}^2 \, \mathrm{d}x_1 \\
&\qquad + 2\epsilon\dot{\bar{x}} \cdot \int_{\mathbb{R}} \left(\mathfrak{a}_2 \nabla_\perp \xi - \mathfrak{b}_1 \left(\mathcal{G}(\eta)^{-1}\nabla_\perp \xi\right)'\right) \dot{\eta}\, \mathrm{d}x_1 \\
&\qquad + \epsilon^2 \dot{\bar{x}}^T \left(\int_{\mathbb{R}} \nabla_\perp \xi \odot \mathcal{G}(\eta)^{-1}\nabla_\perp\xi \, \mathrm{d}x_1\right)\dot{\bar{x}} \\
&\qquad + 2\epsilon\dot{\bar{y}} \cdot \int_{\mathbb{R}} \left(\mathfrak{a}_2 \nabla_\perp \zeta - \mathfrak{b}_1 \left(\mathcal{G}(\eta)^{-1}\nabla_\perp \zeta\right)'\right) \dot{\eta}\, \mathrm{d}x_1 \\
&\qquad + \epsilon^2 \dot{\bar{y}}^T \left(\int_{\mathbb{R}} \nabla_\perp \zeta \odot \mathcal{G}(\eta)^{-1}\nabla_\perp\zeta \, \mathrm{d}x_1\right)\dot{\bar{y}} \\
&\qquad + 2\epsilon^2 \dot{\bar{x}}^T \left(\int_{\mathbb{R}} \nabla_\perp \xi \odot \mathcal{G}(\eta)^{-1}\nabla_\perp\zeta \, \mathrm{d}x_1\right)\dot{\bar{y}},
\end{aligned}$$

where $x \odot y = (x \otimes y + y \otimes x)/2$ is the symmetric outer product. Next, we examine more closely the term involving $D_v^2 E_c$ in (3.22). We calculate that

$$\begin{aligned}
\left\langle D_\eta^2 E_c(u_m)\dot{\eta}, \dot{\eta}\right\rangle &= \int_{\mathbb{R}} \mathfrak{a}_2\dot{\eta}\mathcal{G}(\eta)(\mathfrak{a}_2\eta)\, \mathrm{d}x_1 + \int_{\mathbb{R}} \left(g + \epsilon\mathfrak{b}_1\nabla_\top\Theta_{x_2} + \mathfrak{a}_2\mathfrak{b}_1'\right)\dot{\eta}^2\, \mathrm{d}x_1 \\
&\quad + \int_{\mathbb{R}} \frac{b}{\langle \eta'\rangle^3}(\dot{\eta}')^2\, \mathrm{d}x_1,
\end{aligned}$$

while

$$\begin{aligned}
\nabla_{\bar{x}}\langle D_\eta E_c(u_m), \dot{\eta}\rangle &= \epsilon\int_{\mathbb{R}} (\mathfrak{a}_2\nabla_\perp\xi - \mathfrak{b}_1\nabla_\top\xi)\dot{\eta}\, \mathrm{d}x_1, \\
\nabla_{\bar{y}}\langle D_\eta E_c(u_m), \dot{\eta}\rangle &= \epsilon\int_{\mathbb{R}} (\mathfrak{a}_2\nabla_\perp\zeta - \mathfrak{b}_1\nabla_\top\zeta)\dot{\eta}\, \mathrm{d}x_1.
\end{aligned}$$

Similarly, evaluating the Hessian of E_c with respect to $(\bar{x}, \bar{y})$ gives

$$\begin{aligned}
D_{\bar{x}}^2 E_c(u_m) &= 2\epsilon^2 D_{\bar{x}}^2\Gamma^* - \epsilon\int_{\mathbb{R}} (\mathcal{G}(\eta)\varphi_m D_{\bar{x}}^2\Theta + \varphi_m' D_{\bar{x}}^2\Gamma)|_S\, \mathrm{d}x_1 + \epsilon^2\int_{\mathbb{R}} \nabla_\perp\xi\odot\xi\, \mathrm{d}x_1 \\
&\quad - \frac{\epsilon^2}{2}\int_{\mathbb{R}} (\nabla_\perp\Theta D_{\bar{x}}^2\Theta + \nabla_\top\Theta D_{\bar{x}}^2\Gamma)|_S\, \mathrm{d}x_1, \\
D_{\bar{y}}^2 E_c(u_m) &= 2\epsilon^2 D_{\bar{y}}^2\Gamma^* - \epsilon\int_{\mathbb{R}} (\mathcal{G}(\eta)\varphi_m D_{\bar{y}}^2\Theta + \varphi_m' D_{\bar{y}}^2\Gamma)|_S\, \mathrm{d}x_1 + \epsilon^2\int_{\mathbb{R}} \nabla_\perp\zeta\odot\zeta\, \mathrm{d}x_1 \\
&\quad - \frac{\epsilon^2}{2}\int_{\mathbb{R}} (\nabla_\perp\Theta D_{\bar{y}}^2\Theta + \nabla_\top\Theta D_{\bar{y}}^2\Gamma)|_S\, \mathrm{d}x_1,
\end{aligned}$$

and

$$\nabla_{\bar{x}}\nabla_{\bar{y}}E_c(u_m) = \epsilon^2\nabla_{\bar{x}}\nabla_{\bar{y}}\Gamma^* + \frac{\epsilon^2}{2}\int_{\mathbb{R}}\nabla_\perp(\xi\odot\zeta)\,\mathrm{d}x_1.$$

