SPALTENSTEIN VARIETIES OF PURE DIMENSION

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(Communicated by Kailash C. Misra)

In memory of my uncle Renyi Huang

ABSTRACT. We show that Spaltenstein varieties of classical groups are pure dimensional when the Jordan-type of the nilpotent element involved is an even or odd partition. We further show that they are Lagrangian in the partial resolutions of the associated nilpotent Slodowy slices, from which their dimensions are known to be one half of the dimension of the partial resolution minus the dimension of the nilpotent orbit. The results are then extended to the σ-quiver-variety setting.

1. INTRODUCTION

1.1. **Spaltenstein varieties.** Let G be a complex reductive group. Fix a parabolic subgroup P of G and a nilpotent element x in $\mathrm{Lie}(G)$. The Spaltenstein variety of the triple (G, P, x) is defined to be

$$X_x^P = \{gP \in G/P | g^{-1}xg \in \mathfrak{n}_{\mathrm{Lie}(P)}\},$$

where $\mathfrak{n}_{\mathrm{Lie}(P)}$ is the nilpotent radical of $\mathrm{Lie}(P)$. When P is a Borel subgroup, a Spaltenstein variety is more commonly referred to as a Springer fiber [Spr76]. In general, a Spaltenstein variety is neither smooth nor irreducible. So an immediate question of substantial interest is if it is pure dimensional, that is, if the dimensions of irreducible components of an X_x^P are the same. It was answered in the affirmative in the following two fundamental cases by Spaltenstein [Sp76, Sp77] in the 1970s, and independently by Steinberg [St74] for case (a) when G is a general linear group:

(a) P is a Borel subgroup.
(b) G is a general linear group.

Spaltenstein further provided an example in [Sp82, 11.6] showing that the variety X_x^P is not always pure dimensional for a nilpotent element in $\mathfrak{so}_8$ of Jordan-type $(1, 2^2, 3)$. This example is recalled in Section 4, together with a few more in [Sp82] where X_x^P can be described explicitly with fresh light casted upon it. Beyond Steinberg and Spaltenstein's results, little is known on the pure dimensionality of X_x^P.

   In this paper, we shall prove the following.

Theorem A. *The Spaltenstein variety X_x^P is pure dimensional if*

    (c) *G is classical and the Jordan-type of x is an even or odd partition, i.e., of the form $1^{\mathbf{w_1}} 3^{\mathbf{w_3}} 5^{\mathbf{w_5}} \cdots$ or $2^{\mathbf{w_2}} 4^{\mathbf{w_4}} 6^{\mathbf{w_6}} \cdots$.*

Our approach is to study Spaltenstein varieties in the context of symplectic geometry.

1.2. **Symplectic geometry and $\mathbb{C}^*$-action.** As is generally known, complex symplectic geometry provides a new and conceptual way to understand the pure dimensionality of a complex variety. Precisely, there is the following remarkable result, whose proof can be found in the proof of Proposition 5.4.7 in [G09]. Note that we work in the setting of complex algebraic geometry.

Theorem B. *Suppose that $p : \widetilde{Y} \to Y$ is a proper morphism from a smooth symplectic algebraic variety, with algebraic symplectic 2-form, to an affine variety. Suppose further that both varieties admit a $\mathbb{C}^*$-action, compatible with p. If the following two conditions hold:*

    • *the $\mathbb{C}^*$-action provides a contraction of Y to its fixed point locus $Y^{\mathbb{C}^*}$,*
    • *the $\mathbb{C}^*$-action on $\widetilde{Y}$ has weight 1 on the symplectic form ω on $\widetilde{Y}$, i.e.,*

$$t^* \omega = t\omega, \quad \forall t \in \mathbb{C}^*,$$

then the fiber $p^{-1}(Y^{\mathbb{C}^*})$, or rather its associated reduced scheme, is Lagrangian in $\widetilde{Y}$.

Being Lagrangian implies that $p^{-1}(Y^{\mathbb{C}^*})$ is pure dimensional, provided that $\widetilde{Y}$ is so, and moreover its dimension is one half of that of $\widetilde{Y}$.

It is exactly the framework of Theorem B that Spaltenstein variety is put under and that the proof of Theorem A falls out, which we shall discuss in more detail as follows.

1.3. **Slodowy slices and their partial resolutions.** Retaining the setting in Section 1.1, the cotangent bundle $T^*(G/P)$ of G/P yields a partial resolution of singularities of the closure of a nilpotent orbit $\mathcal{O}_e$ in $\mathrm{Lie}(G)$ for a Richardson element e:

$$\pi_P' : T^*(G/P) \to \overline{\mathcal{O}}_e.$$

Here the terminology "partial" refers to the fact that the restriction of π_P' to the orbit $\mathcal{O}_e$ is generically finite, but not isomorphic, in general. When P is a Borel, the morphism π_P' is the Springer resolution to the nilcone of G and a genuine resolution of singularities. On the other hand, fixing an $\mathfrak{sl}_2(\mathbb{C})$-triple (x, y, h) in $\mathrm{Lie}(G)$, one can consider the Slodowy slice $S_x := x + \ker \mathrm{ad}(y)$ (see [Sl80]). We set $S_{e,x} = \overline{\mathcal{O}}_e \cap S_x$ and $\widetilde{S}_{e,x} = (\pi_P')^{-1}(S_{e,x})$ (so that $S_{e,x}$ is nonempty if and only if $x \in \overline{\mathcal{O}}_e$). The above map π_P' restricts to a partial resolution of the nilpotent Slodowy slice $S_{e,x}$:

(1) $$\pi_P : \widetilde{S}_{e,x} \to S_{e,x} \quad \text{with } \pi_P^{-1}(x) = X_x^P.$$

Again when P is a Borel, the morphism π is a genuine resolution of singularities. The cotangent bundle $T^*(G/P)$ carries a canonical symplectic structure, i.e., a closed 2-form, and from which the variety $\widetilde{S}_{e,x}$ inherits one, say ω, as well. The variety $S_{e,x}$ is clearly an affine variety. Thanks to [BM83, Corollary 3.5 b)], it is known that

(2) $$\dim X_x^P \leq \frac{1}{2} \dim \widetilde{S}_{e,x}.$$

In the cases (a) and (b) in Section 1.1, the above inequality becomes an equality and X_x^P is Lagrangian in $\widetilde{S}_{e,x}$. We shall show that the same holds for the case (c) in Theorem A. Moreover, $\widetilde{S}_{e,x}$ is pure dimensional in general: it is a reduced complete intersection in $T^*(G/P)$ of dimension $\dim T^*(G/P) - \dim \mathcal{O}_x$ (see [G08, Corollary 1.3.8]). Therefore we actually have a stronger version of Theorem A.

