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RATES OF CONVERGENCE IN W_p^2-NORM FOR THE MONGE–AMPÈRE EQUATION*

MICHAEL NEILAN† AND WUJUN ZHANG‡

Abstract. We develop discrete W_p^2-norm error estimates for the Oliker–Prussner method applied to the Monge–Ampère equation. This is obtained by extending discrete Alexandroff estimates and showing that the contact set of a nodal function contains information on its second-order difference. In addition, we show that the size of the complement of the contact set is controlled by the consistency of the method. Combining both observations, we show that the error estimate $\|u-u_h\|_{W^2_{f,p}}(\mathcal{N}_h^I)$ converges in order $O(h^{1/p})$ if $p > d$ and converges in order $O(h^{1/d}\ln(\frac{1}{h})^{1/d})$ if $p \le d$, where $\|\cdot\|_{W^2_{f,p}(\mathcal{N}_h^I)}$ is a weighted W_p^2-type norm, and the constant $C > 0$ depends on $\|u\|_{C^{3,1}(\bar{\Omega})}$, the dimension d, and the constant p. Numerical examples are given in two space dimensions and confirm that the estimate is sharp in several cases.

Key words. W_p^2 error estimate, Monge–Ampère equation, discrete Alexandroff maximum principle

AMS subject classifications. 65N12, 65N15, 35B50, 35D30, 35J96

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1. Introduction. In this paper we develop discrete W_p^2 error estimates for numerical approximations of the Monge–Ampère equation with Dirichlet boundary conditions:

$$\det\left(D^2u\right) = f \qquad \text{in } \Omega, \tag{1.1a}$$
$$u = 0 \qquad \text{on } \partial\Omega \tag{1.1b}$$

with given function $f \in C(\bar{\Omega})$ satisfying $\underline{f} \le f \le \bar{f}$ in $\bar{\Omega}$ for some positive constants $\underline{f}, \bar{f}$. Here, D^2u denotes the Hessian matrix of u. The domain $\Omega \subset \mathbb{R}^d$ is assumed to be bounded and uniformly convex. We seek a solution to (1.1) in the class of convex functions, which ensures ellipticity of the problem and its unique solvability [16].

Because of its wide array of applications in, e.g., differential geometry, optimal mass transport, and meteorology, several numerical methods have been developed for the Monge–Ampère problem. These methods can be roughly divided into two categories, namely, monotone methods and nonmonotone methods. The monotone methods include finite difference schemes [27, 15, 7, 22] and semi-Lagrangian schemes [14]. The convergence of this class of methods requires minimal regularity of the true solution, and the theoretical tools are based on discrete maximum/comparison principles and the theory developed in the foundational work [4, 19, 20]. On the other hand, while this framework is robust with respect to the (lack of) smoothness of the solution, these convergence results often come without explicit rates, and empirical evidence suggests that these methods are low-order. In addition to theoretical

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†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260 (neilan@pitt.edu).
‡Department of Mathematics, Rutgers University, Piscataway, NJ 08854 (wujun@math.rutgers.edu).

convergence results, nonlinear solvers (e.g., Picard, Perron, and Howard iterations) have been constructed and analyzed which are robust with respect to the initial guess [26, 27, 14].

For the nonmonotone methods, their convergence is proved provided that the exact solution is sufficient regular (e.g., H^2-regularity) and the mesh sufficiently fine; however, numerical evidence suggests that this regularity restriction might not be needed in practice. In addition, nonlinear solvers (e.g., Newton's method) only converge provided the initial guess is sufficiently close to the exact discrete solution. These methods are generally higher-order and relatively easy to implement on existing computing software. Examples of nonmonotone methods are finite element type methods such as the vanishing moment method [13], C^1 finite element methods [6, 3], and C^0 penalty methods [8, 23, 2]. We also refer the interested reader to a review of numerical methods for fully nonlinear elliptic equations [24].

The method we analyze in this paper is due to Oliker and Prussner [28, 7, 22] for the Monge–Ampère problem, and a variant of it is applied to the optimal transport problem in [1, 21, 17]. The method falls into the category of monotone methods and it is based on a geometric notion of generalized solutions called Alexandroff solutions. In this setting, the determinant of the Hessian matrix of u in (1.1a) is interpreted as the measure of the subdifferential of u; see [16]. The method proposed in [28] simply poses this solution concept onto the space of nodal functions and enforces the geometric condition implicitly given in (1.1a) at a finite number of points. Namely, the method seeks a nodal function u_h satisfying the Dirichlet boundary conditions on boundary nodes, and

$$|\partial u_h(x_i)| = f_i$$

at all interior grid points x_i. Here, $\partial u_h(x_i)$ denotes that the subdifferential of u_h at x_i, $|\cdot|$ is the d-dimensional Lebesgue measure, $f_i \approx h^d f(x_i)$, and h is the mesh parameter. The existence and uniqueness of the method and convergence to the Alexandroff solution are shown in [28].