Substituting the above results into the expression (3.22), we arrive at

$$\begin{aligned}\langle D^2V_c(v)\dot{v},\dot{v}\rangle = &\int_{\mathbb{R}}(g+\mathfrak{b}_2'\mathfrak{b}_1)\dot{\eta}^2\,\mathrm{d}x_1 - \int_{\mathbb{R}}\left(\frac{b}{\langle\eta'\rangle^3}\dot{\eta}'\right)'\dot{\eta}\,\mathrm{d}x_1 - \int_{\mathbb{R}}\dot{\eta}\mathcal{M}\dot{\eta}\,\mathrm{d}x_1\\ &+2\epsilon\dot{\bar{x}}\cdot\int_{\mathbb{R}}\dot{\eta}\mathfrak{b}_1\nabla_\top(\mathcal{G}(\eta)^{-1}\nabla_\perp\xi-\xi)\,\mathrm{d}x_1\\ &+2\epsilon\dot{\bar{y}}\cdot\int_{\mathbb{R}}\dot{\eta}\mathfrak{b}_1\nabla_\top(\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta-\zeta)\,\mathrm{d}x_1\\ &+\dot{\bar{x}}^T\left(D^2_{\bar{x}}E_c(u_m)-\epsilon^2\int_{\mathbb{R}}\nabla_\perp\xi\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\xi\,\mathrm{d}x_1\right)\dot{\bar{x}}\\ &+\dot{\bar{y}}^T\left(D^2_{\bar{y}}E_c(u_m)-\epsilon^2\int_{\mathbb{R}}\nabla_\perp\zeta\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta\,\mathrm{d}x_1\right)\dot{\bar{y}}\\ &+\dot{\bar{x}}^T\left(\nabla_{\bar{x}}\nabla_{\bar{y}}E_c(u_m)-2\epsilon^2\int_{\mathbb{R}}\nabla_\perp\xi\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta\,\mathrm{d}x_1\right)\dot{\bar{y}}.\end{aligned} \tag{3.24}$$

Thus, inspecting (3.24), we see that the claimed quadratic form representation holds with the operator A defined as follows:

$$A_{11}\dot{\eta} := (g+\mathfrak{b}_2'\mathfrak{b}_1)\dot{\eta} - \left(\frac{b}{\langle\eta'\rangle^3}\dot{\eta}'\right)' - \mathcal{M}\dot{\eta}, \tag{3.25a}$$

$$A_{13}\dot{\bar{x}} := \epsilon\mathfrak{b}_1\nabla_\top(\mathcal{G}(\eta)^{-1}\nabla_\perp\xi-\xi)\cdot\dot{\bar{x}}, \tag{3.25b}$$

$$A_{13}^*\dot{\eta} := \epsilon\int_{\mathbb{R}}\dot{\eta}\mathfrak{b}_1\nabla_\top(\mathcal{G}(\eta)^{-1}\nabla_\perp\xi-\xi)\,\mathrm{d}x_1, \tag{3.25c}$$

$$A_{14}\dot{\bar{y}} := \epsilon\mathfrak{b}_1\nabla_\top(\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta-\zeta)\cdot\dot{\bar{y}}, \tag{3.25d}$$

$$A_{14}^*\dot{\eta} := \epsilon\int_{\mathbb{R}}\dot{\eta}\mathfrak{b}_1\nabla_\top(\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta-\zeta)\,\mathrm{d}x_1, \tag{3.25e}$$

$$A_{33} := D^2_{\bar{x}}E_c(u_m) - \epsilon^2\int_{\mathbb{R}}\nabla_\perp\xi\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\xi\,\mathrm{d}x_1, \tag{3.25f}$$

$$A_{44} := D^2_{\bar{y}}E_c(u_m) - \epsilon^2\int_{\mathbb{R}}\nabla_\perp\zeta\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta\,\mathrm{d}x_1, \tag{3.25g}$$

$$A_{34} = A_{43} := \nabla_{\bar{x}}\nabla_{\bar{y}}E_c(u_m) - \epsilon^2\int_{\mathbb{R}}\nabla_\perp\xi\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta\,\mathrm{d}x_1. \tag{3.25h}$$

This finishes the proof of Lemma 3.7. ∎

The next lemma verifies that the second variation of the augmented Hamiltonian E_c has an extension to the energy space $\mathbb{X}$, which enables us to assign meaning to its spectrum as an operator on $\mathbb{X}$.