Theorem C. *If G is a classical group and the Jordan-type of x is an even or odd partition, then the Spaltenstein variety X_x^P is Lagrangian in $\widetilde{S}_{e,x}$ in* (9), *and hence of pure dimension $\frac{1}{2}\dim T^*(G/P) - \frac{1}{2}\dim \mathcal{O}_x$.*

With the above discussion, the proof of Theorem C (and hence Theorem A) finally boils down to a search of the desired $\mathbb{C}^*$-actions for $\widetilde{S}_{e,x}$ and $S_{e,x}$ to apply Theorem B.

Both varieties $\widetilde{S}_{e,x}$ and $S_{e,x}$ admit a natural $\mathbb{C}^*$-action induced from the $\mathfrak{sl}_2(\mathbb{C})$-triple (x, y, h) so that π_P is $\mathbb{C}^*$-equivariant. Moreover the $\mathbb{C}^*$-action provides a contraction of $S_{e,x}$ to $\{x\}$, its $\mathbb{C}^*$-fixed point ([G08, 1.4]). However, the $\mathbb{C}^*$-action on the symplectic structure ω has weight 2 instead of weight 1, as required in Theorem B. This defect is expected in light of Spaltenstein's example: there is no uniform $\mathbb{C}^*$-action on $\widetilde{S}_{e,x}$ and $S_{e,x}$ for all e and x satisfying all conditions in Theorem B.

Instead we obtain the desired $\mathbb{C}^*$-actions in the setting of Nakajima quiver varieties [N94, N98] and their variants in [Li19], from which this paper grew.

1.4. $\mathbb{C}^*$-action on Nakajima varieties.
Thanks to the works of Nakajima [N94] and Maffei [M05], the proper map π_P for G being a general linear group has an incarnation as Nakajima quiver varieties attached to a type-A quiver

$$(3) \qquad \pi_A : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})_A \to \mathfrak{M}_1(\mathbf{v}, \mathbf{w})_A.$$

Here $\mathbf{v}$ and $\mathbf{w}$ are tuples of integers determined by the Jordan-types of the Richardson element e and x, respectively, and ζ is a generic parameter used for the stability condition. The orientation induces intrinsically a $\mathbb{C}^*$-action on the quiver varieties $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})_A$ and $\mathfrak{M}_1(\mathbf{v}, \mathbf{w})_A$. This action satisfies all conditions in Theorem B and hence provides a conceptual proof of the pure dimensionality of X_x^P for G being a general linear group, i.e., case 1.1(b).

If G is classical, i.e., an orthogonal or symplectic group, the map π_P admits a quiver description $\pi_{\sigma,A}$, as a restriction of π_A, in the recent work [Li19], with $\widetilde{S}_{e,x}$ and $S_{e,x}$ realized as the fixed-point loci $\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w})_A$ (resp., $\mathfrak{S}_1(\mathbf{v}, \mathbf{w})_A$) of Nakajima varieties $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})_A$ (resp., $\mathfrak{M}_1(\mathbf{v}, \mathbf{w})_A$) under a specific involution σ:

$$(4) \qquad \pi_{\sigma,A} : \mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w})_A \to \mathfrak{S}_1(\mathbf{v}, \mathbf{w})_A.$$

The $\mathbb{C}^*$-actions on Nakajima varieties cannot be compatible with the involution in general, again due to Spaltenstein's example. The crucial observation is that the place where the $\mathbb{C}^*$-action and the involution σ is compatible is where X_x^P is Lagrangian. To this end, we show that the tuple $\mathbf{w}$ under the conditions in Theorem C are the compatible places for the $\mathbb{C}^*$-action and the involution, hence providing a proof of Theorems C and A finally.

The arguments are indeed not restricted to type-A graphs. We are able to establish a result that is valid for all Dynkin graphs. We drop the subscript A in (3) and (4) to denote the morphism between Nakajima varieties of a fixed Dynkin graph.

Theorem D. *Assume that* $\mathbf{w}_i\mathbf{w}_j = 0$ *if there is an edge joining i and j. Then the fiber of the $\mathbb{C}^*$-fixed point under π_σ is Lagrangian in $\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w})$.*

The main content of the paper is the study of the compatibility of the $\mathbb{C}^*$-action and the automorphism σ in order to prove Theorem D. When the signature c^0 of the diagram isomorphism in the automorphism σ is -1, we can drop the assumption on $\mathbf{w}$ in Theorem D and this more general result is stated in Theorem E.

1.5. **Layout of the paper.** In Section 2, we recall Nakajima varieties and their σ variants. In Section 3, we study the compatibility of $\mathbb{C}^*$-action with the various isomorphisms in the definition of σ-quiver varieties. In Section 4, we reproduce Spaltenstein's examples in [Sp82, 11.6, 11.8] with new observations on being Lagrangian.

2. Preliminaries on quiver varieties

In the section, we recall briefly Nakajima varieties [N94, N98] and their σ variants in [Li19]. Our treatment follows closely Sections 1-4 in [Li19].

2.1. **Nakajima varieties.** Let Γ be a Dynkin graph. Let I and H be the vertex and arrow set, respectively. For each arrow h, let $\mathbf{o}(h)$ and $\mathbf{i}(h)$ be its outgoing and incoming vertex. There is an involution on the arrow set $^-: H \to H$, $h \mapsto \bar{h}$ such that $\mathbf{o}(\bar{h}) = \mathbf{i}(h)$ and $\mathbf{i}(\bar{h}) = \mathbf{o}(h)$. Let $V = \bigoplus_{i \in I} V_i$ and $W = \bigoplus_{i \in I} W_i$ be two finite dimensional I-graded vector spaces over the complex field $\mathbb{C}$ of dimension vectors $\mathbf{v} = (\mathbf{v}_i)_{i \in I}$ and $\mathbf{w} = (\mathbf{w}_i)_{i \in I}$, respectively. The framed representation space of the graph Γ in $V \oplus W$ is

$$(5) \qquad \mathbf{M}(\mathbf{v}, \mathbf{w}) = \bigoplus_{h \in H} \mathrm{Hom}(V_{\mathbf{o}(h)}, V_{\mathbf{i}(h)}) \oplus \bigoplus_{i \in I} \mathrm{Hom}(W_i, V_i) \oplus \mathrm{Hom}(V_i, W_i).$$

