Journal of Pure and Applied Algebra 224 (2020) 106227

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

On canonical bases for the Letzter algebra $\mathbf{U}^\imath(\mathfrak{sl}_2)$

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ARTICLE INFO

Article history:
Received 28 March 2019
Received in revised form 29 August 2019
Available online 10 September 2019
Communicated by D. Nakano

MSC:
Primary: 17B37; secondary: 14L35

ABSTRACT

A canonical basis was constructed by Wang and the author in [6] inside Letzter's coideal subalgebra in quantum $\mathfrak{sl}_2$. In this article, we provide an explicit description for the canonical bases and show that the bases coincide with the one defined algebraically by Bao-Wang in [1].

1. Introduction

Let $\mathbf{U}^\imath \equiv \mathbf{U}^\imath(\mathfrak{sl}_2)$ be Letzter's coideal subalgebra of quantum $\mathfrak{sl}_2$ corresponding to the symmetric pair $(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C})$ ([5]). As a subalgebra of quantum $\mathfrak{sl}_2$, $\mathbf{U}^\imath$ is generated by the sum $\mathbf{E} + v\mathbf{KF} + \mathbf{K}$ of standard generators, and hence can be identified with the polynomial ring $\mathbb{Q}(v)[t]$. In [1] and [6], two distinguished bases, called $\imath$ canonical bases, are constructed inside the modified form of $\mathbf{U}^\imath$ via algebraic and geometric approaches respectively. The modified form of $\mathbf{U}^\imath$ is isomorphic to a direct sum of two copies of $\mathbf{U}^\imath \cong \mathbb{Q}(v)[t]$ itself. Under this isomorphism, an explicit and elegant formula, as a polynomial in t, of algebraic basis elements is conjectured in [1] and proved in [3]. The purpose of this short paper is to show that the geometric basis in [6] admits the same description and, consequently, that the two bases coincide. Notice that the proofs are within the scope of [6] and [2], whose notations shall be adopted here.

2. The description

Set $[\![n]\!] = (v^n - v^{-n})/(v - v^{-1})$ and $[\![n]\!]! = \prod_{i=1}^n [\![i]\!]$. Let $\mathbf{U}^\jmath(\mathfrak{sl}_3)$ be an associative algebra over $\mathbb{Q}(v)$ generated by e, f, k, k^{-1} and subject to the following defining relations.

$$kk^{-1} = 1,\ ke = v^3 ek,\ kf = v^{-3} fk,$$

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[1] Partially supported by the National Science Foundation grant DMS-1801915.

https://doi.org/10.1016/j.jpaa.2019.106227
0022-4049/

$$e^2f - [\![2]\!]efe + fe^2 = -[\![2]\!]e(vk + v^{-1}k^{-1}),$$
$$f^2e - [\![2]\!]fef + ef^2 = -[\![2]\!](vk + v^{-1}k^{-1})f.$$

Let $e^{(n)} = e^n/[\![n]\!]!$ and $f^{(n)} = f^n/[\![n]\!]!$. By an induction argument and making use the above inhomogeneous Serre relations, we have the following formula in $\mathbf{U}^\jmath(\mathfrak{sl}_3)$.

Lemma 1. *We have* $fe^{(n+1)} = e^{(n)}(fe - vef - [\![n]\!](v^nk + v^{-n}k^{-1})) + v^{n+1}e^{(n+1)}f$.

Let $\mathbf{S}^\jmath_{3,d}$ be $\mathbf{S}^\jmath$ in [2, 3.1] for $n = 1$ and $\mathbf{e}, \mathbf{f}, \mathbf{k} = \mathbf{d}_1\mathbf{d}_2^{-1}$ its generators. The assignments $e \mapsto \mathbf{e}$, $f \mapsto \mathbf{f}$ and $k^{\pm 1} \mapsto \mathbf{k}^{\pm 1}$ define an algebra homomorphism $\mathbf{U}^\jmath(\mathfrak{sl}_3) \to \mathbf{S}^\jmath_{3,d}$. We set

$$A_{a,b} = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{bmatrix}, \quad a, b \in \mathbb{N} := \{0, 1, 2, \cdots\}.$$

