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Positivity of Canonical Bases Under Comultiplication

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We show the positivity of the canonical basis for a modified quantum affine $\mathfrak{sl}_n$ under the comultiplication. Moreover, we establish the positivity of the i-canonical basis in [21] with respect to the coideal subalgebra structure.

1 Introduction

The geometric study of the modified quantum $\mathfrak{sl}_n$ via perverse sheaves on partial flag varieties of type A is initiated in the work of Beilinson *et al.* [3]. It is then generalized to quantum affine $\mathfrak{sl}_n$ by Ginzburg–Vasserot [15] and Lusztig [23, 24], independently, by considering the geometry of affine partial flag varieties of type A. This line of research is culminated in the work of Schiffmann–Vasserot [27] and McGerty [25] showing that the canonical basis of modified quantum affine $\mathfrak{sl}_n$ defined geometrically via transfer maps can be identified with the one defined algebraically by Lusztig [22] (see also Kashiwara [17]). Consequently, the positivity conjecture [22, 25.4.2] of the structure constants of the canonical basis of quantum affine $\mathfrak{sl}_n$ with respect to multiplication follows.

In a remarkable work of Bao–Wang [2], a quantum-Schur-like duality relating a type-B/C Hecke algebra and a coideal subalgebra of quantum $\mathfrak{sl}_n$ defined by Letzter in [20] is obtained, and moreover an $\imath$-canonical basis for the representations of the coideal subalgebras is constructed. The desires to geometrize Bao–Wang's work and to describe the convolution algebras of certain perverse sheaves on partial flag varieties

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of classical type lead to the work [1], where the approach in [3] is revived and adapted to give a geometric construction of the (modified) coideal subalgebra of quantum $\mathfrak{gl}_n$ and a *stably* canonical basis by using certain perverse sheaves of partial flag varieties of type B/C. Since a modified coideal subalgebra of quantum $\mathfrak{gl}_n$ can be regarded as a direct sum of infinitely many copies of its $\mathfrak{sl}_n$ version, one obtains infinitely many stably canonical bases of the modified coideal subalgebra of quantum $\mathfrak{sl}_n$. As a result, the $\imath$-canonical basis of the tensor space in the duality in [2] admits a geometric incarnation as certain intersection cohomology complexes.

Despite of many favorable properties of the stably canonical bases of modified quantum $\mathfrak{gl}_n$ and its coideal subalgebras, they do not admit positivity with respect to multiplication; see [21]. Instead, a new basis, called the i-canonical basis, of the modified coideal subalgebra of quantum $\mathfrak{sl}_n$ is constructed in *loc. cit.* following the spirit of [24] and [25] (see also [27]). This basis can be regarded as an asymptotical version of the stably canonical basis since they coincide asymptotically ([21]). It is further shown that the i-canonical basis does admit three positivities with respect to the multiplication, a bilinear form of geometric origin in *loc. cit.* and its action on the $\imath$-canonical basis of a tensor space.

In this article, we establish three more positivities of i-canonical bases, in addition to the previous ones in *loc. cit.* (see also [11]), mainly with respect to the coideal subalgebra structure. Precisely, let $\mathbb{B}$ be the canonical basis of modified quantum $\mathfrak{sl}_n$, say $\dot{\mathbb{U}}$, and $\mathbb{B}^{\mathrm{i}}$ its coideal analog in the modified coideal subalgebra $\dot{\mathbb{U}}^{\mathrm{i}}$. Note that notations in the introduction are slightly different from the main body of the paper. There is a natural algebra homomorphism $\Delta^{\mathrm{i}} : \dot{\mathbb{U}}^{\mathrm{i}} \to (\dot{\mathbb{U}}^{\mathrm{i}} \otimes \dot{\mathbb{U}})^\wedge$, where the target is a certain variant of the tensor $\dot{\mathbb{U}}^{\mathrm{i}} \otimes \dot{\mathbb{U}}$, which is an idempotented version of the coideal structure coming from the coproduct of quantum $\mathfrak{sl}_n$. (See (43) and (61) for definitions.) In particular, if $a \in \mathbb{B}^{\mathrm{i}}$, one has

$$\Delta^{\mathrm{i}}(a) = \sum_{b \in \mathbb{B}^{\mathrm{i}}, c \in \mathbb{B}} n_a^{b,c} b \otimes c, \quad n_a^{b,c} \in \mathbb{Z}[v, v^{-1}].$$

The positivity with respect to the idempotented coideal structure further says that

Positivity A Theorems 4.3.1 and 5.2.1 The structure constant $n_a^{b,c}$ is in $\mathbb{Z}_{\geq 0}[v, v^{-1}]$.

A degenerate version of Δ^{i} induces an imbedding $\mathrm{i} : \dot{\mathbb{U}}^{\mathrm{i}} \to (\dot{\mathbb{U}})^\wedge$, which reflects the subalgebra structure of the ordinary coideal subalgebra in quantum $\mathfrak{sl}_n$.

(See (52) and (62).) The positivity with respect to the subalgebra structure says that (see Theorems 4.4.1 and 5.2.2)

Positivity B Joint with W. Wang. If $\mathfrak{i}(a) = \sum_{b\in\mathbb{B}} g_{b,a} b$, $\forall a \in \mathbb{B}^{\mathfrak{i}}$, then $g_{b,a} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

As a 2nd degeneration of $\Delta^{\mathfrak{i}}$, we make a direct connection between the geometric type A duality of [14] and type B/C duality of [1], which reveals yet another positivity:

Positivity C Theorems 3.5.3 The $\imath$-canonical basis in a tensor space is a positive sum of the canonical basis in the same tensor space.

As is shown, these positivities are boiled down to a geometric interpretation of the coideal structure coming from the comultiplication of quantum $\mathfrak{sl}_n$. To this end, we also establish a geometric realization of the comultiplication of quantum *affine* $\mathfrak{sl}_n$, and we obtain the following positivity on quantum affine $\mathfrak{sl}_n$.

Positivity D Theorems 6.4.2 The canonical basis of modified quantum affine $\mathfrak{sl}_n$ admits positivity with respect to the idempotented comultiplication.

The proof of the positivity result on quantum affine $\mathfrak{sl}_n$ consists of two parts since the geometrically defined comultiplication on the affine Schur algebra level is a composition of a hyperbolic localization [4] and a twist of a certain v-power. The positivity on the former is well known by [4], (see also [24] and [27]), while we show that in the latter it sends a canonical basis to a canonical basis up to a v-power. Note that the 2nd step is trivial in the ordinary quantum $\mathfrak{sl}_n$ case, but nontrivial in the affine case as far as we can see: because at some point, we have to invoke the multiplication formula of a semisimple generator of Du–Fu [5], for which we provide a new geometric proof. These arguments are contained in the 1st and last sections, with the 1st section devoted to quantum $\mathfrak{sl}_n$ and the last one to its affine version.

The argument of the proof on quantum affine $\mathfrak{sl}_n$ also applies with modifications to the various positivities of the i-canonical basis, which occupies the last three sections. The 3rd section treats the results on the i-Schur-algebra level, and the 4th section lifts the results on the i-Schur-algebra level to the projective limit level for n being odd. The last section collects similar results for n even. The $\jmath$ transfer maps used in [21] are constructed geometrically in these sections and the proof of [21, Lemma 4.3] is in Proposition 3.6.1.

Note that we work over the partial flag varieties of type B for the i-canonical basis and following the treatment of type A in [24] and [25]. One can obtain the same results via partial flag varieties of type C by using the principle in [1].

In [8], we shall construct and investigate geometrically the i-canonical basis of modified coideal subalgebras of quantum affine $\mathfrak{sl}_n$ among others.

We refer to [7] and [9] for the interactions of type D partial flag varieties, coideal subalgebras, and type D duality. In a forthcoming paper, we will present a type D picture similar to the positivity results on i-canonical basis in this paper.

2 Positivity for Quantum $\mathfrak{sl}_n$

In this section, we shall present a proof of the positivity of the canonical basis of quantum $\mathfrak{sl}_n$ with respect to comultiplication.

2.1 Convolution

Let G be a group, and $\mathcal{X}$ a G-set with finitely many G-orbits. The G-action on $\mathcal{X}$ thus induces a diagonal G-action on the product $\mathcal{X} \times \mathcal{X}$. Let $\mathcal{A}$ be a unital commutative ring. We consider the set $\mathcal{A}_G(\mathcal{X} \times \mathcal{X})$ of all $\mathcal{A}$-valued G-invariant functions on $\mathcal{X} \times \mathcal{X}$ supported on finitely many G-orbits. Assume that any G-orbit $\mathcal{O}$ in $\mathcal{X} \times \mathcal{X}$ has the property that the set $\mathcal{X}^x_{\mathcal{O}} = \{y \in \mathcal{X} | (x, y) \in \mathcal{O}\}$ is finite for one and hence any fixed x in $\mathcal{X}$. Then $\mathcal{A}_G(\mathcal{X} \times \mathcal{X})$ is a free $\mathcal{A}$-module with a basis indexed by the G-orbits in $\mathcal{X} \times \mathcal{X}$, and further an associative $\mathcal{A}$-algebra with multiplication as follows. For any $f_1, f_2 \in \mathcal{A}_G(\mathcal{X} \times \mathcal{X})$, the function $f_1 * f_2$ is defined by

$$f_1 * f_2(x_1, x_3) = \sum_{x_2 \in \mathcal{X}} f_1(x_1, x_2) f_2(x_2, x_3), \quad \forall\, x_1, x_3 \in \mathcal{X}. \tag{1}$$

Let $\mathbf{1}$ be the characteristic function of the diagonal $\{(x, x) | x \in \mathcal{X}\}$. Since G acts on $\mathcal{X}$ with finitely many orbits, then, by definition, $\mathbf{1}$ is the unit of the algebra $(\mathcal{A}_G(\mathcal{X} \times \mathcal{X}), *)$. For convenience, we will simply use the notation $\mathcal{A}_G(\mathcal{X} \times \mathcal{X})$ for the algebra $(\mathcal{A}_G(\mathcal{X} \times \mathcal{X}), *)$.

2.2 Quantum Schur algebras

Let $\mathbb{F}_q$ be a finite field of q elements and of odd characteristic. Let

$$\mathbf{v} = \sqrt{q}, \quad \mathcal{A} = \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]. \tag{2}$$

We fix a pair (n, d) of irrelevant positive integers. Consider the set X_d of n-step partial flags in a fixed d-dimensional vector space $\mathbb{F}_q^d$ over $\mathbb{F}_q$ of the form

$$V = \left(0 \equiv V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n \equiv \mathbb{F}_q^d\right).$$

Denote by $\mathbf{G}_d = \mathrm{GL}(\mathbb{F}_q^d)$ the general linear group over $\mathbb{F}_q$ of rank d. Let $\mathbf{G}_d$ act from the left on the set X_d and diagonally on $X_d \times X_d$.

Let $|W|$ denote the dimension of the vector space W over $\mathbb{F}_q$. To a pair (V, V'), we can associate an $n \times n$ matrix $M = (m_{ij})$ with coefficients in $\mathbb{Z}_{\geq 0}$ by

$$m_{ij} = \left| \frac{V_i \cap V'_j}{V_{i-1} \cap V'_j + V_i \cap V'_{j-1}} \right|, \quad \forall i, j \in [1, n]. \tag{3}$$

Let Ξ_d be the set of all matrices obtained this way. Any matrix M in Ξ_d can be characterized by the property that $m_{ij} \in \mathbb{Z}_{\geq 0}$ and $\sum_{1 \leq i,j \leq n} m_{ij} = d$. It is shown in [3] that the set Ξ_d parameterizes the $\mathbf{G}_d$-orbits in $X_d \times X_d$. Let η_M be the characteristic function of the $\mathbf{G}_d$-orbit in $X_d \times X_d$ indexed by M, for any $M \in \Xi_d$.

By the general setting in Section 2.1, we have a unital associative algebra

$$\mathbf{S}_d \equiv \mathcal{A}_{\mathrm{G}_d}(X_d \times X_d) = \mathrm{Span}_{\mathcal{A}}\{\eta_M | M \in \Xi_d\}. \tag{4}$$

It is well known that the algebra $\mathbf{S}_d$ is the $\mathbf{v}$-*Schur algebra* of type $\mathbf{A}_{n-1}$ ([3]).

The definitions of these objects depend on the integer n, but it is suppressed since it never changes, except at Section 5 where we use notations $X_{d,n}$, $\mathbf{S}_{d,n}$, etc.

Let v be an indeterminate, and consider the Laurent polynomial ring

$$\mathbb{A} = \mathbb{Z}[v, v^{-1}].$$

Recall from [3] that one has a generic version $\mathbb{S}_d$ of $\mathbf{S}_d$ so that $\mathbf{S}_d = \mathcal{A} \otimes_{\mathbb{A}} \mathbb{S}_d$, where $\mathcal{A}$ is regarded as an $\mathbb{A}$-module with v acting as $\mathbf{v}$. More precisely, $\mathbb{S}_d$ is a free $\mathbb{A}$-module spanned by the symbols ζ_M, $\forall M \in \Xi_d$, such that $\mathcal{A} \otimes_{\mathbb{A}} \zeta_M = \eta_M$. The multiplication on $\mathbb{S}_d$ is defined so that if $\zeta_{M_1}\zeta_{M_2} = \sum_{M \in \Xi_d} c^M_{M_1,M_2}(v)\zeta_M$, $c^M_{M_1,M_2}(v) \in \mathbb{A}$, then $\eta_{M_1}\eta_{M_2} = \sum_{M \in \Xi_d} c^M_{M_1,M_2}(v)|_{v=\mathbf{v}}\eta_M$ in $\mathbf{S}_d$.

By the sheaf-function correspondence [19], to prove a statement on the level of the algebra $\mathbb{S}_d$, it suffices to prove it in $\mathbf{S}_d$. We shall apply this principle freely in what follows.

2.3 Coproduct on $\mathbb{S}_d$

In this section, we define an algebra homomorphism, which we call coproduct, from $\mathbb{S}_d$ to $\mathbb{S}_{d'} \otimes \mathbb{S}_{d''}$, $d' + d'' = d$, and as it suggested, the coproduct becomes the genuine coproduct of the quantum $\mathfrak{sl}_n$ when taking d to ∞. As indicated, our computation will be over $\mathbb{F}_q$, and so we introduce its specialization version at first.

Now consider a triple (d, d', d'') of positive integers such that $d' + d'' = d$. We fix an isomorphism of vector spaces $\mathbb{F}_q^{d'} \oplus \mathbb{F}_q^{d''} \simeq \mathbb{F}_q^{d}$. Let π' be the projection of $\mathbb{F}_q^d$ to $\mathbb{F}_q^{d'}$. Let π'' be the operation of intersection with $\mathbb{F}_q^{d''}$, that is, $\pi''(W) = W \cap \mathbb{F}_q^{d''}$ for any subspace W in $\mathbb{F}_q^d$. Given a flag V in X_d, the notations $\pi'(V)$ and $\pi''(V)$ are thus meaningful. For any $(V', V'') \in X_{d'} \times X_{d''}$, we set

$$Z_{V',V''} = \left\{V \in X_d | \pi'(V) = V', \pi''(V) = V''\right\}.$$

We can identify $\mathbf{S}_{d'} \otimes \mathbf{S}_{d''}$ with the algebra $\mathcal{A}_{G_{d'} \times G_{d''}}(X_{d'} \times X_{d'} \times X_{d''} \times X_{d''})$. We define

$$\widetilde{\Delta} : \mathbf{S}_d \to \mathbf{S}_{d'} \otimes \mathbf{S}_{d''} \tag{5}$$

by $\widetilde{\Delta}(f)(V', \tilde{V}', V'', \tilde{V}'') = \sum_{\tilde{V} \in Z_{\tilde{V}', \tilde{V}''}} f(V, \tilde{V})$, for any quadruple $(V', \tilde{V}', V'', \tilde{V}'') \in X_{d'} \times X_{d'} \times X_{d''} \times X_{d''}$ where V is a fixed element in $Z_{V',V''}$. It can be shown that the definition is independent of the choice of V. By [24, 2.2], which is credited back to Grojnowski, we know that the map $\widetilde{\Delta}$ in (5) is a well-defined algebra homomorphism over $\mathcal{A}$.

By using the monomial basis in [3, Theorem 3.10], one can show that $\mathbb{S}_d$ admits a linear map $\widetilde{\mathbb{\Delta}}$ such that it descends to $\widetilde{\Delta}$ after a specialization.

We use the notation $W_1 \overset{a}{\subset} W_2$ to denote $W_1 \subset W_2$ and $\dim W_2/W_1 = a$. Similarly, we define the notation $W_1 \overset{a}{\supset} W_2$. We define the following functions in $\mathbf{S}_d$. For any $i \in [1, n-1]$, $a \in [1, n]$,

$$\mathbf{E}_i(V, V') = \begin{cases} \mathbf{v}^{-|V'_{i+1}/V'_i|}, & \text{if } V_i \overset{1}{\subset} V'_i, V_j = V_{j'}, \forall j \neq i; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{F}_i(V, V') = \begin{cases} \mathbf{v}^{-|V'_i/V'_{i-1}|}, & \text{if } V_i \overset{1}{\supset} V'_i, V_j = V_{j'}, \forall j \neq i; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

$$\mathbf{H}_a^{\pm 1}(V, V') = \mathbf{v}^{\pm |V_a/V_{a-1}|} \delta_{V,V'}, \quad \forall V, V' \in X_d.$$

$$\mathbf{K}_i^{\pm 1} = \mathbf{H}_{i+1}^{\pm 1} \mathbf{H}_i^{\mp 1}.$$

Notices that if the subscript d is replaced by d' or d'', the functions defined above are in $\mathbf{S}_{d'}$ or $\mathbf{S}_{d''}$, respectively, and will be denoted by $\mathbf{H}'_a$, $\mathbf{K}'_i$, $\mathbf{E}'_i$, $\mathbf{F}'_i$ or $\mathbf{H}''_a$, $\mathbf{K}''_i$, $\mathbf{E}''_i$, and $\mathbf{F}''_i$. (This convention will be used in any similar situation appearing later.) We shall also use the same notation to denote the corresponding element in $\mathbb{S}_d$, which is a sum of certain ζ_M up to a twist. The following lemma is due to Lusztig [24, Lemma 1.6].

Lemma 2.3.1. For any $i \in [1, n-1]$, we have the following formulas in $\mathbb{S}_d$.

$$\widetilde{\Delta}(\mathbf{E}_i) = \mathbf{E}'_i \otimes \mathbf{H}''_{i+1} + \mathbf{H}'^{-1}_{i+1} \otimes \mathbf{E}''_i,\ \widetilde{\Delta}(\mathbf{F}_i) = \mathbf{F}'_i \otimes \mathbf{H}''^{-1}_i + \mathbf{H}'_i \otimes \mathbf{F}''_i,\ \widetilde{\Delta}(\mathbf{K}_i) = \mathbf{K}'_i \otimes \mathbf{K}''_i.$$

$\mathbf{E}_i$ and $\mathbf{F}_i$ correspond to F_i and E_i in [23, 2.4], respectively. $\forall M = (m_{ij}) \in \Xi_d$, we set

$$\mathrm{ro}(M) = \left(\sum_{j=1}^{n} m_{ij}\right)_{1\le i\le n} \quad \text{and} \quad \mathrm{co}(M) = \left(\sum_{i=1}^{n} m_{ij}\right)_{1\le j\le n},$$

which lie in the set

$$\Lambda_{d,n} = \left\{\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n_{\ge 0} | a_1 + \cdots + a_n = d\right\}. \tag{7}$$

Then we have a decomposition

$$\mathbb{S}_d = \oplus_{\mathbf{b},\mathbf{a}\in\Lambda_{d,n}} \mathbb{S}_d(\mathbf{b}, \mathbf{a}), \quad \mathbb{S}_d(\mathbf{b}, \mathbf{a}) = \mathrm{span}_{\mathbb{A}}\{\zeta_M | \mathrm{ro}(M) = \mathbf{b}, \mathrm{co}(M) = \mathbf{a}\}.$$

Let

$$\widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} : \mathbb{S}_d(\mathbf{b}, \mathbf{a}) \to \mathbb{S}_{d'}(\mathbf{b}', \mathbf{a}') \otimes \mathbb{S}_{d''}(\mathbf{b}'', \mathbf{a}'')$$

be the linear map obtained from $\widetilde{\Delta}$ by restricting $\widetilde{\Delta}$ to the subspace $\mathbb{S}_d(\mathbf{b}, \mathbf{a})$ and projecting down to the component $\mathbb{S}_{d'}(\mathbf{b}', \mathbf{a}') \otimes \mathbb{S}_{d''}(\mathbf{b}'', \mathbf{a}'')$. Then we have

$$\widetilde{\Delta} = \oplus \widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''},$$

where the sum runs over $\mathbf{b}, \mathbf{a}, \mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''$ such that $\mathbf{a}, \mathbf{b} \in \Lambda_{d,n}$, $\mathbf{a}', \mathbf{b}' \in \Lambda_{d',n}$, $\mathbf{a}'', \mathbf{b}'' \in \Lambda_{d'',n}$, and $\mathbf{b} = \mathbf{b}' + \mathbf{b}''$ and $\mathbf{a} = \mathbf{a}' + \mathbf{a}''$. We set

$$\Delta_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} = \mathbf{v}^{\sum_{1\le i\le j\le n} b'_i b''_j - a'_i a''_j} \widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}, \quad \Delta = \oplus \Delta_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}. \tag{8}$$

The following is a refinement of Lemma 2.3.1.

Proposition 2.3.2. The linear map $\mathbb{\Delta}$ in (8) is an algebra homomorphism. Moreover,

$$\mathbb{\Delta}(\mathbf{E}_i) = \mathbf{E}'_i \otimes \mathbf{K}''_i + 1 \otimes \mathbf{E}''_i,\ \mathbb{\Delta}(\mathbf{F}_i) = \mathbf{F}'_i \otimes 1 + \mathbf{K}'^{-1}_i \otimes \mathbf{F}''_i,\ \mathbb{\Delta}(\mathbf{K}_i) = \mathbf{K}'_i \otimes \mathbf{K}''_i, \forall i. \tag{9}$$

Proof. It is straightforward to see that $\mathbb{\Delta}$ is an algebra homomorphism. We proceed to the proof of the equalities in the proposition. Suppose that a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ satisfies the conditions that $b'_k = a'_k - \delta_{k,i} + \delta_{k,i+1}$ and $b''_k = a''_k$ for some i and for all $1 \leq k \leq n$. Then

$$\sum_{1\leq i\leq j\leq n} b'_i b''_j - a'_i a''_j = \sum_{1\leq k\leq j\leq n} (b'_k - a'_k) a''_j = -\sum_{i\leq j\leq n} a''_j + \sum_{i+1\leq j\leq n} a''_j = -a''_i.$$

So if $(V', \tilde{V}', V'', \tilde{V}'') \in X_{d'} \times X_{d'} \times X_{d''} \times X_{d''}$, then

$$\mathbb{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}(V', \tilde{V}', V'', \tilde{V}'') = v^{-a''_i} \mathbf{E}'_i \otimes \mathbf{H}''_{i+1}(V', \tilde{V}', V'', \tilde{V}'') = \mathbf{E}'_i \otimes \mathbf{K}''_i(V', \tilde{V}', V'', \tilde{V}'').$$

On the other hand, if $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ is a quadruple subject to $b'_k = a'_k$ and $b''_k = a''_k - \delta_{i,k} + \delta_{i+1,k}$ for some i and for all $1 \leq k \leq n$, then $\sum_{1\leq i\leq j\leq n} b'_i b''_j - a'_i a''_j = a'_{i+1}$. Thus, if $(V', \tilde{V}', V'', \tilde{V}'') \in X_{d'} \times X_{d'} \times X_{d''} \times X_{d''}$, then

$$\mathbb{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}(V', \tilde{V}', V'', \tilde{V}'') = v^{a'_{i+1}} \mathbf{H}'^{-1}_{i+1} \otimes \mathbf{E}''_i(V', \tilde{V}', V'', \tilde{V}'') = 1 \otimes \mathbf{E}''_i(V', \tilde{V}', V'', \tilde{V}'').$$

Altogether, we have $\mathbb{\Delta}(\mathbf{E}_i) = \mathbf{E}'_i \otimes \mathbf{K}''_i + 1 \otimes \mathbf{E}''_i$, which is the 1st equality in the lemma.

If the quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ satisfies that $b'_k = a'_k + \delta_{k,i} - \delta_{k,i+1}$ and $b''_k = a''_k$ for some i and for all $1 \leq k \leq n$, then the twist $\sum_{1\leq i\leq j\leq n} b'_i b''_j - a'_i a''_j$ is equal to a''_i. So after the twist, it makes the 1st term $\mathbf{F}'_i \otimes \mathbf{H}''^{-1}_i$ of $\widetilde{\mathbb{\Delta}}(\mathbf{F}_i)$ in Lemma 2.3.1 into $\mathbf{F}'_i \otimes 1$. Meanwhile, if $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ is a quadruple subject to $b'_k = a'_k$ and $b''_k = a''_k + \delta_{i,k} - \delta_{i+1,k}$ for some i and for all $1 \leq k \leq n$, then $\sum_{1\leq i\leq j\leq n} b'_i b''_j - a'_i a''_j = -a'_{i+1}$. Hence, after the twist, the 2nd term $\mathbf{H}'_i \otimes \mathbf{F}''_i$ in $\widetilde{\mathbb{\Delta}}(\mathbf{F}_i)$ becomes $\mathbf{K}'^{-1}_i \otimes \mathbf{F}''_i$. This verifies the 2nd equality in the lemma.

Since the twist is zero if $\mathbf{b}' = \mathbf{a}'$ and $\mathbf{b}'' = \mathbf{a}''$, the 3rd equality holds. ■

Remark 2.3.3. If we write $(\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i)$ as $(\mathbf{F}_i, \mathbf{E}_i, \mathbf{K}_i^{-1})$, we have the conventional coproduct.

For the rest of this subsection, we give a 2nd interpretation of $\widetilde{\Delta}$ to be used in the proof of Proposition 2.3.6. We can decompose X_d as follows:

$$X_d = \sqcup_{\mathbf{a}\in\Lambda_{d,n}} X_d(\mathbf{a}), \quad X_d(\mathbf{a}) = \{V \in X_d \,|\, |V_i/V_{i-1}| = a_i, \forall 1 \le i \le n\}.$$

Fix $V \in X_d(\mathbf{b})$ and set $P_{\mathbf{b}} = \mathrm{Stab}_{\mathbf{G}_d}(V)$. Then $P_{\mathbf{b}}$ acts via $\mathbf{G}_d$ on $X_d(\mathbf{b})$. Consider

$$i_{\mathbf{b},\mathbf{a}} : X_d(\mathbf{a}) \to X_d(\mathbf{b}) \times X_d(\mathbf{a}), \quad \tilde{V} \mapsto (V, \tilde{V}).$$

It induces a bijection between $P_{\mathbf{b}}$-orbits in the domain and $\mathbf{G}_d$-orbits in the range of $i_{\mathbf{b},\mathbf{a}}$. Hence, the pullback (restriction)

$$i^*_{\mathbf{b},\mathbf{a}} : \mathcal{A}_{\mathbf{G}_d}(X_d(\mathbf{b}) \times X_d(\mathbf{a})) \to \mathcal{A}_{P_{\mathbf{b}}}(X_d(\mathbf{a})) \tag{10}$$

of the imbedding $i_{\mathbf{b},\mathbf{a}}$ is an isomorphism of $\mathcal{A}$-modules.

Recall now that we fix a triple (V, V', V'') in the definition of $\widetilde{\Delta}$ in (5). We assume that $V \in X_d(\mathbf{b})$, $V' \in X_{d'}(\mathbf{b}')$, and $V'' \in X_{d''}(\mathbf{b}'')$ so that $\mathbf{b}' + \mathbf{b}'' = \mathbf{b}$. We also define $P_{\mathbf{b}'}$ and $P_{\mathbf{b}''}$ similar to $P_{\mathbf{b}}$. Thus, we have similar isomorphisms

$$i^*_{\mathbf{b}',\mathbf{a}'} : \mathcal{A}_{\mathbf{G}_{d'}}(X_{d'}(\mathbf{b}') \times X_{d'}(\mathbf{a}')) \to \mathcal{A}_{P_{\mathbf{b}'}}(X_{d'}(\mathbf{a}')),$$
$$i^*_{\mathbf{b}'',\mathbf{a}''} : \mathcal{A}_{\mathbf{G}_{d''}}(X_{d''}(\mathbf{b}'') \times X_{d''}(\mathbf{a}'')) \to \mathcal{A}_{P_{\mathbf{b}''}}(X_{d''}(\mathbf{a}'')).$$

Consider the subset of $X_d(\mathbf{a})$:

$$X^+_{\mathbf{a},\mathbf{a}',\mathbf{a}''} = \left\{\tilde{V} \in X_d(\mathbf{a}) | \pi'(\tilde{V}) \in X_{d'}(\mathbf{a}'), \pi''(\tilde{V}) \in X_{d''}(\mathbf{a}'')\right\}. \tag{11}$$

Then we have the following diagram:

$$X_d(\mathbf{a}) \xleftarrow{\iota} X^+_{\mathbf{a},\mathbf{a}',\mathbf{a}''} \xrightarrow{\pi} X_{d'}(\mathbf{a}') \times X_{d''}(\mathbf{a}''),$$

where ι is the natural inclusion and $\pi(\tilde{V}) = (\pi'(\tilde{V}), \pi''(\tilde{V}))$. Thus, the composition of the pullback ι^* of ι followed by the pushforward $\pi_!$ of π defines a linear map

$$\pi_!\iota^* : \mathcal{A}_{P_{\mathbf{b}}}(X_d(\mathbf{a})) \to \mathcal{A}_{P_{\mathbf{b}'} \times P_{\mathbf{b}''}}(X_{d'}(\mathbf{a}') \times X_{d''}(\mathbf{a}'')), \tag{12}$$

where $\pi_!$ is defined by $\pi_!(f)(\tilde{V}', \tilde{V}'') = \sum_{x \in X^+_{\mathbf{a},\mathbf{a}',\mathbf{a}''}: \pi(x) = (\tilde{V}', \tilde{V}'')} f(x)$, for all $\tilde{V}', \tilde{V}''$. Clearly, we have an isomorphism of $\mathcal{A}$-modules

$$\mathcal{A}_{P_{\mathbf{b}'} \times P_{\mathbf{b}''}}(X_{d'}(\mathbf{a}') \times X_{d''}(\mathbf{a}'')) \cong \mathcal{A}_{P_{\mathbf{b}'}}(X_{d'}(\mathbf{a}')) \otimes \mathcal{A}_{P_{\mathbf{b}''}}(X_{d''}(\mathbf{a}'')).$$

The following lemma makes connection between $\widetilde{\Delta}$ and $\pi_! \iota^*$.