While the convergence of monotone methods is ensured under the framework in [4, 19], the study of rate of convergence of these methods remains largely open for the Monge–Ampère equation. Recently, a pointwise error estimate of the Oliker–Prussner scheme is established in [26] and a coming paper [31]. There it is shown that, if the exact convex solution to (1.1) is sufficiently smooth, and if the nodes are translation invariant, then the error is of (optimal) order $\mathcal{O}(h^2)$ in the L_∞-norm. We note that standard scaling arguments based on this estimate yield $\mathcal{O}(1)$ errors in W_p^2, i.e., no convergence. Generalities of these L_∞ estimates, depending on solution regularity, are also given in [26]. However, in many applications, the variable of primary interest is the gradient map $\boldsymbol{\nabla} u$, instead the scale function u. For example, for the optimal transport problems, the gradient $\boldsymbol{\nabla} u$ yields the optimal mapping which minimizes the L_2-cost to transport one measure to another. Therefore, it is desirable to get an error estimate for u in a W_p^1-norm. While one might derive the W_p^1 estimate for the L_∞ error estimate by an inverse inequality and obtain $\|u - u_h\|_{W_p^1(\Omega)} \le O(h)$, such an estimate is suboptimal as observed by numerical experiments; see section 6. Recently H^1 error estimates with rate $\mathcal{O}(h^{1/2})$ have been established for the optimal transport problem in [5]. The arguments given there are quite different from ours.

Our contribution in this paper is to develop a discrete W_p^2 error estimates for all $p \in [1, \infty)$. The idea is inspired by the PDE work [10, 12]. Let

$$\delta_e v(x_i) := \frac{v(x_i + he) - 2v(x_i) + v(x_i - he)}{|e|^2 h^2}$$

be the second-order difference operator of a nodal or continuous function v in the direction $e \in \mathbb{Z}^d$ at a node x_i, where $|e|$ denotes the Euclidean norm of e. The (weighted) W_p^2-norm of a nodal function v with respect to direction e on a set of nodes S is given by

$$\|v\|_{W^2_{f,p}(S)} := \left(\sum_{x_i \in S} f_i |\delta_e v(x_i)|^p \right)^{1/p}$$

with f_i given by (2.7). The main result of the paper, precisely given in Theorem 5.3, is the estimate

$$\|u - u_h\|_{W^2_{f,p}(\mathcal{N}_h^I)} \leq \begin{cases} Ch^{1/p} & \text{if } p > d, \\ Ch^{1/d} \ln\left(\frac{1}{h}\right)^{1/d} & \text{if } p \leq d. \end{cases}$$

Similar to the arguments in [26], operator consistency of the method is one of the results we use. However, as Alexandroff maximum principles are inherently restricted to the L_∞-norm, there is no hope that the techniques given in [26] will yield error estimates in W_p^2. Instead, we first make an observation that the contact set of a nodal function contains useful information about its second-order difference, Lemma 4.1. Based on this observation, we establish the key stability result in Proposition 4.1 and show that the size of the complement of the contact set is controlled by the consistency error of the method. Along with a decomposition of nodal functions in terms of its level sets (Lemma 5.1), we use these technical tools to construct the W_p^2 error estimates stated above.

Another application of our results is to combine the Oliker–Prussner method with a higher-order scheme. The aforementioned convergence results of a higher-order scheme given in [23] require that the initial guess and the exact solution are sufficiently close in a W_p^2-norm. The Oliker–Prussner scheme, as we prove in this paper, can be used as a convergent initial guess within a higher-order scheme. We will explore this idea in a coming paper. We mention that the solution of the Oliker–Prussner method can be solved by introducing a suitable triangulation of the nodal set and by applying Newton's iteration. At each iteration $k \geq 1$, the triangulation may be altered to ensure that the piecewise linear interpolation of the nodal solution u_h^k is convex. The procedure of changing triangulation is completely local and can be efficiently implemented. We refer to [26] for these implementation details and to [18] for alternative nonlinear solvers.

The organization of the paper is as follows. In the next section, we state the Oliker–Prussner method and state some preliminary results. In section 3 we give operator consistency results of the scheme. Section 4 gives stability results with respect to the second-order difference operators, and in section 5 we provide W_p^2 error estimates. Finally, we end the paper with some numerical experiments in section 6.