LEMMA 3.8 (extension of D^2E_c). *For all $v\in\mathbb{V}_{1,3,4}\cap\mathcal{O}_{1,3,4}$, there exists a self-adjoint operator $H_c(v)\in\mathrm{Lin}(\mathbb{X},\mathbb{X}^*)$ such that*

$$\langle D^2E_c(u_m(v))\dot{u},\dot{w}\rangle_{\mathbb{V}^*\times\mathbb{V}} = \langle H_c(v)\dot{u},\dot{w}\rangle_{\mathbb{X}^*\times\mathbb{X}} \tag{3.26}$$

for all $\dot{u}, \dot{w} \in \mathbb{V}$ *with*

$$H_c(v)\dot{u} = \begin{pmatrix} \mathrm{Id}_{\mathbb{X}_1^*} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{\mathbb{X}_2^*} \\ 0 & \mathrm{Id}_{\mathbb{R}^2} & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{\mathbb{R}^2} & 0 \end{pmatrix} \begin{pmatrix} A(v) + \mathcal{L}(v)^*\mathcal{G}(\eta)^{-1}\mathcal{L}(v) & -\mathcal{L}(v)^* \\ -\mathcal{L}(v) & \mathcal{G}(\eta) \end{pmatrix} \begin{bmatrix} \dot{v} \\ \dot{\varphi} \end{bmatrix},$$

where $\mathcal{L}(v)$ *and* $A(v)$ *are defined in Lemmas* 3.6 *and* 3.7, *respectively. The adjoint* $\mathcal{L}(v)^* \in \mathrm{Lin}(\mathbb{X}_2; \mathbb{X}_{1,3,4}^*)$ *is given by*

$$\mathcal{L}(v)^*\dot{\varphi} = (\mathfrak{a}_2\mathcal{G}(\eta)\dot{\varphi} - \mathfrak{b}_1\dot{\varphi}', \epsilon\langle\nabla_\perp(\xi+\zeta), \dot{\varphi}\rangle),$$

and we have

$$\begin{aligned} \langle H_c(v)\dot{u}, \dot{u}\rangle_{\mathbb{X}^*\times\mathbb{X}} &= \langle A(v)\dot{v}, \dot{v}\rangle_{\mathbb{X}_{1,3,4}^*\times\mathbb{X}_{1,3,4}} \\ &\quad + \big\langle \mathcal{G}(\eta)(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}\dot{v}), \dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}\dot{v}\big\rangle_{\mathbb{X}_2^*\times\mathbb{X}_2} \end{aligned} \tag{3.27}$$

for all $\dot{u} \in \mathbb{X}$.

Proof. It is straightforward to see that

$$\langle D_\varphi D_v E_c(u_m(v))\dot{v}, \dot{\varphi}\rangle_{\mathbb{V}_2^*\times\mathbb{V}_2} = -\int_{\mathbb{R}} \dot{v}\mathcal{L}(v)\dot{v}\,\mathrm{d}x_1$$

holds for all $\dot{v} \in \mathbb{V}_{1,3,4}$ and $\dot{\varphi} \in \mathbb{V}_2$. Because of symmetry, it suffices to consider only the diagonal entries. For all $\dot{u} \in \mathbb{V}$, Lemmas 3.6 and 3.7 give

$$\begin{aligned} \langle D^2E_c(u_m(v))\dot{u}, \dot{u}\rangle_{\mathbb{V}^*\times\mathbb{V}} &= \langle D_v^2E_c(u_m(v))\dot{v}, \dot{v}\rangle + 2\langle D_\varphi D_v E_c(u_m(v))\dot{v}, \dot{\varphi}\rangle \\ &\quad + \langle D_\varphi^2 E_c(u_m(v))\dot{\varphi}, \dot{\varphi}\rangle \\ &= \langle A(v)\dot{v}, \dot{v}\rangle_{\mathbb{X}_{1,3,4}^*\times\mathbb{X}_{1,3,4}} \\ &\quad + \int_{\mathbb{R}} \Big[(\mathcal{L}(v)\dot{v})\mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v} - 2\dot{\varphi}\mathcal{L}(v)\dot{v} + \dot{\varphi}\mathcal{G}(\eta)\dot{\varphi}\Big]\,\mathrm{d}x_1 \\ &= \langle A(v)\dot{v}, \dot{v}\rangle_{\mathbb{X}_{1,3,4}^*\times\mathbb{X}_{1,3,4}} - \int_{\mathbb{R}} \mathcal{L}(v)\dot{v}(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})\,\mathrm{d}x_1 \\ &\quad + \int_{\mathbb{R}} \dot{\varphi}\mathcal{G}(\eta)(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})\,\mathrm{d}x_1. \end{aligned} \tag{3.28}$$

Using the fact that $\mathcal{G}(\eta)$ and $\mathcal{G}(\eta)^{-1}$ are self-adjoint operators, the integral is equal to

$$\begin{aligned} &\int_{\mathbb{R}} \Big[-\mathcal{L}(v)\dot{v}(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v}) + (\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})\mathcal{G}(\eta)\dot{\varphi}\Big]\,\mathrm{d}x_1 \\ &\quad = \int_{\mathbb{R}} \mathcal{G}(\eta)(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})\,\mathrm{d}x_1. \end{aligned}$$

Substituting this into (3.28) yields our desired result. ∎

We finish this section by characterizing the spectrum of the linearized augmented Hamiltonian at U_c for $0 < |c| \ll 1$. In particular, we show that Assumption 6 (spectrum) in [32] is satisfied.