Lemma 2.3.4. We have the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{A}_{\mathrm{G}_d}(X_d(b) \times X_d(a)) & \xrightarrow{\;i^*_{b,a}\;} & \mathcal{A}_{P_b}(X_d(a)) \\
\Big\downarrow \widetilde{\Delta}_{b',a',b'',a''} & & \Big\downarrow \pi_! \iota^* \\
\begin{array}{c} \mathcal{A}_{\mathrm{G}_{d'}}(X_{d'}(b') \times X_{d'}(a')) \\ \otimes \\ \mathcal{A}_{\mathrm{G}_{d''}}(X_{d''}(b'') \times X_{d''}(a'')) \end{array} & \xrightarrow{\;i^*_{b',a'} \otimes i^*_{b'',a''}\;} & \mathcal{A}_{P_{b'}}(X_{d'}(a')) \otimes \mathcal{A}_{P_{b''}}(X_{d''}(a'')).
\end{array}$$

Proof. For any $f \in \mathcal{A}_{\mathrm{G}_d}(X_d(\mathbf{b}) \times X_d(\mathbf{a}))$ and $(\widetilde{V}', \widetilde{V}'') \in X_{d'}(\mathbf{a}') \times X_{d''}(\mathbf{a}'')$, we have

$$\pi_! \iota^* i^*_{\mathbf{b},\mathbf{a}}(f)(\widetilde{V}', \widetilde{V}'') = \sum_{\widetilde{V} \in Z_{\widetilde{V}', \widetilde{V}''}} i^*_{\mathbf{b},\mathbf{a}}(f)(\widetilde{V}) = \sum_{\widetilde{V} \in Z_{\widetilde{V}', \widetilde{V}''}} f(V, \widetilde{V})$$

$$= \widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}(f)(V', \widetilde{V}', V'', \widetilde{V}'') = (i^*_{\mathbf{b}',\mathbf{a}'} \otimes i^*_{\mathbf{b}'',\mathbf{a}''}) \circ \widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}(f)(\widetilde{V}', \widetilde{V}'').$$

The lemma is thus proved. ■

Remark 2.3.5. π is a vector bundle of rank $\sum_{1 \le i < j \le n} a'_i a''_j$ (compare the twist in (8)).

Recall the canonical basis $\{\{B\} | B \in \Xi_d\}$ of $\mathbb{S}_d$ from [3].

Proposition 2.3.6. If $\Delta_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}(\{B\}) = \sum_{B' \in \Xi_{d'}, B'' \in \Xi_{d''}} c_B^{B',B''} \{B'\} \otimes \{B''\}$, then $c_B^{B',B''} \in \mathbb{Z}_{\ge 0}[v, v^{-1}]$.

Proof. To establish the latter positivity, we switch from the finite field $\mathbb{F}_q$ to its algebraic closure $\overline{\mathbb{F}_q}$. Let $\overline{\mathbf{G}}_d$ be the general linear group over $\overline{\mathbb{F}_q}$ whose $\mathbb{F}_q$-points form $\mathbf{G}_d$. Similarly, we define an algebraic variety $\overline{\mathbf{X}}_d(\mathbf{a})$ over $\overline{\mathbb{F}_q}$ for $X_d(\mathbf{a})$. We set $\overline{\mathbf{G}}_m = \mathrm{GL}(1, \overline{\mathbb{F}_q})$. For $d' + d'' = d$, we fix an isomorphism $\overline{\mathbb{F}_q}^d \cong \overline{\mathbb{F}_q}^{d'} \oplus \overline{\mathbb{F}_q}^{d''}$. Via the isomorphism, we fix an imbedding $\overline{\mathbf{G}}_m \to \overline{\mathbf{G}}_d$ defined by $t \mapsto (1_{\overline{\mathbb{F}_q}^{d'}}, t1_{\overline{\mathbb{F}_q}^{d''}})$. Thus, $\overline{\mathbf{G}}_m$ acts on $\overline{\mathbf{X}}_d(\mathbf{a})$ via the imbedding. It is straightforward to see that the fixed-point set of $\overline{\mathbf{G}}_m$ in $\overline{\mathbf{X}}_d(\mathbf{a})$ is $\sqcup_{\mathbf{a}'+\mathbf{a}''=\mathbf{a}} \overline{\mathbf{X}}_{d'}(\mathbf{a}') \times \overline{\mathbf{X}}_{d''}(\mathbf{a}'')$. Moreover, the attracting set of $\overline{\mathbf{X}}_{d'}(\mathbf{a}') \times \overline{\mathbf{X}}_{d''}(\mathbf{a}'')$, that is, those points x such that $\lim_{t\to 0} t.x \in \overline{\mathbf{X}}_{d'}(\mathbf{a}') \times \overline{\mathbf{X}}_{d''}(\mathbf{a}'')$, is exactly the algebraic variety whose $\mathbb{F}_q$-point is $X^+_{\mathbf{a},\mathbf{a}',\mathbf{a}''}$ in (11). Thus, the linear map $\pi_! \iota^*$ in (12) is the function version of the hyperbolic localization functor attached to the data $(\overline{\mathbf{X}}_d(\mathbf{a}), \overline{\mathbf{G}}_m)$ in [4]. On the other hand, the function $i^*_{\mathbf{b},\mathbf{a}}(\{A\}_d)$ is nothing but the function version of the intersection cohomology complex attached to the $P_{\mathbf{b}}$-orbit in $X_d(\mathbf{a})$ indexed by A. Now the result in [4] says that a hyperbolic localization functor sends a simple perverse sheaf to a semisimple complex. Therefore, we have the positivity for the generic version of $\pi_! \iota^*$, hence for the generic version of $\widetilde{\Delta}$ and therefore the proposition. ■

Remark 2.3.7. The positivity for the algebra structure of $\mathbb{S}_d$ is proved by Green in [13].

2.4 Transfer map

Let

$$\chi : \mathbb{S}_n \to \mathcal{A} \tag{13}$$

be the algebra homomorphism defined by $\chi(\zeta_M) = \det(M)$, for all $M \in \Xi_n$. (Here d is taken to be n.) Let $\xi : \mathbb{S}_{d-n} \to \mathbb{S}_{d-n}$ be the $\mathbb{A}$-algebra isomorphism defined by

$$\xi(\zeta_M) = v^{-\sum_{i=1}^{n}(n+1-i)(b_i - a_i)} \zeta_M, \quad \forall \zeta_M \in \mathbb{S}_{d-n}(\mathbf{b}, \mathbf{a}).$$

The transfer map

$$\phi_{d,d-n} : \mathbb{S}_d \to \mathbb{S}_{d-n}, \quad \forall d \geq n \tag{14}$$

is defined to be the composition $\mathbb{S}_d \xrightarrow{\Delta} \mathbb{S}_{d-n} \otimes \mathbb{S}_n \xrightarrow{\xi \otimes \chi} \mathbb{S}_{d-n} \otimes \mathbb{A} = \mathbb{S}_{d-n}$. The following lemma is quoted from [24, Lemma 1.10].

Proposition 2.4.1. There is a unique algebra homomorphism

$$\phi_{d,d-n} : \mathbb{S}_d \to \mathbb{S}_{d-n}$$

such that $\phi_{d,d-n}(\mathbf{E}_i) = \mathbf{E}'_i$, $\phi_{d,d-n}(\mathbf{F}_i) = \mathbf{F}'_i$, $\phi_{d,d-n}(\mathbf{K}_i^{\pm 1}) = \mathbf{K}_i'^{\pm 1}$, $\forall 1 \leq i \leq n-1$.

2.5 Positivity for $\dot{\mathbb{U}}$

By definition, the quantum $\mathfrak{sl}_n$, denoted by $\mathbb{U} \equiv \mathbb{U}(\mathfrak{sl}_n)$, is an associative algebra over $\mathbb{Q}(v)$ generated by the generators:

$$\mathbb{E}_i, \mathbb{F}_i, \mathbb{K}_i, \mathbb{K}_i^{-1}, \quad \forall 1 \leq i \leq n-1,$$

and subject to the following defining relations. For $1 \le i, j \le n-1$,

$$
\begin{aligned}
\mathbb{K}_i\mathbb{K}_i^{-1} &= \mathbb{K}_i^{-1}\mathbb{K}_i = 1,\\
\mathbb{K}_i\mathbb{K}_j &= \mathbb{K}_j\mathbb{K}_i,\\
\mathbb{K}_i\mathbb{E}_j &= v^{2\delta_{i,j}-\delta_{i,j+1}-\delta_{i,j-1}}\mathbb{E}_j\mathbb{K}_i,\\
\mathbb{K}_i\mathbb{F}_j &= v^{-2\delta_{i,j}+\delta_{i,j+1}+\delta_{i,j-1}}\mathbb{F}_j\mathbb{K}_i,\\
\mathbb{E}_i\mathbb{F}_j - \mathbb{F}_j\mathbb{E}_i &= \delta_{i,j}\frac{\mathbb{K}_i - \mathbb{K}_i^{-1}}{v-v^{-1}},\\
\mathbb{E}_i^2\mathbb{E}_j + \mathbb{E}_j\mathbb{E}_i^2 &= (v+v^{-1})\mathbb{E}_i\mathbb{E}_j\mathbb{E}_i, \text{ if } |i-j|=1,\\
\mathbb{F}_i^2\mathbb{F}_j + \mathbb{F}_j\mathbb{F}_i^2 &= (v+v^{-1})\mathbb{F}_i\mathbb{F}_j\mathbb{F}_i, \text{ if } |i-j|=1,\\
\mathbb{E}_i\mathbb{E}_j &= \mathbb{E}_j\mathbb{E}_i, \text{ if } |i-j|\neq 1,\\
\mathbb{F}_i\mathbb{F}_j &= \mathbb{F}_j\mathbb{F}_i, \text{ if } |i-j|\neq 1.
\end{aligned}
\tag{15}
$$

Moreover, $\mathbb{U}$ admits a Hopf algebra structure, whose comultiplication is defined by

$$
\mathbb{\Delta}(\mathbb{E}_i) = \mathbb{E}_i \otimes \mathbb{K}_i + 1 \otimes \mathbb{E}_i,\ \mathbb{\Delta}(\mathbb{F}_i) = \mathbb{F}_i \otimes 1 + \mathbb{K}_i^{-1} \otimes \mathbb{F}_i,\ \mathbb{\Delta}(\mathbb{K}_i) = \mathbb{K}_i \otimes \mathbb{K}_i,\ \forall i. \tag{16}
$$

Remark 2.5.1. If one rewrites $\mathbb{E}_i, \mathbb{F}_i$, and $\mathbb{K}_i$ as E_i, F_i, and $K_iK_{i+1}^{-1}$, respectively, then the resulting presentation is a subalgebra of the quantum $\mathfrak{gl}_n$ used in [1, 4.3].

It is well known from [3] that there is a surjective algebra homomorphism

$$
\phi_d : \mathbb{U} \to {}_{\mathbb{Q}(v)}\mathbb{S}_d, \mathbb{E}_i \mapsto \mathbf{E}_i, \mathbb{F}_i \mapsto \mathbf{F}_i, \mathbb{K}_{\pm i} \mapsto \mathbf{K}_{\pm i}, \quad \forall 1 \le i \le n-1, \tag{17}
$$

where ${}_{\mathbb{Q}(v)}\mathbb{S}_d$ is the algebra obtained from $\mathbb{S}_d$ in Section 2.2 by extending the ground ring $\mathbb{A}$ to $\mathbb{Q}(v)$. By using Proposition 2.3.2, (16) and tracing along the generators, we obtain the following commutative diagram.

$$
\begin{array}{ccc}
\mathbb{U} & \xrightarrow{\Delta} & \mathbb{U} \otimes \mathbb{U} \\
{\scriptstyle \phi_d}\downarrow & & \downarrow{\scriptstyle \phi_{d'}\otimes\phi_{d''}} \\
{}_{\mathbb{Q}(v)}\mathbb{S}_d & \xrightarrow{\Delta} & {}_{\mathbb{Q}(v)}\mathbb{S}_{d'} \otimes {}_{\mathbb{Q}(v)}\mathbb{S}_{d''},
\end{array}
\tag{18}
$$

where $d' + d'' = d$ and $\mathbb{\Delta}$ for ${}_{\mathbb{Q}(v)}\mathbb{S}_d$ is defined as in (8).

Define an equivalence relation $\sim$ on $\mathbb{Z}^n$ by $\mu \sim \nu$ if and only if $\mu - \nu = p(1, \cdots, 1)$ for some $p \in \mathbb{Z}$. Let

$$\mathbb{X} = \mathbb{Z}^n / \sim$$

be the set of all equivalence classes. Let $\overline{\mu}$ denote the equivalence class of $\mu \in \mathbb{Z}^n$. Let

$$\mathbb{Y} = \left\{ \nu \in \mathbb{Z}^n | \sum_{1 \le i \le n} \nu_i = 0 \right\}.$$

Then the standard dot product on $\mathbb{Z}^n$ induces a pairing $\cdot : \mathbb{Y} \times \mathbb{X} \to \mathbb{Z}$. Set $I = \{1, \cdots, n-1\}$. We define two injective maps $I \to \mathbb{Y}$, $\quad I \to \mathbb{X}$, by $i \mapsto -e_i + e_{i+1}, i \mapsto -\overline{e_i} + \overline{e_{i+1}}, \quad \forall 1 \le i \le n-1$, respectively, where e_i is the i-th standard basis element in $\mathbb{Z}^n$. We thus obtain a root datum of type $\mathbf{a}_{n-1}$ in [22, 2.2]. It is both $\mathbb{X}$-regular and $\mathbb{Y}$-regular.

Following [22, 23.1.1], $\mathbb{U}$ admits a decomposition $\mathbb{U} = \oplus_{\nu \in \mathbb{Z}[I]} \mathbb{U}(\nu)$ defined by

$$\mathbb{U}(\nu')\mathbb{U}(\nu'') \subseteq \mathbb{U}(\nu' + \nu''), \ \mathbb{K}_{\pm i} \in \mathbb{U}(0), \ \mathbb{E}_i \in \mathbb{U}(i), \ \mathbb{F}_i \in \mathbb{U}(-i).$$

For a triple ν', ν'', ν in $\mathbb{Z}[I]$ such that $\nu' + \nu'' = \nu$, we can have a linear map

$$\Delta_{\nu', \nu''} : \mathbb{U}(\nu) \to \mathbb{U}(\nu') \otimes \mathbb{U}(\nu''),$$

obtained from Δ by restricting to $\mathbb{U}(\nu)$ and projecting to $\mathbb{U}(\nu') \otimes \mathbb{U}(\nu'')$. Moreover, the restriction of ϕ_d in (17) to $\mathbb{U}(\nu)$ induces a linear map, still denoted by ϕ_d,

$$\phi_d : \mathbb{U}(\nu) \to \oplus_{\overline{\mathbf{b}} - \overline{\mathbf{a}} = \nu} \ {}_{\mathbb{Q}(v)}\mathbb{S}_d(\mathbf{b}, \mathbf{a}),$$

where $\mathbb{Z}[I]$ is treated as a subset in $\mathbb{X}$ via the imbedding $I \to \mathbb{X}$.

Lemma 2.5.2. The commutative diagram (18) can be refined to the following commutative diagram, where $\Delta_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}$ is defined similar to (8).

$$\begin{array}{ccc} \mathbb{U}(\nu) & \xrightarrow{\Delta_{\nu', \nu''}} & \mathbb{U}(\nu') \otimes \mathbb{U}(\nu'') \\ \phi_d \downarrow & & \downarrow \phi_{d'} \otimes \phi_{d''} \\ \oplus_{\overline{b} - \overline{a} = \nu} \ {}_{\mathbb{Q}(v)}\mathbb{S}_d(b, a) & \xrightarrow{\oplus \Delta_{b', a', b'', a''}} & \oplus_{\substack{b' + b'' = b, a' + a'' = a \\ \overline{b'} - \overline{a'} = \nu', \overline{b''} - \overline{a''} = \nu''}} \ {}_{\mathbb{Q}(v)}\mathbb{S}_{d'}(b', a') \otimes \ {}_{\mathbb{Q}(v)}\mathbb{S}_{d''}(b'', a''). \end{array} \quad (19)$$

Now set

$$\dot{\mathbb{U}} = \oplus_{\overline{\mu},\overline{\lambda}\in\mathbb{X}}\ {}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}},$$

$${}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}} = \mathbb{U}/\left(\sum_{1\le i\le n-1}\left(\mathbb{K}_i - v^{-\mu_i+\mu_{i+1}}\right)\mathbb{U} + \sum_{1\le i\le n-1}\mathbb{U}\left(\mathbb{K}_i - v^{-\lambda_i+\lambda_{i+1}}\right)\right).$$

This is the modified/idempotented form of $\mathbb{U}$ defined in [22, 23.1.1], see also [3]. Recall from [22, 23.1.5], the comultiplication $\mathbb{\Delta}$ induces a linear map

$$\mathbb{\Delta}_{\overline{\mu'},\overline{\lambda'},\overline{\mu''},\overline{\lambda''}} : {}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}} \to {}_{\overline{\mu'}}\mathbb{U}_{\overline{\lambda'}} \otimes {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}, \tag{20}$$

and makes the following diagram commutative.

$$\begin{array}{ccc} \mathbb{U}(\nu) & \xrightarrow{\Delta_{\nu',\nu''}} & \mathbb{U}(\nu')\otimes\mathbb{U}(\nu'') \\ {\scriptstyle \pi_{\overline{\mu},\overline{\lambda}}}\downarrow & & \downarrow{\scriptstyle \pi_{\overline{\mu'},\overline{\lambda'}}\otimes\pi_{\overline{\mu''},\overline{\lambda''}}} \\ {}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}} & \xrightarrow{\Delta_{\overline{\mu'},\overline{\lambda'},\overline{\mu''},\overline{\lambda''}}} & {}_{\overline{\mu'}}\mathbb{U}_{\overline{\lambda'}}\otimes {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}, \end{array} \tag{21}$$

where $\overline{\mu} - \overline{\lambda} = \nu$, $\overline{\mu'} - \overline{\lambda'} = \nu'$, $\overline{\mu''} - \overline{\lambda''} = \nu''$, and $\pi_{\overline{\mu},\overline{\lambda}}$ is the projection from $\mathbb{U}$ to ${}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}}$.

We write $1_{\overline{\lambda}} = \pi_{\overline{\lambda},\overline{\lambda}}(1)$. It is well known that $\dot{\mathbb{U}}$ and ${}_{\mathbb{Q}(v)}\mathbb{S}_d$ are $\mathbb{U}$-bimodules. So the notations $\mathbb{E}_i 1_{\overline{\lambda}}$ and $\mathbb{F}_i 1_{\overline{\lambda}}$ in $\dot{\mathbb{U}}$ are meaningful, and so are $\mathbb{E}_i\zeta_M$, $\mathbb{F}_i\zeta_M$ in ${}_{\mathbb{Q}(v)}\mathbb{S}_d$ where the notation ζ_M is from Section 2.2. Recall from [24] (see also [21]) that there is a surjective algebra homomorphism $\widetilde{\phi}_d : \dot{\mathbb{U}} \to {}_{\mathbb{Q}(v)}\mathbb{S}_d$ defined by

$$\widetilde{\phi}_d(1_{\overline{\lambda}}) = \begin{cases} \zeta_{M_{\mathbf{a}}}, & \text{if } \overline{\lambda} = \overline{\mathbf{a}}, \text{ for some } \mathbf{a} \in \Lambda_{d,n}, \\ 0, & \text{o.w.} \end{cases}$$

$$\widetilde{\phi}_d(\mathbb{E}_i 1_{\overline{\lambda}}) = \begin{cases} \mathbb{E}_i\zeta_{M_{\mathbf{a}}}, & \text{if } \overline{\lambda} = \overline{\mathbf{a}}, \text{ for some } \mathbf{a} \in \Lambda_{d,n}, \\ 0, & \text{o.w.} \end{cases}$$

$$\widetilde{\phi}_d(\mathbb{F}_i 1_{\overline{\lambda}}) = \begin{cases} \mathbb{F}_i\zeta_{M_{\mathbf{a}}}, & \text{if } \overline{\lambda} = \overline{\mathbf{a}}, \text{ for some } \mathbf{a} \in \Lambda_{d,n}, \\ 0, & \text{o.w.} \end{cases}$$

where $M_{\mathbf{a}}$ is the diagonal matrix with diagonal $\mathbf{a}$. Further, $\widetilde{\phi}_d$ induces a linear map:

$$\widetilde{\phi}_d : {}_{\overline{\mathbf{b}}}\mathbb{U}_{\overline{\mathbf{a}}} \to {}_{\mathbb{Q}(v)}\mathbb{S}_d(\mathbf{b},\mathbf{a}).$$

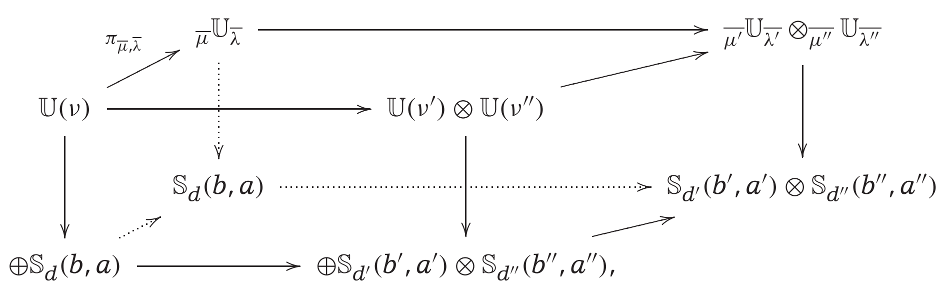
By definition, we have the following lemma.

Lemma 2.5.3. If $\overline{\mu} = \overline{\mathbf{b}}$, $\overline{\lambda} = \overline{\mathbf{a}}$ and $\overline{\mu} - \overline{\lambda} = \nu$, then the following diagram is commutative.

$$\begin{array}{ccc} \mathbb{U}(\nu) & \longrightarrow & {}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}} \\ \phi_d \big\downarrow & & \widetilde{\phi}_d \big\downarrow \\ \oplus_{\overline{b}-\overline{a}=\nu}\ {}_{\mathbb{Q}(v)}\mathbb{S}_d(b,a) & \longrightarrow & {}_{\mathbb{Q}(v)}\mathbb{S}_d(b,a), \end{array} \tag{22}$$

where the bottom row is the natural projection.

Note $\overline{\mathbf{b}} - \overline{\mathbf{a}} \in \mathbb{Z}[I] \subseteq \mathbb{X}$. By piecing together (19), (21), and (22), we have the cube:



(23)

where each of the $\mathbb{S}$ in the bottom square has a subscript $\mathbb{Q}(v)$ on the left. In (23), the front square is (19), the top square is (21), the two side squares are (22), and the commutativity of the bottom square is obvious. Since $\pi_{\overline{\mu},\overline{\lambda}}$ is surjective and each square is commutative except the one in the back, we have immediately the following proposition by diagram chasing.

Proposition 2.5.4. The square in the back of the cube (23) is commutative.

$$\begin{array}{ccc} {}_{\overline{b}}\mathbb{U}_{\overline{a}} & \xrightarrow{\Delta_{\overline{b'},\overline{a'},\overline{b''},\overline{a''}}} & {}_{\overline{b'}}\mathbb{U}_{\overline{a'}} \otimes {}_{\overline{b''}}\mathbb{U}_{\overline{a''}} \\ \widetilde{\phi}_d \big\downarrow & & \big\downarrow \widetilde{\phi}_{d'} \otimes \widetilde{\phi}_{d''} \\ {}_{\mathbb{Q}(v)}\mathbb{S}_d(b,a) & \xrightarrow{\Delta_{b',a',b'',a''}} & {}_{\mathbb{Q}(v)}\mathbb{S}_{d'}(b',a') \otimes {}_{\mathbb{Q}(v)}\mathbb{S}_{d''}(b'',a''). \end{array} \tag{24}$$

By using Proposition 2.5.4, we can prove the following positivity with respect to the comultiplication. Let $\mathbb{B}$ be the canonical basis of $\dot{\mathbb{U}}$ defined in [22, 25.2.4].

Theorem 2.5.5. Let $b \in \mathbb{B} \cap {}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}}$. If $\Delta_{\overline{\mu'},\overline{\lambda'},\overline{\mu''},\overline{\lambda''}}(b) = \sum_{b',b''\in\mathbb{B}} \hat{m}_b^{b',b''} b' \otimes b''$, then $\hat{m}_b^{b',b''} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

Proof. Let $\mathcal{I} = \{(b', b'')|\hat{m}_b^{b',b''} \neq 0\}$. Clearly, $\#\mathcal{I} < \infty$. By [25, Proposition 7.8], we can find d, d' and d'' large enough such that

$$\widetilde{\phi}_d(b) = \{B\}_d,\ \widetilde{\phi}_{d'}(b') = \{B'\}_{d'},\ \widetilde{\phi}_{d''}(b'') = \{B''\}_{d''}, \quad \forall (b', b'') \in \mathcal{I},$$

where $\{B\}_d$, $\{B'\}_{d'}$ and $\{B''\}_{d''}$ are certain canonical basis elements in $\mathbb{S}_d$, $\mathbb{S}_{d'}$ and $\mathbb{S}_{d''}$, respectively. Then by (24), we have

$$(\widetilde{\phi}_{d'} \otimes \widetilde{\phi}_{d''})\Delta_{\overline{\mathbf{b}'},\overline{\mathbf{a}'},\overline{\mathbf{b}''},\overline{\mathbf{a}''}}(b) = \sum_{(b',b'')\in\mathcal{I}} \hat{m}_b^{b',b''} \{B'\}_{d'} \otimes \{B''\}_{d''} = \Delta_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}(\{B\}_d). \tag{25}$$

By comparing $\Delta_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$ in 2.3.6 with (25), $\hat{m}_b^{b',b''} = c_B^{B',B''}$ and hence the theorem follows. ■

Remark 2.5.6. Theorem 2.5.5 was first proved by Grojnowski in an unpublished paper. In Section 6, we shall extend this result to the affine $\mathfrak{sl}_n$ case. For all symmetric Cartan data, the positivity is conjectured in [22, Conjecture 25.4.2].

3 Coideal Structure for the ${}_J$Schur Algebras

In this section, we define the copruduct on the ${}_J$Schur algebra level and show that it gives rise to the transfer map used in [21]. We shall also show that the coproduct degenerates to an imbedding of a ${}_J$Schur algebra to an ordinary Schur algebra and establish a direct connection of the type A geometric duality of Grojnowski–Lusztig [14] and the type B/C geometric duality in [1].

3.1 The ${}_J$Schur algebra $\mathbb{S}_d^J$

In this section, we assume that n and D are odd, that is,

$$n = 2r + 1 \quad \text{and} \quad D = 2d + 1.$$

We fix a non-degenerate symmetric bilinear form $Q^J : \mathbb{F}_q^D \times \mathbb{F}_q^D \to \mathbb{F}_q$. Let $W^\perp$ stand for the orthogonal complement of the vector subspace W in $\mathbb{F}_q^D$ with respect to the form Q^J. By convention, W is called isotropic if $W \subseteq W^\perp$. Recall the set X_d from Section 2.2. Consider the subset X_d^J of X_d defined by

$$X_d^J = \{V \in X_d | V_i = V_j^\perp,\ \text{if } i + j = n\}.$$

Let $\mathbf{G}_d^J$ be the orthogonal group attached to Q^J, that is,

$$\mathrm{G}_d^J = \left\{ g \in \mathrm{G}_d | Q^J(gu, gu') = Q^J(u, u'),\ \forall u, u' \in \mathbb{F}_q^D \right\}.$$

The group G_d^J acts from the left on X_d^J. It induces a diagonal action on $X_d^J \times X_d^J$. By the general construction in Section 2.1, we have a unital associative algebra

$$\mathbf{S}_d^J \equiv \mathcal{A}_{\mathbf{G}_d^J}\left(X_d^J \times X_d^J\right). \tag{26}$$

This is the algebra first appeared in [1]. See also [12] and [6].

Recall the definition of $\mathbf{v} = \sqrt{q}$ from (2) and $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$. Recall from [1] that one can construct an associative algebra $\mathbb{S}_d^J$ over $\mathbb{A}$ such that

$$\mathbf{S}_d^J = \mathcal{A} \otimes_{\mathbb{A}} \mathbb{S}_d^J,$$

where $\mathcal{A}$ is regarded as an $\mathbb{A}$-module with v acting as $\mathbf{v}$. Let us make the algebra $\mathbb{S}_d^J$ more precise. Recall Ξ_d from Section 2.2. Consider the set

$$\Xi_d^J = \left\{ M \in \Xi_d | m_{ij} = m_{n+1-i,n+1-j}, \forall 1 \le i, j \le n \right\}. \tag{27}$$

Then $\mathbb{S}_d^J$ is a free $\mathbb{A}$-module with basis ζ_M^J for any $M \in \Xi_d^J$ whose multiplication is defined by the condition that if $\zeta_{M_1}^J \zeta_{M_2}^J = \sum_{M \in \Xi_d^J} h_{M_1,M_2}^M(v) \zeta_M^J$, where $h_{M_1,M_2}^M(v) \in \mathbb{A}$, then $\eta_{M_1}^J \eta_{M_2}^J = \sum_{M \in \Xi_d^J} h_{M_1,M_2}^M(v)|_{v=\mathbf{v}} \eta_M^J$, in $\mathbf{S}_d^J$, where η_M^J is the characteristic function of the $\mathbf{G}_d^J$-orbit in $X_d^J \times X_d^J$ indexed by M via (3). Let

$$\Lambda_{d,n}^J = \left\{ \mathbf{a} \in \Lambda_{2d+1,n} | a_i = a_{n+1-i},\ \forall 1 \le i \le n \right\}.$$

It is clear that $\mathrm{ro}(M), \mathrm{co}(M) \in \Lambda_{d,n}^J$ for all $M \in \Xi_d^J$. Let $\mathbb{S}_d^J(\mathbf{b}, \mathbf{a}) = \mathrm{span}_{\mathbb{A}}\{\zeta_M | \mathrm{ro}(M) = \mathbf{b}, \mathrm{co}(M) = \mathbf{a}\}$ for $\mathbf{b}, \mathbf{a} \in \Lambda_{d,n}^J$. We have $\mathbb{S}_d^J(\mathbf{c}, \mathbf{b}')\mathbb{S}_d^J(\mathbf{b}, \mathbf{a}) \subseteq \delta_{\mathbf{b}',\mathbf{b}} \mathbb{S}_d^J(\mathbf{c}, \mathbf{a})$.