2. Preliminaries.

2.1. Nodal set and nodal function. Let $\mathcal{N}_h$ be a set of nodes in the domain $\bar{\Omega}$. We denote the set of interior nodes $\mathcal{N}_h^I := \mathcal{N}_h \cap \Omega$, the set of boundary nodes $\mathcal{N}_h^B := \mathcal{N}_h \cap \partial\Omega$, and the nodal set

$$\mathcal{N}_h = \mathcal{N}_h^I \cup \mathcal{N}_h^B.$$

To ensure that the interior node is not too close to the boundary $\partial\Omega$, we require that

$$\operatorname{dist}(z, \partial\Omega) \geq \frac{h}{2} \quad \text{for any nodes } z \in \mathcal{N}_h^I. \tag{2.1}$$

Such a nodal set can be obtained by removing the nodes whose distance to $\partial\Omega$ is less than $h/2$. We assume that the nodal set is *translation invariant*, i.e., there exist a point $b \in \mathbb{R}^d$ and a basis $\{e_i\}_{i=1}^d$ in $\mathbb{R}^d$ such that any interior node $z \in \mathcal{N}_h^I$ can be written as

$$z = b + \sum_{i=1}^{d} h z_i e_i \quad \text{for some integers } z_i \in \mathbb{Z}. \tag{2.2}$$

Since the basis e_i can be transformed into the canonical basis in $\mathbb{R}^d$ under a linear transformation, hereafter to simplify the presentation, we will assume that $\mathcal{N}_h^I = b + h\mathbb{Z}^d$. We also make the following additional assumption on the boundary nodal set $\mathcal{N}_h^B$:

$$\operatorname{dist}\left(x, \mathcal{N}_h^B\right) \le h/2 \qquad \forall x \in \partial\Omega. \tag{2.3}$$

We say the nodal spacing of $\mathcal{N}_h$ is h. It is worth mentioning that one can construct a translation invariant $\mathcal{N}_h$ on a curved domain Ω. In fact, for a nodal set $\mathcal{N}_h$ to be translation invariant, we only require the interior nodal set $\mathcal{N}_h^I$ to satisfy (2.2), while no such requirement is made on the boundary nodes.

Associated with the nodes is a simplicial triangulation $\mathcal{T}_h$ with vertices $\mathcal{N}_h$. We denote by h_T the diameter of $T \in \mathcal{T}_h$, and by ρ_T the diameter of the largest inscribed ball in T. We assume that the triangulation is shape-regular, i.e., there exists $\sigma > 0$ such that

$$\frac{h_T}{\rho_T} \le \sigma \qquad \forall T \in \mathcal{T}_h.$$

We denote by $\{\phi_i\}_{i=1}^n$ with $n = \#\mathcal{N}_h^I$ the canonical piecewise linear hat functions associated with $\mathcal{T}_h$. Namely, the function $\phi_i \in C(\bar{\Omega})$ is a piecewise linear polynomial with respect to $\mathcal{T}_h$ and is uniquely determined by the condition $\phi_i(x_j) = \delta_{i,j}$ (Kronecker delta) for all $x_j \in \mathcal{N}_h^I$ and $\phi_i(x_j) = 0$ for all $x_j \in \mathcal{N}_h^B$. We denote by ω_i the support of ϕ_i, i.e., the patch of elements in $\mathcal{T}_h$ that have x_i as a vertex.

A function defined on $\mathcal{N}_h$ is called a nodal function, and we denote the space of nodal functions by $\mathcal{M}_h$. For a nodal function g with values $\{g_i\}_{x_i \in \mathcal{N}_h}$, and for a subset of nodal points $\mathcal{C} \subset \mathcal{N}_h$, we set the discrete ℓ^d norm as

$$\|g\|_{\ell^d(\mathcal{C})} := \left(\sum_{x_i \in \mathcal{C}} |g_i|^d \right)^{1/d}.$$

We say that a nodal function $u_h \in \mathcal{M}_h$ is convex if, for all $x_i \in \mathcal{N}_h^I$, there exists a supporting hyperplane L of u_h, i.e.,

$$L(x_j) \le u_h(x_j) \quad \forall x_j \in \mathcal{N}_h \text{ and } L(x_i) = u(x_i).$$

The convex envelope of u_h is the function $\Gamma(u_h) \in C(\bar{\Omega})$ given by

$$\Gamma(u_h)(x) = \sup_L \{L(x) \text{ is affine}: \ L(x_i) \le u_h(x_i) \ \forall x_i \in \mathcal{N}_h\}.$$

Finally, we denote by $N_h : C(\bar{\Omega}) \to \mathcal{M}_h$ the nodal interpolant satisfying $N_h v(x_i) = v(x_i)$ for all $x_i \in \mathcal{N}_h$. It is easy to see that if v is a convex function on $\bar{\Omega}$, then $N_h v$ is a convex nodal function.

2.2. The Oliker–Prussner method. To motivate the method introduced in [28], we first introduce the notion of an Alexandroff solution to the Monge–Ampère equation (1.1). To this end, note that if the solution to (1.1) is strictly convex, and if $u \in C^2(\Omega)$, then a change of variables reveals that

$$\int_E f\,dx = \int_E \det\left(D^2 u\right)\,dx = \int_{\nabla u(E)} dx = |\nabla u(E)| \quad \text{for all Borel } E \subset \Omega,$$

where $|\nabla u(E)|$ denote the d-dimensional Lebesgue measure of $\nabla u(E) = \{\nabla u(x) : x \in E\}$. To extend this identity to a larger class of functions, we introduce the subdifferential of the function u at the point x_0 as

$$\partial u(x_0) = \left\{p \in \mathbb{R}^d : \ u(x) \geq u(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega\right\}.$$