THEOREM 3.9 (spectrum). *Fix any choice of* $0 < \rho_0 < a_0$ *subject to the compatibility condition* (2.3)*, and consider the family of traveling wave solutions* $\{U_c\}_{c\in\mathcal{I}}$ *as in* (3.17)*. Then, perhaps upon shrinking* $\mathcal{I}$*, it holds that for all* $c \in \mathcal{I}$*,* $I^{-1}H_c$ *has one negative eigenvalue,* 0 *is in the spectrum, and the rest of the spectrum* $\Sigma_c \subset (0,\infty)$ *is bounded away from* 0*.*

Proof. From the asymptotic information (2.9) furnished by Theorem 1.1 we infer that

$$\mathfrak{a}_1 = O(|\epsilon|^3), \quad \mathfrak{a}_2 = O(|\epsilon|^3), \quad \mathfrak{b}_1 = \mathfrak{a}_1 - c + o(\epsilon^2) = O(|\epsilon|), \quad \mathfrak{b}_2 = \mathfrak{a}_2 + o(\epsilon^2) = O(|\epsilon|).$$

Then from Lemmas 3.7 and 3.8, we can write

$$H_c = \begin{pmatrix} g - b\partial_{x_1}^2 & 0 & 0 \\ 0 & |\partial_{x_1}| & 0 \\ 0 & 0 & \frac{\epsilon^2}{4\pi}\mathcal{A} \end{pmatrix} + R_c \in \mathrm{Lin}(\mathbb{X}, \mathbb{X}^*),$$

where

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{33} & \mathcal{A}_{34} \\ \mathcal{A}_{43} & \mathcal{A}_{44} \end{pmatrix} := \begin{pmatrix} -\alpha & 0 & \alpha & 0 \\ 0 & \delta_1 + \alpha & 0 & -\beta \\ \alpha & 0 & -\alpha & 0 \\ 0 & -\beta & 0 & \delta_2 + \alpha \end{pmatrix}$$

with

$$\alpha := \frac{\gamma_1\gamma_2}{2}\left(\frac{1}{\rho^2} - \frac{1}{a^2}\right), \quad \beta := \frac{\gamma_1\gamma_2}{2}\left(\frac{1}{\rho^2} + \frac{1}{a^2}\right), \quad \delta_1 := \frac{\gamma_1^2}{(a-\rho)^2}, \quad \delta_2 := \frac{\gamma_2^2}{(a+\rho)^2}.$$

Here, the remainder term R_c is $O(|\epsilon|^3)$ in $\mathrm{Lin}(\mathbb{X}, \mathbb{X}^*)$, and $I^{-1}R_c$ is self-adjoint on $\mathbb{X}$.

It follows that $I^{-1}H_c|_{\epsilon=0}$ is a self-adjoint operator on $\mathbb{X}$ for which zero is an eigenvalue of multiplicity 4, and the remainder of the spectrum is strictly positive. By general results from perturbation theory of self-adjoint operators (see, for example, [14, Chapter 5, Theorem 4.10]), it follows that for $0 < |\epsilon| \ll 1$, $I^{-1}H_c$ will have four real eigenvalues in a neighborhood of 0, and the rest of the spectrum $\Sigma_c \subset (0,\infty)$.

The key point is where the small eigenvalues are situated, and to determine that we must look more closely at the matrix $\mathcal{A}$. In particular, direct computation confirms that it has the eigenvalues

$$0, \quad -2\alpha, \quad \frac{2\alpha + \delta_1 + \delta_2 + \sqrt{(\delta_1 - \delta_2)^2 + 4\beta^2}}{2}, \quad \frac{2\alpha + \delta_1 + \delta_2 - \sqrt{(\delta_1 - \delta_2)^2 + 4\beta^2}}{2}.$$

We know that 0 is in the spectrum of $I^{-1}H_c$ due to translation invariance. Clearly, $-2\alpha < 0$, and the third eigenvalue above is positive. We claim that the last eigenvalue is also positive. Indeed,

$$2\alpha + \delta_1 + \delta_2 - \sqrt{(\delta_1 - \delta_2)^2 + 4\beta^2} > 0$$

is equivalent to

$$\alpha^2 + \alpha\delta_1 + \alpha\delta_2 + \delta_1\delta_2 - \beta^2 > 0.$$

Using the compatibility condition (1.8), we compute

$$\begin{aligned} \alpha^2 + \alpha\delta_1 + \alpha\delta_2 + \delta_1\delta_2 - \beta^2 &= \frac{\gamma_1\gamma_2}{2}\left(\frac{1}{\rho^2} - \frac{1}{a^2}\right)\left(\frac{\gamma_1^2}{(a-\rho)^2} + \frac{\gamma_2^2}{(a+\rho)^2}\right) - \frac{\gamma_1^2\gamma_2^2}{a^2\rho^2} \\ &= \frac{2(a+\rho)}{(a-\rho)(a^2 + a\rho + \rho^2)^2}\gamma_1^4 > 0. \end{aligned}$$