As usual, we are interested in the results on the generic level, while their proofs will be reduced to the finite field setting.

3.2 Coideal structure for $\mathbb{S}_d^J$

We set $\mathcal{D} = \mathbb{F}_q^D$. We need the following auxiliary lemma.

Lemma 3.2.1. Suppose that $\mathcal{D}''$ is an isotropic subspace of $\mathcal{D}$ and $L = (L_i | 0 \le i \le n) \in X_d^J$. Then we can find a pair (T, W) of subspaces in $\mathcal{D}$ such that

(a) $\mathcal{D} = \mathcal{D}'' \oplus T \oplus W$, $(\mathcal{D}'')^\perp = \mathcal{D}'' \oplus T$,

(b) W is isotropic and $T \perp W$,

(c) There exists bases $\{z_1, \cdots, z_s\}$ and $\{w_1, \cdots, w_s\}$ of $\mathcal{D}''$ and W, respectively, such that $Q^J(z_i, w_j) = \delta_{ij}$ for any $i, j \in [1, s]$,

(d) $L_i = (L_i \cap \mathcal{D}'') \oplus (L_i \cap T) \oplus (L_i \cap W)$, for any $1 \le i \le n-1$.

Proof. Assume that $n = 3$. We can use an induction process to find a subspace $T' \subset (\mathcal{D}'')^{\perp}$ such that $(\mathcal{D}'')^{\perp} = \mathcal{D}'' \oplus T'$ and

$$L_i \cap (\mathcal{D}'')^{\perp} = (L_i \cap \mathcal{D}'') \oplus (L_i \cap T'), \quad \forall 1 \le i \le n-1.$$

Moreover, the restriction of the bilinear form Q^J to T' is automatically non-degenerate. Next, we can find a subspace $W_1 \subseteq L_1$ such that $L_1 = (L_1 \cap (\mathcal{D}'')^{\perp}) \oplus W_1$. Similarly, we can find subspaces U_2 and T_2 such that

$$L_2 \cap \mathcal{D}'' = (L_1 \cap \mathcal{D}'') \oplus U_2, \quad L_2 \cap T' = (L_1 \cap T') \oplus T_2.$$

Via the natural projection $L_2 \to L_2/L_1$, we can regard $U_2 \oplus T_2$ as subspaces in L_2/L_1. Now L_2/L_1 inherits a non-degenerate bilinear form from that of $\mathcal{D}$. Moreover, $U_2 \oplus T_2$ is the orthogonal complement of U_2 with respect to the form on L_2/L_1. By a well-known fact, say [16, Theorem 6.11], we can find an isotropic subspace W_2' such that $L_2/L_1 = U_2 \oplus T_2 \oplus W_2'$, $T_2 \perp W_2'$, and $\dim U_2 = \dim W_2'$. Furthermore, the restriction of the form to $U_2 + W_2'$ is non-degenerate. Now take a subspace W_2 in L_2 such that it gets sent to W_2' via the projection map. Then by comparing the dimensions, we have

$$L_2 = (L_2 \cap (\mathcal{D}'')^{\perp}) \oplus (W_1 \oplus W_2).$$

It is clear that $W_1 \oplus W_2$ is an isotropic subspace in L_2 and $(W_1 \oplus W_2) \perp (L_2 \cap T')$.

Note that T' is not necessarily perpendicular to $W_1 \oplus W_2$. We consider the subspace $\mathcal{D}'' \oplus T' \oplus W_1 \oplus W_2$. We can find a subspace U_1 in V'' such that $U_1 \cap (L_2 \cap \mathcal{D}'') = \{0\}$ and the restriction of the bilinear form to $U_1 \oplus W_1$ is non-degenerate. The latter implies that we can find bases $\{u_1, \cdots, u_s\}$ and $\{w_1, \cdots, w_s\}$ in U_1 and W_1, respectively, such that $(v_i, w_j) = \delta_{ij}$. Recall that we have bases $\{u_{r+i} | 1 \le i \le s_1\}$ and $\{w_{r+i} | 1 \le i \le s_1\}$ for U_2 and W_2 such that $(u_{r+i}, w_{r+j}) = \delta_{ij}$. Fix a basis $\{t_i'\}$ for T' such that $\{t_i'\} \cap (L_2 \cap T')$ is a basis of $L_2 \cap T'$. Let T be the subspace spanned by the elements $t_i = t_i' - \sum_{1 \le j \le s+s_1} (t_i', w_j) u_j$. We thus have $T \perp (W_1 \oplus W_2)$ and T satisfies all properties T' has with respect to the flag L.

By [16, Theorem 6.11], we can extend $W_1 \oplus W_2$ to a subspace W satisfying the required properties, by extending the subspace $(\mathcal{D}'')^{\perp} \oplus W_1 \oplus W_2$ to the whole space $\mathcal{D}$.

So the pair (T, W) satisfies the desired properties. The lemma follows for $n = 3$. For general n, it can be shown by a similar argument inductively. ■

Suppose $\mathcal{D}''$ is an isotropic subspace of $\mathcal{D}$ of dimension d''. Set $\mathcal{D}' = (\mathcal{D}'')^{\perp}/\mathcal{D}''$, and denote by D' its dimension $D - 2d'' = 2d' + 1$. Thus, $\mathcal{D}'$ admits a non-degenerate bilinear form induced from that of $\mathcal{D}$. Given any subspace $C \subseteq D$, it induces a subspace $\pi^{\natural}(C) \in \mathcal{D}'$:

$$\pi^{\natural}(C) = \frac{C \cap (\mathcal{D}'')^{\perp} + \mathcal{D}''}{\mathcal{D}''}.$$

Recall the operation π'' from Section 2.3. For any $L \in X^{J}_{d}$, we have that $\pi^{\natural}(L) \in X^{J}_{d'}$ and $\pi''(L) \in X_{d''}$. For any pair $(L'', L') \in X_{d''} \times X^{J}_{d'}$, we set

$$Z^{J}_{L',L''} = \left\{ L \in X^{J}_{d} | \pi^{\natural}(L) = L', \pi''(L) = L'' \right\}. \tag{28}$$

We also set $\tilde{Z}$ to be the set of all pairs (T, W) subject to the conditions (1), (2), and (3) in Lemma 3.2.1. To a pair $(T, W) \in \tilde{Z}$, we have an isomorphism $\pi : T \to \mathcal{D}'$. Define a map $\tilde{Z} \to Z^{J}_{L',L''}$ by sending (T, W) to $L^{T,W}$, where

$$L^{T,W}_{i} = L''_{i} \oplus \pi^{-1}(L'_{i}) \oplus (L''_{n-i})^{\#}, \ (L''_{n-i})^{\#} = \left\{ w \in W | (w, L''_{n-i}) = 0 \right\}, \quad \forall 1 \le i \le n.$$

By Lemma 3.2.1, we see that the map $\tilde{Z} \to Z^{J}_{L',L''}$ is surjective. Let

$$\mathcal{U} = \left\{ g \in \mathrm{G}^{J}_{d} | g(v) = v, \forall v \in \mathcal{D}'', g(v_1) - v_1 \in \mathcal{D}'', \forall v_1 \in (\mathcal{D}'')^{\perp} \right\}.$$

Clearly $\mathcal{U}$ acts on $\tilde{Z}$ and $Z^{J}_{L',L''}$. Moreover, it can be checked that $\mathcal{U}$ acts transitively on $\tilde{Z}$ and is compatible with the surjective map $\tilde{Z} \to Z^{J}_{L',L''}$. Therefore, we have the following lemma, analogous to [24, Lemma 1.4].

Lemma 3.2.2. The group $\mathcal{U}$ acts transitively on the set $Z^{J}_{L',L''}$.

Recall $\mathbf{S}_d$ from (4). We are ready to define the comulitiplication Δ^{J}. This is a map

$$\widetilde{\Delta}^{J} : \mathbf{S}^{J}_{d} \to \mathbf{S}^{J}_{d'} \otimes \mathbf{S}_{d''}, \quad \forall d' + d'' = d, \tag{29}$$

defined by $\widetilde{\Delta}^{J}(f)(L', \check{L}', L'', \check{L}'') = \sum_{\check{L} \in Z^{J}_{\check{L}',\check{L}''}} f(L, \check{L}), \quad \forall L', \check{L}' \in X^{J}_{d'}, L'', \check{L}'' \in X_{d''}$, where L is a fixed element in $Z^{J}_{L',L''}$ (See (28) for notations). By Lemma 3.2.2, we see that the definition of $\widetilde{\Delta}^J$ is independent of the choice of L. Moreover, by an argument exactly the same way as that of Proposition 1.5 in [24], the map $\widetilde{\Delta}^J$ is an algebra homomorphism.

By using the monomial basis in [1, Theorem 3.10], one can show that $\mathbb{S}_d^J$ admits an algebra homomorphism $\widetilde{\mathbb{\Delta}}^J$, which descends to $\widetilde{\Delta}^J$ when specialized to finite fields.

For any $i \in [1, r]$, $a \in [1, r+1]$, we define the following functions in $\mathbf{S}_d^J$

$$\mathbf{e}_i(L, L') = \begin{cases} \mathbf{v}^{-|L'_{i+1}/L'_i|}, & \text{if } L_i \overset{1}{\subset} L'_i, L_j = L'_j, \forall j \in [1, r]\backslash\{i\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{f}_i(L, L') = \begin{cases} \mathbf{v}^{-|L'_i/L'_{i-1}|}, & \text{if } L_i \overset{1}{\supset} L'_i, L_j = L'_j, \forall j \in [1, r]\backslash\{i\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{H}_a^{\pm 1}(L, L') = \mathbf{v}^{\pm|L_a/L_{a-1}|}\delta_{L,L'}, \quad \mathbf{k}_i^{\pm 1} = \mathbf{H}_{i+1}^{\pm 1}\mathbf{H}_i^{\mp 1},$$

for any $L, L' \in X_d^J$. We write $\mathbf{e}'_i, \mathbf{f}'_i$, and $\mathbf{H}'_{\pm i}$ for the elements in $\mathbf{S}_{d'}^J$ analogous to $\mathbf{e}_i$, $\mathbf{f}_i$, and $\mathbf{H}_{\pm i}$ in $\mathbf{S}_d^J$, respectively. Similarly, we use the notations $\mathbf{E}''_i$, $\mathbf{F}''_i$, and $\mathbf{K}''_i$ for $1 \le i \le n-1$, and $\mathbf{H}''_{\pm i}$, for $1 \le i \le n$, for elements in $\mathbf{S}_{d''}$ defined in Section 2.3. We use the same notations for the corresponding elements in $\mathbb{S}_d^J$.

Proposition 3.2.3. For any $i \in [1, r]$, we have

$$\widetilde{\mathbb{\Delta}}^J(\mathbf{e}_i) = \mathbf{e}'_i \otimes \mathbf{H}''_{i+1}\mathbf{H}''^{-1}_{n-i} + \mathbf{H}'^{-1}_{i+1} \otimes \mathbf{E}''_i\mathbf{H}''^{-1}_{n-i} + \mathbf{H}'_{i+1} \otimes \mathbf{F}''_{n-i}\mathbf{H}''_{i+1}.$$

$$\widetilde{\mathbb{\Delta}}^J(\mathbf{f}_i) = \mathbf{f}'_i \otimes \mathbf{H}''^{-1}_i\mathbf{H}''_{n+1-i} + \mathbf{H}'_i \otimes \mathbf{F}''_i\mathbf{H}''_{n+1-i} + \mathbf{H}'^{-1}_i \otimes \mathbf{E}''_{n-i}\mathbf{H}''^{-1}_i.$$

$$\widetilde{\mathbb{\Delta}}^J(\mathbf{K}_i) = \mathbf{K}'_i \otimes \mathbf{K}''_i\mathbf{K}''^{-1}_{n-i}.$$

Proof. As before, we only need to check the equalities over $\mathbf{S}_d^J$. By definition, we have

$$\widetilde{\Delta}^J(\mathbf{e}_i)(L', \check{L}', L'', \check{L}'') = \mathbf{v}^{-|\check{L}_{i+1}/\check{L}_i|}\#S,$$

where $S = \{\check{L} \in Z^J_{\check{L}', \check{L}''} | L_i \subset \check{L}_i, |\check{L}_i/L_i| = 1, L_j = \check{L}_j, \forall 1 \le j \ne i \le r\}$. The set S is nonempty only when the quadruple $(L', \check{L}', L'', \check{L}'')$ is in one of the following three cases.

(i) $L'_i \subset \check{L}'_i$, $|\check{L}'_i/L'_i| = 1$, $L'_j = \check{L}'_j$, for all $1 \le j \ne i \le r$, $L''_j = \check{L}''_j$, for all j.

(ii) $L'_j = \check{L}'_j$, for all j, $L''_i \subset \check{L}''_i$, $|\check{L}''_i/L''_i| = 1$, $L''_j = \check{L}''_j$ for all $j \ne i$.

(iii) $L'_j = \check{L}'_j$, for all j, $L''_{n-i} \supset \check{L}''_{n-i}$, $|L''_{n-i}/\check{L}''_{n-i}| = 1$, $L''_j = \check{L}''_j$ for all $j \ne n-i$.

We now compute the number $\#S$ in case (i). This amounts to count all possible lines $\langle u \rangle$, spanned by the vector u, such that $L_i + \langle u \rangle$ is in S. Since we want $L_i + \langle u \rangle \subseteq L_{i+1}$, we should find u in L_{i+1}. Since we need $\pi^{\natural}(L_i + \langle u\rangle) = \check{L}'_i$, we need to find those u such that $\pi(u) = u'$, where u' is a fixed element in $\mathcal{D}'$ such that $\check{L}'_i = L'_i + \langle u'\rangle$. Fix a pair (T, W) in $\mathcal{D}$ such that it satisfies all conditions in Lemma 3.2.1 with respect to the flag L. In particular, $L_{i+1} = L''_{i+1} \oplus (L_{i+1} \cap T) \oplus (L_{i+1} \cap W)$. Since T gets identified with $\mathcal{D}'$ via the canonical projection, there is a unique t in T sending to u'. So we need to look for those u such that at component $L_{i+1} \cap T$, $u = t$, and at component $L_{i+1} \cap W$, $u = 0$. Thus, u is of the form $t + w$ where $w \in L''_{i+1}$. Since adding w by any vector in L''_i does not change the resulting space $L_i + \langle u\rangle$, we see that the freedom of choice for w is L''_{i+1} mod L''_i, that is, L''_{i+1}/L''_i. So we see that the value of $\widetilde{\Delta}^J(\mathbf{e}_i)(L', \check{L}', L'', \check{L}'')$ is equal to

$$\mathbf{v}^{-|\check{L}_{i+1}/\check{L}_i|} q^{|L''_{i+1}/L''_i|} = \mathbf{v}^{-|\check{L}'_{i+1}/\check{L}'_i|} \mathbf{v}^{-|\check{L}''_{n-i}/\check{L}''_{n-i-1}|+|\check{L}''_{i+1}/\check{L}''_i|} = \left(\mathbf{e}'_i \otimes \mathbf{H}''^{-1}_{n-i} \mathbf{H}''_{i+1}\right)(L', \check{L}', L'', \check{L}''),$$

where we use $|\check{L}_{i+1}/\check{L}_i| = |\check{L}''_{i+1}/\check{L}''_i| + |\check{L}'_{i+1}/\check{L}'_i| + |\check{L}''_{n-i}/\check{L}''_{n-i-1}|$.

For case (ii), S consists of only one element, that is, the $\check{L}$ such that $\check{L}_j = L_j$ for $1 \le j \ne i \le r$, and $\check{L}_i = L_i + \check{L}''_i$. (Since $\check{L}''_i \subseteq L''_{n-i}$, $\check{L}_i$ is isotropic.) So the value of $\widetilde{\Delta}^J(\mathbf{e}_i)(L', \check{L}', L'', \check{L}'')$ in case (ii) is equal to

$$\mathbf{v}^{-|\check{L}_{i+1}/\check{L}_i|} = \mathbf{v}^{-|\check{L}'_{i+1}/\check{L}'_i|} \mathbf{v}^{-|\check{L}''_{i+1}/\check{L}''_i|} \mathbf{v}^{-|\check{L}''_{n-i}/\check{L}''_{n-i-1}|} = \left(\mathbf{H}'^{-1}_{i+1} \otimes \mathbf{E}''_i \mathbf{H}''^{-1}_{n-i}\right)(L', \check{L}', L'', \check{L}'').$$

For case (iii), we need to consider two situations, that is, $i = r$ or $i \ne r$. For $i = r$, the set S gets identified with the set $S_r = \{l \in L_{r+1}/L_r : \check{U}_{r+1} \subset l^{\perp}, U_{r+1} \not\subset l^{\perp}\}$, via $\check{L} \mapsto \check{L}_r/L_r$, where $\check{U}_{r+1} = (\check{L}''_{r+1} + L_r)/L_r$ and $U_{r+1} = (L''_{r+1} + L_r)/L_r$. Set

$$\tilde{S}_r = \{W \subset L_{r+1}/L_r | W \text{ isotropic}, \ \check{U}_{r+1} \subset W, \dim W/\check{U}_{r+1} = 1, W + U_{r+1} \text{ not isotropic}\}.$$

We define a map $S_r \to \tilde{S}_r$ by $l \mapsto l + \check{U}_{r+1}$. It is clear that this is a surjective map and its fiber is isomorphic to $\check{U}_{r+1}$. The set $\tilde{S}_r$ can be broken into the difference of the two sets

$$\begin{aligned} \tilde{S}_r = \{&W | W \text{ isotropic } \check{U}_{r+1} \subset W, \dim W/\check{U}_{r+1} = 1\} \\ &- \{W | W \text{ isotropic } \check{U}_{r+1} \subset W, \dim W/\check{U}_{r+1} = 1, W + U_{r+1} \text{ isotropic}\}. \end{aligned}$$

For the 1st set, its order is equal to $\frac{q^{|L_{r+1}/L_r| - 2|\check{L}''_{r+1}/\check{L}''_r| - 1} - 1}{q-1}$, because $\check{U}_{r+1} \simeq \check{L}''_{r+1}/\check{L}''_r$. For the 2nd set, it is the union of $\{W = U_{r+1}\}$ and the subset $\{\dim W + U_{r+1}/\check{U}_{r+1} = 2\}$. The latter has a surjection onto the set of isotropic lines in $U^{\perp}_{r+1}/U_{r+1}$ with fiber $\mathbb{F}_q$, via

$W \mapsto W + U_{r+1}/U_{r+1}$. Thus, the order of the 2nd set is $1 + q\frac{q^{|L_{r+1}/L_r|-2|L''_{r+1}/L''_r|-1}-1}{q-1}$. So

$$\#S = \#\check{U}_{r+1}\left(\frac{q^{|L_{r+1}/L_r|-2|\check{L}''_{r+1}/\check{L}''_r|-1}-1}{q-1} - 1 - q\frac{q^{|L_{r+1}/L_r|-2|L''_{r+1}/L''_r|-1}-1}{q-1}\right)$$
$$= q^{|\check{L}''_{r+1}/\check{L}''_r|+|\check{L}'_{r+1}/\check{L}'_r|}.$$

So we see that the value of $\widetilde{\Delta}^J(\mathbf{e}_i)(L', \check{L}', L'', \check{L}'')$, for $i = r$ in case (iii), is equal to

$$\mathbf{v}^{-|\check{L}_{r+1}/\check{L}_r|}q^{|\check{L}''_{r+1}/\check{L}''_r|+|\check{L}'_{r+1}/\check{L}'_r|} = \mathbf{v}^{|\check{L}'_{r+1}/\check{L}'_r|} = (\mathbf{H}'_{i+1} \otimes \mathbf{F}''_{n-i}\mathbf{H}''_{i+1})(L', \check{L}', L'', \check{L}'').$$

For $i \neq r$, the set S gets identified with the set S' of isotropic lines l in L_{n-i}/L_i such that

$$l \subseteq L_{i+1}/L_i, \check{U} \subset l^\perp, U \not\subset l^\perp,$$

where $\check{U} = \check{L}''_{n-i} + L_i/L_i$ and $U = L''_{n-i} + L_i/L_i$. Notice S' is the difference of the two sets:

$$S' = \{l | l \subseteq L_{i+1}/L_i, \check{U} \subset l^\perp\} - \{l | l \subseteq L_{i+1}/L_i, U \subset l^\perp\}.$$

We can use a similar arguments as in (i) to compute the two sets and we get

$$\#S = \frac{q^{|L''_{i+1}/L''_i|+|L'_{i+1}/L'_i|+1}-1}{q-1} + \frac{q^{|L''_{i+1}/L''_i|+|L'_{i+1}/L'_i|}-1}{q-1} = q^{|L''_{i+1}/L''_i|+|L'_{i+1}/L'_i|}.$$

So we see that the value of $\widetilde{\Delta}^J(\mathbf{e}_i)(L', \check{L}', L'', \check{L}'')$, for $i \neq r$ in case (iii), is equal to

$$\mathbf{v}^{-|\check{L}_{i+1}/\check{L}_i|}q^{|L''_{i+1}/L''_i|+|L'_{i+1}/L'_i|} = \mathbf{v}^{|\check{L}'_{i+1}/\check{L}'_i|}\mathbf{v}^{-|\check{L}''_{n-i}/\check{L}''_{n-i-1}|+|\check{L}''_{i+1}/\check{L}''_i|}$$
$$= (\mathbf{H}'_{i+1} \otimes \mathbf{F}''_{n-i}\mathbf{H}''_{i+1})(L', \check{L}', L'', \check{L}'').$$

We have the 1st identity.

Next, we determine $\widetilde{\Delta}^J(\mathbf{f}_i)$. By definition, we have

$$\widetilde{\Delta}^J(\mathbf{f}_i)(L', \check{L}', L'', \check{L}'') = \mathbf{v}^{-|\check{L}_i/\check{L}_{i-1}|}\#R, \tag{30}$$

where $R = \{\check{L} \in Z^J_{\check{L}', \check{L}''} | L_i \supset \check{L}_i, |L_i/\check{L}_i| = 1, L_j = \check{L}_j, \forall 1 \leq j \leq r, j \neq r\}$. Now the set R is empty unless the quadruple $(L', \check{L}', L'', \check{L}'')$ is in one of the following cases.

(iv) $L'_i \supset \check{L}'_i$, $|L'_i/\check{L}'_i| = 1$, $L'_j = \check{L}'_j$, $\forall 1 \leq j \leq r, j \neq i$, $L''_j = \check{L}''_j$ for all j.

(v) $L'_j = \check{L}'_j$ for all j, $L''_i \supset \check{L}''_i$, $|L''_i/\check{L}''_i| = 1$, $L''_j = \check{L}''_j$, $\forall 1 \leq j \neq i \leq n$.

(vi) $L'_j = \check{L}'_j$ for all j, $L''_{n-i} \subset \check{L}''_{n-i}$, $|\check{L}''_{n-i}/L''_{n-i}| = 1$, $L''_j = \check{L}''_j$, $\forall 1 \leq j \neq n-i \leq n$.

In these cases, $\check{L}$ differs from L only at i and $n-i$. Thus, we can identify $\check{L}$ with $\check{L}_i$.

In case (iv), to count the number of elements in R, we break it into two steps. We first determine all possible choices of $\check{L}_i \cap (\mathcal{D}'')^\perp$ for $\check{L}_i \in R$. Since $\check{L}''_i = L''_i$, we have only one choice, that is, $\tilde{L}_i = L''_i + T$, where T is any subspace (of dimension $|\check{L}'_i|$) in $L_i \cap (\mathcal{D}'')^\perp$ maps onto $\check{L}'_i$ via the canonical projection. We next want to determine the number of choices of $W \subseteq L_i$ such $W + \tilde{L}_i \in R$. We first observe that if $\check{L} \in R$, then the projection, say $\check{L}'''_i$, of $\check{L}_i$ to $\mathcal{D}/(\mathcal{D}'')^\perp$ is the same as that of L_i. Since $|L_i \cap (\mathcal{D}'')^\perp/\check{L}_i \cap (\mathcal{D}'')^\perp| = 1$ and $L_{i-1} \subseteq \check{L}_i$, we see that all possible choices for $W \in L_i$ such that $W + \tilde{L}_i \in R$ is bijective to the space

$$\check{L}'''_i/L'''_{i-1} \simeq (\check{L}''_{n-i})^\#/(\check{L}''_{n-i+1})^\# \simeq \check{L}''_{n-i+1}/\check{L}''_{n-i},$$

where L'''_{i-1} is the projection of L_{i-1} to $\mathcal{D}/(\mathcal{D}'')^\perp$. Thus, in case (iv), the left-hand side of (30) is equal to $\mathbf{v}^{-|\check{L}_i/\check{L}_{i-1}|} q^{|\check{L}''_{n-i+1}/\check{L}''_{n-i}|} = \mathbf{f}'_i \otimes \mathbf{H}''^{-1}_i \mathbf{H}''_{n+1-i}(L', \check{L}', L'', \check{L}'')$.

In case (v), to build a subspace $\check{L}_i$ in L_i such that it is in R, there are $\check{L}'_i/\check{L}'_{i-1}$ choices to build the component $\tilde{L}_i = \check{L}_i \cap (\mathcal{D}'')^\perp$. This is done by using a similar argument as in the 1st step of case (iv) since $L''_i \supset \check{L}''_i$ and $|L''_i/\check{L}''_i| = 1$. By a similar argument as the step two in case (iv), we see that the number of choices for a subspace W in L_i such that $W + \tilde{L}_i \in R$ is again $\check{L}''_{n-i+1}/\check{L}''_{n-i}$ for a fixed subspace from the 1st step. Thus, the value of (30) in case (v) is equal to

$$\mathbf{v}^{-|\check{L}_i/\check{L}_{i-1}|} q^{|\check{L}'_i/\check{L}'_{i-1}|} q^{|\check{L}''_{n-i+1}/\check{L}''_{n-i}|} = \mathbf{H}'_i \otimes \mathbf{F}''_i \mathbf{H}''_{n+1-i}(L', \check{L}', L'', \check{L}'').$$

In case (vi), there is only one element in R. First of all, $\check{L}_i \cap (\mathcal{D}'')^\perp = L_i \cap (\mathcal{D}'')^\perp$ for $\check{L}_i \in R$. Second of all, by fixing a decomposition of $\mathcal{D} = \mathcal{D}'' \oplus T \oplus W$ as in Lemma 3.2.1, we see that if $\check{L}_i \in R$, then the projection of $\check{L}_i$ to $\mathcal{D}/(\mathcal{D}'')^\perp$ is $(\check{L}''_{n-i})^\#$. Thus, by an argument similar to the 2nd step of case (iv), we see that there is only one $\check{L}_i$ in R. This implies that the value of (30) in case (vi) is equal to $\mathbf{v}^{-|\check{L}_i/\check{L}_{i-1}|} = \mathbf{H}'^{-1}_i \otimes \mathbf{E}''_{n-i} \mathbf{H}''^{-1}_i (L', \check{L}', L'', \check{L}'')$. We see that the 2nd identity follows from the above computations.

Finally the last identity follows from the definitions and $\widetilde{\Delta}^J(\mathbf{H}_i) = \mathbf{H}'_i \otimes \mathbf{H}''_i \mathbf{H}''_{n+1-i}$. ■

Corollary 3.2.4. We have $(1 \otimes \widetilde{\mathbb{\Delta}})\widetilde{\mathbb{\Delta}}^J = (\widetilde{\mathbb{\Delta}}^J \otimes 1)\widetilde{\mathbb{\Delta}}^J$.

The corollary follows by checking if the relation holds for generators, which is immeidate.

3.3 Renormalization

Given a pair $(\mathbf{b}, \mathbf{a})$ in $\Lambda_{d,n}$ for $n = 2r + 1$ (see (7)), we set

$$\begin{aligned} u(\mathbf{b}, \mathbf{a}) &= \frac{1}{2}\left(\sum_{i+j\geq n+1} b_i b_j - a_i a_j + \sum_{i\geq r+1} a_i - b_i\right) \\ &= \sum_{i>j, i+j\geq n+1} b_i b_j - a_i a_j + \frac{1}{2}\left(\sum_{i\geq r+1} b_i^2 - a_i^2 + a_i - b_i\right) \in \mathbb{Z}. \end{aligned} \tag{31}$$

The coproduct $\widetilde{\Delta}^J$ in (29) can be decomposed as

$$\widetilde{\Delta}^J = \oplus \widetilde{\Delta}^J_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''},$$

where $\widetilde{\Delta}^J_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}$ is the component from $\mathbb{S}^J_d(\mathbf{b}, \mathbf{a})$ to $\mathbb{S}^J_d(\mathbf{b}', \mathbf{a}') \otimes \mathbb{S}(\mathbf{b}'', \mathbf{a}'')$ such that

$$b_i = b'_i + b''_i + b''_{n+1-i}, \quad a_i = a'_i + a''_i + a''_{n+1-i}, \quad \forall 1 \leq i \leq n.$$

We renormalize $\widetilde{\Delta}^J$ in (29) as follows:

$$\Delta^J = \oplus_{\mathbf{b}, \mathbf{a}, \mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''} \Delta^J_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}, \tag{32}$$

where $\Delta^J_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''} = v^{\sum_{1\leq i\leq j\leq n} b'_i b''_j - a'_i a''_j} v^{u(\mathbf{b}'', \mathbf{a}'')} \widetilde{\Delta}^J_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}$. Note that Δ^J is again an algebra homomorphism, due to the fact that $u(\mathbf{c}, \mathbf{a}) = u(\mathbf{c}, \mathbf{b}) + u(\mathbf{b}, \mathbf{a})$. By a straightforward computation based on Proposition 3.2.3, we have

Proposition 3.3.1. For any $i \in [1, r]$,

$$\Delta^J(\mathbf{e}_i) = \mathbf{e}'_i \otimes \mathbf{K}''_i + 1 \otimes \mathbf{E}''_i + \mathbf{K}'_i \otimes \mathbf{F}''_{n-i}\mathbf{K}''_i.$$

$$\Delta^J(\mathbf{f}_i) = \mathbf{f}'_i \otimes \mathbf{K}''_{n-i} + \mathbf{k}'^{-1}_i \otimes \mathbf{K}''_{n-i}\mathbf{F}''_i + 1 \otimes \mathbf{E}''_{n-i}.$$

$$\Delta^J(\mathbf{K}_i) = \mathbf{K}'_i \otimes \mathbf{K}''_i \mathbf{K}''^{-1}_{n-i}.$$

Proof. Fix an $i \in [1, r]$. Assume that we have a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ such that $b'_k = a'_k - \delta_{k,i} + \delta_{k,i+1} + \delta_{k,n-i} - \delta_{k,n+1-i}$ and $b''_k = a''_k$ for all $k \in [1, n]$. We have $u(\mathbf{b}'', \mathbf{a}'') = 0$ and $\sum_{1\leq k\leq j\leq n} b'_k a''_j - a'_k a''_j = -a''_i + a''_{n-i}$. So, after the twist, the 1st term on the right of $\widetilde{\Delta}^J(\mathbf{e}_i)$ in Proposition 3.2.3 becomes $\mathbf{e}'_i \otimes \mathbf{H}''_{i+1}\mathbf{H}''^{-1}_{n-i}|_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''} v^{-a''_i + a''_{n-i}} = \mathbf{e}'_i \otimes \mathbf{K}''_i|_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}$ where the notation $f|_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}$ is the restriction of f to $X^J_{d'}(\mathbf{b}') \times X^J_{d'}(\mathbf{a}') \times X_{d''}(\mathbf{b}'') \times X_{d''}(\mathbf{a}'')$.