Thus, $\partial u(x_0)$ is the set of supporting hyperplanes of the graph of u at x_0. If u is strictly convex and smooth, then $\partial u(x_0) = \{\nabla u(x_0)\}$, and the same calculation as above shows that

$$\int_E f\,dx = |\partial u(E)| \quad \text{for all Borel } E \subset \Omega. \tag{2.4}$$

Definition 2.1. *A convex function $u \in C(\bar{\Omega})$ is an* Alexandroff solution *to* (1.1) *provided that $u = 0$ on $\partial\Omega$ and* (2.4) *is satisfied.*

The method introduced in [28] simply poses this solution concept onto the space of nodal functions. To do so, the definition of the subdifferential is extended to the spaces of nodal functions in the natural way:

$$\partial u_h(x_i) = \left\{p \in \mathbb{R}^d : \ u(x_j) \geq u_h(x_i) + p \cdot (x_j - x_j)\ \forall x_j \in \mathcal{N}_h\right\}. \tag{2.5}$$

The subdifferential of a convex nodal function u_h defined above is simple to characterize. The convex function $\Gamma(u_h)$ is continuous and piecewise linear with respect to a simplicial partition of Ω. The subdifferential ∂u_h at a node z is just the convex hull of the piecewise gradients $\nabla\Gamma(u_h)|_T$ for all simplices T that have z as a vertex; see Figure 1 for a pictorial description and [25, 24] for further details. Thus, the subdifferential ∂u_h can be viewed as a map between the nodes and these polytopal cells.

The discrete method is to find a convex nodal function u_h with $u_h = 0$ on $\mathcal{N}_h^B$ and

$$|\partial u_h(x_i)| = f_i \qquad \forall x_i \in \mathcal{N}_h^I, \tag{2.6}$$

where

$$f_i = \int_\Omega f(x)\phi_i(x)\,dx = \int_{\omega_i} f(x)\phi_i(x)\,dx. \tag{2.7}$$

Remark 2.1. The existence and uniqueness of a solution to (2.6) are given in [28, 26].

2.3. Brunn–Minkowski inequality and subdifferential of convex functions. In this subsection, we develop a few techniques which will be useful in establishing the error estimate. We start with the celebrated Brunn–Minkowski inequality which relates the volumes of compact sets of $\mathbb{R}^d$.

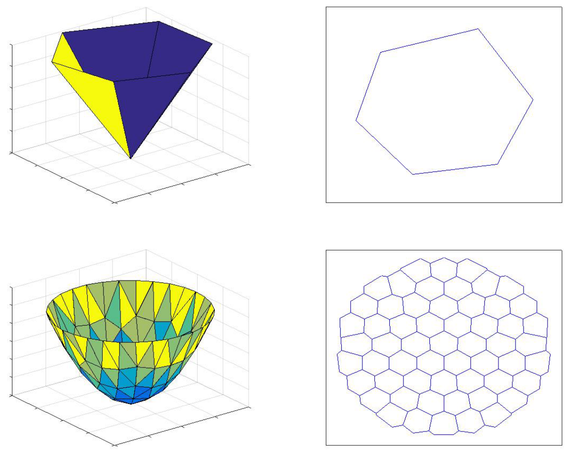

FIG. 1. *Left: The graph of the convex envelope of a nodal function on a coarse (top) and fine (bottom) grid. Right: The convex hulls of the piecewise gradients of the convex envelopes on the respective grids. These polygonal cells characterize the subdifferential of the nodal function.*

PROPOSITION 2.1 (Brunn–Minkowski inequality). *Let A and B be two nonempty compact subsets of $\mathbb{R}^d$ for $d \geq 1$. Then the following inequality holds:*

$$|A+B|^{1/d} \geq |A|^{1/d} + |B|^{1/d},$$

where $A+B$ denotes the Minkowski sum:

$$A+B := \left\{v+w \in \mathbb{R}^d : v \in A \text{ and } w \in B\right\}.$$

Next, we make the following observation on the sum of two subdifferential sets.

LEMMA 2.2 (see [26, Lemma 2.3]). *Let u_h and v_h be two convex nodal functions. Then there holds*

$$\partial u_h(x_i) + \partial v_h(x_i) \subset \partial(u_h+v_h)(x_i)$$

for all $x_i \in \mathcal{N}_h^I$.

Proof. Let p_1 and p_2 be in $\partial u_h(x_i)$ and $\partial v_h(x_i)$, respectively. By the definition of subdifferential (2.5), we have

$$\begin{aligned} p_1 \cdot (x_j - x_i) &\leq u_h(x_j) - u_h(x_i) \quad \forall x_j \in \mathcal{N}_h, \\ p_2 \cdot (x_j - x_i) &\leq v_h(x_j) - v_h(x_i) \quad \forall x_j \in \mathcal{N}_h. \end{aligned}$$

Adding both inequalites, we obtain

$$(p_1+p_2)\cdot(x_j-x_i) \leq (u_h+v_h)(x_j) - (u_h+v_h)(x_i) \quad \forall x_j \in \mathcal{N}_h.$$

This shows that $p_1+p_2 \in \partial(u_h+v_h)(x_i)$. ∎

Combining both estimates, we derive the following result.