In total, then we have proved that $\mathcal{A}$ has a unique negative real eigenvalue. For $0 < |\epsilon| \ll 1$, the contribution of $I^{-1}R_c$ is perturbative as it is $O(|\epsilon|^3)$ in $\mathrm{Lin}(\mathbb{X})$. We therefore may conclude that $I^{-1}H_c$ possess a unique negative eigenvalue. ∎

3.3. Proof of Theorem 1.4. At this point, we have verified all of the hypotheses of Varholm, Wahlén, and Walsh [32, section 2]. Applying [32, Theorems 2.4 and 2.6], we conclude that the stability or instability of the traveling wave is determined by the so-called *moment of instability.* That is, let $d = d(c)$ be the scalar-valued function that results from evaluating E_c along the family $\{U_c\}$:

$$d(c) := E_c(U_c) = E(U_c) - cP(U_c). \tag{3.29}$$

Under the above hypotheses, the sign of $d''(c)$ determines precisely whether U_c sits at a minimum or saddle of the energy constrained to a level set of the momentum, and this in turn implies stability or instability.

Differentiating d gives the identity

$$d'(c) = \left\langle DE(U_c) - cDP(U_c), \frac{\mathrm{d}U_c}{\mathrm{d}c} \right\rangle - P(U_c) = -P(U_c).$$

Using the expressions for the momentum P, $\bar{x}_2$, and $\bar{y}_2$, we can compute

$$d'(c) = \epsilon\gamma_1(-a+\rho) - \epsilon\gamma_2(-a-\rho) - \int_{\mathbb{R}} \eta(\varphi' + \epsilon\nabla_{\top}\Theta)\,\mathrm{d}x_1.$$

Differentiating once more yields

$$\begin{aligned} d''(c) = \epsilon\gamma_1\partial_c(-a+\rho) + \epsilon\gamma_2\partial_c(a+\rho) \\ - \int_{\mathbb{R}} \Big((\partial_c\eta)(\varphi' + \epsilon\nabla_{\top}\Theta) + \eta\,\partial_c(\varphi' + \epsilon\nabla_{\top}\Theta) \Big)\,\mathrm{d}x_1. \end{aligned}$$

Recalling the definition of $\mathscr{T}$ in (2.8) and using the compatibility (1.8) and variations for a and ρ in (2.10), we obtain

$$\begin{aligned} d''(c) &= -\gamma_1(a_{\tilde{c}} - \rho_{\tilde{c}}) + \gamma_2(a_{\tilde{c}} + \rho_{\tilde{c}}) + O(\epsilon^3) \\ &= -\frac{\gamma_1}{\det\mathscr{T}}\left(-\frac{\gamma_2^0}{2\pi(a_0+\rho_0)^2} + \frac{-\gamma_1^0+\gamma_2^0}{4\pi\rho_0^2} + \frac{\gamma_1^0+\gamma_2^0}{4\pi a_0^2} \right) \\ &\quad + \frac{\gamma_2}{\det\mathscr{T}}\left(\frac{\gamma_1^0}{2\pi(a_0-\rho_0)^2} + \frac{-\gamma_1^0+\gamma_2^0}{4\pi\rho_0^2} - \frac{\gamma_1^0+\gamma_2^0}{4\pi a_0^2} \right) + O(\epsilon^3) \\ &= \frac{\gamma_1^2}{2\pi\det\mathscr{T}} \frac{6a_0\rho_0^2}{(a_0+\rho_0)(a_0-\rho_0)^2(a_0^2+a_0\rho_0+\rho_0^2)} + O(\epsilon^3). \end{aligned}$$

Thus, since $\det\mathscr{T} < 0$, we conclude that $d''(c) < 0$ for $|\epsilon| \ll 1$ and $c = O(\epsilon)$. Hence, [32, Theorem 2.6] tells us that the corresponding water waves $\{U_c\}$ constructed in Theorem 1.1 are orbitally unstable.

3.4. Stability of a dipole in a half-plane. To understand the physical meaning behind Theorem 1.4, it is useful to reconsider the classical finite-dimensional problem of a dipole moving in the lower half-plane (2.11). This has a (much simpler) Hamiltonian formulation with energy

$$E(\bar{x},\bar{y}) = \Gamma^*(\bar{x},\bar{y}) = \frac{\gamma_1}{2}\Big(\Gamma_1^*(\bar{x}) + \Gamma_2(\bar{x}) + \Gamma_2^*(\bar{x})\Big) - \frac{\gamma_2}{2}\Big(\Gamma_1(\bar{y}) + \Gamma_1^*(\bar{y}) + \Gamma_2^*(\bar{y})\Big)$$

and Poisson map

$$\begin{pmatrix} \frac{1}{\epsilon\gamma_1}\mathcal{J} & 0 \\ 0 & -\frac{1}{\epsilon\gamma_2}\mathcal{J} \end{pmatrix},$$

where recall that Γ^* and $\mathcal{J}$ were introduced earlier in section 3.1. Likewise, the (linear) momentum is given by

$$P(\bar{x}, \bar{y}) := -\epsilon\gamma_1\bar{x}_2 + \epsilon\gamma_2\bar{y}_2.$$

We have already commented in section 2.4 that there exists a family of steady wave solutions to this system. One can again study their stability using GSS. In this case, it is elementary to compute that the second variation of the corresponding augmented Hamiltonian will give exactly $\frac{\epsilon^2}{4\pi}\mathcal{A}$ at leading order. Comparing this to Theorem 3.9, we see that the spectrum for the water wave problem is simply a perturbation of the half-plane system.