Assume that we have a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ such that $b'_k = a'_k$ and $b''_k = a''_k - \delta_{k,i} + \delta_{k,i+1}$ for all $k \in [1, n]$. Then we have $\sum_{1\le k\le j\le n} b'_k b''_j - a'_k a''_j = a'_{i+1}$ and $u(\mathbf{b}'', \mathbf{a}'') = a''_{n-i}$. Thus, after the twist, the 2nd term on the right of $\widetilde{\Delta}^J(\mathbf{e}_i)$ in Proposition 3.2.3 is equal to $\mathbf{H}'^{-1}_{i+1} \otimes \mathbf{E}''_i \mathbf{H}''^{-1}_{n-i}|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} v^{a'_{i+1}+a''_{n-i}} = 1 \otimes \mathbf{E}''_i|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$.

Assume that we have a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ such that $b'_k = a'_k$ and $b''_k = a''_k + \delta_{k,n-i} - \delta_{k,n+1-i}$ for all $k \in [1, n]$. Then we have $\sum_{1\le k\le j\le n} b'_k b''_j - a'_k a''_j = -a'_i$ and $u(\mathbf{b}'', \mathbf{a}'') = -a''_i + \delta_{i,r+1} = -a''_i$, where the latter equality is due to $i \in [1, r]$. Hence, after the twist, the 3rd term on the right of $\widetilde{\Delta}^J(\mathbf{e}_i)$ in Proposition 3.2.3 is equal to $\mathbf{H}'_{i+1} \otimes \mathbf{F}''_{n-i}\mathbf{H}''_{i+1} v^{-a'_i-a''_i}|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} = \mathbf{k}'_i \otimes \mathbf{F}''_{n-i}\mathbf{K}''_i|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$.

The 1st equality in the proposition follows from the above analysis.

Assume that we have a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ such that $b'_k = a'_k + \delta_{k,i} - \delta_{k,i+1} - \delta_{k,n-i} + \delta_{k,n+1-i}$ and $b''_k = a''_k$ for all $k \in [1, n]$. Then we have $\sum_{1\le k\le j\le n} b'_k b''_j - a'_k a''_j = a''_i - a''_{n-i}$ and $u(\mathbf{b}'', \mathbf{a}'') = 0$. So, after the twist, the 1st term on the right of $\widetilde{\Delta}^J(\mathbf{f}_i)$ in Proposition 3.2.3 becomes $\mathbf{f}'_i \otimes \mathbf{H}''^{-1}_i \mathbf{H}''_{n+1-i} v^{a''_i - a''_{n-i}}|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} = \mathbf{f}'_i \otimes \mathbf{K}''_{n-i}|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$.

Assume that we have a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ such that $b'_k = a'_k$ and $b''_k = a''_k + \delta_{k,i} - \delta_{k,i+1}$ for all $k \in [1, n]$. Then we have $\sum_{1\le k\le j\le n} b'_k b''_j - a'_k a''_j = -a'_{i+1}$ and $u(\mathbf{b}'', \mathbf{a}'') = -a_{n-i} + \delta_{i,r}$. So, after the twist, the 2nd term on the right of $\widetilde{\Delta}^J(\mathbf{f}_i)$ in Proposition 3.2.3 becomes $\mathbf{H}'_i \otimes \mathbf{F}''_i \mathbf{H}''_{n+1-i} v^{-a'_{i+1}-a_{n-i}+\delta_{i,r}}|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} = \mathbf{k}'^{-1}_i \otimes v^{\delta_{i,r}} \mathbf{F}''_i \mathbf{K}''_{n-i}|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} = \mathbf{k}'^{-1}_i \otimes \mathbf{K}''_{n-i}\mathbf{F}''_i|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$.

Assume that we have a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ such that $b'_k = a'_k$ and $b''_k = a''_k - \delta_{k,n-i} + \delta_{k,n+1-i}$ for all $k \in [1, n]$. Then we have $\sum_{1\le k\le j\le n} b'_k b''_j - a'_k a''_j = a'_{n+1-i} = a'_i$ and $u(\mathbf{b}'', \mathbf{a}'') = a''_i$. So, after the twist, the 3rd term on the right of $\widetilde{\Delta}^J(\mathbf{f}_i)$ in Proposition 3.2.3 becomes $\mathbf{H}'^{-1}_i \otimes \mathbf{E}''_{n-i}\mathbf{H}''^{-1}_i v^{a'_i+a''_i}|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} = 1 \otimes \mathbf{E}''_{n-i}|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$.

The above analysis implies the 2nd equality in the proposition. Since the twist will not affect the original term when $\mathbf{b}' = \mathbf{a}'$ and $\mathbf{b}'' = \mathbf{a}''$, we have the 3rd equality. ■

Recall the canonical basis $\{\{M\} | M \in \Xi^J_d\}$ of $\mathbb{S}^J_d$ from [1, 3.6]. We have the following positivity result of the canonical basis of $\mathbb{S}^J_d$ with respect to the coproduct Δ^J.

Proposition 3.3.2. If $\Delta^J(\{M\}) = \sum_{M'\in\Xi^J_{d'}, M''\in\Xi_{d''}} h^{M',M''}_M \{M'\}\otimes\{M''\}$, then we have $h^{M',M''}_M \in \mathbb{Z}_{\ge 0}[v, v^{-1}]$.

Proof. The proof is similar to that of Proposition 2.3.6. We consider the orthogonal group $\overline{\mathbf{G}}^J_d$ and the isotropic flag variety $\overline{\mathbf{X}}^J_d(\mathbf{a})$ over $\overline{\mathbb{F}_q}$, whose $\mathbb{F}_q$-points are exactly $\mathbf{G}^J_d$ and $X^J_d(\mathbf{a})$, respectively. The linear form Q^J can be extended naturally to a form $\mathbf{Q}^J$ on $\overline{\mathbb{F}_q}^D$. Suppose that $\mathcal{D}''$ is isotropic with respect to $\mathbf{Q}^J$. We can fix a decomposition $\overline{\mathbb{F}_q}^D =$

$\mathcal{D}'' \oplus T \oplus W$ such that Lemma 3.2.1 (a)–(c) hold. With respect to the bases in Lemma 3.2.1 (c) and a fixed basis of T, we can further assume that the associated matrix of $\mathbf{Q}^J$ is of the form $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, since $\mathbf{Q}^J$ is defined over an algebraic closed field. Recall that $\overline{\mathbf{G}}_m = \mathrm{GL}(1, \overline{\mathbb{F}_q})$. We define an imbedding $\overline{\mathbf{G}}_m \to \overline{\mathbf{G}}^J_d$ by $t \mapsto \begin{pmatrix} 0 & 0 & t.1 \\ 0 & 1 & 0 \\ t^{-1}.1 & 0 & 0 \end{pmatrix}$ where the 1s denote the identity matrix of the desired rank. Then the $\overline{\mathbf{G}}_m$-fixed-point set of $\overline{\mathbf{X}}^J_d(\mathbf{a})$ consists of all flags L such that $L_i = (L_i \cap \mathcal{D}'') \oplus (L_i \cap T) \oplus (L_i \cap W)$ for all i, hence is $\sqcup_{(\mathbf{a}',\mathbf{a}'')\vdash \mathbf{a}} \overline{\mathbf{X}}^J_{d'}(\mathbf{a}') \times \overline{\mathbf{X}}_{d''}(\mathbf{a}'')$. Furthermore, the attracting set of $\overline{\mathbf{X}}^J_{d'}(\mathbf{a}') \times \overline{\mathbf{X}}_{d''}(\mathbf{a}'')$, for all $(\mathbf{a}', \mathbf{a}'') \vdash \mathbf{a}$, is the algebraic variety $\overline{\mathbf{X}}^{J+}_{\mathbf{a},\mathbf{a}',\mathbf{a}''}$ whose $\mathbb{F}_q$-point set is $X^{J+}_{\mathbf{a},\mathbf{a}',\mathbf{a}''} = \{L \in X^J_d | (\pi^\natural(L), \pi''(L)) \in X^J_{d'}(\mathbf{a}') \times X_{d''}(\mathbf{a}'')\}$. Thus, we have

$$\overline{X}^J_d(a) \xleftarrow{\iota_J} \overline{X}^{J+}_{a,a',a''} \xrightarrow{\pi^J} \overline{X}^J_{d'}(a') \times \overline{X}_{d''}(a''),$$

where the 1st arrow is an inclusion and the 2nd is induced by the definitions. Arguing in a similar way as Lemma 2.3.4, we see that $\Delta^J_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$ is the function version of the hyperbolic localization functor $\pi^J_! \iota^*_J$. Now applying Braden's [4] result, we are done. ■

Remark 3.3.3. Note that the rank of the vector bundle π^J is

$$\sum_{1\le i<j\le n} a'_i a''_j + \frac{1}{2}\left(\sum_{i+j>n+1} a''_i a''_j - \sum_{i>r+1} a''_i\right).$$

This provides an explanation of the twist in (32).

Moreover, we have the following coassociative property.

Proposition 3.3.4. $(1 \otimes \mathbb{A})\mathbb{A}^J = (\mathbb{A}^J \otimes 1)\mathbb{A}^J$. More precisely, we have the following commutative diagram for the quadruple d, d', d'', d''' such that $d = d' + d'' + d'''$

$$\begin{array}{ccc} \mathbb{S}^J_d & \xrightarrow{\Delta^J} & \mathbb{S}^J_{d'} \otimes \mathbb{S}_{d''+d'''} \\ {\scriptstyle \Delta^J}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ \mathbb{S}^J_{d'+d''} \otimes \mathbb{S}_{d'''} & \xrightarrow{\Delta^J\otimes 1} & \mathbb{S}^J_{d'} \otimes \mathbb{S}_{d''} \otimes \mathbb{S}_{d'''}. \end{array}$$

3.4 The imbedding $\jmath_d : \mathbb{S}^J_d \to \mathbb{S}_d$

In this section, we set $d' = 0$ and $d'' = d$, then the coproduct Δ^J in (29) becomes Δ^J : $\mathbb{S}^J_d \to \mathbb{S}^J_0 \otimes \mathbb{S}_d$. Observe that $\mathbb{S}^J_0$ consists of only one basis element, so we have $\mathbb{S}^J_0 \simeq \mathbb{A}$. Thus, the coproduct Δ^J becomes the following algebra homomorphism, denoted by $\jmath_d$,

$$\jmath_d : \mathbb{S}^J_d \to \mathbb{S}_d. \tag{33}$$

The following corollary is by Proposition 3.3.1, $\mathbf{e}_i = 0$, $\mathbf{f}_i = 0$, and $\mathbf{K}'_i = v^{\delta_{i,r}}$ in $\mathbb{S}^J_0$.

Proposition 3.4.1. There is a unique algebra imbedding

$$\jmath_d : \mathbb{S}^J_d \to \mathbb{S}_d$$

such that $\jmath_d(\mathbf{e}_i) = \mathbf{E}_i + \mathbf{K}_i\mathbf{F}_{n-i}$, $\jmath_d(\mathbf{f}_i) = \mathbf{F}_i\mathbf{K}_{n-i} + \mathbf{E}_{n-i}$, $\jmath_d(\mathbf{K}_i) = v^{\delta_{i,r}}\mathbf{K}_i\mathbf{K}^{-1}_{n-i}$, $\forall 1 \leq i \leq r$.

Proof. By Proposition 3.3.1, we have

$$\jmath_d(\mathbf{e}_i) = \mathbf{e}'_i \otimes \mathbf{K}''_i + 1 \otimes \mathbf{E}''_i + \mathbf{K}'_i \otimes \mathbf{F}''_{n-i}\mathbf{K}''_i = 0 + \mathbf{E}''_i + v^{\delta_{i,r}}\mathbf{F}''_{n-i}\mathbf{K}''_i = \mathbf{E}''_i + \mathbf{K}''_i\mathbf{F}''_{n-i},$$

which is the 1st identity if we skip the superscripts. The remaining two are obtained in exactly the same manner, and hence skipped. ■

Remark 3.4.2. The homomorphism $\jmath_d$ matches with the imbedding $\jmath$ in [1, Proposition 4.5]. The only difference is an involution ω on $\mathbb{U}$ defined by $(\mathbb{E}_i, \mathbb{F}_i, \mathbb{K}_i) \mapsto (\mathbb{F}_i, \mathbb{E}_i, \mathbb{K}_i^{-1})$.

By Propositions 3.3.2 and 3.4.1, we have the following corollary.

Corollary 3.4.3. Let $\{B\}$ be a canonical basis element in $\mathbb{S}^J_d$. If $\jmath_d(\{B\}) = \sum g_{B,A}\{A\}$, where the sum is over the set of canonical basis elements $\{A\}$ in $\mathbb{S}_d$, then $g_{B,A} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

We will need to the following lemma later.

Lemma 3.4.4. The map $\jmath_d$ in (33) is injective.

Proof. Recall from [1, Theorem 3.10] that $\mathbb{S}^J_d$ has a monomial basis m^J_A indexed by $A \in \Xi^J_d$ (which is denoted m_A therein). It is enough to show that the set $\{\jmath_d(m^J_A) | A \in \Xi^J_d\}$

is linearly independent in $\mathbb{S}_d$. We set

$$\deg(1_\lambda)=0,\ \deg(\mathbf{e}_i 1_\lambda)=i,\ \deg(\mathbf{f}_i 1_\lambda)=n-i,\quad \forall\lambda\in\Lambda^{\jmath}_{d,n},1\le i\le r.$$

Similarly, we define

$$\deg(1_\lambda)=0,\ \deg(\mathbf{E}_i 1_\lambda)=i,\ \deg(\mathbf{F}_i 1_\lambda)=-i,\quad \forall\lambda\in\Lambda_{d,n},1\le i\le n.$$

We write $\nu'<\nu$ if $\nu'_i\le\nu_i$ for all i and $\nu'_{i_0}<\nu_{i_0}$ for some i_0. Suppose that $\deg(m^{\jmath}_A)=\nu\in\mathbb{Z}_{\ge 0}[I]$. By Proposition 3.4.1, we have

$$\jmath_d(m^{\jmath}_A)\in\oplus_{\overline{\mathbf{b}}-\overline{\mathbf{a}}=\nu}\mathbb{S}_d(\mathbf{b},\mathbf{a})\oplus\oplus_{\overline{\mathbf{d}}-\overline{\mathbf{c}}<\nu}\mathbb{S}_d(\mathbf{d},\mathbf{c}).$$

For $A=(a_{ij})\in\Xi^{\jmath}_d$, we set

$$\Xi_d(A)=\{B=(b_{ij})\in\Xi_d|b_{ij}=0,\forall i<j,b_{ij}=a_{ij},\forall i>j,\mathrm{co}(B)\vdash\mathrm{co}(A)\},$$

where $\mathbf{b}\vdash\mathbf{a}$ if $b_i+b_{n+1-i}+\delta_{i,r+1}=a_i$ for all $1\le i\le n$. By Proposition 3.4.1, we see that

$$\jmath_d(m^{\jmath}_A)=\sum_{B\in\Xi_d(A)}m_B+\text{lower terms},$$

where m_B denotes the monomial attached to B in [3, Proposition 3.9] and "lower term" is the remaining summand in $\oplus_{\overline{\mathbf{d}}-\overline{\mathbf{c}}<\nu}\mathbb{S}_d(\mathbf{d},\mathbf{c})$. Now suppose that we have

$$\sum_{A\in\Xi^{\jmath}_d}c_A\jmath_d(m^{\jmath}_A)=0,\quad c_A\in\mathcal{A}.$$

Let $\mathcal{M}$ be the set of maximal $\nu\in\mathbb{Z}[I]$ in the set $\{\deg(m^{\jmath}_A)|A\in\Xi^{\jmath}_d\}$ with respect to the natural partial order in $\mathbb{Z}[I]$, that is, $\nu'\le\nu$ if and only if $\nu'_i\le\nu_i$ for all i. We have

$$0=\sum_{A\in\Xi^{\jmath}_d}c_A\jmath_d(m^{\jmath}_A)=\sum_{A:\deg(m^{\jmath}_A)\in\mathcal{M}}c_A\jmath_d(m^{\jmath}_A)+\text{lower term}.$$

So we have $\sum_{A:\deg(m^{\jmath}_A)\in\mathcal{M}}c_A\jmath_d(m^{\jmath}_A)=0$. By [3,Proposition 3.9] and the fact that $\Xi_d(A)\cap\Xi_d(A')=\emptyset$ if $A\ne A'$, the set $\{\sum_{B\in\Xi_d(A)}m_B\}$, where A runs over all matrices in $\Xi^{\jmath}_d$ such that $\deg(m^{\jmath}_A)\in\mathcal{M}$, is linearly independent in $\mathbb{S}_d$. Thus, $c_A=0$ for all $A\in\Xi^{\jmath}_d$ such that $\deg(m^{\jmath}_A)\in\mathcal{M}$. Inductively, $c_A=0$ for all $A\in\Xi^{\jmath}_d$. Therefore, the set $\{\jmath_d(m^{\jmath}_A)|A\in\Xi^{\jmath}_d\}$ is linearly independent. Lemma is proved. ■

The following is nothing but a special case of Proposition 3.3.4.

Corollary 3.4.5. Suppose that $d'+d''=d$. We have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{S}^J_d & \xrightarrow{\Delta^J} & \mathbb{S}^J_{d'} \otimes \mathbb{S}_{d''} \\ {\scriptstyle J_d}\downarrow & & \downarrow{\scriptstyle J_{d'}\otimes 1} \\ \mathbb{S}_d & \xrightarrow{\Delta} & \mathbb{S}_{d'} \otimes \mathbb{S}_{d''}. \end{array}$$

Remark 3.4.6. $\mathbb{S}^J_d$ can be regarded as a "coideal" subalgebra of $\mathbb{S}_d$ in view of Lemma 3.4.4 and Corollary 3.4.5.

3.5 Type A duality versus type B duality

In this section, we use the algebra homomorphism $J_{d,\mathbf{v}}$, the specialization of J_d to $v=\mathbf{v}$, to establish a direct connection between the geometric type A duality in [14] and the geometric type B duality in [1].

For any nonnegative integers a, b, we write 1^a0^b for the sequence $(1,\cdots,1,0,\cdots,0)$ containing a copies of 1's and b copies of 0's. Similarly, we can define $1^a0^b1^c$, etc.

Recall X_d, $X_d(\mathbf{b})$ for $\mathbf{b}\in\Lambda_{d,n}$ from Sections 2.2 and 2.3. We set

$$\mathbf{T}_{d,n}=\mathcal{A}_{G_d}(X_d\times X_d(1^d)),\quad \text{and} \quad \mathbf{H}_{A_d}=\mathcal{A}_{G_d}(X_d(1^d)\times X_d(1^d)).$$

By [14], we know that $\mathbf{H}_{A_d}$ is a Hecke algebra of type A_d and $\mathbf{T}_{d,n}$ is a tensor space $\mathbf{V}_n^{\otimes d}$ where $\mathbf{V}_n$ is a free $\mathcal{A}$-module of rank n. Now the standard convolution defines commuting actions of $\mathbf{S}_d$ and $\mathbf{H}_{A_d}$ on $\mathbf{T}_{d,n}$ from the left and the right, respectively:

$$\mathbf{S}_d\times\mathbf{T}_{d,n}\to\mathbf{T}_{d,n}\overset{\psi}{\leftarrow}\mathbf{T}_{d,n}\times\mathbf{H}_{A_d}. \tag{34}$$

Moreover, the two actions centralize each other.

We shall recall a similar picture in [1] if the X_d is replaced by its J-analog. Recall X^J_d, $X^J_d(\mathbf{b})$ for $\mathbf{b}\in\Lambda^J_{d,n}$ from Sections 3.1 and 3.3. We set

$$\mathbf{T}^J_{d,n}=\mathcal{A}_{G^J_d}(X^J_d\times X^J_d(1^{2d+1}))\quad\text{and}\quad \mathbf{H}_{B_d}=\mathcal{A}_{G^J_d}\left(X^J_d(1^{2d+1})\times X^J_d(1^{2d+1})\right). \tag{35}$$

Then $\mathbf{T}^{J}_{d,n}$ is also isomorphic to the tensor space $\mathbf{V}_n^{\otimes d}$, and there is the following diagram of commuting actions.

$$\mathbf{S}^{J}_{d} \times \mathbf{T}^{J}_{d,n} \to \mathbf{T}^{J}_{d,n} \leftarrow \mathbf{T}^{J}_{d,n} \times \mathbf{H}_{B_d}. \tag{36}$$

A slight variant of the imbedding $\jmath_{d,\mathbf{v}}$ yields the following linear map:

$$\zeta'_{d,\mathbf{b},\mathbf{v}} : \mathcal{A}_{\mathrm{G}^{J}_{d}}(X^{J}_{d}(\mathbf{b}) \times X^{J}_{d}(1^{2d+1})) \to \oplus_{\mathbf{b}''\models\mathbf{b},\mathbf{a}''\models 1^{2d+1}} \mathcal{A}_{\mathrm{G}_d}(X_d(\mathbf{b}'') \times X_d(\mathbf{a}'')), \forall \mathbf{b} \in \Lambda^{J}_{d,n},$$

where $\mathbf{b}'' \models \mathbf{b}$ stands for $b_i = b''_i + b''_{n+1-i} + \delta_{i,r+1}$ for all i. For $\mathbf{a}'', \mathbf{b}'' \models 1^{2d+1}$, we set

$$\mathbf{T}^{\mathbf{a}''}_{d,n} = \mathcal{A}_{\mathrm{G}_d}(X_d \times X_d(\mathbf{a}'')) \quad \text{and} \quad {}^{\mathbf{b}''}\mathbf{H}_{A_d} = \mathcal{A}_{\mathrm{G}_d}(X_d(\mathbf{b}'') \times X_d(1^d)).$$

Let $\zeta_{d,\mathbf{b},\mathbf{v}}$ be the composition of $\zeta'_{\mathbf{b},\mathbf{v}}$ with the projection to the components of $\mathbf{a}'' = 1^d 0^{d+1}$:

$$\zeta_{d,\mathbf{b},\mathbf{v}} : \mathcal{A}_{\mathrm{G}^{J}_{d}}(X^{J}_{d}(\mathbf{b}) \times X^{J}_{d}(1^{2d+1})) \to \oplus_{\mathbf{b}''\models\mathbf{b}} \mathcal{A}_{\mathrm{G}_d}(X_d(\mathbf{b}'') \times X_d(1^d)),$$

where we identify $X_d(1^d 0^{d+1})$ with $X_d(1^d)$. Summing over all $\mathbf{b} \in \Lambda^{J}_{d,n}$, we get a linear map

$$\zeta_{d,\mathbf{v}} \equiv \oplus_{\mathbf{b}\in\Lambda_{d,n}} \zeta_{d,\mathbf{b},\mathbf{v}} : \mathbf{T}^{J}_{d,n} \to \mathbf{T}_{d,n}.$$

Take $n = 2d+1$, $\mathbf{b} = 1^{2d+1}$, we obtain a linear map

$$\zeta^{1}_{d,\mathbf{v}} : \mathbf{H}_{B_d} \to \oplus_{\mathbf{b}''\models 1^{2d+1}} {}^{\mathbf{b}''}\mathbf{H}_{A_d},$$

which is not necessarily an algebra homomorphism. Note that we identify $\mathbf{T}^{1^d 0^{d+1}}_{d,n}$ and ${}^{1^d 0^{d+1}}\mathbf{H}_{A_d}$ with $\mathbf{T}_{d,n}$ and $\mathbf{H}_{A_d}$, respectively. We thank W. Wang for pointing out a mistake in a previous version of the following proposition.

Proposition 3.5.1. We have the following commutative diagram relating the geometric type A duality with the geometric type B duality.

$$\begin{array}{ccccccc}
S^{J}_{d} \times T^{J}_{d,n} & \longrightarrow & T^{J}_{d,n} & \longleftarrow & T^{J}_{d,n} \times H_{B_d} & & \\
\downarrow{\scriptstyle \jmath_{d,v}\times\zeta_{d,v}} & & \downarrow{\scriptstyle \zeta_{d,v}} & & \downarrow{\scriptstyle \zeta_{d,v}\times\zeta^{1}_{d,v}} & & \\
S_d \times T_{d,n} & \longrightarrow & T_{d,n} & \underset{\psi_1}{\longleftarrow} & \oplus_{a''\models 1^{2d+1}} T^{a''}_{d,n} \times \oplus_{b''\models 1^{2d+1}} {}^{b''}H_{A_d} & \underset{\psi_2}{\longleftarrow} & T_{d,n} \times H_{A_d}.
\end{array}$$

where ψ_2 is the natural imbedding and $\psi_2\psi_1$ is the ψ in (34).

We can describe the linear map $\zeta_{d,\mathbf{v}}$ explicitly. Let $\Pi_{d,n}$ be the set of $n\times d$ matrices A such that $a_{ij}\in\{0,1\}$ and $\sum_{1\le i\le n}a_{ij}=1$ for all $1\le j\le d$. Then we have

$$\mathbf{T}_{d,n}=\mathrm{span}_{\mathcal{A}}\left\{ {}^{a}[A]|A\in\Pi_{d,n}\right\}, \tag{37}$$

where ${}^{a}[A]=v^{d_A}\zeta_A$ and $d_A=\sum_{i\ge k,j<l}a_{ij}a_{kl}$.

Let $\Pi^{J}_{d,n}$ be the subset of $\Pi_{2d+1,n}$ such that $a_{ij}=a_{n+1-i,2d+2-j}$ for all $1\le i\le n$ and $1\le j\le 2d+1$. (In particular, we have $a_{r+1,d+1}=1$.) We have

$$\mathbf{T}^{J}_{d,n}=\mathrm{span}_{\mathcal{A}}\{[A]|A\in\Pi^{J}_{d,n}\}, \tag{38}$$

where $[A]=v^{\ell_A}\zeta^{J}_A$ and $\ell_A=\frac{1}{2}\left(\sum_{i\ge j,k<l}a_{ij}a_{kl}-\sum_{i\ge n+1,d+1>j}a_{ij}\right)$.

Let J_m be the $m\times m$ matrix whose (i,j)-th component is $\delta_{i,n+1-j}$ for all $1\le i,j\le m$. To a matrix $A\in\Pi_{d,n}$, we define a matrix

$$A^J=(A|\epsilon_{r+1}|J_nAJ_d),$$

where ϵ_{r+1} is the column vector whose entries are zero except at $r+1$, which is 1. Then the assignment $A\mapsto A^J$ defines a bijection $\Pi_{d,n}\overset{\simeq}{\to}\Pi^{J}_{d,n}$.

Proposition 3.5.2. $\zeta_{d,\mathbf{v}}([A^J])={}^{a}[A]$, for all $A\in\Pi_{d,n}$.