LEMMA 2.3. *Let u_h and v_h be two convex nodal functions defined on $\mathcal{N}_h$ and $\mathcal{C}_h$ be the lower contact set of $(u_h - v_h)$:*

$$\mathcal{C}_h := \left\{x_i \in \mathcal{N}_h^I : \ \Gamma(u_h - v_h)(x_i) = (u_h - v_h)(x_i)\right\}.$$

Then for any node $x_i \in \mathcal{C}_h$*,*

$$|\partial\Gamma(u_h - v_h)(x_i)|^{1/d} \le |\partial u_h(x_i)|^{1/d} - |\partial v_h(x_i)|^{1/d}. \tag{2.8}$$

Proof. The proof of this result is implicitly given in [26, Proposition 4.3], but we give it here for completeness.

The definition of the convex envelope and the subdifferential shows that

$$\partial\Gamma(u_h - v_h)(x_i) \subset \partial(u_h - v_h)(x_i)$$

for all $x_i \in \mathcal{C}_h$. Applying Lemma 2.2 then yields

$$\partial v_h(x_i) + \partial\Gamma(u_h - v_h)(x_i) \subset \partial v_h(x_i) + \partial(u_h - v_h)(x_i) \subset \partial u_h(x_i).$$

An application of the Brunn–Minkowski inequality (cf. Lemma 2.1) gets

$$\begin{aligned} |\partial v_h(x_i)|^{1/d} + |\partial\Gamma(u_h - v_h)(x_i)|^{1/d} &\le |\partial v_h(x_i) + \partial\Gamma(u_h - v_h)(x_i)|^{1/d} \\ &\le |\partial u_h(x_i)|^{1/d}. \end{aligned}$$

Rearranging terms we obtain (2.8). ∎

We also note that the numerical method (2.6) has a discrete comparison principle. Here, we refer to [26] for a proof.

LEMMA 2.4 (discrete comparison principle [26, Corollary 4.4]). *Let* $v_h, w_h \in \mathcal{M}_h$ *satisfy* $v_h(x_i) \ge w_h(x_i)$ *for all* $x_i \in \mathcal{N}_h^B$ *and* $|\partial v_h(x_i)| \le |\partial w_h(x_i)|$ *for all* $x_i \in \mathcal{N}_h^I$*. Then*

$$v_h(x_i) \ge w_h(x_i) \qquad \forall x_i \in \mathcal{N}_h.$$

3. Consistency of the Oliker–Prussner method. In this section, we state the consistency of the method (2.6) given in [31]. The result shows that the relative consistency error is of order $\mathcal{O}(h^2)$ away from the boundary and of order $\mathcal{O}(1)$ in a $\mathcal{O}(h)$ region of the boundary.

LEMMA 3.1. *Let* $\mathcal{N}_h$ *be a translation invariant nodal set defined on the domain* Ω*. If* $u \in C^{k,\alpha}(\bar\Omega)$ *is a convex function with* $0 < \lambda I \le D^2 u \le \Lambda I$ *and* $2 \le k+\alpha \le 4$*, there holds, for* $\operatorname{dist}(x_i, \partial\Omega) \ge Rh$*,*

$$\big||\partial N_h u(x_i)| - f_i\big| \le Ch^{k+\alpha+d-2}, \tag{3.1}$$

where R *depends on* λ *and* Λ*. Moreover, there holds for* $\operatorname{dist}(x_i, \partial\Omega) \le Rh$*,*

$$\big|\partial N_h u(x_i) - f_i\big| \le Ch^d.$$

Remark 3.1. The regularity of f and $\partial\Omega$, the strict convexity of Ω, and the positivity of f guarantee that the convex solution to (1.1) enjoys the regularity $u \in C^{k,\alpha}(\bar\Omega)$. For example, if $f \in C^{k-2,\alpha}(\bar\Omega)$ and Ω is smooth, then the solutions satisfy $u \in C^{k,\alpha}(\bar\Omega)$ [16, 9, 29]

Remark 3.2. We note that if the boundary nodes also form part of the regular lattice (e.g., on a rectangular domain/lattice), then the consistency estimate could hold up to the boundary, i.e., in this case estimate (3.1) holds for all $x_i \in \mathcal{N}_h$.

Thanks to the consistency error of the method, Lemma 3.1, an L_∞-error estimate is derived in [26, 11] which states the following.