Similarly, in the proof of Theorem 1.4, we found that

$$d''(c) = -\frac{\mathrm{d}}{\mathrm{d}c}P(U_c).$$

But, the leading order term in the momentum for the water wave is exactly the momentum of the half-plane dipole.

In that sense, the infinite-dimensional system (2.3) is a small perturbation of the finite-dimensional system of point vortices with a rigid lid (2.11). Therefore, it is reasonable to see that the family of water waves U_c is orbitally unstable.

Appendix A. Steady and unsteady equations. For the convenience of the reader, in this appendix we derive the nonlocal formulations for the water wave with a finite dipole problem (2.3).

Using the definitions of φ in (1.6) and $\mathcal{G}(\eta)$ in (2.2), we obtain

$$\nabla\Phi = \frac{1}{\langle\eta'\rangle^2}\begin{pmatrix} 1 & -\eta' \\ \eta' & 1 \end{pmatrix}\begin{pmatrix} \varphi' \\ \mathcal{G}(\eta)\varphi \end{pmatrix} = \frac{1}{\langle\eta'\rangle^2}\begin{pmatrix} \varphi' - \eta'\mathcal{G}(\eta)\varphi \\ \eta'\varphi' + \mathcal{G}(\eta)\varphi \end{pmatrix}. \tag{A.1}$$

Combining with the definitions of ψ in (2.1) gives

$$\begin{pmatrix} \mathcal{G}(\eta)\varphi \\ \varphi' \end{pmatrix} = \begin{pmatrix} \psi' \\ -\mathcal{G}(\eta)\psi \end{pmatrix}. \tag{A.2}$$

Then from the incompressible Euler equations (1.2), using the splitting $v = \nabla\Phi + \epsilon\nabla\Theta$ and separating the irrotational and rotational parts, we can derive the *unsteady equation for velocity potential* on S

$$\begin{aligned} \partial_t\varphi = &-\frac{1}{2\langle\eta'\rangle^2}\left((\varphi')^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi - (\mathcal{G}(\eta)\varphi)^2\right) - \epsilon\partial_t\Theta + \epsilon\varphi'\partial_{x_2}\Gamma - \frac{\epsilon^2}{2}|\nabla\Gamma|^2 \\ &- g\eta + b\frac{\eta''}{\langle\eta'\rangle^3}. \end{aligned} \tag{A.3}$$

Using the relation (A.2), we also have the relation on S:

$$\begin{aligned} \partial_t\varphi = &-\frac{1}{2\langle\eta'\rangle^2}\left((\mathcal{G}(\eta)\psi)^2 + 2\eta'\psi'\mathcal{G}(\eta)\psi - (\psi')^2\right) - \epsilon\partial_t\Theta - \epsilon\mathcal{G}(\eta)\psi\,\partial_{x_2}\Gamma - \frac{\epsilon^2}{2}|\nabla\Gamma|^2 \\ &- g\eta + b\frac{\eta''}{\langle\eta'\rangle^3}. \end{aligned} \tag{A.4}$$

For the traveling water waves, the *steady equation for velocity potential* on S is

$$\begin{aligned}0 = &-\frac{c}{\langle\eta'\rangle^2}(\varphi' - \eta'\mathcal{G}(\eta)\varphi) + c\epsilon\partial_{x_2}\Gamma + \frac{1}{2\langle\eta'\rangle^2}\left[(\varphi')^2 + (\mathcal{G}(\eta)\varphi)^2\right]\\ &+ \frac{\epsilon}{\langle\eta'\rangle^2}\left[-\varphi'\nabla_\perp\Gamma + \mathcal{G}(\eta)\varphi\ \nabla_\top\Gamma\right] + \frac{\epsilon^2}{2}|\nabla\Gamma|^2 + g\eta - b\frac{\eta''}{\langle\eta'\rangle^3},\end{aligned} \tag{A.5}$$

and the *steady equation for stream function* on S is

$$\begin{aligned}0 = &\frac{c}{\langle\eta'\rangle^2}(\psi' + \eta'\mathcal{G}(\eta)\psi) + c\epsilon\partial_{x_2}\Gamma + \frac{1}{2\langle\eta'\rangle^2}\left[(\psi')^2 + (\mathcal{G}(\eta)\psi)^2\right]\\ &+ \frac{\epsilon}{\langle\eta'\rangle^2}\left[\mathcal{G}(\eta)\psi\nabla_\perp\Gamma + \psi'\ \nabla_\top\Gamma\right] + \frac{\epsilon^2}{2}|\nabla\Gamma|^2 + g\eta - b\frac{\eta''}{\langle\eta'\rangle^3}.\end{aligned} \tag{A.6}$$

Appendix B. Variations of the energy and momentum. Finally, in this appendix we record the first and second Fréchet derivatives of the energy and momentum.