Proof. Suppose that $\mathrm{ro}(A^J)=\mathbf{b}$. We set $\mathbf{a}''=1^d0^{d+1}$, $\mathbf{b}'=0^r1^10^r$, and $\mathbf{a}'=0^d1^10^d$. Then by the definition of $\zeta_{d,\mathbf{v}}$, we have

$$\zeta_{d,\mathbf{v}}([A^J])=\mathbf{v}^{t_{\mathbf{b}''}}\widetilde{\Delta}^{J}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}([A^J]), \tag{39}$$

where

$$t_{\mathbf{b}''}=\sum_{1\le i\le j\le n}b'_ib''_j-\sum_{1\le i\le j\le 2d+1}a'_ia''_i+\frac{1}{2}\left(\sum_{i+j\ge n+1}b''_ib''_j-\sum_{i\ge r+1}b''_i-\sum_{i+j\ge 2d+1}a''_ia''_j+\sum_{i\ge d+1}a''_i\right).$$

Note that the following formula $t_{\mathbf{b}''}$ is compatible with the twist in (32), since we need to rescale from n components to $2d+1$ components for $\mathbf{a}'$ and $\mathbf{a}''$. Now using the fact that

$\mathbf{a}'' = 1^d 0^{d+1}$, $\mathbf{b}' = 0^r 1^1 0^r$, and $\mathbf{a}' = 0^d 1^1 0^d$, the twist $t_{\mathbf{b}''}$ can be simplified to

$$t_{\mathbf{b}''} = \frac{1}{2}\left(\sum_{i+j\geq n+1} b''_i b''_j - \sum_{i\geq r+1} b''_i\right).$$

By the definition of A^J, we can also simplify the numeric ℓ_{A^J} as follows:

$$\begin{aligned} \ell_{A^J} &= \frac{1}{2}\left(\sum_{i\geq k, j<l} a^J_{ij} a^J_{kl} - \sum_{i\geq r+1, d+1>j} a^J_{ij}\right), \quad A^J = \left(a^J_{ij}\right) \\ &= \frac{1}{2}\left(\left(\sum_{i\geq k, j<l<d+1} + \sum_{i\geq k, j<d+1\leq l} + \sum_{i\geq k, d+1\leq j<l}\right) a^J_{ij} a^J_{kl} - \sum_{i\geq r+1, d+1>j} a_{ij}\right). \end{aligned} \tag{40}$$

The 1st sum simplifies to $\sum_{i\geq k, j<l<d+1} a_{ij} a_{kl}$. The 3rd sum simplifies to

$$\sum_{i\geq k, d+1\leq j<l} a^J_{ij} a^J_{kl} = \sum_{i\geq k, j<d+1\leq l} a_{n+1-i, 2d+2-j} a_{n+1-k, 2d+2-l} = \sum_{i\geq k, j<l<d+1} a_{ij} a_{kl} + \sum_{i\geq r+1, d+1>j} a_{ij}.$$

The 2nd sum is reduced to

$$\sum_{i\geq k, j<d+1\leq l} a^J_{ij} a^J_{kl} = \sum_{i+k\geq n+1, j, l<d+1} a_{ij} a_{kl} + \sum_{i\geq r+1, d+1>j} a_{ij} = 2t_{\mathrm{ro}(A)}.$$

So we get $t_{\mathbf{b}''} - \ell_{A^J} = -d_A + t_{\mathbf{b}''} - t_{\mathrm{ro}(A)}$. Thus, the identity (39) can be rewritten as

$$\zeta_{d,\mathbf{v}}([A^J]) = v^{-d_A + t_{\mathbf{b}''} - t_{\mathrm{ro}(A)}} \widetilde{\Delta}^J_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}(\zeta^J_{A^J}),$$

where $\zeta^J_{A^J}$ denote the characteristic function attached to the $\mathbf{G}^J_d$-orbit indexed by A^J.

Recall that $\mathbf{a}'' = 1^d 0^{d+1}$. This implies that for any $\tilde{L}'' \in X_d(\mathbf{a}'')$, we have $Z^J_{\tilde{L}', \tilde{L}''}$ consists of only one point, that is, the flag $\tilde{L}$ such that $\tilde{L}_i = \tilde{L}''_i$ for all $i \leq r$ and $\tilde{L}_i = (\tilde{L}''_{n+1-i})^{\perp}$ for all $i \geq r+1$. Furthermore, if $(L'', \tilde{L}'') \in \mathcal{O}_A$, then $(L, \tilde{L}) \in \mathcal{O}_{A^J}$ for any $L \in Z^J_{L', L''}$ and $\tilde{L} \in Z^J_{\tilde{L}', \tilde{L}''}$ because $L_i \cap \tilde{L}_j = L''_i \cap \tilde{L}''_j$, $\forall j \leq d$. Hence, we have $\widetilde{\Delta}^J_{\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}''}(\zeta^J_{A^J}) = \delta_{\mathbf{b}'', \mathrm{ro}(A)} \zeta_A$. The proposition is proved. ■

By Proposition 3.5.2, we have

Theorem 3.5.3. For all $A \in \Pi_{d,n}$, we have $\zeta_{d,\mathbf{v}}(\{A^J\}) = {}^a\{A\} + \sum_{B^J \prec A^J, \mathrm{ro}(B) \neq \mathrm{ro}(A)} c_{B,A}{}^a\{B\}$ where $c_{B,A} \in \mathbb{Z}_{\geq 0}[\mathbf{v}, \mathbf{v}^{-1}]$.

Recall the parabolic Kazhdan–Lusztig polynomials P_{B^J,A^J} and $P_{B,A}$ of type B_d and A_d, respectively. If $\mathrm{ro}(B) = \mathrm{ro}(A)$, the remainder in the above theorem vanishes, so we have

Corollary 3.5.4. $P_{B^J,A^J} = P_{B,A}$ if $\mathrm{ro}(B) = \mathrm{ro}(A)$.

More generally, we have the following commutative diagram of algebras.

$$\begin{array}{ccc} H_{B_d} = \mathcal{A}_{\mathrm{G}^J_d}(X^J_d(1^{2d+1}) \times X^J_d(1^{2d+1})) & \xrightarrow{\zeta_{d,v}} & \oplus_{b'',a'' \models 1^{2d+1}} \mathcal{A}_{\mathrm{G}_d}(X_d(b'') \times X_d(a'')) \\ \downarrow & & \downarrow \\ S^J_{d,2d+1} & \xrightarrow[j_{d,v}]{} & S_{d,2d+1}. \end{array}$$

3.6 Transfer maps on $\mathbb{S}^J_d$

The transfer map

$$\phi^J_{d,d-n} : \mathbb{S}^J_d \to \mathbb{S}^J_{d-n}$$

is defined to be the composition $\mathbb{S}^J_d \xrightarrow{\widetilde{\Delta}^J} \mathbb{S}^J_{d-n} \otimes \mathbb{S}_n \xrightarrow{1\otimes\chi} \mathbb{S}^J_{d-n} \otimes \mathbb{A} \equiv \mathbb{S}^J_{d-n}$, where χ is in (13). It is clear that $\phi^J_{d,d-n}$ is an algebra homomorphism. Moreover, we have

Proposition 3.6.1. $\phi^J_{d,d-n}(\mathbf{e}_i) = \mathbf{e}'_i$, $\phi^J_{d,d-n}(\mathbf{f}_i) = \mathbf{f}'_i$, and $\phi^J_{d,d-n}(\mathbf{k}_i^{\pm 1}) = \mathbf{k}_i'^{\pm 1}$, $\forall i \in [1, r]$.

Proof. By definitions, we have $\chi(\mathbf{E}''_i) = 0$, $\chi(\mathbf{F}''_i) = 0$, and $\chi(\mathbf{H}''_i) = v$. So we have

$$\begin{aligned} \phi^J_{d,d-n}(\mathbf{e}_i) &= \mathbf{e}'_i \chi(\mathbf{H}''_{i+1}\mathbf{H}''^{-1}_{n-i}) + \mathbf{H}'^{-1}_{i+1}\chi(\mathbf{E}''_i\mathbf{H}''^{-1}_{n-i}) + \mathbf{H}'_{i+1}\chi(\mathbf{F}''_{n-i}\mathbf{H}''_{i+1}) = \mathbf{e}'_i, \\ \phi^J_{d,d-n}(\mathbf{f}_i) &= \mathbf{f}'_i \chi(\mathbf{H}''^{-1}_i\mathbf{H}''_{n+1-i}) + \mathbf{H}'_i\chi(\mathbf{F}''_i\mathbf{H}''_{n+1-i}) + \mathbf{H}'^{-1}_i\chi(\mathbf{E}''_{n-i}\mathbf{H}''^{-1}_i) = \mathbf{f}'_i, \\ \phi^J_{d,d-n}(\mathbf{k}_i) &= \mathbf{K}'_i\chi(\mathbf{K}''_i\mathbf{K}''^{-1}_{n+1-i}) = \mathbf{k}'_i. \end{aligned}$$

The lemma is proved. ∎

Together with [1, Theorem 3.10], we have

Corollary 3.6.2. The homomorphism $\phi^J_{d,d-n}$ is surjective.

4 Positivity for the Modified Coideal Subalgebra $\dot{\mathbb{U}}^\jmath$

4.1 The coideal subalgebra $\mathbb{U}^\jmath$

By definition, $\mathbb{U}^\jmath \equiv \mathbb{U}^\jmath(\mathfrak{sl}_n)$ is an associative algebra over $\mathbb{Q}(v)$ generated by $\mathbb{e}_i$, $\mathbb{f}_i$, $\mathbb{k}_{\pm i}$ for $1 \leq i \leq r$ and subject to the following defining relations. For any $1 \leq i,j \leq r$ and $a_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$,

$$\begin{aligned}
\mathbb{k}_i\mathbb{k}_j &= \mathbb{k}_j\mathbb{k}_i, \\
\mathbb{k}_i\mathbb{k}_i^{-1} &= \mathbb{k}_i^{-1}\mathbb{K}_i = 1, \\
\mathbb{k}_i\mathbb{e}_j &= v^{a_{ij}+\delta_{i,r}\delta_{j,r}}\mathbb{e}_j\mathbb{k}_i, \\
\mathbb{k}_i\mathbb{f}_j &= v^{-a_{ij}-\delta_{i,r}\delta_{j,r}}\mathbb{f}_j\mathbb{k}_i, \\
\mathbb{e}_i\mathbb{f}_j - \mathbb{f}_j\mathbb{e}_i &= \delta_{i,j}\frac{\mathbb{k}_i - \mathbb{k}_i^{-1}}{v - v^{-1}}, && \text{if} \quad (i,j) \neq (r,r), \\
\mathbb{e}_r^2\mathbb{f}_r + \mathbb{f}_r\mathbb{e}_r^2 &= (v+v^{-1})(\mathbb{e}_r\mathbb{f}_r\mathbb{e}_r - \mathbb{e}_r(v\mathbb{k}_r + v^{-1}\mathbb{k}_r^{-1})), \\
\mathbb{f}_r^2\mathbb{e}_r + \mathbb{e}_r\mathbb{f}_r^2 &= (v+v^{-1})(\mathbb{f}_r\mathbb{e}_r\mathbb{f}_r - (v\mathbb{k}_r + v^{-1}\mathbb{k}_r^{-1})\mathbb{f}_r), \\
\mathbb{e}_i\mathbb{e}_j &= \mathbb{e}_j\mathbb{e}_i, && \text{if } |i-j| > 1, \\
\mathbb{f}_i\mathbb{f}_j &= \mathbb{f}_j\mathbb{f}_i, && \text{if } |i-j| > 1, \\
\mathbb{e}_i^2\mathbb{e}_j + \mathbb{e}_j\mathbb{e}_i^2 &= (v+v^{-1})\mathbb{e}_i\mathbb{e}_j\mathbb{e}_i, && \text{if } |i-j| = 1, \\
\mathbb{f}_i^2\mathbb{f}_j + \mathbb{f}_j\mathbb{f}_i^2 &= (v+v^{-1})\mathbb{f}_i\mathbb{f}_j\mathbb{f}_i, && \text{if } |i-j| = 1.
\end{aligned}$$

Recall the algebra $\mathbb{U}$ from Section 2.5 from [1, Proposition 4.5], see also [2], we have an injective algebra homomorphism

$$\jmath : \mathbb{U}^\jmath \to \mathbb{U},$$

defined by

$$\jmath(\mathbb{e}_i) = \mathbb{E}_i + \mathbb{K}_i\mathbb{F}_{n-i}, \ \jmath(\mathbb{f}_i) = \mathbb{F}_i\mathbb{K}_{n-i} + \mathbb{E}_{n-i}, \ \jmath(\mathbb{k}_i) = v^{\delta_{i,r}}\mathbb{K}_i\mathbb{K}_{n-i}^{-1}, \quad \forall 1 \leq i \leq r.$$

Here $n = 2r+1$. By composing $\jmath$ with Δ in (16), we have an algebra homomorphism $\Delta^\jmath : \mathbb{U}^\jmath \to \mathbb{U}^\jmath \otimes \mathbb{U}$ defined by

$$\Delta^\jmath(\mathbb{e}_i) = \mathbb{e}_i \otimes \mathbb{K}_i + 1 \otimes \mathbb{E}_i + \mathbb{k}_i \otimes \mathbb{F}_{n-i}\mathbb{K}_i,$$

$$\Delta^\jmath(\mathbb{f}_i) = \mathbb{f}_i \otimes \mathbb{K}_{n-i} + \mathbb{k}_i^{-1} \otimes \mathbb{K}_{n-i}\mathbb{F}_i + 1 \otimes \mathbb{E}_{n-i},$$

$$\Delta^\jmath(\mathbb{k}_i) = \mathbb{k}_i \otimes \mathbb{K}_i\mathbb{K}_{n-i}^{-1}, \quad \forall 1 \leq i \leq r.$$

4.2 The algebra $\dot{\mathbb{U}}^\jmath$

On $\mathbb{Z}^n$, we define an equivalence relation "$\approx$" by $\mu \approx \lambda$ if and only if $\mu - \lambda = m(2, \cdots, 2)$ for some $m \in \mathbb{Z}$. Let $\widehat{\mu}$ denote the equivalence class of μ with respect to $\approx$. Consider the following subset in the set $\mathbb{Z}^n/\approx$ of equivalence classes.

$$\mathbb{X}^\jmath = \{\widehat{\mu} \in \mathbb{Z}^n/\approx \,|\mu_i = \mu_{n+1-i}, \forall 1 \le i \le n, \mu_{r+1} \text{ is odd}\}. \tag{41}$$

We define

$$\dot{\mathbb{U}}^\jmath = \oplus_{\widehat{\mu},\widehat{\lambda} \in \mathbb{X}^\jmath}\ {}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}},$$
$${}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}} = \mathbb{U}^\jmath / \left(\sum_{1 \le i \le r} (\mathbb{k}_i - v^{-\mu_i+\mu_{i+1}})\mathbb{U}^\jmath + \sum_{1 \le i \le r} \mathbb{U}^\jmath (\mathbb{k}_i - v^{-\lambda_i+\lambda_{i+1}}) \right).$$

The algebra $\dot{\mathbb{U}}^\jmath$ is the modified form of $\mathbb{U}^\jmath$ (see [1, 4.6] for the $\mathfrak{gl}_n$ version). Let

$$\pi_{\widehat{\mu},\widehat{\lambda}} : \mathbb{U}^\jmath \to {}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}}$$

be the natural projection.

Recall the set $\mathbb{X}^\jmath$ from (41) and s_i the i-standard basis element of $\mathbb{Z}^n$. We define an abelian group structure on $\mathbb{X}^\jmath$ by $\widehat{\mu} + \widehat{\lambda} = \widehat{\pi}$, with $\pi = \mu + \lambda - s_{r+1}$. We set

$$I^\jmath = \{1, \cdots, r\}.$$

The assignment $i \mapsto -s_i + s_{i+1} + s_{n-i} - s_{n+1-i} + s_{r+1}, \forall i \in I^\jmath$, defines an embedding of abelian groups $\mathbb{Z}[I^\jmath] \hookrightarrow \mathbb{X}^\jmath$. We shall identify elements in $\mathbb{Z}[I^\jmath]$ with their images in $\mathbb{X}^\jmath$. Then the algebra $\mathbb{U}^\jmath$ in Section 4.1 admits a $\mathbb{Z}[I^\jmath]$-graded decomposition $\mathbb{U}^\jmath = \oplus_{\omega \in \mathbb{Z}[I^\jmath]} \mathbb{U}^\jmath(\omega)$ defined by

$$\mathbb{e}_i \in \mathbb{U}^\jmath(i),\ \mathbb{f}_i \in \mathbb{U}^\jmath(-i),\ \mathbb{k}_i^{\pm 1} \in \mathbb{U}^\jmath(0),\ \mathbb{U}^\jmath(\omega)\mathbb{U}^\jmath(\omega') \subseteq \mathbb{U}^\jmath(\omega + \omega'), \quad \forall i \in I^\jmath.$$

Let ${}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}}(\omega) = \pi_{\widehat{\mu},\widehat{\lambda}}(\mathbb{U}^\jmath(\omega))$. By a standard argument, we have

Lemma 4.2.1. ${}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}}(\omega) = 0$ unless $\widehat{\mu} - \widehat{\lambda} = \omega \in \mathbb{Z}[I^\jmath] \subseteq \mathbb{X}^\jmath$.

4.3 Positivity with respect to $\mathbb{\Delta}^J$

We introduce the following notations to simplify the presentation. For any $\mu, \mu', \mu'' \in \mathbb{Z}^n$, we write

$$(\mu', \mu'') \vdash \mu, \tag{42}$$

if and only if $\mu'_i + \mu''_i + \mu''_{n+1-i} = \mu_i$, for all $1 \le i \le n$. If $\mu' = s_{r+1}$, we simply write $\mu'' \vdash \mu$.

Assume that $(\mu', \mu'') \vdash \mu$, then by definition we have

$$\begin{aligned} \mathbb{\Delta}^J(\Bbbk_i - v^{-\mu_i+\mu_{i+1}}) =& (\Bbbk_i - v^{-\mu'_i+\mu'_{i+1}}) \otimes \mathbb{K}_i\mathbb{K}^{-1}_{n-i} + v^{-\mu'_i+\mu'_{i+1}} \otimes (\mathbb{K}_i - v^{-\mu''_i+\mu''_{i+1}})\mathbb{K}^{-1}_{n-i} \\ &+ v^{-\mu'_i-\mu''_i+\mu'_{i+1}+\mu''_{i+1}} \otimes \left(\mathbb{K}^{-1}_{n-i} - v^{\mu''_{n-i}-\mu''_{n+1-i}}\right). \end{aligned}$$

This induces a unique linear map

$$\mathbb{\Delta}^J_{\widehat{\mu'},\widehat{\lambda'},\overline{\mu''},\overline{\lambda''}} : {}_{\widehat{\mu}}\mathbb{U}^J_{\widehat{\lambda}} \to {}_{\widehat{\mu'}}\mathbb{U}^J_{\widehat{\lambda'}} \otimes {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}, \quad \forall (\mu', \mu'') \vdash \mu, (\lambda', \lambda'') \vdash \lambda$$

such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{U}^J & \xrightarrow{\Delta^J} & \mathbb{U}^J \otimes \mathbb{U} \\ \downarrow & & \downarrow \\ {}_{\widehat{\mu}}\mathbb{U}^J_{\widehat{\lambda}} & \xrightarrow{\Delta^J_{\widehat{\mu'},\widehat{\lambda'},\overline{\mu''},\overline{\lambda''}}} & {}_{\widehat{\mu'}}\mathbb{U}^J_{\widehat{\lambda'}} \otimes {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}. \end{array} \tag{43}$$

Recall $\mathbb{B}$ is the canonical basis for $\dot{\mathbb{U}}$. Let $\mathbb{B}^J$ be the canonical basis for $\dot{\mathbb{U}}^J$ defined in [21].

Theorem 4.3.1. Let $a \in \mathbb{B}^J$. If $\mathbb{\Delta}^J_{\widehat{\mu'},\widehat{\lambda'},\overline{\mu''},\overline{\lambda''}}(a) = \sum_{b\in\mathbb{B}^J, c\in\mathbb{B}} n_a^{b,c} b \otimes c$, then $n_a^{b,c} \in \mathbb{Z}_{\ge 0}[v, v^{-1}]$.

The rest of this section is devoted to the proof of Theorem 4.3.1.

For $\omega, \omega' \in \mathbb{Z}[I^J]$ and $\nu \in \mathbb{Z}[I]$, we write

$$(\omega', \nu) \models \omega, \tag{44}$$

if and only if $\omega'_i + \nu_i - \nu_{n-i} = \omega_i$, for all $1 \le i \le r$. If $\omega' = 0$, we simply write $\nu \models \omega$. By the definition of $\mathbb{\Delta}^J$, we have

Lemma 4.3.2. For any $\omega \in \mathbb{Z}[I^J]$, $\mathbb{\Delta}^J(\mathbb{U}^J(\omega)) \subseteq \oplus_{(\omega',\nu)\models\omega}\mathbb{U}^J(\omega') \otimes \mathbb{U}(\nu)$.

The following lemma is a refinement of (43), and follows from Lemmas 4.3.2 and 4.2.1.

Lemma 4.3.3. Assume that $\widehat{\mu}, \widehat{\lambda}, \widehat{\mu'}, \widehat{\lambda'} \in \mathbb{X}^J$, $\omega, \omega' \in \mathbb{Z}[I^J]$, $\overline{\mu''}, \overline{\lambda''} \in \mathbb{X}$, and $\nu \in \mathbb{Z}[I]$ such that $\widehat{\mu} - \widehat{\lambda} = \omega, \widehat{\mu'} - \widehat{\lambda'} = \omega', \overline{\mu''} - \overline{\lambda''} = \nu, (\mu', \mu'') \vdash \mu, (\lambda', \lambda'') \vdash \lambda, (\omega', \nu) \models \omega$. The following diagram commutes.

$$\begin{array}{ccc}
\mathbb{U}^J(\omega) & \xrightarrow{\Delta^J_{\omega,\nu}} & \mathbb{U}^J(\omega') \otimes \mathbb{U}(\nu) \\
\pi_{\widehat{\mu},\widehat{\lambda}} \downarrow & & \downarrow \pi_{\widehat{\mu'},\widehat{\lambda'}} \otimes \pi_{\overline{\mu''},\overline{\lambda''}} \\
{}_{\widehat{\mu}}\mathbb{U}^J_{\widehat{\lambda}} & \xrightarrow{\Delta^J_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}} & {}_{\widehat{\mu'}}\mathbb{U}^J_{\widehat{\lambda'}} \otimes {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}.
\end{array}$$

The assignment of sending generators of $\mathbb{U}^J$ to the respective generators of ${}_{\mathbb{Q}(v)}\mathbb{S}^J_d$ defines an algebra homomorphism, denoted by

$$\phi^J_d : \mathbb{U}^J \to {}_{\mathbb{Q}(v)}\mathbb{S}^J_d. \tag{45}$$

Moreover, this algebra homomorphism is compatible with the gradings. In particular,

$$\phi^J_d(\mathbb{U}^J(\omega)) \subseteq \oplus_{\mathbf{b},\mathbf{a} \in \Lambda^J_{d,n} : \bar{\mathbf{b}} - \bar{\mathbf{a}} = \omega} \, {}_{\mathbb{Q}(v)}\mathbb{S}^J_d(\mathbf{b}, \mathbf{a}), \quad \forall \omega \in \mathbb{Z}[I^J]. \tag{46}$$

On the other hand, we have

$$\Delta^J({}_{\mathbb{Q}(v)}\mathbb{S}^J_d(\mathbf{b}, \mathbf{a})) \subseteq \bigoplus_{\substack{(\mathbf{b}',\mathbf{b}'') \vdash \mathbf{b} \\ (\mathbf{a}',\mathbf{a}'') \vdash \mathbf{a}}} {}_{\mathbb{Q}(v)}\mathbb{S}^J_{d'}(\mathbf{b}', \mathbf{a}') \otimes {}_{\mathbb{Q}(v)}\mathbb{S}_{d''}(\mathbf{b}'', \mathbf{a}''), \tag{47}$$

where $d = d' + d''$. By Lemma 4.3.2, (46), and (47), we have the following lemma.

Lemma 4.3.4. Assume that $\mathbf{b}, \mathbf{a} \in \Lambda^J_{d,n}$, $\omega, \omega' \in \mathbb{Z}[I^J]$, $\nu \in \mathbb{Z}[I]$ such that $(\omega', \nu) \models \omega$. The following diagram commutes.

$$\begin{array}{ccc}
\mathbb{U}^J(\omega) & \xrightarrow{\Delta^J_{\omega',\nu}} & \mathbb{U}^J(\omega') \otimes \mathbb{U}(\nu) \\
\phi^J_d \downarrow & & \downarrow \phi^J_{d'} \otimes \phi_{d''} \\
\oplus_{\substack{b,a \in \Lambda^J_{d,n} \\ \widehat{b} - \widehat{a} = \omega}} {}_{\mathbb{Q}(v)}\mathbb{S}^J_d(b, a) & \xrightarrow{\Delta^J} & \oplus_{\substack{(b',b'') \vdash b, (a',a'') \vdash a \\ \widehat{b'} - \widehat{a'} = \omega', \overline{b''} - \overline{a''} = \nu}} {}_{\mathbb{Q}(v)}\mathbb{S}^J_{d'}(b', a') \otimes {}_{\mathbb{Q}(v)}\mathbb{S}_{d''}(b'', a'').
\end{array}$$

Recall from [21] that we have an algebra homomorphism

$$\widetilde{\phi}^J_d : \dot{\mathbb{U}}^J \to {}_{\mathbb{Q}(v)}\mathbb{S}^J_d \tag{48}$$

defined by

$$\widetilde{\phi}^J_d(1_{\widehat{\lambda}}) = \begin{cases} \zeta^J_{M_{\mathbf{a}}}, & \text{if } \widehat{\lambda} = \widehat{\mathbf{a}},\ \mathbf{a} \in \Lambda^J_{d,n}, \\ 0, & \text{o.w.} \end{cases}$$
$$\widetilde{\phi}^J_d(\mathbb{e}_i 1_{\widehat{\lambda}}) = \begin{cases} \mathbf{e}_i\zeta^J_{M_{\mathbf{a}}}, & \text{if } \widehat{\lambda} = \widehat{\mathbf{a}},\ \mathbf{a} \in \Lambda^J_{d,n}, \\ 0, & \text{o.w.} \end{cases} \quad \widetilde{\phi}^J_d(\mathbb{f}_i 1_{\widehat{\lambda}}) = \begin{cases} \mathbf{F}_i\zeta^J_{M_{\mathbf{a}}}, & \text{if } \widehat{\lambda} = \widehat{\mathbf{a}},\ \mathbf{a} \in \Lambda^J_{d,n}, \\ 0, & \text{o.w.} \end{cases} \tag{49}$$

By restricting to ${}_{\widehat{\mu}}\mathbb{U}^J_{\widehat{\lambda}}$, it induces a linear map

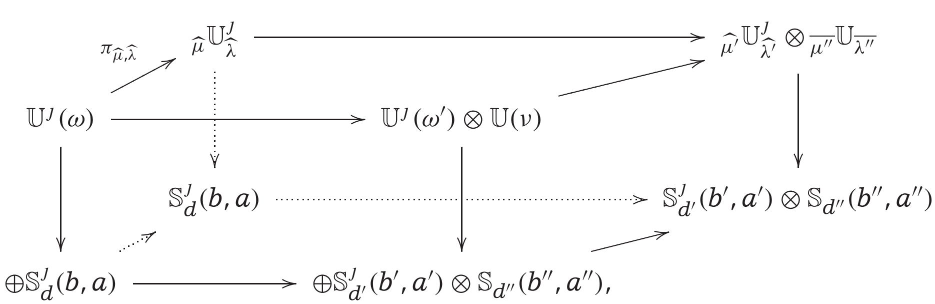
$$\widetilde{\phi}^J_d : {}_{\widehat{\mu}}\mathbb{U}^J_{\widehat{\lambda}} \to {}_{\mathbb{Q}(v)}\mathbb{S}^J_d(\mathbf{b}, \mathbf{a}), \text{ if } \widehat{\mu} = \widehat{\mathbf{b}}, \widehat{\lambda} = \widehat{\mathbf{a}}.$$

In particular, we have the following lemma.

Lemma 4.3.5. Suppose $\mathbf{b}^0, \mathbf{a}^0 \in \Lambda^J_{d,n}$ satisfy $\widehat{\mathbf{b}^0} - \widehat{\mathbf{a}^0} = \omega \in \mathbb{Z}[I^J]$. The following diagram is commutative, where the arrow in the bottom is the natural projection.

$$\begin{array}{ccc} \mathbb{U}^J(\omega) & \xrightarrow{\pi_{\widehat{b^0},\widehat{a^0}}} & {}_{\widehat{b^0}}\mathbb{U}^J_{\widehat{a^0}} \\ {\scriptstyle \widetilde{\phi}^J_d}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\phi}^J_d} \\ \oplus_{\widehat{b}-\widehat{a}=\omega}\ {}_{\mathbb{Q}(v)}\mathbb{S}^J_d(b,a) & \longrightarrow & {}_{\mathbb{Q}(v)}\mathbb{S}^J_d(b^0,a^0). \end{array} \tag{50}$$

By putting together Lemmas 4.3.4, 4.3.3, and (50), we have the following cube.



(51)

where the sum on the bottom left is overall $\mathbf{b}, \mathbf{a} \in \Lambda^\jmath_{d,n}$ such that $\widehat{\mathbf{b}} - \widehat{\mathbf{a}} = \omega$, while the sum on the bottom right is over $\mathbf{b}', \mathbf{a}' \in \Lambda^\jmath_{d',n}$ and $\mathbf{b}'', \mathbf{a}'' \in \Lambda_{d'',n}$ such that $(\mathbf{b}', \mathbf{b}'') \vdash \mathbf{b}$, $(\mathbf{a}', \mathbf{a}'') \vdash \mathbf{a}$, $\widehat{\mathbf{b}'} - \widehat{\mathbf{a}'} = \omega$, and $\overline{\mathbf{b}''} - \overline{\mathbf{a}''} = \nu$.

From (51) and the surjectivity of $\pi_{\widehat{\mu},\widehat{\lambda}}$, we have the following proposition.

Proposition 4.3.6. The square in the back of (51) is commutative.

$$\begin{array}{ccc} {}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}} & \longrightarrow & {}_{\widehat{\mu'}}\mathbb{U}^\jmath_{\widehat{\lambda'}} \otimes {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}} \\ \downarrow & & \downarrow \\ \mathbb{S}^\jmath_d(b,a) & \longrightarrow & \mathbb{S}^\jmath_{d'}(b',a') \otimes \mathbb{S}_{d''}(b'',a''). \end{array}$$

By using Proposition 4.3.6, we deduce Theorem 4.3.1 via a similar way for Theorem 2.5.5.

4.4 Positivity with respect to $\jmath$

Notice that for $\mu, \mu'' \in \mathbb{Z}^n$ such that $\mu'' \vdash \mu$, we have

$$\jmath((\Bbbk_i - v^{-\mu_i+\mu_{i+1}})\mathbb{U}^\jmath) \subseteq \sum_{1\le i\le n} (\mathbb{K}_i - v^{-\mu''_i+\mu''_{i+1}})\mathbb{U}.$$

Similarly, for any $\lambda, \lambda'' \in \mathbb{Z}^n$ such that $\lambda'' \vdash \lambda$ we have

$$\jmath(\mathbb{U}^\jmath(\Bbbk_i - v^{-\lambda_i+\lambda_{i+1}})) \subseteq \sum_{1\le i\le n} \mathbb{U}(\mathbb{K}_i - v^{-\lambda''_i+\lambda''_{i+1}}).$$

The above observations induce a linear map

$$\jmath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}} : {}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}} \longrightarrow {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}, \quad \forall \mu'' \vdash \mu, \lambda'' \vdash \lambda \tag{52}$$

such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{U}^\jmath & \xrightarrow{\ \jmath\ } & \mathbb{U} \\ {\scriptstyle \pi_{\widehat{\mu},\widehat{\lambda}}}\downarrow & & \downarrow{\scriptstyle \pi_{\overline{\mu''},\overline{\lambda''}}} \\ {}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}} & \xrightarrow{\jmath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}} & {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}. \end{array} \tag{53}$$

Theorem 4.4.1. Let $b \in \mathbb{B}^\jmath$. If $\jmath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}(b) = \sum_{a\in\mathbb{B}} g_{b,a} a$, then $g_{b,a} \in \mathbb{Z}_{\ge 0}[v, v^{-1}]$.