PROPOSITION 3.2. *Let Ω be uniformly convex and $\mathcal{N}_h^I$ be translation invariant. Suppose further that the boundary nodes satisfy* (2.1)*, that $f \geq \underline{f} > 0$, and that the convex solution to* (1.1) *satisfies $u \in C^{k,\alpha}(\bar{\Omega})$ for some $2 \leq k + \alpha \leq 4$ and $0 < \lambda I \leq D^2 u \leq \Lambda I$. Then the numerical solution to the discrete Monge–Ampère equation* (2.6) *satisfies*

$$\|u_h - N_h u\|_{L_\infty(\mathcal{N}_h)} \leq C h^{k+\alpha-2} \|u\|_{C^{k,\alpha}(\bar{\Omega})},$$

where $\|v_h\|_{L_\infty(\mathcal{N}_h)} := \max_{x_i \in \mathcal{N}_h} |v_h(x_i)|$.

We note that if $u \in C^{3,1}(\bar{\Omega})$, then the optimal order of the L_∞ error is $\mathcal{O}(h^2)$. By this L_∞ error estimate and the assumption (2.1) that the boundary node is at least $h/2$ away from the boundary, we immediately deduce that $|\delta_e(N_h u - u_h)(x_i)|$ is bounded. This observation will be useful in the following sections when we investigate the discrete W_p^2 error estimate.

4. Stability of the second-order difference of the Oliker–Prussner method. Given two solutions u_h and v_h of the discrete Monge–Ampère equations

$$|\partial u_h(x_i)| = f_i \quad \text{and} \quad |\partial v_h(x_i)| = g_i \quad \text{with} \quad u_h = v_h = 0 \quad \text{on } \partial\Omega,$$

our goal in this section is to control the second-order difference of the error function $v_h - u_h$ in terms of the consistency error $f_i^{1/d} - g_i^{1/d}$. We define a set of relative error τ as

$$E_\tau = \left\{ x_i \in \mathcal{N}_h,\ \delta_e(v_h - u_h)(x_i) \geq \tau \delta_e v_h(x_i) \quad \text{for some vector } e \in \mathbb{Z}^d \right\}.$$

The main result of the section is to show that a measure of E_τ is controlled by the consistency error $f_i^{1/d} - g_i^{1/d}$ in ℓ^d-norm for any $\tau > 0$. The precise statement is in Proposition 4.1.

We start with an observation that the contact set of a nodal function contains interesting information on its second-order difference.

LEMMA 4.1 (estimate of second-order difference). *Given two convex nodal functions v_h and u_h defined on the nodal set $\mathcal{N}_h$, let*

$$w_\epsilon = u_h - (1-\epsilon) v_h \quad \text{and} \quad w^\epsilon = v_h - (1-\epsilon) u_h$$

for some $0 < \epsilon \leq 1$ and the contact sets

$$\mathcal{C}_\epsilon := \{x_i \in \mathcal{N}_h, \quad w_\epsilon(x_i) = \Gamma w_\epsilon(x_i)\}, \tag{4.1}$$

$$\mathcal{C}^\epsilon := \{x_i \in \mathcal{N}_h, \quad w^\epsilon(x_i) = \Gamma w^\epsilon(x_i)\}. \tag{4.2}$$

If a node $x_i \in \mathcal{C}_\epsilon \cap \mathcal{C}^\epsilon$, then

$$-\epsilon \delta_e v_h(x_i) \leq \delta_e(u_h - v_h)(x_i) \leq \frac{\epsilon}{1-\epsilon} \delta_e v_h(x_i) \tag{4.3}$$

for any vector $e \in \mathbb{Z}^d$.

Proof. We observe that if a node is in the contact set $x_i \in \mathcal{C}_\epsilon$, then the second-order difference of w_ϵ satisfies $\delta_e w_\epsilon(x_i) \geq \delta_e \Gamma w_\epsilon(x_i) \geq 0$ for any vector $e \in \mathbb{Z}^d$. Hence,

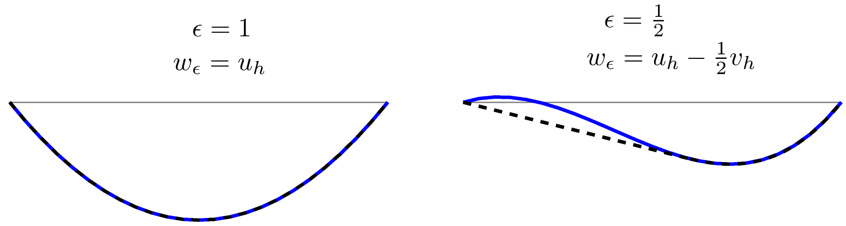

FIG. 2. *A pictorial description of Remark* 4.1 *with $w_\epsilon = u_h - (1-\epsilon)v_h$. The graph of w_ϵ is depicted in blue, and the graph of its convex envelope is given by the dashed black line. If $\epsilon = 1$, then $w_\epsilon = u_h$ is convex, and the noncontact set is empty (left). Otherwise for $\epsilon \in [0,1)$, w_ϵ is not necessarily convex, and the noncontact set is nonempty (right).*

for any node $x_i \in \mathcal{C}_\epsilon$, we have

$$\delta_e(u_h - v_h)(x_i) \geq -\epsilon \delta_e v_h(x_i). \tag{4.4}$$

This inequality yields a lower bound of the second-order difference.