When V and W shall be highlighted, we write $\mathbf{M}(V, W)$ for $\mathbf{M}(\mathbf{v}, \mathbf{w})$. An element in $\mathbf{M}(\mathbf{v}, \mathbf{w})$ is denoted by $\mathbf{x} \equiv (x, p, q) \equiv (x_h, p_i, q_i)_{h \in H, i \in I}$ where x_h is in $\mathrm{Hom}(V_{\mathbf{o}(h)}, V_{\mathbf{i}(h)})$, p_i in $\mathrm{Hom}(W_i, V_i)$, and q_i in $\mathrm{Hom}(V_i, W_i)$. Let $\varepsilon^0 : H \to \{\pm 1\}$ be an orientation function such that $\varepsilon^0(h) + \varepsilon^0(\bar{h}) = 0 \; \forall h \in H$. To a point $\mathbf{x} \in \mathbf{M}(\mathbf{v}, \mathbf{w})$, we set

$$(6) \qquad a_i(\mathbf{x}) = (q_i, x_h)_{h:\mathbf{o}(h)=i} \quad \text{and} \quad b_i(\mathbf{x}) = (p_i, \varepsilon^0(\bar{h})x_h)_{h:\mathbf{i}(h)=i}.$$

The space $\mathbf{M}(\mathbf{v}, \mathbf{w})$ admits a symplectic structure with respect to ε^0 given by

$$(7) \qquad \omega(\mathbf{x}, \mathbf{x}') = \mathrm{trace}\left(\sum_{i \in I} b_i(\mathbf{x})a_i(\mathbf{x}') - q_i p_i'\right), \quad \forall \mathbf{x}, \mathbf{x}' \in \mathbf{M}(\mathbf{v}, \mathbf{w}).$$

Let $\mathrm{G}_\mathbf{v} = \prod_{i \in I} \mathrm{GL}(V_i)$ act on $\mathbf{M}(\mathbf{v}, \mathbf{w})$ from the left as follows. For all $g = (g_i)_{i \in I} \in \mathrm{G}_\mathbf{v}$ and $\mathbf{x} \in \mathbf{M}(\mathbf{v}, \mathbf{w})$, we define $g.\mathbf{x} = \mathbf{x}' \equiv (x_h', p_i', q_i')$ where $x_h' = g_{\mathbf{i}(h)} x_h g_{\mathbf{o}(h)}^{-1}$, $p_i' = g_i p_i$, and $q_i' = q_i g_i^{-1}$ for all $h \in H$ and $i \in I$. Let

$$\mu_\mathbb{C} : \mathbf{M}(\mathbf{v}, \mathbf{w}) \to \mathrm{Lie}(\mathrm{G}_\mathbf{v})^*$$

be the moment map associated to the $\mathrm{G}_\mathbf{v}$-action on the symplectic space $\mathbf{M}(\mathbf{v}, \mathbf{w})$. After identifying $\mathrm{Lie}(\mathrm{G}_\mathbf{v}) = \bigoplus_{i \in I} \mathfrak{gl}(V_i)$ with its dual $\mathrm{Lie}(\mathrm{G}_\mathbf{v})^*$ via the trace form, the ith component of $\mu_\mathbb{C}$ is given by $\mu_\mathbb{C}^{(i)}(\mathbf{x}) = b_i(\mathbf{x})a_i(\mathbf{x})$.

Let $[\mathbf{x}]$ denote the $\mathrm{G}_\mathbf{v}$-orbit of $\mathbf{x}$ in $\mathbf{M}(\mathbf{v}, \mathbf{w})$.

Fix an embedding $\mathbb{C}^I \to \mathrm{Lie}(\mathrm{G_v})$ by $(\zeta_{\mathbb{C}}^{(i)})_{i\in I} \mapsto (\zeta_{\mathbb{C}}^{(i)}\mathrm{Id}_{V_i})_{i\in I}$ for all $\zeta_{\mathbb{C}} = (\zeta_{\mathbb{C}}^{(i)})_{i\in I} \in \mathbb{C}^I$. Let $\Lambda_{\zeta_{\mathbb{C}}}(\mathbf{v}, \mathbf{w})$ be the fiber $\mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$. The group $\mathrm{G_v}$ acts on $\Lambda_{\zeta_{\mathbb{C}}}(\mathbf{v}, \mathbf{w})$.

Let $\xi = (\xi_i)_{i\in I} \in \mathbb{Z}^I$. Fixing an element $x = (x_h)_{h\in H}$ in the first component of $\mathbf{M}(\mathbf{v}, \mathbf{w})$ and an I-graded subspace $S = (S_i)_{i\in}$ of V, we say that S is x-invariant if $x_h(S_{\mathbf{o}(h)}) \subseteq S_{\mathbf{i}(h)}$ for all $h \in H$. A point $\mathbf{x} = (x, p, q)$ in $\mathbf{M}(\mathbf{v}, \mathbf{w})$ is called ξ-semistable if the following two stability conditions are satisfied. For any I-graded subspaces S and T of V of dimension $\mathbf{s}$ and $\mathbf{t}$, respectively,

(S1)   if S is x-invariant and $S \subseteq \ker q$, then $\xi \cdot \mathbf{s} \le 0$,

(S2)   if T is x-invariant and $T \supseteq \mathrm{im}\, p$, then $\xi \cdot \mathbf{t} \le \xi \cdot \mathbf{v}$.

Let $\Lambda_{\zeta_{\mathbb{C}}}^{\xi\text{-}ss}(\mathbf{v}, \mathbf{w})$ be the $\mathrm{G_v}$-invariant set of all ξ-semistable points in $\Lambda_{\zeta_{\mathbb{C}}}(\mathbf{v}, \mathbf{w})$.