Let $\mathbf{j}_d = [A_{d,0}]$, an idempotent in $\mathbf{S}^\jmath_{3,d}$ ([2, 3.17]). We consider the subalgebra $\mathbf{S}^\imath_{2,d} = \mathbf{j}_d\mathbf{S}^\jmath_{3,d}\mathbf{j}_d$ and its integral form ${}_\mathcal{A}\mathbf{S}^\imath_{2,d}$ with $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ ([6, 4.1]). Let

$$\mathbf{t}_d = \left(\mathbf{fe} + \frac{\mathbf{k} - \mathbf{k}^{-1}}{v - v^{-1}}\right)\mathbf{j}_d \in \mathbf{S}^\imath_{2,d}.$$

Lemma 2. *In* $\mathbf{S}^\imath_{2,d}$*, one has* $\mathbf{f}^{(n)}\mathbf{e}^{(n)}\mathbf{j}_d = (\mathbf{t}_d + [\![d-1]\!])(\mathbf{t}_d + [\![d-3]\!]) \cdots (\mathbf{t}_d + [\![d-2n+1]\!])/[\![n]\!]!$.

Proof. When $n = 1$, this is the defining relation for $\mathbf{t}_d$ in [2, Remark 5.3]. Assume the statement holds for n. In light of the geometric feature of the generators in [2, 3.1], we have $\mathbf{fj}_d = 0$ and $\mathbf{k}^{\pm 1}\mathbf{j}_d = v^{\pm(1-d)}$. Together with Lemma 1, we deduce that

$$\begin{aligned}
\mathbf{f}^{(n+1)}\mathbf{e}^{(n+1)}\mathbf{j}_d &= \frac{1}{[\![n+1]\!]}\mathbf{f}^{(n)}\mathbf{fe}^{(n+1)}\mathbf{j}_d = \frac{1}{[\![n+1]\!]}\mathbf{f}^{(n)}\mathbf{e}^{(n)}(\mathbf{fe} - [\![n]\!](v^nv^{1-d} + v^{-n}v^{d-1}))\mathbf{j}_d \\
&= \mathbf{f}^{(n)}\mathbf{e}^{(n)}(\mathbf{t}_d + [\![d-1]\!] - [\![n]\!](v^nv^{1-d} + v^{-n}v^{d-1}))\mathbf{j}_d/[\![n+1]\!] \\
&= \mathbf{f}^{(n)}\mathbf{e}^{(n)}(\mathbf{t}_d + [\![(d-1) - 2n]\!])\mathbf{j}_d/[\![n+1]\!] \\
&= (\mathbf{t}_d + [\![d-1]\!])(\mathbf{t}_d + [\![d-3]\!]) \cdots (\mathbf{t}_d + [\![d-2n-1]\!])/[\![n+1]\!]!.
\end{aligned}$$

Lemma follows by induction. □

Lemma 2 provides a characterization of $\mathbf{S}^\imath_{2,d}$ as follows.

Proposition 3. *The algebra* $\mathbf{S}^\imath_{2,d}$ *is isomorphic to the quotient algebra of* $\mathbb{Q}(v)[t]$ *by the ideal generated by the polynomial* $(t + [\![d-1]\!])(t + [\![d-3]\!]) \cdots (t + [\![-d-1]\!])$.