To derive the upper bound, we apply a similar argument above to the function w^ϵ and derive

$$\delta_e(v_h - u_h)(x_i) \geq -\epsilon \delta_e u_h(x_i)$$

for any node $x_i \in \mathcal{C}^\epsilon$. A simple algebraic manipulation yields

$$\delta_e(u_h - v_h)(x_i) \leq \frac{\epsilon}{1-\epsilon} \delta_e v_h(x_i). \tag{4.5}$$

Combining both the lower bound (4.4) and upper bound (4.5), we obtain the desired estimate. ∎

Remark 4.1. The lemma above shows that we have control of the error $\delta_e(u_h - v_h)$ on the contact sets $\mathcal{C}_\epsilon$ and $\mathcal{C}^\epsilon$. Define the set E_τ to be

$$E_\tau = \left\{ x_i \in \mathcal{N}_h,\ \delta_e(v_h - u_h)(x_i) \geq \tau \delta_e v_h(x_i) \quad \text{for some vector } e \in \mathbb{Z}^d \right\}, \tag{4.6}$$

where $\tau = \epsilon/(1-\epsilon)$. Then the proof of Lemma 4.1 shows that E_τ is contained in the noncontact set

$$S_\epsilon := \mathcal{N}_h \setminus \mathcal{C}_\epsilon. \tag{4.7}$$

Analogously,

$$\begin{aligned} E^\tau &:= \left\{ x_i \in \mathcal{N}_h,\ \delta_e(u_h - v_h)(x_i) \geq \tau \delta_e v_h(x_i) \quad \text{for some vector } e \in \mathbb{Z}^d \right\} \\ &\subset S^\epsilon := \mathcal{N}_h \setminus \mathcal{C}^\epsilon. \end{aligned}$$

In the next step, we estimate the cardinality of S_ϵ. Heuristically, if $\epsilon = 1$, then $w_\epsilon = u_h$ which is a convex nodal function, and so we have $S_\epsilon = \emptyset$. As ϵ decreases to zero, the function w_ϵ becomes "less convex" and the cardinality $\#(S_\epsilon)$ increases; see Figure 2. Therefore, our next goal is to estimate how fast $\#(S_\epsilon)$ increases as $\epsilon \to 0$. The following lemma shows that this is controlled by the consistency error of the method.

PROPOSITION 4.1. *Let u_h and v_h be two convex nodal functions satisfying $u_h = v_h$ on $\mathcal{N}_h^B$, $u_h \leq v_h$ in $\mathcal{N}_h^I$, and*

$$|\partial u_h(x_i)| = f_i, \quad \textit{and} \quad |\partial v_h(x_i)| = g_i \tag{4.8}$$

for all $x_i \in \mathcal{N}_h^I$. For any subset $S \subset \mathcal{N}_h^I$, let

$$\mu(S) = \sum_{x_i \in S} f_i \quad \text{and} \quad \nu_\tau(S) = \sum_{x_i \in S} \left(f_i^{1/d} + \frac{1}{\tau} e_i^{1/d} \right)^d, \tag{4.9}$$

where $e_i^{1/d} = (f_i^{1/d} - g_i^{1/d}$. Then

$$\mu(E_\tau) \le \mu(S_\epsilon) \le \nu_\tau(\mathcal{C}_\epsilon) - \mu(\mathcal{C}_\epsilon), \tag{4.10}$$

where $\mathcal{C}_\epsilon$ is given by (4.1), S_ϵ is given by (4.7), and $\tau = \epsilon/(1-\epsilon)$. Consequently, there holds

$$\mu(E_\tau) \le \mu(S_\epsilon) \le \tau^{-1} C_f \left\| e^{1/d} \right\|_{\ell^d(\mathcal{C}_\epsilon)} \tag{4.11}$$

with $C_f = d\|f^{1/d}\|_{\ell^d(\mathcal{N}_h^I)}^{d-1}$.

Proof. Since $E_\tau \subset S_\epsilon$ by Remark 4.1, we only need to show that

$$\mu(S_\epsilon) \le \nu_\tau(\mathcal{C}_\epsilon) - \mu(\mathcal{C}_\epsilon), \quad \text{and} \quad \mu(S_\epsilon) \le \tau^{-1} C_f \left\| e^{1/d} \right\|_{\ell^d(\mathcal{C}_\epsilon)}.$$

We first show that

$$\sum_{x_i \in \mathcal{N}_h^I} \epsilon \partial u_h(x_i) \subset \sum_{x_i \in \mathcal{N}_h^I} \partial \Gamma w_\epsilon(x_i), \tag{4.12}$$

where $w_\epsilon = u_h - (1-\epsilon) v_h$. Since $u_h \le v_h$ in $\mathcal{N}_h^I$ and $u_h = v_h$ on $\mathcal{N}_h^B$, we get

$$w_\epsilon \le \epsilon u_h \text{ in } \mathcal{N}_h^I \quad \text{and} \quad w_\epsilon = \epsilon u_h \text{ on } \mathcal{N}_h^B.$$

Taking a convex envelope on both sides of the inequality, we obtain

$$\Gamma w_\epsilon(x) \le \epsilon \Gamma u_h(x) \quad \text{in } \Omega \text{ and} \quad \Gamma w_\epsilon(x) = \epsilon \Gamma u_h(x) \quad \text{on } \partial\Omega. \tag{4.13}$$

Since $u_h = \Gamma u_h$ on $\mathcal{N}_h$ due to the convexity of u_h, the inequality (4.13) implies (4.12).