Let $\mathbf{C} = (c_{ij})_{i,j\in I}$ be the Cartan matrix of the graph Γ. We set

$$R_+ = \{\gamma \in \mathbb{N}^I \mid {}^t\gamma \mathbf{C}\gamma \le 2\}\backslash\{0\},$$
$$R_+(\mathbf{v}) = \{\gamma \in R_+ \mid \gamma_i \le \mathbf{v}_i \forall i \in I\},$$
$$D_\gamma = \{a \in \mathbb{C}^I \mid a \cdot \gamma = 0\}.$$

A parameter $\zeta = (\xi, \zeta_{\mathbb{C}}) \in \mathbb{Z}^I \times \mathbb{C}^I$ is called generic if it satisfies $\xi \in \mathbb{Z}^I \backslash \bigcup_{\gamma\in R_+(\mathbf{v})} D_\gamma$ or $\zeta_{\mathbb{C}} \in \mathbb{C}^I \backslash \bigcup_{\gamma\in R_+(\mathbf{v})} D_\gamma$. From now on, we assume that ζ is generic. When ζ is generic, the group $\mathrm{G_v}$ acts freely on $\Lambda_{\zeta_{\mathbb{C}}}^{\xi\text{-}ss}(\mathbf{v}, \mathbf{w})$. Following Nakajima [N94, N98], we define the quiver-variety attached to the data $(\Gamma, \varepsilon^0, \mathbf{v}, \mathbf{w}, \zeta)$ to be

(8) $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) = \Lambda_{\zeta_{\mathbb{C}}}^{\xi\text{-}ss}(\mathbf{v}, \mathbf{w})/\mathrm{G_v}, \quad \zeta \equiv (\xi, \zeta_{\mathbb{C}}) \in \mathbb{Z}^I \times \mathbb{C}^I$ generic.

Let $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ be the affinization of $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$, with which is equipped a projective morphism $\pi : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$. Let $\mathfrak{M}_1(\mathbf{v}, \mathbf{w})$ be the image of $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ under π so that π factors through a proper map under the same notation, which is (3) in type A:

(9) $\pi : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_1(\mathbf{v}, \mathbf{w}).$

The variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ is smooth and symplectic with the latter induced from $\mathbf{M}(\mathbf{v}, \mathbf{w})$.

2.2. σ-quiver varieties.

In this section, we recall σ-quiver varieties from [Li19].

2.2.1. *Reflection functors.* Recall the Cartan matrix $\mathbf{C} = (c_{ij})$. For each $i \in I$, we define a bijection $s_i : \mathbb{Z}^I \to \mathbb{Z}^I$ by $s_i(\xi) = \xi'$ where $\xi'_j = \xi_j - c_{ji}\xi_i$, $\xi = (\xi_j)_{j\in I}$, $\xi' = (\xi'_j)_{j\in I} \in \mathbb{Z}^I$. Let $\mathcal{W}$ be the Weyl group generated by s_i for all $i \in I$.

Let $s_i *_\mathbf{w} \mathbf{v}$ denote the vector whose j-component is $\mathbf{v}_j$ if $j \ne i$ and whose ith component is $\mathbf{w}_i + \sum_{h:\mathbf{o}(h)=i} \mathbf{v}_{\mathbf{i}(h)} - \mathbf{v}_i$.

The reflection functor S_i of Nakajima, Lusztig, and Maffei [L00, M02, N03] associated to the simple reflection s_i is defined to be

$$S_i : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{s_i(\zeta)}(s_i *_\mathbf{w} \mathbf{v}, \mathbf{w}), [\mathbf{x}] \mapsto [\mathbf{x}'] \quad \text{if } \xi_i < 0 \text{ or } \zeta_{\mathbb{C}}^{(i)} \ne 0,$$

where the pair $([\mathbf{x}], [\mathbf{x}'])$ satisfies the conditions (R1)-(R4) as follows. Let V' be

a vector space of dimension $s_i *_{\mathbf{w}} \mathbf{v}$ such that $V_j' = V_j$ if $j \neq i$ and $U_i = W_i \oplus \bigoplus_{h \in H: \mathbf{o}(h) = i} V_{\mathbf{i}(h)}$

(R1) $\qquad 0 \longrightarrow V_i' \xrightarrow{a_i(\mathbf{x}')} U_i \xrightarrow{b_i(\mathbf{x})} V_i \longrightarrow 0 \quad$ is exact,

(R2) $\qquad a_i(\mathbf{x})b_i(\mathbf{x}) - a_i(\mathbf{x}')b_i(\mathbf{x}') = \zeta_{\mathbb{C}}'^{(i)}, \quad \zeta_{\mathbb{C}}' = s_i(\zeta_{\mathbb{C}}),$

(R3) $\qquad x_h = x_h', p_j = p_j', q_j = q_j' \qquad$ if $\mathbf{o}(h) \neq i, \mathbf{i}(h) \neq i$ and $j \neq i,$

(R4) $\qquad \mu_j(\mathbf{x}) = \zeta_{\mathbb{C}}^{(j)}, \mu_j(\mathbf{x}') = \zeta_{\mathbb{C}}'^{(j)} \qquad$ if $j \neq i.$

Since $(s_i(\xi))_i > 0$ if $\xi_i < 0$, we can define the reflection S_i when $\xi_i > 0$ by switching the roles of $\mathbf{x}$ and $\mathbf{x}'$. So if $\boldsymbol{\omega} = s_{i_1}s_{i_2}\cdots s_{i_l} \in \mathcal{W}$ and ζ is generic, the reflection functor $S_{\boldsymbol{\omega}}$ is defined to be the composition of the S_i's:

(10) $\qquad S_{\boldsymbol{\omega}} = S_{i_1}S_{i_2}\cdots S_{i_l} : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{\boldsymbol{\omega}(\zeta)}(\boldsymbol{\omega} *_{\mathbf{w}} \mathbf{v}, \mathbf{w}),$

where $\boldsymbol{\omega} *_{\mathbf{w}} \mathbf{v}$ is a composition of $s_{i_j} *_{\mathbf{w}} \mathbf{v}$'s.

2.2.2. *The transpose* τ. To any linear transformation $T : E \to E'$ between two vector spaces, each equipped with a nondegenerate bilinear form $(-,-)_E$ and $(-,-)_{E'}$, we define its right adjoint $T^* : E' \to E$ by the rule

$$(T(e), e')_{E'} = (e, T^*(e'))_E \quad \forall e \in E, e' \in E'.$$

There is an isomorphism $\operatorname{Hom}(E, E') \cong \operatorname{Hom}(E', E)$ defined by $T \mapsto T^*$.