The proof of Theorem 4.4.1 is a degenerate version of the proof of Theorem 4.3.1. For the sake of completeness, we provide it here. The following lemma is due to the fact that

$$\jmath(\mathbb{e}_i) \in \mathbb{U}(i) + \mathbb{U}(-(n-i)),\ \jmath(\mathbb{f}_i) \in \mathbb{U}(-i) + \mathbb{U}(n-i),\ \jmath(\mathbb{k}_i^{\pm 1}) \in \mathbb{U}(0).$$

Lemma 4.4.2. For any $\omega \in \mathbb{Z}[I^\jmath]$, $\jmath(\mathbb{U}^\jmath(\omega)) \subseteq \oplus_{\nu \models \omega} \mathbb{U}(\nu)$.

From Lemmas 4.4.2 and 4.2.1, we have the following refinement of (53).

Lemma 4.4.3. Assume that $\widehat{\mu}, \widehat{\lambda} \in \mathbb{X}^\jmath$, $\omega \in \mathbb{Z}[I^\jmath]$, $\overline{\mu''}, \overline{\lambda''} \in \mathbb{X}$, and $\nu \in \mathbb{Z}[I]$ such that

$$\widehat{\mu} - \widehat{\lambda} = \omega, \overline{\mu''} - \overline{\lambda''} = \nu, \mu'' \vdash \mu, \lambda'' \vdash \lambda, \nu \models \omega.$$

The following diagram commutes.

$$\begin{array}{ccc}
\mathbb{U}^\jmath(\omega) & \xrightarrow{\jmath_{\omega,\nu}} & \mathbb{U}(\nu) \\
{\scriptstyle \pi_{\widehat{\mu},\widehat{\lambda}}}\big\downarrow & & \big\downarrow {\scriptstyle \pi_{\overline{\mu''},\overline{\lambda''}}} \\
{}_{\widehat{\mu}}\mathbb{U}^\jmath_{\widehat{\lambda}} & \xrightarrow{\jmath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}} & {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}},
\end{array}$$

where $\jmath_{\omega,\nu}$ is the one induced from $\jmath$ by restricting to $\mathbb{U}^\jmath(\omega)$ and projecting down to $\mathbb{U}(\nu)$.

Note that we have

$$J_d({}_{\mathbb{Q}(v)}\mathbb{S}^\jmath_d(\mathbf{b}, \mathbf{a})) \subseteq \bigoplus_{\mathbf{b}'' \vdash \mathbf{b}, \mathbf{a}'' \vdash \mathbf{a}} {}_{\mathbb{Q}(v)}\mathbb{S}_d(\mathbf{b}'', \mathbf{a}''). \tag{54}$$

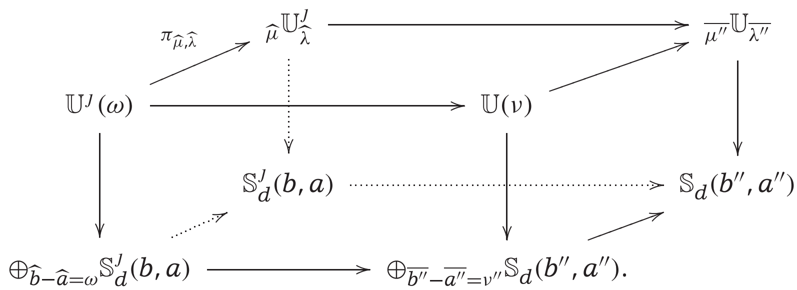
By Lemma 4.4.2, (46), and (54), we have the following lemma.

Lemma 4.4.4. Assume that $\mathbf{b}, \mathbf{a} \in \Lambda^\jmath_{d,n}$, $\omega \in \mathbb{Z}[I^\jmath]$, $\nu \in \mathbb{Z}[I]$ such that $\nu_i - \nu_{n-i} = \omega_i$ for any $i \in I^\jmath$. The following diagram commutes.

$$\begin{array}{ccc}
\mathbb{U}^\jmath(\omega) & \xrightarrow{\jmath_{\omega,\nu}} & \mathbb{U}(\nu) \\
{\scriptstyle \phi^\jmath_d}\big\downarrow & & \big\downarrow {\scriptstyle \phi_d} \\
\oplus_{\substack{b,a \in \Lambda^\jmath_{d,n} \\ \widehat{b}-\widehat{a}=\omega}} {}_{\mathbb{Q}(v)}\mathbb{S}^\jmath_d(b,a) & \xrightarrow{J_d} & \oplus_{b'',a'' \in \Lambda_{d,n}:(\star)} {}_{\mathbb{Q}(v)}\mathbb{S}_d(b'',a''),
\end{array}$$

where the condition ($\star$) is $\overline{\mathbf{b}''} - \overline{\mathbf{a}''} = \nu$, $\mathbf{b}'' \vdash \mathbf{b}$, and $\mathbf{a}'' \vdash \mathbf{a}$.

By putting together Lemmas 4.4.4, 4.4.3, and (50), we have the following cube.



(55)

From (55) and the surjectivity of $\pi_{\widehat{\mu},\widehat{\lambda}}$, we have the following proposition.

Proposition 4.4.5. The square in the back of (55) is commutative.

$$\begin{array}{ccc} {}_{\widehat{\mu}}\mathbb{U}^J_{\widehat{\lambda}} & \xrightarrow{\jmath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}} & {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}} \\ \downarrow & & \downarrow \\ \mathbb{S}^J_d(b,a) & \xrightarrow{\jmath_d} & \mathbb{S}_d(b'',a''). \end{array}$$

Recall that for any $b \in \mathbb{B}^J$, we suppose that

$$\jmath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}(b) = \sum_{a\in\mathbb{B}} g_{b,a} a,$$

where $g_{b,a} \in \mathbb{Z}[v, v^{-1}]$ is zero except for finitely many terms. Let $S = \{a | g_{b,a} \neq 0\}$. Since the set S is finite, we can find a large enough d using [21] and [25] such that

$$\phi^J_d(b) = \{B\}_d, \quad \phi_d(a) = \{A\}_d, \quad \forall a \in S,$$

where $\{B\}_d$ and $\{A\}_d$ are certain canonical basis elements in $\mathbb{S}^J_d$ and $\mathbb{S}_d$, respectively. Applying ϕ^J_d and Lemma 4.4.5, we have

$$\jmath_d(\{B\}_d) = \jmath_d(\phi^J_d(b)) = \phi_d \jmath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}(b) = \sum_{a\in\mathbb{B}} g_{b,a}\phi_d(a) = \sum_{a\in\mathbb{B}} g_{b,a}\{A\}_d.$$

By comparing the above with Corollary 3.4.3, we have $g_{b,a} = g_{B,A}$. So we have $g_{b,a} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$ by Corollary 3.4.3. Theorem 4.4.1 follows.

4.5 The imbedding $\tilde{\jmath}$

For any pair $(\widehat{\mu}, \widehat{\lambda})$ in $\mathbb{X}^{\jmath}$, we define

$$J_{\widehat{\mu},\widehat{\lambda}} \equiv \prod J_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}} : {}_{\widehat{\mu}}\mathbb{U}^{\jmath}_{\widehat{\lambda}} \longrightarrow \prod {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}},$$

where the product runs over all $\overline{\mu''}, \overline{\lambda''}$ in $\mathbb{X}$ such that $\mu'' \vdash \mu$ and $\lambda'' \vdash \lambda$. We set

$$\tilde{\jmath} \equiv \bigoplus_{\widehat{\mu},\widehat{\lambda} \in \mathbb{X}^{\jmath}} J_{\widehat{\mu},\widehat{\lambda}} : \dot{\mathbb{U}}^{\jmath} \longrightarrow \bigoplus_{\widehat{\mu},\widehat{\lambda} \in \mathbb{X}^{\jmath}} \prod {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}.$$

Proposition 4.5.1. The map $\tilde{\jmath}$ is injective.

Proof. It suffices to show that for any nonzero element x in ${}_{\widehat{\mu}}\mathbb{U}^{\jmath}_{\widehat{\lambda}}$, there is $\overline{\mu''}$ and $\overline{\lambda''}$ such that $J_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}(x)$ is nonzero. Suppose that $\widehat{\mu} - \widehat{\lambda} = \omega$. Let us pick an element $u \in \mathbb{U}^{\jmath}(\omega)$ such that $\pi_{\widehat{\mu},\widehat{\lambda}}(u) = x$. Since $\jmath$ is injective, we have $\jmath(u) \neq 0 \in \oplus_{\nu \in \mathbb{Z}[I]} \mathbb{U}(\nu)$. Thus, there is ν such that the ν-component $\jmath(u)_\nu$ of $\jmath(u)$ is nonzero. It is well known (see [24]) that we can then find a large enough d such that $\phi_d(\jmath(u)_\nu) \neq 0$. In particular, there is a pair $\mathbf{b}''$, $\mathbf{a}''$ in $\Lambda_{d,n}$ such that the $(\mathbf{b}'', \mathbf{a}'')$-component of $\phi_d(\jmath(u)_\nu)$ is nonzero. Take $\overline{\mu''} = \overline{\mathbf{b}''}$ and $\overline{\lambda''} = \overline{\mathbf{a}''}$. By chasing along the cube (55), we see immediately that $J_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}(x) \neq 0$. ■

Remark 4.5.2. $\tilde{\jmath}$ can be regarded as an idempotented version of $\jmath$.

5 $\imath$-Version

In this section, we show the positivity of the i-canonical basis of the modified coideal subalgebra of quantum $\mathfrak{sl}_\ell$ for ℓ even. Since the arguments are more or less the same as the n odd situation, the presentation will be brief.

5.1 $\imath$-Schur algebras and related results

Recall $n = 2r + 1$ and $D = 2d + 1$. We set

$$\ell = n - 1.$$

Recall $\Xi^{\jmath}_d$ from (27). Let $\Xi^{\imath}_d = \{A \in \Xi^{\jmath}_d | a_{r+1,j} = \delta_{j,r+1}, a_{i,r+1} = \delta_{i,r+1}\}$. Let $\mathbf{j} = \sum [A]_d$ where the sum runs over all diagonal matrices in $\Xi^{\imath}_d$. Let $\mathbb{S}^{\imath}_{d,\ell} = \mathbf{j}\mathbb{S}^{\jmath}_{d,n}\mathbf{j}$. It is a subalgebra in $\mathbb{S}^{\jmath}_{d,n}$ and admits a basis $[A]_d$ for all $A \in \Xi^{\imath}_d$. In particular, $\mathbb{S}^{\imath}_{d,\ell}$ contains the following.

$$\check{\mathbf{e}}_{i,d} = \mathbf{j}\mathbf{e}_i\mathbf{j}, \quad \check{\mathbf{f}}_{i,d} = \mathbf{j}\mathbf{f}_i\mathbf{j}, \quad \check{\mathbf{k}}_{i,d} = \mathbf{j}\mathbf{k}_i\mathbf{j}, \quad \forall i \in [1, r-1], \quad \check{\mathbf{h}}_{a,d} = \mathbf{j}\mathbf{h}_a\mathbf{j}, \quad \forall a \in [1, r], \check{\mathbf{t}}_d = \mathbf{j}\left(\mathbf{f}_r\mathbf{e}_r + \frac{\mathbf{k}_r - \mathbf{k}_r^{-1}}{v - v^{-1}}\right)\mathbf{j}. \tag{56}$$

Similarly, we consider the subset $\Xi_{d,\ell}$ of Ξ_d defined by the condition $a_{r+1,j} = 0$ and $a_{i,r+1} = 0$ for all i, j. Let $\mathbf{J} = \sum [A]_d$ where the sum runs over all diagonal matrices $A \in \Xi_{d,\ell}$. Then $\mathbb{S}_{d,\ell} = \mathbf{J}\mathbb{S}_d\mathbf{J}$ is a subalgebra of $\mathbb{S}_d$ with a basis $[A]_d$ indexed by $\Xi_{d,\ell}$. $\mathbb{S}_{d,\ell}$ contains the following.

$$\check{\mathbf{E}}_{i,d} = \begin{cases} \mathbf{J}\mathbf{E}_i\mathbf{J}, & \text{if } i \in [1, r-1], \\ \mathbf{J}\mathbf{E}_{r+1}\mathbf{E}_r\mathbf{J}, & \text{if } i = r, \\ \mathbf{J}\mathbf{E}_{i+1}\mathbf{J}, & \text{if } i \in [r+1, \ell-1]. \end{cases} \qquad \check{\mathbf{F}}_{i,d} = \begin{cases} \mathbf{J}\mathbf{F}_i\mathbf{J}, & \text{if } i \in [1, r-1], \\ \mathbf{J}\mathbf{F}_r\mathbf{F}_{r+1}\mathbf{J}, & \text{if } i = r, \\ \mathbf{J}\mathbf{F}_{i+1}\mathbf{J}, & \text{if } i \in [r+1, \ell-1]. \end{cases}$$

$$\check{\mathbf{K}}_{i,d} = \begin{cases} \mathbf{J}\mathbf{K}_i\mathbf{J}, & \text{if } i \in [1, r-1], \\ \mathbf{J}\mathbf{K}_r\mathbf{K}_{r+1}\mathbf{J}, & \text{if } i = r, \\ \mathbf{J}\mathbf{K}_{i+1}\mathbf{J}, & \text{if } i \in [r+1, \ell-1]. \end{cases} \qquad \check{\mathbf{H}}_{a,d} = \begin{cases} \mathbf{J}\mathbf{H}_a\mathbf{J}, & \text{if } a \in [1, r], \\ \mathbf{J}\mathbf{H}_{a+1}\mathbf{J}, & \text{if } a \in [r+1, \ell]. \end{cases}$$

Notice that we have $\mathbf{J}\mathbf{H}_{r+1}\mathbf{J} = 1$ and $\check{\mathbf{K}}_{r,d} = \check{\mathbf{H}}^{-1}_{r,d}\check{\mathbf{H}}_{r+1,d}$. Denote by $\widetilde{\mathbb{\Delta}}^{\jmath}$ the generic version of $\widetilde{\Delta}^{\jmath}$, that is, the unique map such that $\mathcal{A} \otimes_{\mathbb{A}} \widetilde{\mathbb{\Delta}}^{\jmath} = \widetilde{\Delta}^{\jmath}$.

Lemma 5.1.1. Let $d' + d'' = d$. $\widetilde{\mathbb{\Delta}}^{\jmath}(\mathbb{S}^{\imath}_{d,\ell}) \subseteq \mathbb{S}^{\imath}_{d',\ell} \otimes \mathbb{S}_{d'',\ell}$. Moreover, for all $i \in [1, r-1]$

$$\begin{aligned}
\widetilde{\mathbb{\Delta}}^{\jmath}(\check{\mathbf{e}}_{i,d}) &= \check{\mathbf{e}}_{i,d'} \otimes \check{\mathbf{H}}_{i+1,d''}\check{\mathbf{H}}^{-1}_{\ell-i,d''} + \check{\mathbf{h}}^{-1}_{i+1,d'} \otimes \check{\mathbf{E}}_{i,d''}\check{\mathbf{H}}^{-1}_{\ell-i,d''} + \check{\mathbf{h}}_{i+1,d'} \otimes \check{\mathbf{F}}_{\ell-i,d''}\check{\mathbf{H}}_{i+1,d''}.\\
\widetilde{\mathbb{\Delta}}^{\jmath}(\check{\mathbf{f}}_{i,d}) &= \check{\mathbf{f}}_{i,d'} \otimes \check{\mathbf{H}}^{-1}_{i,d''}\check{\mathbf{H}}_{\ell+1-i,d''} + \check{\mathbf{h}}_{i,d'} \otimes \check{\mathbf{F}}_{i,d''}\check{\mathbf{H}}_{\ell+1-i,d''} + \check{\mathbf{h}}^{-1}_{i,d'} \otimes \check{\mathbf{E}}_{\ell-i,d''}\check{\mathbf{H}}^{-1}_{i,d''}.\\
\widetilde{\mathbb{\Delta}}^{\jmath}(\check{\mathbf{k}}_{i,d}) &= \check{\mathbf{k}}_{i,d'} \otimes \check{\mathbf{K}}_{i,d''}\check{\mathbf{K}}^{-1}_{\ell-i,d''}.\\
\widetilde{\mathbb{\Delta}}^{\jmath}(\check{\mathbf{t}}_{d}) &= \check{\mathbf{t}}_{d'} \otimes \check{\mathbf{K}}_{r,d''} + v^2\check{\mathbf{k}}^{-1}_{r,d'} \otimes \check{\mathbf{H}}_{r+1,d''}\check{\mathbf{F}}_{r,d''} + v^{-2}\check{\mathbf{k}}_{r,d'} \otimes \check{\mathbf{H}}^{-1}_{r,d''}\check{\mathbf{E}}_{r,d''}.
\end{aligned} \tag{57}$$

Proof. For convenience, we shall drop the subscript d and replace d', d'' by superscript $'$ and $''$ respectively in the proof. The 1st three equalities are from definitions and

$\widetilde{\Delta}^J(\mathbf{j}) = \mathbf{j}' \otimes \mathbf{J}''$. We now show the last one. By using $\mathbf{j}\mathbf{f}_r\mathbf{j} = 0$ and $\mathbf{j}\mathbf{e}_r\mathbf{j} = 0$, we have

$$\begin{aligned}\widetilde{\Delta}^J(\mathbf{j}\mathbf{f}_r\mathbf{e}_r\mathbf{j}) = {} & \mathbf{j}\mathbf{f}_r\mathbf{e}_r\mathbf{j} \otimes \mathbf{J}\mathbf{H}_r^{-1}\mathbf{H}_{r+2}\mathbf{J} + \mathbf{j}\mathbf{h}_r\mathbf{h}_{r+1}^{-1}\mathbf{j} \otimes \mathbf{J}\mathbf{F}_r\mathbf{H}_{r+2}\mathbf{E}_r\mathbf{H}_{r+1}^{-1}\mathbf{J}\\ & + \mathbf{j}\mathbf{h}_r\mathbf{h}_{r+1}\mathbf{j} \otimes \mathbf{J}\mathbf{F}_r\mathbf{H}_{r+2}\mathbf{F}_{r+1}\mathbf{H}_{r+1}\mathbf{J} + \mathbf{j}\mathbf{h}_r^{-1}\mathbf{h}_{r+1}\mathbf{j} \otimes \mathbf{J}\mathbf{E}_{r+1}\mathbf{H}_r^{-1}\mathbf{F}_{r+1}\mathbf{H}_{r+1}\mathbf{J}\\ & + \mathbf{j}\mathbf{h}_r^{-1}\mathbf{h}_{r+1}^{-1}\mathbf{j} \otimes \mathbf{J}\mathbf{E}_{r+1}\mathbf{H}_r^{-1}\mathbf{E}_r\mathbf{H}_{r+1}^{-1}\mathbf{J}.\end{aligned}$$

We observe that $\mathbf{j}\mathbf{h}_r^{-1}\mathbf{h}_{r+1}\mathbf{j} = \check{\mathbf{k}}_r$ and $\mathbf{j}\mathbf{h}_r\mathbf{h}_{r+1}\mathbf{j} = v^2\check{\mathbf{k}}_r^{-1}$. We further observe that

$$\begin{aligned}\mathbf{J}\mathbf{F}_r\mathbf{H}_{r+2}\mathbf{E}_r\mathbf{H}_{r+1}^{-1}\mathbf{J} &= \check{\mathbf{H}}_{r+1}\frac{\check{\mathbf{H}}_r - \check{\mathbf{H}}_r^{-1}}{v - v^{-1}}, \mathbf{J}\mathbf{F}_r\mathbf{H}_{r+2}\mathbf{F}_{r+1}\mathbf{H}_{r+1}\mathbf{J}\\ &= \check{\mathbf{H}}_{r+1}\check{\mathbf{F}}_r, \mathbf{J}\mathbf{E}_{r+1}\mathbf{H}_r^{-1}\mathbf{F}_{r+1}\mathbf{H}_{r+1}\mathbf{J}\\ &= \check{\mathbf{H}}_r^{-1}\frac{\check{\mathbf{H}}_{r+1} - \check{\mathbf{H}}_{r+1}^{-1}}{v - v^{-1}}, \mathbf{J}\mathbf{E}_{r+1}\mathbf{H}_r^{-1}\mathbf{E}_r\mathbf{H}_{r+1}^{-1}\mathbf{J} = \check{\mathbf{H}}_r^{-1}\check{\mathbf{E}}_r.\end{aligned}$$

So we have

$$\widetilde{\Delta}^J(\check{\mathbf{t}}) = \check{\mathbf{t}}' \otimes \check{\mathbf{K}}_r'' + v^2\check{\mathbf{k}}_r'^{-1} \otimes \check{\mathbf{H}}_{r+1}''\check{\mathbf{F}}_r'' + v^{-2}\check{\mathbf{k}}_r' \otimes \check{\mathbf{H}}_r''^{-1}\check{\mathbf{E}}_r'' + R,$$

where the remainder R is equal to

$$\begin{aligned}&-\frac{\check{\mathbf{k}}_r - \check{\mathbf{k}}_r^{-1}}{v - v^{-1}} \otimes \check{\mathbf{K}}_r + \check{\mathbf{k}}_r^{-1} \otimes \check{\mathbf{H}}_{r+1}\frac{\check{\mathbf{H}}_r - \check{\mathbf{H}}_r^{-1}}{v - v^{-1}} + \check{\mathbf{k}}_r \otimes \check{\mathbf{H}}_r^{-1}\frac{\check{\mathbf{H}}_{r+1} - \check{\mathbf{H}}_{r+1}^{-1}}{v - v^{-1}}\\ &+ \frac{\check{\mathbf{k}}_r \otimes \check{\mathbf{H}}_r\check{\mathbf{H}}_{r+1}^{-1} - \check{\mathbf{k}}_r^{-1} \otimes \check{\mathbf{H}}_r\check{\mathbf{H}}_{r+1}}{v - v^{-1}}.\end{aligned}$$

We combine the terms with $\check{\mathbf{k}}_r$ together and we get zero. So is the case when we combine the terms with $\check{\mathbf{k}}_r^{-1}$. Hence, R is zero. Therefore, we have the last equality in the lemma. ■

We define the transfer map

$$\phi^\imath_{d,d-\ell} : \mathbb{S}^\imath_{d,\ell} \to \mathbb{S}^\imath_{d-\ell,\ell} \tag{58}$$

to be the composition $\mathbb{S}^\imath_{d,\ell} \xrightarrow{\widetilde{\Delta}^J} \mathbb{S}^\imath_{d-\ell,\ell} \otimes \mathbb{S}_{\ell,\ell} \xrightarrow{1\times\chi_\ell} \mathbb{S}^\imath_{d-\ell,\ell}$, where $\chi_\ell : \mathbb{S}_{\ell,\ell} \to \mathbb{A}$ is the signed representation. By Lemma 5.1.1, we have

Lemma 5.1.2. $\phi^{\imath}_{d,d-\ell}(\check{\mathbf{e}}_{i,d}) = \check{\mathbf{e}}_{i,d-\ell}$, $\phi^{\imath}_{d,d-\ell}(\check{\mathbf{f}}_{i,d}) = \check{\mathbf{f}}_{i,d-\ell}$, $\phi^{\imath}_{d,d-\ell}(\mathbf{k}^{\pm1}_{i,d}) = \mathbf{k}^{\pm1}_{i,d-\ell}$ and $\phi^{\imath}_{d,d-\ell}(\check{\mathbf{t}}_d) = \check{\mathbf{t}}_{d-\ell}$ for all $i \in [1, r-1]$.

Now we handle the case of $\Delta^{\jmath}$.

Proposition 5.1.3. For $i \in [1, r-1]$,

$$\begin{aligned}
\Delta^{\jmath}(\check{\mathbf{e}}_{i,d}) &= \check{\mathbf{e}}_{i,d'} \otimes \check{\mathbf{K}}_{i,d''} + 1 \otimes \check{\mathbf{E}}_{i,d''} + \check{\mathbf{k}}_{i,d'} \otimes \check{\mathbf{F}}_{\ell-i,d''}\check{\mathbf{K}}_{i,d''}.\\
\Delta^{\jmath}(\check{\mathbf{f}}_{i,d}) &= \check{\mathbf{f}}_{i,d'} \otimes \check{\mathbf{K}}_{\ell-i,d''} + \check{\mathbf{k}}^{-1}_{i,d'} \otimes \check{\mathbf{K}}_{\ell-i,d''}\check{\mathbf{F}}_{i,d''} + 1 \otimes \check{\mathbf{E}}_{\ell-i,d''}.\\
\Delta^{\jmath}(\check{\mathbf{k}}_{i,d}) &= \check{\mathbf{k}}_{i,d'} \otimes \check{\mathbf{K}}_{i,d''}\check{\mathbf{K}}^{-1}_{\ell-i,d''}.\\
\Delta^{\jmath}(\check{\mathbf{t}}_{d}) &= \check{\mathbf{t}}_{d'} \otimes \check{\mathbf{K}}_{r,d''} + 1 \otimes v\check{\mathbf{K}}_{r,d''}\check{\mathbf{F}}_{r,d''} + 1 \otimes \check{\mathbf{E}}_{r,d''}.
\end{aligned} \tag{59}$$

Proof. Again only the last equality is nontrivial, and we drop the subscript d and d', d'' are replaced by $'$ and $''$, respectively. Suppose that we have a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ such that $b'_k = a'_k$ and $b''_k = a''_k$ for all k, then the twists $\sum_{1\le k\le j\le n} b'_k b''_j - a'_k a''_j$ and $u(\mathbf{b}'', \mathbf{a}'')$ are zero. Hence, we have the 1st term $\check{\mathbf{t}} \otimes \check{\mathbf{K}}_r$ after the twist.

Suppose that we have a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ such that $b'_k = a'_k$ and $b''_k = a''_k + \delta_{k,r} - \delta_{k,r+2}$, then we have $\sum_{1\le k\le j\le n} b'_k b''_j - a'_k a''_j = -(a'_{r+2}+1)$ and $u(\mathbf{b}'', \mathbf{a}'') = -a''_r$. So after the twist, we have $v^2 \check{\mathbf{k}}'^{-1}_r \otimes \check{\mathbf{H}}''_{r+1} \check{\mathbf{F}}''_r|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} v^{-(a'_{r+2}+1)-a''_r} = 1 \otimes v\check{\mathbf{K}}_r \check{\mathbf{F}}_r|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$, hence we have the 2nd term.

If a quadruple $(\mathbf{b}', \mathbf{a}', \mathbf{b}'', \mathbf{a}'')$ satisfies $b'_k = a'_k$ and $b''_k = a''_k - \delta_{k,r} + \delta_{k,r+2}$. Then $\sum_{1\le k\le j\le n} b'_k b''_j - a'_k a''_j = a'_{r+2} + 1$ and $u(\mathbf{b}'', \mathbf{a}'') = a''_r - 1$. Thus, adding the twists, there is $v^{-2} \check{\mathbf{k}}'_r \otimes \check{\mathbf{H}}''^{-1}_r \check{\mathbf{E}}''_r|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} v^{a'_{r+2}+1+a''_r-1} = 1 \otimes \check{\mathbf{E}}_r|_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$. Whence we obtain the 3rd term.

By the above analysis, we have the last equality. The proposition is proved. ■

Now we take care of the degenerate version when $d' = 0$ and $d'' = d$. In this case, $\Delta^{\jmath}$ degenerates to an algebra homomorphism

$$\imath_d : \mathbb{S}^{\imath}_{d,\ell} \to \mathbb{S}_{d,\ell},$$

since $\mathbb{S}^{\imath}_{0,n-1} \simeq \mathcal{A}$. Observe that $\check{\mathbf{e}}_{i,0} = 0$, $\check{\mathbf{f}}_{i,0} = 0$, and $\check{\mathbf{k}}_{i,0} = v^{\delta_{i,r}}$, for all $i \in [1, r]$, and $\mathbf{t}_0 = 1$ in $\mathbb{S}^{\imath}_{0,n-1}$ from which the statements in Proposition 5.1.3 now read as follows.

Corollary 5.1.4. For all $i \in [1, r-1]$,

$$\begin{aligned} &\imath_d(\check{\mathbf{e}}_{i,d}) = \check{\mathbf{E}}_{i,d} + \check{\mathbf{K}}_{i,d}\check{\mathbf{F}}_{\ell-i,d}, \quad \imath_d(\check{\mathbf{f}}_{i,d}) = \check{\mathbf{E}}_{\ell-i,d} + \mathbf{K}_{\ell-i,d}\check{\mathbf{F}}_{i,d}, \quad \imath_d(\check{\mathbf{k}}_{i,d}) = \check{\mathbf{K}}_{i,d}\check{\mathbf{K}}^{-1}_{\ell-i,d}. \\ &\imath_d(\check{\mathbf{t}}_d) = \check{\mathbf{E}}_{r,d} + v\check{\mathbf{K}}_{r,d}\check{\mathbf{F}}_{r,d} + \check{\mathbf{K}}_{r,d}. \end{aligned} \tag{60}$$

Proposition 5.1.5. $\imath_d$ is injective.