Proof. The map $t \mapsto \mathbf{t}_d$ defines a surjective algebra homomorphism $\phi^\imath_d : \mathbb{Q}(v)[t] \to \mathbf{S}^\imath_{2,d}$. Due to $\mathbf{f}^{(d+1)}\mathbf{e}^{(d+1)}\mathbf{j}_d = 0$ and Lemma 1, the polynomial above is zero in $\mathbf{S}^\imath_{2,d}$ for $t = \mathbf{t}_d$. So $\phi^\imath_d$ factors through the desired quotient. Clearly the dimensions of $\mathbf{S}^\imath_{2,d}$ and the quotient algebra are the same, so they must be isomorphic. The proposition is thus proved. □

Define an equivalence relation on the set $\{A_{a,b} | a, b \in \mathbb{N}\}$ by $A_{a,b} \sim A_{a',b'}$ if $a \equiv a' \pmod 2$ and $b = b'$. Let $\check{A}_{a,b}$ be the equivalence class of $A_{a,b}$. As a $\mathbb{Q}(v)$-vector space, the modified form $\dot{\mathbf{U}}^\imath$ of $\mathbf{U}^\imath$ is spanned by the canonical basis elements $b_{\check{A}_{0,d}}$ and $b_{\check{A}_{1,d}}$ for $d \in \mathbb{N}$. Let $\dot{\mathbf{U}}^\imath_0 = \mathrm{Span}\{b_{\check{A}_{0,d}}, b_{\check{A}_{1,d+1}} | d \text{ even}\}$ and $\dot{\mathbf{U}}^\imath_1 = \mathrm{Span}\{b_{\check{A}_{0,d}}, b_{\check{A}_{1,d-1}} | d \text{ odd}\}$. Then $\dot{\mathbf{U}}^\imath = \dot{\mathbf{U}}^\imath_0 \oplus \dot{\mathbf{U}}^\imath_1$ as algebras. We have an isomorphism $\dot{\mathbf{U}}^\imath_0 \to \mathbb{Q}(v)[t]$

(resp. $\dot{\mathbf{U}}^\imath_1 \to \mathbb{Q}(v)[t]$) via $b_{\check{A}_{1,1}} \mapsto t$ (resp. $b_{\check{A}_{0,1}} \mapsto t$). The isomorphisms are compatible with $\phi^\imath_d$. We have the following explicit description[2] of geometrically-defined canonical basis elements of $\dot{\mathbf{U}}^\imath$.

Theorem 4. *The canonical basis elements of $\dot{\mathbf{U}}^\imath \equiv \dot{\mathbf{U}}^\imath(\mathfrak{sl}_2)$ in [6] are of the form*

$$b_{\check{A}_{0,d}} = \frac{(t+[\![d-1]\!])(t+[\![d-3]\!])\cdots(t+[\![-d+3]\!])(t+[\![-d+1]\!])}{[\![d]\!]!}, \ \forall d \in \mathbb{N}; \tag{1}$$

$$b_{\check{A}_{1,d+1}} = \frac{t\cdot(t+[\![d-1]\!])(t+[\![d-3]\!])\cdots(t+[\![-d+3]\!])(t+[\![-d+1]\!])}{[\![d+1]\!]!}, \ \forall d \in \mathbb{N}. \tag{2}$$

Proof. Let us denote the polynomial in (1) by $P_{0,d}(t)$ and that in (2) by $P_{1,d+1}(t)$ in the proof. By Lemma 2, we have $\mathbf{f}^{(d)}\mathbf{e}^{(d)}\mathbf{j}_d = P_{0,d}(\mathbf{t}_d)$. Observe that the element $\mathbf{f}^{(d)}\mathbf{e}^{(d)}\mathbf{j}_d$ is a canonical basis element in $\mathbf{S}^\imath_{2,d}$, corresponding to the constant sheaf on the product of maximal isotropic Grassmannians. Indeed, by [2, Thm. 3.7(a)], we have $\mathbf{f}^{(d)}\mathbf{e}^{(d)}\mathbf{j}_d = \sum_{i=0}^d v^{-\binom{i}{2}}[A_{i,d-i}]$. By [6, Prop. 6.3], the canonical basis elements of $\dot{\mathbf{U}}^\imath$ get sent to canonical basis elements in $\mathbf{S}^\imath_{2,d}$ or zero via $\phi^\imath_d$. So $b_{\check{A}_{0,d}} = P_{0,d}(t)$, and (1) holds.