Recall that

$$\mathfrak{a} = (\nabla(\mathcal{H}\varphi))|_{S_t}, \quad \xi = (\Upsilon_{1_{x_1}}, \Xi_{1_{x_2}})^T, \quad \text{and} \quad \zeta = (\Upsilon_{2_{x_1}}, \Xi_{2_{x_2}})^T.$$

Let $\nabla\xi := (\Upsilon_{1_{x_1x_1}}, \Xi_{1_{x_2x_2}})^T$, $\nabla\zeta := (\Upsilon_{2_{x_1x_1}}, \Xi_{2_{x_2x_2}})^T$, and

$$D^2_{\bar{x}}\Theta := \begin{pmatrix}\Upsilon_{1_{x_1x_1}} & \Xi_{1_{x_1x_2}}\\ \Xi_{1_{x_1x_2}} & \Upsilon_{1_{x_2x_2}}\end{pmatrix}, \quad \text{and} \quad D^2_{\bar{y}}\Theta := \begin{pmatrix}\Upsilon_{2_{x_1x_1}} & \Xi_{2_{x_1x_2}}\\ \Xi_{2_{x_1x_2}} & \Upsilon_{2_{x_2x_2}}\end{pmatrix}.$$

Variations of $\boldsymbol{K_0(u)}$. We compute that

$$D_\varphi K_0(u)\dot{\varphi} = \int_{\mathbb{R}} \dot{\varphi}\mathcal{G}(\eta)\varphi\,\mathrm{d}x_1, \quad D_\eta K_0(u)\dot{\eta} = \frac{1}{2}\int_{\mathbb{R}} \varphi\langle D_\eta\mathcal{G}(\eta)\dot{\eta}, \varphi\rangle\,\mathrm{d}x_1,$$

and

$$\begin{aligned}\langle D^2_\varphi K_0(u)\dot{\varphi}, \dot{\varphi}\rangle &= \int_{\mathbb{R}} \dot{\varphi}\mathcal{G}(\eta)\dot{\varphi}\,\mathrm{d}x_1,\\ \langle D_\varphi D_\eta K_0(u)\dot{\varphi}, \dot{\eta}\rangle &= \int_{\mathbb{R}} \dot{\varphi}\langle D_\eta\mathcal{G}(\eta)\dot{\eta}, \varphi\rangle\,\mathrm{d}x_1 = \int_{\mathbb{R}} \dot{\eta}(\mathfrak{a}_1\dot{\varphi}' - \mathfrak{a}_2\mathcal{G}(\eta)\dot{\varphi})\,\mathrm{d}x_1\\ \langle D^2_\eta K_0(u)\dot{\eta}, \dot{\eta}\rangle &= \frac{1}{2}\int_{\mathbb{R}} \varphi\langle\langle D^2_\eta\mathcal{G}(\eta)\dot{\eta}, \dot{\eta}\rangle, \varphi\rangle\,\mathrm{d}x_1 = \int_{\mathbb{R}} (\mathfrak{a}_1'\mathfrak{a}_2\dot{\eta}^2 + \mathfrak{a}_2\dot{\eta}\mathcal{G}(\eta)(\mathfrak{a}_2\dot{\eta}))\,\mathrm{d}x_1.\end{aligned}$$

Variations of $\boldsymbol{K_1(u)}$. Likewise, the first variations of K_1 are

$$\begin{aligned}D_\varphi K_1(u)\dot{\varphi} &= \int_{\mathbb{R}} \dot{\varphi}\nabla_\perp\Theta\,\mathrm{d}x_1, & D_\eta K_1(u)\dot{\eta} &= \int_{\mathbb{R}} \dot{\eta}\varphi'\Theta_{x_1}|_S\,\mathrm{d}x_1,\\ \nabla_{\bar{x}} K_1(u) &= -\int_{\mathbb{R}} \varphi\nabla_\perp\xi\,\mathrm{d}x_1, & \nabla_{\bar{y}} K_1(u) &= -\int_{\mathbb{R}} \varphi\nabla_\perp\zeta\,\mathrm{d}x_1,\end{aligned}$$

and the second are given by

$$\begin{aligned}\langle D_\varphi D_\eta K_1(u)\dot{\eta}, \dot{\varphi}\rangle &= \int_{\mathbb{R}} \dot{\eta}\dot{\varphi}'\Theta_{x_1}|_S\,\mathrm{d}x_1, & \langle D^2_\eta K_1(u)\dot{\eta}, \dot{\eta}\rangle &= \int_{\mathbb{R}} \dot{\eta}^2\varphi'\Theta_{x_1x_2}|_S\,\mathrm{d}x_1,\\ D^2_{\bar{x}} K_1(u) &= \int_{\mathbb{R}} \varphi\nabla_\perp D^2_{\bar{x}}\Theta\,\mathrm{d}x_1, & D^2_{\bar{y}} K_1(u) &= \int_{\mathbb{R}} \varphi\nabla_\perp D^2_{\bar{y}}\Theta\,\mathrm{d}x_1,\\ \nabla_{\bar{x}} D_\eta K_1(u)\dot{\eta} &= -\int_{\mathbb{R}} \dot{\eta}\varphi'(\nabla\xi)|_S\,\mathrm{d}x_1, & \nabla_{\bar{y}} D_\eta K_1(u)\dot{\eta} &= -\int_{\mathbb{R}} \dot{\eta}\varphi'(\nabla\zeta)|_S\,\mathrm{d}x_1,\\ \nabla_{\bar{x}} D_\varphi K_1(u)\dot{\varphi} &= -\int_{\mathbb{R}} \dot{\varphi}\nabla_\perp\xi\,\mathrm{d}x_1, & \nabla_{\bar{y}} D_\varphi K_1(u)\dot{\varphi} &= -\int_{\mathbb{R}} \dot{\varphi}\nabla_\perp\zeta\,\mathrm{d}x_1.\end{aligned}$$