Assume that the ith components V_i and W_i of V and W are equipped with nondegenerate bilinear forms for all $i \in I$. We define an automorphism

$$\tau : \mathbf{M}(\mathbf{v}, \mathbf{w}) \to \mathbf{M}(\mathbf{v}, \mathbf{w}), \quad \mathbf{x} = (x_h, p_i, q_i) \mapsto {}^\tau\mathbf{x} = ({}^\tau x_h, {}^\tau p_i, {}^\tau q_i),$$

where ${}^\tau x_h = \varepsilon(h)x_{\bar{h}}^*$, ${}^\tau p_i = -q_i^*$, and ${}^\tau q_i = p_i^*$ for all $h \in H$ and $i \in I$. This automorphism induces an isomorphism:

(11) $\qquad\qquad \tau : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{-\zeta}(\mathbf{v}, \mathbf{w}).$

2.2.3. *Diagram isomorphism* a. Let a be an automorphism of Γ, i.e., there are automorphisms of vertex and arrow sets, both denoted by a, such that $a(\mathbf{o}(h)) = \mathbf{o}(a(h))$, $a(\mathbf{i}(h)) = \mathbf{i}(a(h))$, and $a(\bar{h}) = \overline{a(h)}$ for all $h \in H$. Assume that a is compatible with the function ε^0 in the following sense. There exists a constant $c^0 \equiv c_{a,\varepsilon^0} \in \{\pm 1\}$ such that

(12) $\qquad\qquad \varepsilon^0(a(h)) = c^0 \cdot \varepsilon^0(h) \; \forall h \in H.$

Let $a(V)$ be the I-graded vector space whose ith component is $V_{a^{-1}(i)}$. The dimension vector of $a(V)$ is $a(\mathbf{v})$ whose i-entry is $\mathbf{v}_{a^{-1}(i)}$. Given any point $\mathbf{x} = (x, p, q) \in \mathbf{M}(V, W)$, we define a point $a(\mathbf{x}) = (a(x), a(p), a(q)) \in \mathbf{M}(a(V), a(W))$ by

$$a(p)_i = p_{a^{-1}(i)}, \; a(q)_i = q_{a^{-1}(i)}, \; a(x)_h = \varepsilon^0(h)^{\frac{1-c^0}{2}} x_{a^{-1}(h)} \quad \forall i \in I, h \in H.$$

It induces a diagram isomorphism on Nakajima varieties:

(13) $\qquad\qquad a : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{a(\zeta)}(a(\mathbf{v}), a(\mathbf{w})).$

2.2.4. *σ-Quiver varieties.* Consider

$$(14) \qquad \sigma := a S_{\boldsymbol{\omega}} \tau : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{-a\boldsymbol{\omega}(\zeta)}(a(\boldsymbol{\omega} *_{\mathbf{w}} \mathbf{v}), a(\mathbf{w})),$$

where τ, $S_{\boldsymbol{\omega}}$ and a are in (11), (10), and (13), respectively. The σ-quiver variety is defined by

$$(15) \qquad \begin{aligned} \mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w}) &= (\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}))^\sigma, \quad \mathfrak{S}_1(\mathbf{v}, \mathbf{w}) = \pi(\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w})) \\ &\text{if } \mathbf{w} = a(\mathbf{w}), \zeta = -a\boldsymbol{\omega}(\zeta), \mathbf{v} = a(\boldsymbol{\omega} *_{\mathbf{w}} \mathbf{v}). \end{aligned}$$

The proper map π restricts to a proper morphism which is (4) in type A:

$$(16) \qquad \pi_\sigma : \mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{S}_1(\mathbf{v}, \mathbf{w}).$$

$\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w})$ has a symplectic structure inherited from that of $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ and $\mathfrak{S}_1(\mathbf{v}, \mathbf{w})$ is an affine variety as a closed subvariety of $\mathfrak{M}_1(\mathbf{v}, \mathbf{w})$.

For the rest of this section, we consider the Dynkin graph of type A_n: $1 \leftrightarrows 2 \leftrightarrows \cdots \leftrightarrows n$. Set $\varepsilon^0(h) = i - j$ if h is an arrow from i to j and $c^0 = 1$. The automorphism a is the identity automorphism. The Weyl group element $\boldsymbol{\omega}$ is the longest Weyl group element. Let $\zeta = (\xi, 0)$ where all components in ξ are 1. For any pair $(\mathbf{v}, \mathbf{w})$, we define a new pair $(\widetilde{\mathbf{v}} = (\widetilde{\mathbf{v}}_i)_{1 \le i \le n}, \widetilde{\mathbf{w}} = (\widetilde{\mathbf{w}}_i)_{1 \le i \le n})$ where

$$(17) \qquad \widetilde{\mathbf{v}}_i = \mathbf{v}_i + \sum_{j \ge i+1} (j - i)\mathbf{w}_j, \quad \widetilde{\mathbf{w}}_i = \delta_{i,1} \sum_{1 \le j \le n} j \mathbf{w}_j \quad \forall 1 \le i \le n.$$

Now set $\mu = (\widetilde{\mathbf{v}}_0 - \widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_1 - \widetilde{\mathbf{v}}_2, \cdots, \widetilde{\mathbf{v}}_{n-1} - \widetilde{\mathbf{v}}_n, \widetilde{\mathbf{v}}_n)$. Let P_μ be a parabolic subgroup of a classical group G whose levi has size indexed by μ. In other words, the isotropic flag variety G/P_μ is the collection of all isotropic flags such that the dimension difference of the ith step flag and $(i+1)$th step flag is $\widetilde{\mathbf{v}}_{i-1} - \widetilde{\mathbf{v}}_i$. Note that P_μ may be empty. Let e_{P_μ} be the associated Richardson element. Let

$$\lambda = 1^{\mathbf{w}_1} 2^{\mathbf{w}_2} \cdots.$$

We write $\tilde{S}_{e_{P_\mu}, x}$ in Section 1.3 as $\tilde{S}_{e_{P_\mu}, \lambda}^{\mathrm{Lie}(G)}$ when the Jordan-type of x is λ. The following result is obtained in [Li19, Corollary 8.3.4].