Now consider the element $P_{1,d+1}(\mathbf{t}_{d+2})$ in $\mathbf{S}^\imath_{2,d+2}$. By rewriting the factor t in $P_{1,d+1}(t)$ as $(t+[\![d+1]\!]) - [\![d+1]\!]$ and combining with the remaining terms, there is

$$P_{1,d+1}(\mathbf{t}_{d+2}) = \mathbf{f}^{(d+1)}\mathbf{e}^{(d+1)}\mathbf{j}_{d+2} - P_{0,d}(\mathbf{t}_{d+2}).$$

Hence $P_{1,d+1}(\mathbf{t}_{d+2})$ is in the integral form ${}_{\mathcal{A}}\mathbf{S}^\imath_{2,d}$. Further, the polynomial $P_{1,d+1}(\mathbf{t}_{d+2})$ can be rewritten as

$$P_{1,d+1}(\mathbf{t}_{d+2}) = P_{0,d}(\mathbf{t}_{d+2}) + \frac{(\mathbf{t}_{d+2}+[\![d-1]\!])(\mathbf{t}_{d+2}+[\![d-3]\!])\cdots(\mathbf{t}_{d+2}+[\![-d-1]\!])}{[\![d+1]\!]!}.$$

With the above expression, Proposition 3 and the property of the transfer map $\phi^\imath_{d+2,d}$ in [6, (6.6)] and [4], we must have

$$\phi^\imath_{d+2,d}(P_{1,d+1}(\mathbf{t}_{d+2})) = P_{0,d}(\mathbf{t}_d). \tag{3}$$

By the definition of $P_{1,d+1}(t)$, there is $P_{1,d+1}(\mathbf{t}_{d+2}) = \frac{1}{[\![d+1]\!]}\mathbf{t}_{d+2} * \{A_{2,d}\}$ and so, in light of the positivity property in [6, Theorem 6.12], it leads to

$$P_{1,d+1}(\mathbf{t}_{d+2}) = \{A_{1,d+1}\} + \sum_{i>1} c_i\{A_{i,d+2-i}\}, \quad c_i \in \mathbb{N}[v,v^{-1}].$$

The positivity of c_i together with (3) and *loc. cit.* (6.6) implies that $c_i = 0$ and so

$$P_{1,d+1}(\mathbf{t}_{d+2}) = \{A_{1,d+1}\} \in \mathbf{S}^\imath_{2,d+2}.$$

Therefore, we have $b_{\check{A}_{1,d+1}} = P_{1,d+1}(t)$ and (2) holds. The proof is finished. □

We conclude the paper with a remark.

Remark 5. (1) The canonical basis of $\mathbf{S}^\imath_{2,d}$ is $\{b_{\check{A}_{0,d-2i}}(\mathbf{t}_d), b_{\check{A}_{1,d-2i-1}}(\mathbf{t}_d) | 0 \le i \le \lfloor d/2 \rfloor\}$ where $b_{\check{A}_{a,b}}$ are regarded as a polynomial under the identifications (1)-(2).

(2) Clearly $b_{\check{A}_{0,1}} * b_{\check{A}_{0,d}} = [\![d+1]\!] b_{\check{A}_{1,d+1}}$, $b_{\check{A}_{0,1}} * b_{\check{A}_{1,d+1}} = [\![d+2]\!] b_{\check{A}_{0,d+2}} + [\![d+1]\!] b_{\check{A}_{0,d}}$.

[2] We thank Weiqiang Wang for pointing out a gap in an early draft, and helpful comments.

(3) The $\mathbf{U}^\imath$ in [1] and [6] differs by an involution $(\mathbf{E},\mathbf{F},\mathbf{K}) \mapsto (\mathbf{F},\mathbf{E},\mathbf{K}^{-1})$ on quantum $\mathfrak{sl}_2$. Hence the canonical bases constructed therein coincide.

(4) A presentation of $\mathbf{S}^\imath_{2,d}$ for $v \mapsto 1$, similar to Proposition 3, is first observed in [7].

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