Proof. This is because $\jmath_d$ is injective by Proposition 3.4.1. ■

Let $\Delta^\imath : \mathbb{S}^\imath_{d,\ell} \to \mathbb{S}^\imath_{d',\ell} \otimes \mathbb{S}_{d'',\ell}$ be the homomorphism induced from $\Delta^\jmath$. By Proposition 3.3.2,

Proposition 5.1.6. Let $M \in \Xi^\imath_d$. If $\Delta^\imath(\{M\}) = \sum_{M' \in \Xi^\imath_{d'}, M'' \in \Xi_{d'',\ell}} h^{M',M''}_M \{M'\} \otimes \{M''\}$, then we have $h^{M',M''}_M \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

Write $\imath_d(\{B\}) = \sum_{A \in \Xi_{d,\ell}} g_{B,A}\{A\}$. By Proposition 5.1.6, we have

Corollary 5.1.7. $g_{B,A} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

Recall $\mathbf{T}^\jmath_{d,n}$ and $\Pi^\jmath_{d,n}$ from (38). Note that $\mathbf{T}^\jmath_{d,n}$ is defined over $\mathcal{A}$, but can be lifted to its generic version $\mathbb{T}^\jmath_{d,n}$. Let $\Pi^\imath_{d,\ell}$ be the subset of $\Pi^\jmath_{d,n}$ defined by $a_{r+1,d+1} = 1$. Let $\mathbb{T}^\imath_{d,\ell}$ be the space of $\mathbb{T}^\jmath_{d,n}$ spanned by $[A]_d$ where $A \in \Pi^\imath_{d,\ell}$. In the same fashion, let $\mathbb{T}_{d,n}$ be the generic version of $\mathbf{T}_{d,n}$ in (37), and let $\Pi_{d,\ell}$ be the subset of $\Pi_{d,n}$ defined by $a_{r+1,d+1} = 1$. Similarly, we have $\mathbb{T}^{\mathbf{a}''}_{d,\ell}$. Let $\mathbb{H}_{A_d}$ and $\mathbb{H}_{B_d}$ be the generic version of the Hecke algebra $\mathbf{H}_{A_d}$ and $\mathbf{H}_{B_d}$ used in Proposition 3.5.1. The following is the $\imath$-analog of Proposition 3.5.1, obtained by restricting the digram therein to the desired subspaces.

Proposition 5.1.8. We have the following commutative diagram.

$$\begin{array}{ccccccc} \mathbb{S}^\imath_{d,\ell} \times \mathbb{T}^\imath_{d,\ell} & \longrightarrow & \mathbb{T}^\imath_{d,\ell} & \longleftarrow & \mathbb{T}^\imath_{d,\ell} \times \mathbb{H}_{B_d} & & \\ \downarrow{\scriptstyle \imath_{d,\ell} \times \zeta_d} & & \downarrow{\scriptstyle \zeta_d} & & \downarrow{\scriptstyle \zeta_d \times \zeta^1_d} & & \\ \mathbb{S}_{d,\ell} \times \mathbb{T}_{d,\ell} & \longrightarrow & \mathbb{T}_{d,\ell} & \xleftarrow[\psi^\imath_1]{} & \bigoplus_{a'' \models 1^{2d+1}} \mathbb{T}^{a''}_{d,\ell} \times \bigoplus_{\substack{b'' \models 1^{2d+1} \\ b''_{d+1} = 0}} {}^{b''}\mathbb{H}_{A_d} & \xleftarrow[\psi^\imath_2]{} & \mathbb{T}_{d,\ell} \times \mathbb{H}_{A_d}. \end{array}$$

Moreover, $\zeta_d([A^\jmath]_d) = [A]_d$ for all $A \in \Pi^\imath_{d,\ell}$.

5.2 Positivity in the projective limit

Consider the projective system $(\mathbb{S}^{\imath}_{d,\ell}, \phi^{\imath}_{d,d-\ell})_{d\in\mathbb{Z}_{\geq 0}}$ of associative algebras. Define an element $\mathbb{e}_i$ in the projective system whose d-th component is $\mathbf{e}_{i,d}$ $\forall d \in \mathbb{Z}_{\geq 0}$. This is well-defined by Lemma 5.1.2. Similarly, define $\mathbb{f}_i$, $\mathbb{t}$ and $\mathbb{k}_i^{\pm 1}$.

Let $\mathbb{U}^{\imath}_{\ell}$ be the subalgebra of the projective system generated by $\mathbb{e}_i$, $\mathbb{f}_i$, $\mathbb{k}_i^{\pm 1}$ for all $i \in [1, r-1]$ and $\mathbb{t}$. By [21], this is a coideal subalgebra of the quantum $\mathfrak{sl}_n$ for n even. A presentation of this algebra by generators and relations can also be found in [21].

Set $\Xi^{\imath}_{\infty,\ell} = \sqcup_{d\in\mathbb{Z}_{\geq 0}} \Xi^{\imath}_{d,\ell}$. We say two matrices are equivalent if they differ by an even multiply of the identity matrix I_ℓ. We denote $\Xi^{\imath}_{\infty,\ell}/\approx$ for the set of equivalence classes.

By [21], we know that $\phi^{\imath}_{d,d-\ell}(\{A\}_d) = \{A - 2I_\ell\}_{d-\ell}$ if the diagonal entries of A are large enough. To an element $\widehat{A} \in \Xi^{\imath}_{\infty,\ell}/\approx$, we define an element $b_{\widehat{A}}$ in the projective system whose d-th component is $\{A + pI_\ell\}_d$ for some p if d is big enough.

Let $\dot{\mathbb{U}}^{\imath} \equiv \dot{\mathbb{U}}^{\imath}_{\ell}$ be the space spanned by $b_{\widehat{A}}$ for $\widehat{A} \in \Xi^{\imath}_{\infty,\ell}/\approx$. By [21], $\dot{\mathbb{U}}^{\imath}$ is an associative algebra, the idempotented version of $\mathbb{U}^{\imath}_{\ell}$ and $b_{\widehat{A}}$ forms the canonical basis $\mathbb{B}^{\imath}$ of $\dot{\mathbb{U}}^{\imath}$ defined in [21]. Let $\mathbb{X}^{\imath}_{\ell}$ be the subset of $\widehat{A} \in \Xi^{\imath}_{\infty,\ell}/\approx$ parametrized by the diagonal matrices. This algebra admits a decomposition $\dot{\mathbb{U}}^{\imath} = \oplus_{\widehat{\mu},\widehat{\lambda}\in\mathbb{X}^{\imath}} {}_{\widehat{\mu}}\mathbb{U}^{\imath}_{\widehat{\lambda}}$ where ${}_{\widehat{\mu}}\mathbb{U}^{\imath}_{\widehat{\lambda}} = b_{\widehat{\mu}}\dot{\mathbb{U}}^{\imath} b_{\widehat{\lambda}}$.

Replace the projective system $(\mathbb{S}^{\imath}_{d,\ell}, \phi^{\imath}_{d,d-\ell})$ by $(\mathbb{S}_{d,\ell}, \phi_{d,d-\ell})$, we can define the elements $\check{\mathbb{E}}_i, \check{\mathbb{F}}_i$ and $\check{\mathbb{K}}_i^{\pm 1}$ in this projective system and they generate over $\mathbb{Q}(v)$ the quantum $\mathfrak{sl}_l$: $\mathbb{U}_\ell$.

Set $\Xi_{\infty,\ell} = \sqcup_{d\in\mathbb{Z}_{\geq 0}} \Xi_{d,\ell}$. We say two matrices are equivalent if they differ by a multiply of the identity matrix I_ℓ. We denote $\Xi_{\infty,\ell}/\sim$ for the set of equivalence classes. To an element $\overline{A} \in \Xi_{\infty,\ell}/\sim$, we define an element $b_{\overline{A}}$ in the projective system whose d-th component is $\{A + pI_\ell\}_d$ for some p if d is big enough. Then the space $\dot{\mathbb{U}}_\ell$ spanned by $b_{\overline{A}}$ for all $\overline{A} \in \Xi_{\infty,\ell}/\sim$ is an associative algebra, the idempotented version of $\mathbb{U}_\ell$ by [25] and $b_{\overline{A}}$ forms the canonical basis $\mathbb{B}_\ell$. Let $\mathbb{X}_\ell$ be the subset of $\Xi_{\infty,\ell}/\sim$ consisting of all diagonal matrices. $\dot{\mathbb{U}}_\ell = \oplus_{\overline{\mu},\overline{\lambda}\in\mathbb{X}_\ell} {}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}}$, where ${}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}} = b_{\overline{\mu}}\dot{\mathbb{U}} b_{\overline{\lambda}}$.

The linear map $\Delta^{\imath}$ on the $\imath$Schur algebra level induces a linear map

$$\Delta^{\imath}_{\widehat{\mu'},\widehat{\lambda'},\overline{\mu''},\overline{\lambda''}} : {}_{\widehat{\mu}}\mathbb{U}^{\imath}_{\widehat{\lambda}} \to {}_{\widehat{\mu'}}\mathbb{U}^{\imath}_{\widehat{\lambda'}} \otimes {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}, \quad \forall (\mu', \mu'') \vdash \mu, (\lambda', \lambda'') \vdash \lambda, \tag{61}$$

where $\vdash$ is defined similar to (42) with row vectors replaced by diagonal matrices. Write $\Delta^{\imath}_{\widehat{\mu'},\widehat{\lambda'},\overline{\mu''},\overline{\lambda''}}(a) = \sum_{b\in\mathbb{B}^{\imath}, c\in\mathbb{B}} n_a^{b,c} b \otimes c$, for all $a \in \mathbb{B}^{\imath}$, then we have the $\imath$-analog of Theorem 4.3.1, whose proof is the same as the Theorem 4.3.1 using Proposition 5.1.6.

Theorem 5.2.1. We have that $n_a^{b,c} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

The linear map $\imath_d$ induces a linear map

$$\imath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}} : {}_{\widehat{\mu}}\mathbb{U}^{\imath}_{\widehat{\lambda}} \longrightarrow {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}, \quad \forall \mu'' \vdash \mu, \lambda'' \vdash \lambda. \tag{62}$$

We have the $\imath$-analog of Theorem 4.4.1 by using Corollary 5.1.7.

Theorem 5.2.2. Let $b \in \mathbb{B}^{\imath}$. If $\imath_{\widehat{\mu},\widehat{\lambda},\overline{\mu''},\overline{\lambda''}}(b) = \sum_{a\in\mathbb{B}} g_{b,a} a$, then $g_{b,a} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

6 Positivity for Quantum Affine $\mathfrak{sl}_n$

In this section, we shall lift the positivity result on quantum $\mathfrak{sl}_n$ in Section 2.5 to its affine analog. As a byproduct, we provide a new proof of the multiplication formula in [5].

6.1 Results from [24]

Following Section 2, fix a pair (d, n) of nonnegative integers. Set

$$\widehat{\Lambda}_{d,n} = \left\{ \boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}}_{\geq 0} | \lambda_i = \lambda_{i+n}, \forall i \in \mathbb{Z}; \sum_{1\leq i\leq n} \lambda_i = d \right\}.$$

Let $\widehat{\Xi}_d$ be the set of all $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{ij})_{i,j\in\mathbb{Z}}$ such that $a_{ij} \in \mathbb{Z}_{\geq 0}$, $a_{ij} = a_{i+n,j+n}$, and $\sum_{1\leq i\leq n; j\in\mathbb{Z}} a_{ij} = d$. To each matrix $A \in \widehat{\Xi}_d$, we can associate $r(A)$ and $c(A)$ in $\widehat{\Lambda}_{d,n}$ by $r(A)_i = \sum_{j\in\mathbb{Z}} a_{ij}$ and $c(A)_j = \sum_{i\in\mathbb{Z}} a_{ij}$ for all $i, j \in \mathbb{Z}$.

We need to switch the ground field from $\mathbb{F}_q$ to the local field $\mathbb{F}_q((\varepsilon))$. Let $\mathbb{F}_q[[\varepsilon]]$ be the subring of $\mathbb{F}_q((\varepsilon))$ of all formal power series over $\mathbb{F}_q$. Suppose that $\mathbf{V}$ is a d-dimensional vector space over $\mathbb{F}_q((\varepsilon))$. A free $\mathbb{F}_q[[\varepsilon]]$-module $\mathcal{L}$ in $\mathbf{V}$ is called a lattice if $\mathbb{F}_q((\varepsilon)) \otimes_{\mathbb{F}_q[[\varepsilon]]} \mathcal{L} = \mathbf{V}$. A lattice chain $\mathbf{L} = (\mathbf{L}_i)_{i\in\mathbb{Z}}$ of period n is a sequence of lattices $\mathbf{L}_i$ in $\mathbf{V}$ such that $\mathbf{L}_i \subseteq \mathbf{L}_{i+1}$ and $\mathbf{L}_i = \varepsilon \mathbf{L}_{i+n}$ for all $i \in \mathbb{Z}$. Let $\widehat{X}_d$ be the collection of all lattice chains in $\mathbf{V}$. Let $\widehat{\mathbf{G}}_d = \mathrm{GL}(\mathbf{V})$ act from the left on $\widehat{X}_d$ in the canonical way. Then we can form the algebra

$$\widehat{\mathbf{S}}_d = \mathcal{A}_{\widehat{\mathbf{G}}_d}(\widehat{X}_d \times \widehat{X}_d),$$

which is the so-called affine $\mathbf{v}$-Schur algebra. It is well known that the $\widehat{\mathbf{G}}_d$-orbits in $\widehat{X}_d \times \widehat{X}_d$ are parameterized by $\widehat{\Xi}_d$ via the assignment $(\mathbf{L}, \mathbf{L}') \mapsto A$, where $a_{ij} = \left| \frac{\mathbf{L}_i \cap \mathbf{L}'_j}{\mathbf{L}_{i-1} \cap \mathbf{L}'_j + \mathbf{L}_i \cap \mathbf{L}'_{j-1}} \right|$ for all $i, j \in \mathbb{Z}$. So we have

$$\widehat{\mathbf{S}}_d = \mathrm{span}_{\mathcal{A}}\{e_A | A \in \widehat{\Xi}_d\},$$

where e_A is the characteristic function of the $\widehat{\mathbf{G}}_d$-orbit indexed by A. Furthermore, we have $\widehat{\mathbf{S}}_d = \oplus_{\mathbf{b},\mathbf{a}\in\widehat{\Lambda}_{d,n}} \widehat{\mathbf{S}}_d(\mathbf{b},\mathbf{a})$ where $\widehat{\mathbf{S}}_d(\mathbf{b},\mathbf{a})$ is spanned by e_A such that $r(A) = \mathbf{b}$ and $c(A) = \mathbf{a}$.

If one lifts the functions to the sheaf level, one gets the generic version $\widehat{\mathbb{S}}_d$ of $\widehat{\mathbf{S}}_d$ such that $\mathcal{A} \otimes_{\mathbb{A}} \widehat{\mathbb{S}}_d = \widehat{\mathbf{S}}_d$. By abuse of notation, we write e_A for the unique function x in $\widehat{\mathbb{S}}_d$ such that $\mathcal{A} \otimes_{\mathbb{A}} x = e_A$ (for all q).

The standard basis of $\widehat{\mathbb{S}}_d$ consists of elements $[A] = v^{-d_A} e_A$ where $d_A = \sum_{\substack{1\le i\le n \\ i\ge k, j<l}} a_{ij}a_{kl}$. Recall the Bruhat order $\preceq$ on $\widehat{\Xi}_d$ from [24]: $A \preceq B$ if and only if

$$\sum_{i\ge r, j\le s} a_{rs} \le \sum_{i\ge r, j\le s} b_{rs}, \quad \forall i<j \in \mathbb{Z}; \qquad \sum_{i\le r, j\ge s} a_{rs} \le \sum_{i\le r, j\ge s} b_{rs}, \quad \forall i>j \in \mathbb{Z}.$$

Following [24], one can associate a bar involution $\bar{\ } : \widehat{\mathbb{S}}_d \to \widehat{\mathbb{S}}_d$ such that $\overline{[A]} = [A] + \sum_{A'\preceq A, A'\neq A} C_{A,A'}[A']$ where $C_{A,A'} \in \mathbb{A}$.

The canonical basis $\{A\}_d$ for all $A \in \widehat{\Xi}_d$ of $\widehat{\mathbb{S}}_d$ is defined by the properties that $\overline{\{A\}_d} = \{A\}_d$ and $\{A\}_d = [A] + \sum_{A'\preceq A, A'\neq A} P_{A,A'}[A']$ where $P_{A,A'} \in v^{-1}\mathbb{Z}[v^{-1}]$.

Let $E^{i,j}$ be the $\mathbb{Z}\times\mathbb{Z}$ matrix whose (k,l)-th entry is 1 if $(k,l) = (i,j)$ mod n, and zero otherwise. For any $i \in \mathbb{Z}$, we define the following elements in $\widehat{\mathbb{S}}_d$:

$$\begin{aligned} \mathbf{E}_i &= \sum_{A-E^{i+1,i}\ \text{diagonal}} [A], \quad \mathbf{F}_i = \sum_{A-E^{i,i+1}\ \text{diagonal}} [A], \\ \mathbf{H}_i^{\pm 1} &= \sum_{A\ \text{diagonal}} v^{\pm c(A)_i}[A], \quad \mathbf{K}_i^{\pm 1} = \mathbf{H}_{i+1}^{\pm 1}\mathbf{H}_i^{\mp 1}, \quad \forall i \in \mathbb{Z}. \end{aligned} \tag{63}$$

By periodicity, we have $\mathbf{E}_i = \mathbf{E}_{i+n}, \mathbf{F}_i = \mathbf{F}_{i+n}, \mathbf{H}_i^{\pm 1} = \mathbf{H}_{i+n}^{\pm 1}$, and $\mathbf{K}_i^{\pm 1} = \mathbf{K}_{i+n}^{\pm 1}$, for all $i \in \mathbb{Z}$. The following lemma is from [24].

Lemma 6.1.1. There is an algebra homomorphism $\widetilde{\Delta} : \widehat{\mathbb{S}}_d \to \widehat{\mathbb{S}}_{d'} \otimes \widehat{\mathbb{S}}_{d''}$ for $d' + d'' = d$ with

$$\widetilde{\Delta}(\mathbf{E}_i) = \mathbf{E}_i' \otimes \mathbf{H}_{i+1}'' + \mathbf{H}_{i+1}'^{-1} \otimes \mathbf{E}_i'', \ \widetilde{\Delta}(\mathbf{F}_i) = \mathbf{F}_i' \otimes \mathbf{H}_i''^{-1} + \mathbf{H}_i' \otimes \mathbf{F}_i'', \ \widetilde{\Delta}(\mathbf{K}_i) = \mathbf{K}_i' \otimes \mathbf{K}_i'', \ \forall i \in \mathbb{Z}.$$

6.2 The coproduct Δ

Recall the algebra homomorphism from Lemma 6.1.1. If $\mathbf{b} = \mathbf{b}' + \mathbf{b}''$ and $\mathbf{a} = \mathbf{a}' + \mathbf{a}''$, let $\widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''} : \widehat{\mathbb{S}}_d(\mathbf{b},\mathbf{a}) \to \widehat{\mathbb{S}}_{d'}(\mathbf{b}',\mathbf{a}') \otimes \widehat{\mathbb{S}}_{d''}(\mathbf{b}'',\mathbf{a}'')$ be the composition of the restriction of $\widehat{\mathbb{S}}_d$

to $\widehat{\mathbb{S}}_d(\mathbf{b},\mathbf{a})$ and the projection to $\widehat{\mathbb{S}}_{d'}(\mathbf{b}',\mathbf{a}')\otimes\widehat{\mathbb{S}}_{d''}(\mathbf{b}'',\mathbf{a}'')$. We set

$$\Delta^{\dagger}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}=v^{\sum_{1\leq i\leq j\leq n}b'_ib''_j-a'_ia''_j}\widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''},\quad \Delta^{\dagger}=\oplus\Delta^{\dagger}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''},\tag{64}$$

where the sum runs over all quadruples $(\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}'')$ where $\mathbf{b}',\mathbf{a}'\in\widehat{\Lambda}_{d',n}$ and $\mathbf{b}'',\mathbf{a}''\in\widehat{\Lambda}_{d'',n}$.

Proposition 6.2.1. The linear map $\Delta^{\dagger}$ in (64) is an algebra homomorphism. Moreover,

$$\Delta^{\dagger}(\mathbf{E}_i)=v^{\delta_{i,n}d''}\mathbf{E}'_i\otimes\mathbf{K}''_i+1\otimes v^{-\delta_{i,n}d'}\mathbf{E}''_i,$$
$$\Delta^{\dagger}(\mathbf{F}_i)=v^{-\delta_{i,n}d''}\mathbf{F}'_i\otimes 1+\mathbf{K}'^{-1}_i\otimes v^{\delta_{i,n}d'}\mathbf{F}''_i,$$
$$\Delta^{\dagger}(\mathbf{K}_i)=\mathbf{K}'_i\otimes\mathbf{K}''_i,\quad \forall i\in[1,n].$$

Proof. The case when $i\in[1,n-1]$ is proved in the same manner as the finite case in Proposition 2.3.2. We now prove the case when $i=n$. Suppose that $\mathbf{b}''=\mathbf{a}''$, and $(\mathbf{b}',\mathbf{a}')$ is chosen such that $b'_i=a'_i-\delta_{i,n}+\delta_{i,1}$ for all $1\leq i\leq n$. Then the twist $\sum_{1\leq i\leq j\leq n}b'_ib''_j-a'_ia''_j$ is equal to $d''-a''_n$. This implies that the term $\mathbf{E}'_i\otimes\mathbf{H}''_{i+1}$ in $\widetilde{\Delta}(\mathbf{E}_i)$ becomes $v^{\delta_{i,n}d''}\mathbf{E}'_i\otimes\mathbf{K}''_i$. For the term $\mathbf{H}'^{-1}_{i+1}\otimes\mathbf{E}''_i$ in $\widetilde{\Delta}(\mathbf{E}_i)$, the twist contributes a'_i-d' for a quadruple $(\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}'')$ such that $\mathbf{b}'=\mathbf{a}'$ and $b''_i=a''_i-\delta_{i,n}+\delta_{i,1}$ for all $1\leq i\leq n$. The formula for $\Delta^{\dagger}(\mathbf{E}_i)$ is proved.

The proof for the formula $\Delta^{\dagger}(\mathbf{F}_i)$ is entirely similar. The formula for $\Delta^{\dagger}(\mathbf{K}_i)$ is obvious. ■

We set

$$\varepsilon_i(A)=\sum_{r\leq i<s}a_{r,s}-\sum_{r>i\geq s}a_{r,s},\quad \forall i\in\mathbb{Z},A\in\widehat{\Xi}_d.\tag{65}$$

We define a linear map

$$\xi_{d,i,c}:\widehat{\mathbb{S}}_d\to\widehat{\mathbb{S}}_d,\quad \forall i,c\in\mathbb{Z},\tag{66}$$

by $\xi_{d,i,c}([A])=v^{c\varepsilon_i(A)}[A]$. By [24], $\xi_{d,i,c}$ is an algebra isomorphism with inverse $\xi_{d,i,-c}$. Set

$$\Delta\!\!\!\!\Delta:\widehat{\mathbb{S}}_d\to\widehat{\mathbb{S}}_{d'}\otimes\widehat{\mathbb{S}}_{d''}\tag{67}$$

to be the composition $\widehat{\mathbb{S}}_d\xrightarrow{\Delta^{\dagger}}\widehat{\mathbb{S}}_{d'}\otimes\widehat{\mathbb{S}}_{d''}\xrightarrow{\xi_{d',n,d''}\otimes\xi_{d'',n,-d'}}\widehat{\mathbb{S}}_{d'}\otimes\widehat{\mathbb{S}}_{d''}$.

Proposition 6.2.2. The linear map Δ in (67) is an algebra homomorphism. Moreover,

$$\Delta(\mathbf{E}_i) = \mathbf{E}_i' \otimes \mathbf{K}_i'' + 1 \otimes \mathbf{E}_i'',\ \Delta(\mathbf{F}_i) = \mathbf{F}_i' \otimes 1 + \mathbf{K}_i'^{-1} \otimes \mathbf{F}_i'',\ \Delta(\mathbf{K}_i) = \mathbf{K}_i' \otimes \mathbf{K}_i'', \quad \forall i \in \mathbb{Z}.$$

Proof. We have $\xi_{d,n,c}(\mathbf{E}_i) = v^{-c\delta_{i,n}}\mathbf{E}_i$, $\xi_{d,n,c}(\mathbf{F}_i) = v^{c\delta_{i,n}}\mathbf{F}_i$, and $\xi_{d,n,c}(\mathbf{K}_i) = \mathbf{K}_i$. Proposition follows from these computations and the formulas in Proposition 6.2.1. ■

6.3 The compatibility of $\xi_{d,i,c}$ and the canonical basis

We have

Theorem 6.3.1. $\xi_{d,i,c}(\{A\}_d) = v^{c\varepsilon_i(A)}\{A\}_d$ where $\xi_{d,i,c}$ is in (66).

Theorem 6.3.1 follows from the following critical observation.

Theorem 6.3.2. Write $\{A\}_d = \sum_{A' \preceq A} P_{A,A'}[A']$. If $P_{A,A'} \neq 0$, then $\varepsilon_i(A) = \varepsilon_i(A')$ for all i.

We make two remarks before we prove Theorem 6.3.2.

Remark 6.3.3. The algebra isomorphism $\prod_{1 \le i \le n} \xi_{d,i,-1}$ is the linear map ξ in [24, 1.7]. In view of Theorem 6.3.1, we have $\xi(\{A\}_d) = v^{-\sum_{1 \le i \le n}\varepsilon_i(A)}\{A\}_d$.

Remark 6.3.4. Even if $A' \prec A$, $\varepsilon_i(A')$ may not be the same as $\varepsilon_i(A)$. For example, take $A' = 2\sum_{1 \le i \le n} E^{i,i}$ and $A = \sum_{1 \le i \le n} E^{i,i} + E^{i,i+1}$. Then we have $A' \prec A$, $\varepsilon_i(A') = 0$ and $\varepsilon_i(A) = 1$ for all $1 \le i \le n$.

The remaining part of this section is devoted to the proof of Theorem 6.3.2. The main ingredient is a connection between the numerical data $\varepsilon_i(A)$ in (65) and the multiplication formulas in [5], which we shall recall and provide a new proof. Before we state the formula, we need to recall a lemma from [26, Section 2.2] as follows.

Lemma 6.3.5. Let V be a finite-dimensional vector space over $\mathbb{F}_q$. Fix a flag $(V_i)_{1 \le i \le n}$ in V such that $|V_i/V_{i-1}| = l_i\ \forall 1 \le i \le n$. The number of subspaces $W \subset V$ such that $|W \cap V_i| = \sum_{j=1}^i a_j\ \forall 1 \le i \le n$ is given by $q^{\sum_{n \ge i > j \ge 1} a_i(l_j - a_j)} \prod_{i=1}^n \begin{bmatrix} l_i \\ a_i \end{bmatrix}$ where $\begin{bmatrix} l_i \\ a_i \end{bmatrix} = \prod_{1 \le j \le a_i} \frac{q^{l_i - j + 1} - 1}{q^j - 1}$.

The following multiplication formula in $\widehat{\mathbb{S}}_d$ is first obtained in [5]. In a forthcoming paper [10], we provide a multiplication formula for affine type C case. To a matrix $T = (t_{ij})_{i,j \in \mathbb{Z}}$, we set $\check{T} = (\check{t}_{ij})_{i,j \in \mathbb{Z}}$ where $\check{t}_{ij} = t_{i-1,j}$.

Proposition 6.3.6. (1) Suppose that $A = (a_{ij}), B = (b_{ij}) \in \widehat{\Xi}_d$ satisfy that $c(B) = r(A)$ and $B - \sum_{i=1}^n \alpha_i E^{i,i+1}$ is diagonal for some $\alpha_i \in \mathbb{Z}_{\geq 0}$. We have

$$e_B * e_A = \sum_T q^{\sum_{1\leq i\leq n, j>l}(a_{ij}-t_{i-1,j})t_{i,l}} \prod_{1\leq i\leq n, j\in\mathbb{Z}} \begin{bmatrix} a_{ij}+t_{ij}-t_{i-1,j} \\ t_{ij} \end{bmatrix} e_{A+T-\check{T}'}$$

where the sum runs over all $T = (t_{ij})$ such that $t_{i+n,j+n} = t_{ij}$ and $r(T)_i = \alpha_i$ for all $1 \leq i \leq n$.

(2) If $C \in \widehat{\Xi}_d$ satisfies that $c(C) = r(A)$ and $C - \sum_{i=1}^n \beta_i E^{i+1,i}$ is diagonal, then

$$e_C * e_A = \sum_T q^{\sum_{1\leq i\leq n, j<l}(a_{ij}-t_{ij})t_{i-1,l}} \prod_{1\leq i\leq n, j\in\mathbb{Z}} \begin{bmatrix} a_{ij}-t_{ij}+t_{i-1,j} \\ t_{i-1,j} \end{bmatrix} e_{A-T+\check{T}'}$$

where the sum runs over all T such that $t_{ij} = t_{i+n,j+n}$ and $r(T)_i = \beta_i$ for all $1 \leq i \leq n$.