Proposition 2.3.   (1) *If W_i is equipped with a symmetric (resp., skew-symmetric) form for i even (resp., odd), then $\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w}) \cong \widetilde{\mathcal{S}}_{e_{P_\mu}, \lambda}^{\mathfrak{o}_{\widetilde{\mathbf{w}}_1}}$ and $\mathfrak{S}_1(\mathbf{v}, \mathbf{w}) \cong \mathcal{S}_{e_{P_\mu}, \lambda}^{\mathfrak{o}_{\widetilde{\mathbf{w}}_1}}$. If forms on W_i are skew-symmetric (resp., symmetric) for i even (resp., odd), then*
(2) $\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w}) \cong \widetilde{\mathcal{S}}_{e_{P_\mu}, \lambda}^{\mathfrak{sp}_{\widetilde{\mathbf{w}}_1}}$ *and* $\mathfrak{S}_1(\mathbf{v}, \mathbf{w}) \cong \mathcal{S}_{e_{P_\mu}, \lambda}^{\mathfrak{sp}_{\widetilde{\mathbf{w}}_1}}$.

3. $\mathbb{C}^*$-ACTION AND THE AUTOMORPHISM σ

In this section we assume that ζ is generic and $\zeta_{\mathbb{C}} = 0$. We study the compatibility of a modified version of a $\mathbb{C}^*$-action in [N94, Section 5] with the automorphism σ. By using these analyses, we then provide proofs for Theorems A-D.

3.1. **Compatibility.** To an orientation ε of H, not necessarily the same as ε^0 in the definition of Nakajima varieties, we can define two $\mathbb{C}^*$-actions on $\mathbf{M}(\mathbf{v}, \mathbf{w})$ in (5). The first one is given by $(t, \mathbf{x}) \mapsto t \circ_\varepsilon \mathbf{x}$ where

$$(18) \qquad t \circ_\varepsilon \mathbf{x} = (t^{\frac{1+\varepsilon(h)}{2}} x_h, p_i, t q_i).$$

The second one is given by $(t, \mathbf{x}) \mapsto t \star_\varepsilon \mathbf{x}$ where

(19) $$t \star_\varepsilon \mathbf{x} = (t^{\frac{1+\varepsilon(h)}{2}} x_h, tp_i, q_i).$$

It is clear that each $\mathbb{C}^*$-action induces a $\mathbb{C}^*$-action on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ in (8), in light of the assumption that $\zeta_{\mathbb{C}} = 0$, but the induced ones on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ coincide as follows so that we do not have to distinguish the two actions on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.

Lemma 3.2. *We have $t \circ_\varepsilon [\mathbf{x}] = t \star_\varepsilon [\mathbf{x}]$ for all $[\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.*

Proof. Let $g = (t.\mathrm{id}_{V_i})_{i \in I}$. Then $g.(t \circ_\varepsilon \mathbf{x}) = t \star_\varepsilon \mathbf{x}$, as required. $\qquad\square$

It is clear that the weight of the symplectic form on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ with respect to this $\mathbb{C}^*$-action is 1, i.e., $\omega(t \circ_\varepsilon [\mathbf{x}], t \circ_\varepsilon [\mathbf{x}']) = t\omega([\mathbf{x}], [\mathbf{x}'])$. Since the graph is Dynkin, the $\mathbb{C}^*$-action provides a contraction from $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, and hence $\mathfrak{M}_1(\mathbf{v}, \mathbf{w})$, to its $\mathbb{C}^*$-fixed point $[0]$.

The following lemma is the compatibility of the transpose τ in Subsection 2.2.2 and the $\mathbb{C}^*$-action.

Lemma 3.3. *We have $\tau(t \circ_\varepsilon [\mathbf{x}]) = t \circ_{-\varepsilon} \tau([\mathbf{x}])$ for all $t \in \mathbb{C}^*$ and $[\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.*

Proof. We write $t \circ_\varepsilon \mathbf{x} = (t \circ_\varepsilon x_h, t \circ_\varepsilon p_i, t \circ_\varepsilon q_i)_{h \in H, i \in I}$ and $\tau(t \circ_\varepsilon [\mathbf{x}]) = [(x'_h, p'_i, q'_i)]$. We have

$$x'_h = \varepsilon(h)(t \circ_\varepsilon x_{\bar{h}})^* = \varepsilon(h).(t^{\frac{1+\varepsilon(\bar{h})}{2}} x_{\bar{h}})^* = t^{\frac{1+\varepsilon(\bar{h})}{2}} \tau(x_h),$$
$$p'_i = -(t \circ_\varepsilon q_i)^* = -(tq_i)^* = t\tau(p_i),$$
$$q'_i = (t \circ_\varepsilon p_i)^* = p_i^* = \tau(q_i).$$

This shows that $(x'_h, p'_i, q'_i) = t \star_{-\varepsilon} \tau(\mathbf{x})$, and the lemma follows readily by Lemma 3.2. $\qquad\square$

Let a be an automorphism of Γ. We assume that the pair (a, ε) is compatible with signature $c \in \{\pm 1\}$; see (12). We have the following compatibility of the automorphism a and the $\mathbb{C}^*$-action.

Lemma 3.4. *Let (a, ε) be a compatible pair with signature c. Then $a(t \circ_\varepsilon [\mathbf{x}]) = t \circ_{c\varepsilon} a([\mathbf{x}])$ for all $t \in \mathbb{C}^*$ and $[\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.*

Proof. We write $t \circ_\varepsilon \mathbf{x} = (t \circ_\varepsilon x_h, t \circ_\varepsilon p_i, t \circ_\varepsilon q_i)_{h \in H, i \in I}$ and $a(t \circ_\varepsilon \mathbf{x}) = (x'_h, p'_i, q'_i)$. We have

$$x'_h = \varepsilon(h)^{\frac{1-c}{2}} t \circ_\varepsilon x_{a^{-1}(h)} = \varepsilon(h)^{\frac{1-c}{2}} t^{\frac{1+\varepsilon(a^{-1}(h))}{2}} x_{a^{-1}(h)} = t^{\frac{1+c\varepsilon(h)}{2}} a(x)_h,$$
$$p'_i = t \circ_\varepsilon p_{a^{-1}(i)} = p_{a^{-1}(i)} = a(p)_i,$$
$$q'_i = t \circ_\varepsilon q_{a^{-1}(i)} = t.q_{a^{-1}(i)} = t.a(q)_i.$$

So $(x'_h, p'_i, q'_i) = t \star_{-\varepsilon} a(\mathbf{x})$. The lemma follows. $\qquad\square$

The following lemma is the compatibility of the reflection functor S_i and the $\mathbb{C}^*$-action.