Variations of $K_2(u)$. It is straightforward to compute that

$$D_\eta K_2(u)\dot\eta = \frac{1}{2}\int_{\mathbb{R}} \dot\eta |(\nabla\Theta)|_S|^2 \,\mathrm{d}x_1,$$

$$\nabla_{\bar x} K_2(u) = \nabla_{\bar x}\Gamma^* - \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\xi\Theta)\,\mathrm{d}x_1, \quad \nabla_{\bar y} K_2(u) = \nabla_{\bar y}\Gamma^* - \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\zeta\Theta)\,\mathrm{d}x_1,$$

and

$$\begin{aligned}
\langle D_\eta^2 K_2(u)\dot\eta,\dot\eta\rangle &= \int_{\mathbb{R}} \dot\eta^2 \Big(\Theta_{x_1}\Theta_{x_1x_2} + \Theta_{x_2}\Theta_{x_2x_2}\Big)\Big|_S \,\mathrm{d}x_1,\\
D_{\bar x}^2 K_2(u) &= 2D_{\bar x}^2\Gamma^* + \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\Theta D_{\bar x}^2\Theta + \xi\xi^T)\,\mathrm{d}x_1,\\
D_{\bar y}^2 K_2(u) &= 2D_{\bar y}^2\Gamma^* + \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\Theta D_{\bar y}^2\Theta + \zeta\zeta^T)\,\mathrm{d}x_1,\\
\nabla_{\bar x}\nabla_{\bar y} K_2(u) &= \nabla_{\bar x}\nabla_{\bar y}\Gamma^* + \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\xi\odot\zeta)\,\mathrm{d}x_1,\\
\nabla_{\bar x} D_\eta K_2(u)\dot\eta &= -\int_{\mathbb{R}} \dot\eta((D_x\xi)\nabla\Theta)|_S\,\mathrm{d}x_1,\\
\nabla_{\bar y} D_\eta K_2(u)\dot\eta &= -\int_{\mathbb{R}} \dot\eta((D_x\zeta)\nabla\Theta)|_S\,\mathrm{d}x_1.
\end{aligned}$$

Variations of $V(u)$. Similarly, we find that

$$\begin{aligned}
D_\eta V(u)\dot\eta &= \int_{\mathbb{R}} \dot\eta\left(g\eta - b\frac{\eta''}{\langle\eta'\rangle^3}\right)\mathrm{d}x_1,\\
\langle D_\eta^2 V(u)\dot\eta,\hat\eta\rangle &= \int_{\mathbb{R}} \left(g\hat\eta\dot\eta + \frac{b}{\langle\eta'\rangle^3}\hat\eta'\dot\eta'\right)\mathrm{d}x_1.
\end{aligned}$$

Variations of $P(u)$. Finally, the first variations of momentum $P(u)$ are given in section 3.1. The second derivatives are as follows:

$$\langle D_\eta D_\varphi P(u)\dot\varphi,\dot\eta\rangle = -\int_{\mathbb{R}} \dot\eta'\dot\varphi\,\mathrm{d}x_1, \qquad \langle D_\eta^2 P(u)\dot\eta,\dot\eta\rangle = \epsilon\int_{\mathbb{R}} \dot\eta^2\Theta_{x_1x_2}|_S\,\mathrm{d}x_1,$$

$$D_{\bar x}^2 P(u) = -\epsilon\int_{\mathbb{R}} \eta'(D_{\bar x}^2\Theta)|_S\,\mathrm{d}x_1, \qquad D_{\bar y}^2 P(u) = -\epsilon\int_{\mathbb{R}} \eta'(D_{\bar y}^2\Theta)|_S\,\mathrm{d}x_1,$$

$$\nabla_{\bar x} D_\eta P(u)\dot\eta = -\epsilon\int_{\mathbb{R}} \dot\eta(\nabla\xi)|_S\,\mathrm{d}x_1, \qquad \nabla_{\bar y} D_\eta P(u)\dot\eta = -\epsilon\int_{\mathbb{R}} \dot\eta(\nabla\zeta)|_S\,\mathrm{d}x_1.$$

Acknowledgments. The author would like to express his sincere gratitude to Samuel Walsh for his continuous support and insightful comments. The author is also grateful to Roberto Camassa, who suggested this problem. Finally, the author is deeply indebted to the referees for many helpful comments and suggestions which significantly improved the paper.

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