Proof. (1) It suffices to show the similar statement in $\widehat{\mathbf{S}}_d$. Let $A' = (a'_{ij})_{i,j\in\mathbb{Z}}$ be a matrix in $\widehat{\Xi}_d$ such that $r(B) = r(A')$ and $c(A) = c(A')$. Let $\mathcal{O}_{A'}$ be the $\widehat{\mathbf{G}}_d$-orbit in $\widehat{X}_d \times \widehat{\mathcal{X}}_d$ indexed by A'. Fix $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_{A'}$, and we denote

$$Z = \{\mathbf{L}'' \in \widehat{X}_d \mid \mathbf{L}_{i-1} \subset \mathbf{L}''_i \overset{\alpha_i}{\subset} \mathbf{L}_i, \forall 1 \leq i \leq n\}.$$

Note that $(\mathbf{L}, \mathbf{L}'') \in \mathcal{O}_B$ if and only if $\mathbf{L}'' \in Z$. Clearly, Z has a partition $Z = \bigsqcup_T Z_T$ where

$$Z_T = \{\mathbf{L}'' \in Z || \mathbf{L}''_i \cap (\mathbf{L}_{i-1} + (\mathbf{L}_i \cap \mathbf{L}'_j))/\mathbf{L}''_i \cap (\mathbf{L}_{i-1} + (\mathbf{L}_i \cap \mathbf{L}'_{j-1}))| = a'_{ij} - t_{ij}, \forall i,j \in \mathbb{Z}\}$$

and the union runs over all T such that $t_{i+n,j+n} = t_{ij}$ and $r(T)_i = \alpha_i$ for all $1 \leq i \leq n$. For each $\mathbf{L}'' \in Z_T$, we have the following identities.

$$\begin{aligned} a_{ij} &= |\mathbf{L}''_i \cap \mathbf{L}'_j/\mathbf{L}''_i \cap \mathbf{L}'_{j-1}| - |\mathbf{L}''_{i-1} \cap \mathbf{L}'_j/\mathbf{L}''_{i-1} \cap \mathbf{L}'_{j-1}|, \\ a'_{ij} &= |\mathbf{L}_i \cap \mathbf{L}'_j/\mathbf{L}_i \cap \mathbf{L}'_{j-1}| - |\mathbf{L}_{i-1} \cap \mathbf{L}'_j/\mathbf{L}_{i-1} \cap \mathbf{L}'_{j-1}|, \\ a'_{ij} - t_{ij} &= |\mathbf{L}''_i \cap \mathbf{L}'_j/\mathbf{L}''_i \cap \mathbf{L}'_{j-1}| - |\mathbf{L}_{i-1} \cap \mathbf{L}'_j/\mathbf{L}_{i-1} \cap \mathbf{L}'_{j-1}|, \end{aligned} \tag{68}$$

where the last identity follows from the definition of Z_T. By (68), we have

$$t_{ij} = |\mathbf{L}_i \cap \mathbf{L}'_j/\mathbf{L}_i \cap \mathbf{L}'_{j-1}| - |\mathbf{L}''_i \cap \mathbf{L}'_j/\mathbf{L}''_i \cap \mathbf{L}'_{j-1}|.$$

Thus, $a'_{ij} - t_{ij} = a_{ij} - t_{i-1,j}$, that is, $A' = A + T - \check{T}$. Summing up the above analysis, we have

$$e_B * e_A(\mathbf{L}, \mathbf{L}') = \sum_{\mathbf{L}'' \in \widehat{\Xi}_d} e_B(\mathbf{L}, \mathbf{L}'')e_A(\mathbf{L}'', \mathbf{L}') = \sum_{\mathbf{L}'' \in Z} e_A(\mathbf{L}'', \mathbf{L}')$$
$$= \sum_T \sum_{\mathbf{L}'' \in \mathbb{Z}_T} e_A(\mathbf{L}'', \mathbf{L}') = \sum_T \# Z_T \; e_{A+T-\check{T}}(\mathbf{L}, \mathbf{L}'). \tag{69}$$

So it is reduced to compute the cardinality of Z_T. For each $i \in [1, n]$, we set $\mathbb{Z}(i) = \{k \in \mathbb{Z} | k \in [1, i] \bmod n\}$ and we define $Z_T^{[1,i]}$ to be the set of all lattice chains $\mathbf{L}'' = (\mathbf{L}''_k)_{k \in \mathbb{Z}(i)}$ such that $\mathbf{L}''_k$ satisfies $|\mathbf{L}_{k-1} + \mathbf{L}''_k \cap \mathbf{L}'_j / \mathbf{L}_{k-1} + \mathbf{L}''_k \cap \mathbf{L}'_{j-1}| = a'_{ij} - t_{ij}$ for all $j \in \mathbb{Z}$. Consider

$$Z_T = Z_T^{[1,n]} \xrightarrow{\pi_n} Z_T^{[1,n-1]} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} Z_T^{[1,1]} \xrightarrow{\pi_1} \bullet,$$

where $\pi_i((\mathbf{L}''_k)_{k \in \mathbb{Z}(i)}) = (\mathbf{L}''_k)_{k \in \mathbb{Z}(i-1)}$ and the equality is due to $\mathbf{L}''_i \cap (\mathbf{L}_{i-1} + (\mathbf{L}_i \cap \mathbf{L}'_j)) = \mathbf{L}_{i-1} + \mathbf{L}''_i \cap \mathbf{L}'_j$ We observe that the fiber of π_i gets identified with the set of subspaces W in $\mathbf{L}_i/\mathbf{L}_{i-1}$ such that $|W \cap (\mathbf{L}_{i-1} + \mathbf{L}_i \cap \mathbf{L}'_j)/\mathbf{L}_{i-1} / W \cap (\mathbf{L}_{i-1} + \mathbf{L}_i \cap \mathbf{L}'_{j-1})/\mathbf{L}_{i-1}| = a'_{ij} - t_{ij}$. Observe that $|(\mathbf{L}_{i-1} + \mathbf{L}_i \cap \mathbf{L}'_j)/\mathbf{L}_{i-1}| - |(\mathbf{L}_{i-1} + \mathbf{L}_i \cap \mathbf{L}'_{j-1})/\mathbf{L}_{i-1}| = a'_{ij}$, and by applying Lemma 6.3.5, we have $\#\pi_i^{-1}(\mathbf{L}^i) = q^{\sum_{l<j}(a'_{ij} - t_{ij})t_{il}} \prod_{j \in \mathbb{Z}} \begin{bmatrix} a'_{ij} \\ t_{ij} \end{bmatrix}$, where $\mathbf{L}^i$ is any element in $Z_T^{[1,i-1]}$. So π_i is surjective with constant fiber. Hence,

$$\# Z_T = \prod_{1 \le i \le n} \# \pi_i^{-1}(\mathbf{L}^i) = q^{\sum_{1 \le i \le n, l<j}(a'_{ij} - t_{ij})t_{il}} \prod_{1 \le i \le n, j \in \mathbb{Z}} \begin{bmatrix} a'_{ij} \\ t_{ij} \end{bmatrix}. \tag{70}$$

The statement (1) follows from (69) and (70).

Let us prove (2). Let A' be a matrix such that $r(A') = r(C)$ and $c(A') = c(A)$. Fix $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_{A'}$. We consider the set $Y = \{\mathbf{L}'' \widehat{X}_d | \mathbf{L}_{i-1} \overset{\beta_{i-1}}{\subseteq} \mathbf{L}''_{i-1} \subseteq \mathbf{L}_i, \forall 1 \le i \le n\}$. Then Y admits a partition $Y = \sqcup Y_T$, where

$$Y_T = \left\{\mathbf{L}'' \in Y \middle| \left|\mathbf{L}_{i-1} + \mathbf{L}''_{i-1} \cap \mathbf{L}'_j / \mathbf{L}_{i-1} + \mathbf{L}''_{i-1} \cap \mathbf{L}'_{j-1}\right| = t_{i-1,j}, \quad \forall i,j \in \mathbb{Z}\right\}.$$

By applying Lemma 6.3.5 and arguing similar to (1), we have

$$\# Y_T = \prod_{1 \le i \le n} q^{\sum_{l>j} t_{i-1,l}(a'_{ij} - t_{i-1,j})} \prod_{j \in \mathbb{Z}} \begin{bmatrix} a'_{ij} \\ t_{i-1,j} \end{bmatrix}.$$

Moreover, for $L'' \in Y_T$ such that $(\mathbf{L}'', \mathbf{L}') \in \mathcal{O}_A$ if and only if $A' = A - T + \check{T}$. Therefore, we have (2). The proposition is thus proved. ∎

Remark 6.3.7. If $n = 1$, Proposition 6.3.6 shows that $e_B * e_A = e_A * e_B$. (Here we use $(e_A * e_B)^t = e_{B^t} e_{A^t}$.) This implies that $\widehat{\mathbb{S}}_d$ is commutative for $n = 1$, which corresponds to the geometric Satake of type A.

Lemma 6.3.8. Suppose that $[B] * [A] = \sum Q_{B,A}^C [C]$. If $B = \sum_{1 \le j \le n} \beta_j E^{jj} + \alpha_j E^{jj+1}$ or $\sum_{1 \le j \le n} \beta_j E^{jj} + \alpha_j E^{j+1j}$, and $Q_{B,A}^C \neq 0$, then $\varepsilon_i(A) + \varepsilon_i(B) = \varepsilon_i(C)$ for all $i \in \mathbb{Z}$.

Proof. Assume that $B = \sum_{1 \le j \le n} \beta_j E^{jj} + \alpha_j E^{jj+1}$. Then we have $\varepsilon_i(B) = \alpha_i$. If $Q_{B,A}^C \neq 0$, then by Proposition 6.3.6 (1), the matrix C is of the form $A + T - \check{T}$. Thus, we have

$$\begin{aligned}\varepsilon_i(C) = \varepsilon_i(A) + \varepsilon_i(T - \check{T}) &= \varepsilon_i(A) + \sum_{r \le i < s} t_{r,s} - \check{t}_{r,s} + \sum_{r > i \ge s} t_{r,s} - \check{t}_{r,s} \\ &= \varepsilon_i(A) + \sum_{i < s} t_{i,s} - \sum_{i \ge s} -\check{t}_{i+1,s} = \varepsilon_i(A) + \sum_{s \in \mathbb{Z}} t_{i,s} = \varepsilon_i(A) + \alpha_i = \varepsilon_i(A) + \varepsilon_i(B).\end{aligned}$$

Therefore, the lemma holds for $B = \sum_{1 \le j \le n} \beta_j E^{jj} + \alpha_j E^{jj+1}$.

For the case when $B = \sum_{1 \le j \le n} \beta_j E^{jj} + \alpha_j E^{j+1j}$, then $\varepsilon_i(B) = -\alpha_i$ and C is of the form $A - T + \check{T}$ if $Q_{B,A}^C \neq 0$ by Proposition 6.3.6 (2). So we have $\varepsilon_i(C) = \varepsilon_i(A) - \varepsilon_i(T - \check{T}) = \varepsilon_i(A) - \alpha_i = \varepsilon_i(A) + \varepsilon_i(B)$. Therefore, the lemma holds in this case. We are done. ∎

Next we introduce a 2nd numerical data. We define

$$\deg_i\left(\sum_{1 \le j \le n} \beta_j E^{jj} + \alpha_j E^{j+1j}\right) = -\alpha_i, \quad \deg_i\left(\sum_{1 \le j \le n} \beta_j E^{jj} + \alpha_j E^{jj+1}\right) = \alpha_i.$$

Suppose that $M = [A_1] * \cdots * [A_m]$ is a *monomial* in $[A_j]$ where A_j is either $\sum_{1 \le k \le n} \beta_{jk} E^{k,k} + \alpha_{jk} E^{k+1,k}$ or $\sum_{1 \le k \le n} \beta_{jk} E^{k,k} + \alpha_{jk} E^{k,k+1}$. We define $\deg_i(M) = \sum_{1 \le j \le m} \deg_i(A_j)$. To the same monomial M, we also define its length $\ell(M)$ to be $\ell(M) = \sum_{1 \le j \le m} \sum_{1 \le k \le n} \alpha_{jk}$. (We define $[A]$ to be a monomial of length zero if A is diagonal.) Then we have

Lemma 6.3.9. Let M be a monomial and write $M = \sum R_A [A]$. If $R_A \neq 0$, then $\deg_i(M) = \varepsilon_i(A)$ for all $i \in \mathbb{Z}$.

Proof. We argue by induction on the length $\ell(M)$ of M. When $\ell(M) = 1$, the lemma follows from the definitions. Assume now that $\ell(M) > 1$ and the lemma holds for any monomial M' such that $\ell(M') < \ell(M)$. We write $M = [A_1] * [M']$, where A is either $\sum_{1\le j\le n} \beta_j E^{j,j} + \alpha_j E^{j+1,j}$ or $\sum_{1\le j\le n} \beta_j E^{j,j} + \alpha_j E^{j,j+1}$, and M' is a monomial of the remaining terms in M. Thus, $\ell(M') < \ell(M)$. Suppose that $M' = \sum R_{A'}[A']$, then we have

$$M = [A_1] * M' = \sum_{A'} R_{A'}[A_1] * [A'] = \sum_{A',B} R_{A'} Q^B_{A_1,A'}[B].$$

If $A_1 = \sum_{1\le j\le n} \beta_j E^{j,j} + \alpha_j E^{j+1,j}$, then by Lemma 6.3.8 and induction hypothesis, we have

$$\deg_i(M) = -\alpha_i + \deg_i(M') = -\alpha_i + \varepsilon_i(A') = \varepsilon_i(B), \quad \text{if } Q^B_{A_1,A'} \neq 0, R_{A'} \neq 0.$$

Similarly, if $A_1 = \sum_{1\le j\le n} \beta_j E^{j,j} + \alpha_j E^{j,j+1}$, then

$$\deg_i(M) = \alpha_i + \deg_i(M') = \alpha_i + \varepsilon_i(A') = \varepsilon_i(B), \quad \text{if } Q^B_{A_1,A'} \neq 0, R_{A'} \neq 0.$$

Lemma follows. ■

By a result in [5] (see also [18]), there exists a monomial M_A such that

$$M_A = [A] + \sum_{A' \prec A} S_{A,A'}[A'], \quad \text{for some } S_{A,A'} \in \mathbb{Z}[v, v^{-1}]. \tag{71}$$

Since $[A]$ forms a basis for $\widehat{\mathbb{S}}_d$, the monomial M_A forms a basis for $\widehat{\mathbb{S}}_d$. In particular,

$$[A] = M_A + \sum_{A' \prec A} R_{A,A'} M_{A'}, \quad \text{for some } R_{A,A'} \in \mathbb{Z}[v, v^{-1}]. \tag{72}$$

Moreover, we have

Lemma 6.3.10. Suppose that $R_{A,A'} \neq 0$ in (71), then $\varepsilon_i(A) = \varepsilon_i(A')$.

Proof. We prove by induction with respect to $\preceq$ in descending order. If $A' = A$, it is trivial. Suppose that for all A'' such that $A' \prec A'' \preceq A$, the statement holds. If $[A']$ appears in $M_{A''}$ for some A'' such that $A' \prec A''$, then $\varepsilon_i(A) = \varepsilon_i(A'') = \deg_i(M_{A''}) = \varepsilon_i(A')$ by induction hypothesis. If $[A']$ does not appear in $M_{A''}$ for all A'' such that $A' \prec A''$, then the coefficient of $[A']$ in the right-hand side of (72) is 0, contradicting to the assumption. We are done. ■

Furthermore,

Lemma 6.3.11. Suppose that $\overline{[A]} = [A] + \sum_{A' \prec A} C_{A,A'}[A']$ for some $C_{A,A'} \in \mathbb{Z}[v, v^{-1}]$. If $C_{A,A'} \neq 0$, then $\varepsilon_i(A) = \varepsilon_i(A')$ for all $i \in \mathbb{Z}$.

Proof. By (72) and Lemma 6.3.10, we have $\overline{[A]} = M_A + \sum_{A' \prec A, \varepsilon_i(A') = \varepsilon_i(A)} \overline{R_{A,A'}} M_{A'}$. By Lemma 6.3.9, we have $M_{A'} = \sum_{A'' \preceq A', \varepsilon_i(A'') = \varepsilon_i(A')} S_{A',A''}[A'']$. The lemma follows by putting the previous two identities together. ■

Finally, we are ready to prove Theorem 6.3.2. We set $\phi = \{A' | P_{A,A'} \neq 0, \varepsilon_i(A') \neq \varepsilon_i(A)\}$. We only need to show that ϕ is empty. Pick an element B in ϕ that is maximal with respect to the partial order $\preceq$. Clearly, we have $B \neq A$. We rewrite $\{A\}$ as follows:

$$\{A\} = P_{A,B}[B] + \left(\sum_{B \prec A'} + \sum_{A' \prec B} + \sum_{A' \not\preceq B, B \not\preceq A'} \right) P_{A,A'}[A'].$$

Apply the bar operation to the above equality, we have

$$\{A\} = \overline{\{A\}} = \overline{P_{A,B}}\,\overline{[B]} + \left(\sum_{B \prec A'} + \sum_{A' \prec B} + \sum_{A' \not\preceq B, B \not\preceq A'} \right) \overline{P_{A,A'}}\,\overline{[A']}.$$

By Lemma 6.3.11, we know that the coefficient of $[B]$ in $\overline{[A']}$ for $B \prec A'$ is zero. Notice $[B]$ will not appear in the rest of the terms, except $\overline{[B]}$. Hence, by comparing the coefficients of $[B]$ in the previous two equalities, we must have $P_{A,B} = \overline{P_{A,B}}$. But $P_{A,B} \in v^{-1}\mathbb{Z}[v^{-1}]$ forces $P_{A,B} = 0$, a contradiction to the definition of ϕ. Hence, ϕ is empty. Theorem 6.3.2 follows.

6.4 Positivity of $\widehat{\Delta}$

We set $\widehat{X}_d(\mathbf{a}) = \{\mathbf{L} \in \widehat{X}_d \,|\, |\mathbf{L}_i / \mathbf{L}_{i-1}| = a_i, \quad \forall i\}$ and $\widehat{P}_{\mathbf{a}} = \mathrm{Stab}_{\widehat{G}_d}(\mathbf{L})$ for a fixed chain $\mathbf{L} \in \widehat{X}_d$. We still have the same commutative diagram as in Lemma 2.3.4.

$$\begin{array}{ccc}
\mathcal{A}_{G_d}(\widehat{X}_d(b) \times \widehat{X}_d(a)) & \xrightarrow{\;i^*_{b,a}\;} & \mathcal{A}_{\widehat{P}_b}(\widehat{X}_d(a)) \\
\Big\downarrow {\scriptstyle \widetilde{\Delta}_{b',a',b'',a''}} & & \Big\downarrow {\scriptstyle \pi_! \iota^*} \\
\begin{array}{c} \mathcal{A}_{G_{d'}}(\widehat{X}_{d'}(b') \times \widehat{X}_{d'}(a')) \\ \otimes \\ \mathcal{A}_{G_{d''}}(\widehat{X}_{d''}(b'') \times \widehat{X}_{d''}(a'')) \end{array} & \xrightarrow{\;i^*_{b',a'} \otimes i^*_{b'',a''}\;} & \mathcal{A}_{\widehat{P}_{b'}}(\widehat{X}_{d'}(a')) \otimes \mathcal{A}_{\widehat{P}_{b''}}(\widehat{X}_{d''}(a'')).
\end{array}$$

So the positivity of $\widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}$ is reduced to the positivity of $\pi_! \iota^*$.

Fix $\mathbf{b}', \mathbf{b}''$ such that $\mathbf{b}' + \mathbf{b}'' = \mathbf{b}$. Let $d' = |\mathbf{b}'|$ and $d'' = |\mathbf{b}''|$. Let $\mathbf{V} = \mathbf{T} \oplus \mathbf{W}$ and $\mathbf{L_b} = \mathbf{L_{b'}} \oplus \mathbf{L_{b''}}$. Thus, we have $\pi'(\mathbf{L_b}) = \mathbf{L_{b'}}, \pi''(\mathbf{L_b}) = \mathbf{L_{b''}}$. Let $\mathbf{L}_i$ be the i-th lattice in $\mathbf{L_a}$. We consider the following subset in $\widehat{X}_d(\mathbf{a})$.

$$Y_{\mathbf{a}}^{\mathbf{L}_0,p} := \{\widetilde{\mathbf{L}} \in \widehat{X}_d(\mathbf{a}) | \varepsilon^p \mathbf{L}_0 \subseteq \widetilde{\mathbf{L}}_0 \subseteq \varepsilon^{-p} \mathbf{L}_0\}, \quad \forall p \in \mathbb{Z}_{\geq 0}.$$

It is well known that $Y^{\mathbf{L}_0,p}$ for various p is a $G_{\mathbf{L_b}}$-invariant algebraic variety over $\overline{\mathbb{F}_q}$ if we replace the ground field $\mathbb{F}_q((\varepsilon))$ by $\overline{\mathbb{F}_q}((\varepsilon))$, which we shall assume now and for the rest of this section. Moreover, there exists a p_0 such that

$$X_A^{\mathbf{L_b}} := \{\mathbf{L}' | (\mathbf{L_b}, \mathbf{L}') \in \mathcal{O}_A\} \subseteq Y^{\mathbf{L}_0,p}, \quad p \geq p_0.$$

Indeed, we have $a_{0,p} = 0$ and $a_{p,0}$ for $p >> 0$ due to the fact that $\sum_{j \in \mathbb{Z}} a_{0,j}, \sum_{i \in \mathbb{Z}} a_{i,0} < \infty$. The 1st condition implies that $\mathbf{L}_0 \subseteq \widetilde{\mathbf{L}}_p$, if $\widetilde{\mathbf{L}} \in X_A^{\mathbf{L}}$, while $\widetilde{\mathbf{L}}_0 \subseteq \mathbf{L}_p$, if $\widetilde{\mathbf{L}} \in X_A^{\mathbf{L}}$, follows from the 2nd. Fix an l such that $p < ln$. Then we have

$$\varepsilon^l \mathbf{L}_0 \subseteq \widetilde{\mathbf{L}}_0 \subseteq \varepsilon^{-l} \mathbf{L}_0.$$

Set $p_0 = l$, then we have $X_A^{\mathbf{L_b}} \subseteq Y^{\mathbf{L}_0,p}$ for $p \geq p_0$.

Now we fix a 1-parameter subgroup of $P_{\mathbf{L_b}}$:

$$\lambda : \mathrm{GL}(1, \overline{\mathbb{F}}_q) \to P_{\mathbf{L_b}}, t \mapsto \begin{pmatrix} 1_{\mathbf{T}} & 0 \\ 0 & t.1_{\mathbf{W}} \end{pmatrix}.$$

The fixed point set $(Y_{\mathbf{a}}^{\mathbf{L}_0,p})^{\mathrm{GL}(1,\overline{\mathbb{F}_q})} = \sqcup_{\mathbf{a}',\mathbf{a}''} Y_{\mathbf{a}'}^{\mathbf{L}_0',p} \times Y_{\mathbf{a}''}^{\mathbf{L}_0'',p}$, and the attracting set associated to $Y_{\mathbf{a}'}^{\mathbf{L}_0',p} \times Y_{\mathbf{a}''}^{\mathbf{L}_0'',p}$ is

$$Y_{\mathbf{a}',\mathbf{a}''}^{\mathbf{L}_0,p} = \left\{ \widetilde{\mathbf{L}} \in Y_{\mathbf{a}}^{\mathbf{L}_0,p} | \pi'(\widetilde{\mathbf{L}}) \in Y_{\mathbf{a}'}^{\mathbf{L}_0',p}, \pi''(\widetilde{\mathbf{L}}) \in Y_{\mathbf{a}''}^{\mathbf{L}_0'',p} \right\}.$$

Hence, we have the following cartesian diagram.

$$\begin{array}{ccccc} Y_a^{L_0,p} & \xleftarrow{\iota_1} & Y_{a',a''}^{L_0,p} & \xrightarrow{\pi_1} & Y_{a'}^{L_0',p} \times Y_{a''}^{L_0'',p} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{X}_d(a) & \xleftarrow{\iota} & \widehat{X}^+_{a,a',a''} & \xrightarrow{\pi} & \widehat{X}_{d'}(a') \times \widehat{X}_{d''}(a''), \end{array}$$

where vertical maps are inclusions and top horizontal maps are induced from bottom ones.

Hence, the positivity of $\pi_! \iota^*$ is boiled down to that of $\pi_{1!}\iota_1^*$, which follows from Braden's [4] work, since all objects involved are in the category of algebraic varieties over $\overline{\mathbb{F}_q}$.

Proposition 6.4.1. If $\widetilde{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}(\{A\}_d) = \sum \widetilde{m}_A^{B,C}\{B\}_d \otimes \{C\}_d$, then $\widetilde{m}_A^{B,C} \in \mathbb{Z}_{\geq 0}[\mathbf{v}, \mathbf{v}^{-1}]$.

Follows is an affinization of Proposition 2.3.6, by Proposition 6.4.1, and Theorem 6.3.1

Theorem 6.4.2. If $\mathbb{\Delta}_{\mathbf{b}',\mathbf{a}',\mathbf{b}'',\mathbf{a}''}(\{A\}_d) = \sum \widehat{m}_A^{B,C}\{B\}_d \otimes \{C\}_d$, then $\widehat{m}_A^{B,C} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

Following [24], the transfer map $\widehat{\phi}_{d,d-n} : \widehat{\mathbb{S}}_d \to \widehat{\mathbb{S}}_{d-n}$ is the composition of

$$\widehat{\mathbb{S}}_d \xrightarrow{\widetilde{\Delta}} \widehat{\mathbb{S}}_{d-n} \otimes \widehat{\mathbb{S}}_n \xrightarrow{\xi^{-1}\otimes\chi} \widehat{\mathbb{S}}_{d-n} \otimes \mathbb{A} \cong \widehat{\mathbb{S}}_{d-n},$$

where ξ is in Remark 6.3.3 and χ is the signed representation of $\widehat{\mathbb{S}}_n$ defined in [24, 1.8].

Note that by [24, 1.12] and an argument similar to [21, 3.3], χ sends a canonical basis element to 1 or 0. By Remark 6.3.3 and Proposition 6.4.1, we have

Corollary 6.4.3. $\widehat{\phi}_{d,d-n}(\{A\}_d) = \sum c_{A,A'}\{A'\}_{d-n}$ where $c_{A,A'} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

6.5 Positivity in quantum affine $\mathfrak{sl}_n$

Let $\tilde{\mathfrak{S}}_n$ be the set of all $\mathbf{a} = (a_i)_{i\in\mathbb{Z}}$ such that $a_i \in \mathbb{Z}$ and $a_i = a_{i+n}$ for all $i \in \mathbb{Z}$. Let $\widehat{\mathbb{Y}} = \{\mathbf{a} \in \tilde{\mathfrak{S}}_n | \sum_{1\leq i\leq n} a_i = 0\}$. Define an equivalence relation $\sim$ on $\tilde{\mathfrak{S}}_n$ by declaring $\mathbf{a} \sim \mathbf{b}$ if there is a z in $\mathbb{Z}$ such that $a_i - b_i = z$ for all i. Let $\widehat{\mathbb{X}}$ be the set $\tilde{\mathfrak{S}}_n/\sim$ of all equivalence classes in $\tilde{\mathfrak{S}}_n$ with respect to $\sim$. Let $\bar{\mathbf{a}}$ denote the class of $\mathbf{a}$. Both $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{Y}}$ admit a natural abelian group structure with the component-wise addition. Moreover, we have a bilinear form

$$\langle -, - \rangle : \widehat{\mathbb{Y}} \times \widehat{\mathbb{X}} \to \mathbb{Z}, \quad \langle \mathbf{b}, \bar{\mathbf{a}} \rangle = \sum_{1\leq i\leq n} b_i a_i.$$

Set $\widehat{I} = \mathbb{Z}/n\mathbb{Z}$. For $i \in \widehat{I}$, we associate an element, still denoted by i, in $\widehat{\mathbb{Y}}$ whose value is 1 for each integer in the equivalence class i and zero otherwise. This defines a map $\widehat{I} \to \widehat{\mathbb{Y}}$. The same map induces a map $\widehat{I} \to \widehat{\mathbb{X}}$, which sends $i \in \widehat{I}$ to the equivalence class of $i \in \widehat{\mathbb{Y}}$ in $\tilde{\mathfrak{S}}_n$. By abuse of notations, we still use i to denote its image in $\widehat{\mathbb{Y}}$. The data $(\widehat{\mathbb{Y}}, \widehat{\mathbb{X}}, \langle -, - \rangle, \widehat{I} \subset \widehat{\mathbb{Y}}, \widehat{I} \subset \widehat{\mathbb{X}})$ is a root datum of affine $\mathbf{A}_{n-1}$, neither $\widehat{\mathbb{X}}$-regular nor $\widehat{\mathbb{Y}}$-regular.

By definition, the quantum affine $\mathfrak{sl}_n$ attached to the above root datum, denoted by $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$, is an associative algebra over $\mathbb{Q}(v)$ generated by the generators: $\mathbb{E}_i, \mathbb{F}_i, \mathbb{K}_\mu$ for all $i \in \widehat{I}, \mu \in \widehat{\mathbb{Y}}$, and subject to the relations $\mathbb{K}_1\mathbb{K}_2 \cdots \mathbb{K}_n = 1$ and (15), for all $i, j \in \widehat{I}$. Note that the 1st defining relation of $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$ is due to the degeneracy of the Cartan datum.

Moreover, $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$ admits a Hopf algebra structure, whose comultiplication is defined by

$$\Delta(\mathbb{E}_i) = \mathbb{E}_i \otimes \mathbb{K}_i + 1 \otimes \mathbb{E}_i,\ \Delta(\mathbb{F}_i) = \mathbb{F}_i \otimes 1 + \mathbb{K}_i^{-1} \otimes \mathbb{F}_i,\ \Delta(\mathbb{K}_i) = \mathbb{K}_i \otimes \mathbb{K}_i, \quad \forall i \in \widehat{I}. \tag{73}$$

Let $\dot{\mathbb{U}}(\widehat{\mathfrak{sl}}_n)$ be Lusztig's idempotented algebra associated to $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$. It is defined similar to that of quantum $\mathfrak{sl}_n$ in Section 2.5. Similar to the finite case, Δ then induces a linear map

$$\Delta_{\overline{\mu'},\overline{\lambda'},\overline{\mu''},\overline{\lambda''}} : {}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}}(\widehat{\mathfrak{sl}}_n) \to {}_{\overline{\mu'}}\mathbb{U}_{\overline{\lambda'}}(\widehat{\mathfrak{sl}}_n) \otimes {}_{\overline{\mu''}}\mathbb{U}_{\overline{\lambda''}}(\widehat{\mathfrak{sl}}_n), \tag{74}$$

where ${}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}}(\widehat{\mathfrak{sl}})$ is defined similar to ${}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}}$ in finite case and $\overline{\mu} = \overline{\mu'} + \overline{\mu''}$, $\overline{\lambda} = \overline{\lambda'} + \overline{\lambda''}$ in $\widehat{X}$.

By the same definition as ϕ_d in (17), we still have an algebra homomorphism

$$\widehat{\phi}_d : \mathbb{U}(\widehat{\mathfrak{sl}}_n) \to \widehat{\mathbb{S}}_d.$$

But this time $\widehat{\phi}_d$ is not surjective anymore. Then the rest of the result in finite case can be transported to affine case. In particular, we have

Theorem 6.5.1. Let $b \in {}_{\overline{\mu}}\mathbb{U}_{\overline{\lambda}}(\widehat{\mathfrak{sl}})$ be a canonical basis element of $\dot{\mathbb{U}}(\widehat{\mathfrak{sl}}_n)$. If $\Delta_{\overline{\mu'},\overline{\lambda'},\overline{\mu''},\overline{\lambda''}}(b) = \sum_{b',b''} \hat{m}_b^{b',b''} b' \otimes b''$, then $\hat{m}_b^{b',b''} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

By [22, 25.2.2], Theorem 6.5.1 remains valid over other root datum of affine type $\mathbf{A}_{n-1}$. We end this section with the following remark.

Remark 6.5.2. Soon after the 1st version of this paper appeared in arXiv, Fu uploaded the paper arXiv:1511.05745 on arXiv proving the positivity result in Theorem 6.5.1.

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