Lemma 3.5. *We have $S_i(t \circ_\varepsilon [\mathbf{x}]) = t \circ_\varepsilon S_i([\mathbf{x}])$ for all $t \in \mathbb{C}^*$ and $[\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.*

Proof. Let $S_i([\mathbf{x}]) = [\mathbf{x}']$. It suffices to show that the pair $(t \circ_\varepsilon \mathbf{x}, t \circ_\varepsilon \mathbf{x}')$ satisfies the conditions (R1)-(R4) in the definition of reflection functors. Recall $a_i(\mathbf{x})$ and $b_i(\mathbf{x})$ from (6). There is

$$a_i(t \circ_\varepsilon \mathbf{x}) = (tq_i, t^{\frac{1+\varepsilon(h)}{2}} x_h)_{h:\mathbf{o}(h)=i}, \quad b_i(t \circ_\varepsilon \mathbf{x}) = (p_i, t^{\frac{1+\varepsilon(h)}{2}} \varepsilon^0(\bar{h}) x_h)_{h:\mathbf{i}(h)=i}.$$

Thus we must have

$$b_i(t \circ_\varepsilon \mathbf{x}) a_i(t \circ_\varepsilon \mathbf{x}') = t b_i(\mathbf{x}) a_i(\mathbf{x}') = 0,$$

Clearly, $b_i(t \circ_\varepsilon \mathbf{x})$ is surjective since $b_i(\mathbf{x})$ is so and $a_i(t \circ_\varepsilon \mathbf{x}')$ is injective since $a_i(\mathbf{x}')$ is so. Hence (R1) holds for the pair $(t \circ_\varepsilon \mathbf{x}, t \circ_\varepsilon \mathbf{x}')$. Similarly, there is

$$a_i(t \circ_\varepsilon \mathbf{x}) b_i(t \circ_\varepsilon \mathbf{x}) - a_i(t \circ_\varepsilon \mathbf{x}') b_i(t \circ_\varepsilon \mathbf{x}') = t\left(a_i(\mathbf{x}) b_i(\mathbf{x}) - a_i(\mathbf{x}') b_i(\mathbf{x}')\right) = 0.$$

This shows that the pair $(t \circ_\varepsilon \mathbf{x}, t \circ_\varepsilon \mathbf{x}')$ satisfies (R2). The condition (R3) for $(t \circ_\varepsilon \mathbf{x}, t \circ_\varepsilon \mathbf{x}')$ is clearly followed from definition. The condition (R4) for $(t \circ_\varepsilon \mathbf{x}, t \circ_\varepsilon \mathbf{x}')$ can be proved in a similar way as that of (R2). The lemma thus follows. $\square$

By combining Lemmas 3.3, 3.4, and 3.5, we have the following proposition.

Proposition 3.6. *Let (a, ε) be a compatible pair with signature c. Then we have*

$$\sigma(t \circ_\varepsilon [\mathbf{x}]) = t \circ_{-c\varepsilon} \sigma([\mathbf{x}]) \forall t \in \mathbb{C}^*, [\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}).$$

From Proposition 3.6 and the above analysis, we readily have the following.

Proposition 3.7.   (1) *If $t \circ_\varepsilon [\mathbf{x}] = t \circ_{-c\varepsilon} [\mathbf{x}]$ for all $t \in \mathbb{C}^*$ and for all $[\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$, then the $\mathbb{C}^*$-action in (18) on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ induces a $\mathbb{C}^*$-action on $\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w})$ such that the weight of the symplectic form ω on $\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w})$ is 1 with respect to this $\mathbb{C}^*$-action.*
   (2) *If $t \circ_\varepsilon [\mathbf{x}] = t \circ_{-c\varepsilon} [\mathbf{x}]$ for all $t \in \mathbb{C}^*$ and $[\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$, then the $\mathbb{C}^*$-action provides a contraction of $\mathfrak{S}_1(\mathbf{v}, \mathbf{w})$ to its fixed point $\mathfrak{S}_1(\mathbf{v}, \mathbf{w})^{\mathbb{C}^*}$ consisting of a single point $[0]$.*

The following proposition provides compatible cases sufficient to prove our theorems.

Proposition 3.8.   (1) *If $c = -1$, then $t \circ_\varepsilon [\mathbf{x}] = t \circ_{-c\varepsilon} [\mathbf{x}]$ $\forall t \in \mathbb{C}^*$ and $[\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.*
   (2) *Assume $c = 1$ and $\mathbf{w}_i \mathbf{w}_j = 0$ if i and j are joined by an edge. Let $I = I^1 \sqcup I^0$ be a partition satisfying the following conditions:*
     • *For all $i \in I^0$, we have $\mathbf{w}_i = 0$.*
     • *For all $h \in \varepsilon^{-1}(1)$, we have $\mathbf{o}(h) \in I^1$ and $\mathbf{i}(h) \in I^0$.*
   Then $t \circ_\varepsilon [\mathbf{x}] = t \circ_{-c\varepsilon} [\mathbf{x}]$ for all $t \in \mathbb{C}^*$ and $[\mathbf{x}] \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.

Proof. The first statement is obvious. Let $c = 1$. It is enough to show that $g.(t \circ_\varepsilon \mathbf{x}) = t \star_{-c\varepsilon} \mathbf{x}$. Let κ_i be the parity of i, i.e., $\kappa_i = 1$ if $i \in I^1$ and $\kappa_i = 0$ if $i \in I^0$. Let $g_\kappa = (t^{\kappa_i} \mathrm{id}_{V_i})_{i \in I} \in G_{\mathbf{v}}$. Then we have the following computations:

$$\begin{aligned}
g_\kappa.(t \circ_\varepsilon x_h) &= t^{-1}(t \circ_\varepsilon x_h) = x_h & \text{if } h \in \varepsilon^{-1}(1), \\
g_\kappa.(t \circ_\varepsilon x_h) &= t(t \circ_\varepsilon x_h) = t x_h & \text{if } h \in \varepsilon^{-1}(-1), \\
g_\kappa.(t \circ_\varepsilon p_i) &= t(t \circ_\varepsilon p_i) = t p_i & \text{if } i \in I^1, \\
g_\kappa.(t \circ_\varepsilon q_i) &= t^{-1}(t \circ_\varepsilon q_i) = q_i & \text{if } i \in I^1.
\end{aligned}$$

Since $p_i = 0, q_i = 0$ for all $i \in I^0$, the above computation shows that $g_\kappa.(t \circ_\varepsilon \mathbf{x}) = t \star_{-c\varepsilon} \mathbf{x}$. The proof is thus finished. $\square$

3.9. **The proof of Theorems A, C, and D.** Since Γ is a Dynkin graph, hence bipartite, so we can find a partition of I such that the first condition in Proposition 3.8 holds. Now set ε to be the unique orientation such that the second condition in Proposition 3.8 is valid. Since $c = 1$, we see that the automorphism a is compatible with the orientation ε. In this case, the results in Proposition 3.7 are true and so Theorem B is applicable and from which Theorem D follows.

In light of Proposition 2.3, Theorem C, and hence Theorem A, follows from Theorem D. Note that we must show that all parabolic subgroups, up to conjugations, appear in the setting of Proposition 2.3. But this is already observed in Maffei's work [M05, Theorem 8].

The proof of Theorems A, C, and D is finished.

3.10. **A generalization of Theorem D.** In Proposition 3.8, there is no assumption on $\mathbf{w}$ when $c = -1$, which is not stated in Theorem D, and the above argument works in this more general case as well. Let us record this more general result here.

Theorem E. *Let* (a, ε^0) *be a compatible pair of signature* $c = -1$. *Then the fiber* $(\pi_\sigma)^{-1}([0])$ *is Lagrangian in* $\mathfrak{S}_\zeta(\mathbf{v}, \mathbf{w})$.

4. Spaltenstein's examples

In this section, we discuss examples in [Sp82, 11.6, 11.8], except 11.8 c). We show that X_x^P is Lagrangian in all these examples, except the counterexample in [Sp82, 11.6].

4.1 [Sp82, 11.6]. Let us fix a basis $\{e_i\}_{1 \le i \le 8}$ of $\mathbb{C}^8$. Let $B(-, -)$ be the bilinear form defined by $B(e_i, e_j) = \delta_{i, 9-j}$ for all $1 \le i, j \le 8$, so that the associated symmetric matrix is the anti-diagonal identity matrix. Let $G = \mathrm{SO}_8(\mathbb{C})$ be the special orthogonal group of $B(-, -)$ and let $\mathfrak{so}_8(\mathbb{C})$ be its Lie algebra. Let x be an element of the form

$$x = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & -x_1 \end{bmatrix}, \; x_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; x_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then it is clear that x is of Jordan-type $(1, 2^2, 3)$ and is a nilpotent element in $\mathfrak{so}_8(\mathbb{C})$. Let G/P be the isotropic flag variety of isotropic subspaces $F_2 \subseteq F_3$ in $\mathbb{C}^8$ such that $\dim F_2 = 2$ and $\dim F_3 = 3$. Then the Spaltenstein variety X_x^P of the triple $(\mathrm{SO}_8(\mathbb{C}), P, x)$ is the subvariety of G/P consisting of elements $(F_2 \subset F_3)$ such that $x(F_2) = 0, x(F_3) \subseteq F_2, x(F_3^\perp) \subseteq F_3$. There is a partition of $X_x^P = X_3 \sqcup X_2$ where

$$X_3 = \{(F_2 \subseteq F_3) \in X_x^P | e_3 \in F_2\}, \; X_2 = \{(F_2 \subseteq F_3) \in X_x^P | e_3 \notin F_2\}.$$

One can check that X_3 and X_2 are irreducible of dimension 3 and 2, respectively. Indeed, for a fixed flag F_2 in X_3, the freedom of F_3 is $\mathrm{OGr}(1, 4)$, the Grassmannian of isotropic lines in $\mathbb{C}^4$. The dimension of $\mathrm{OGr}(1, 4)$ is 2, hence the dimension of X_3 is 3. For a fixed flag F_2 in X_2, there is a unique flag F_3, i.e., $F_3 = \langle F_2, e_3 \rangle$. Thus the dimension of X_2 is 2.

So the irreducible components of X_x^P are X_3 of dimension 3 and the closure of X_2 in X_x^P of dimension 2. Hence X_x^P is not pure dimensional.

Let Q be a parabolic subgroup such that G/Q is the isotropic flag varieties of all flags $F_1 \subset F_2 \subset F_4$ such that $\dim F_i = i$. From [Sp82, 11.6], X_x^Q is irreducible and of dimension 3. Let e_Q be the Richardson element associated to Q. Then it can be shown that $\dim \tilde{S}_{e_Q,x} = 6$, hence X_x^Q is Lagrangian in $\tilde{S}_{e_Q,x}$. This example is not in the cases (a)-(c) in the introduction.

4.2 [Sp82, 11.8. a)]. If G is of type A_n (resp., D_n; E_6; E_7; E_8), $\dim X_x^B = 2$, with B a Borel, and P is minimal, then X_x^P is a union of projective lines in a configuration of type A_{n-2} (resp., A_1 or D_{n-2}, the last is only possible if $n \geq 5$; A_5; D_6; E_7). The condition $\dim X_x^B = 2$ implies that $\dim \mathcal{O}_x = \dim T^*G/B - 4$ and P is minimal implies that $\dim T^*G/P = \dim T^*G/B - 2$. So the dimension of $\tilde{S}_{e_P,x}$ is 2, and thus X_x^P is Lagrangian in $\tilde{S}_{e_P,x}$.

4.3 [Sp82, 11.8. b)]. Let $G = \mathrm{SO}_7(\mathbb{C})$, with x of type $(3, 1^4)$ and G/P a maximal isotropic Grassmannian. Then X_x^P is a disjoint union of two projective lines. By Theorem C, X_x^P is Lagrangian in $\tilde{S}_{e_P,x}$.

4.4 [Sp82, 11.8. d)]. Let $G = \mathrm{Sp}_{4n+2}(\mathbb{C})$ and let G/P be a partial flag variety obtained from the complete flag by dropping the $(2i+1)$th step for all $0 \leq i \leq n$, and let x be a nilpotent of type $((2n)^2, 2^1)$. Then X_x^P is a union of $2n+1$ projective lines subject to certain conditions. From Theorem C, X_x^P is Lagrangian in $\tilde{S}_{e_P,x}$.

4.5. By the rectangular symmetry in [Li19], one can produce more examples from previous subsections. For example, the corresponding case in Section 4.1 for (G, P, x) is $G' = \mathrm{Sp}_{12}(\mathbb{C})$, P' is chosen such that G'/P' is isomorphic to the isotropic flag varieties of $(F_2 \subset F_5)$ with $\dim F_i = i$, and x' is of Jordan-type $(2, 3^2, 4)$. Then $X_{x'}^{P'} \cong X_x^P$ is not pure dimensional.

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