Taking measure on both sides of (4.12) and substituting (4.8) yields

$$\epsilon^d \sum_{x_i \in \mathcal{N}_h^I} f_i = \epsilon^d \sum_{x_i \in \mathcal{N}_h^I} |\partial u_h(x_i)| \le \sum_{x_i \in \mathcal{C}_\epsilon} |\partial \Gamma w_\epsilon(x_i)|.$$

In view of the convexity of the measure of the subidfferential (2.8),

$$|\partial \Gamma w_\epsilon(x_i)|^{1/d} \le \left| f_i^{1/d} - (1-\epsilon) g_i^{1/d} \right|.$$

Therefore, we infer that

$$\epsilon^d \mu\left(\mathcal{N}_h^I\right) = \epsilon^d \sum_{x_i \in \mathcal{N}_h^I} f_i \le \sum_{x_i \in \mathcal{C}_\epsilon} \left| f_i^{1/d} - (1-\epsilon) g_i^{1/d} \right|^d.$$

Thus, subtracting $\epsilon^d \mu(\mathcal{C}_\epsilon)$, we obtain

$$\epsilon^d \mu(S_\epsilon) = \epsilon^d \sum_{x_i \in S_\epsilon} f_i \le \sum_{x_i \in \mathcal{C}_\epsilon} \left(\left| \epsilon f_i^{1/d} + (1-\epsilon) e_i^{1/d} \right|^d - \epsilon^d f_i \right).$$

Therefore, dividing ϵ^d, we obtain

$$\mu(S_\epsilon) \leq \nu_\tau(\mathcal{C}_\epsilon) - \mu(\mathcal{C}_\epsilon).$$

To derive the estimate (4.11), we first see that (4.10) is equivalent to

$$\left\| f^{1/d} \right\|_{\ell^d(\mathcal{N}_h^I)} \leq \left\| f^{1/d} + \tau^{-1} e^{1/d} \right\|_{\ell^d(\mathcal{C}_\epsilon)},$$

and therefore $\|f^{1/d}\|_{\ell^d(\mathcal{N}_h^I)} - \|f^{1/d}\|_{\ell^d(\mathcal{C}_\epsilon)} \leq \tau^{-1}\|e^{1/d}\|_{\ell^d(\mathcal{C}_\epsilon)}$ by the Minkowski inequality. From this estimate and the inequality $a^d - b^d \leq da^{d-1}(a-b)$ for $a \geq b$, we derive

$$\begin{aligned}
\mu(S_\epsilon) &= \left\| f^{1/d} \right\|^d_{\ell^d(\mathcal{N}_h^I)} - \left\| f^{1/d} \right\|^d_{\ell^d(\mathcal{C}_\epsilon)} \\
&\leq d \left\| f^{1/d} \right\|^{d-1}_{\ell^d(\mathcal{N}_h^I)} \left(\left\| f^{1/d} \right\|_{\ell^d(\mathcal{N}_h^I)} - \left\| f^{1/d} \right\|_{\ell^d(\mathcal{C}_\epsilon)} \right) \\
&\leq C_f \tau^{-1} \left\| e^{1/d} \right\|_{\ell^d(\mathcal{C}_\epsilon)}.
\end{aligned}$$

□

5. W_p^2-estimate of the method.

To establish W_p^2-estimates of the method, we first introduce an estimate of the discrete L_1 norm of a nodal function in terms of its level sets.

LEMMA 5.1. *Let s_h be a bounded nodal function with $|s_h(x_i)| \leq M$ for some $M > 0$. Then, for any $\sigma > 0$,*

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |s_h(x_i)| \leq \sigma \sum_{k=0}^{N} \mu(A_k), \tag{5.1}$$

where

$$A_k := \left\{ x_i \in \mathcal{N}_h^I : \ |s_h(x_i)| \geq k\sigma \right\},$$

$\mu(\cdot)$ is given by (4.9), and $N = \lceil M/\sigma \rceil$.

Remark 5.1. Roughly speaking, Lemma 5.1 gives a relation between Riemann and Lebesgue sums. For example, if $f_i = h^d$ for all i, then the left-hand side of (5.1) yields a discrete Riemann integral of s_h ("areas of vertical bars"), and the right-hand side is an approximation of a discrete Lebesgue integral of s_h ("areas of horizontal bars"); see Figure 3.

Proof. Set

$$P_k := \left\{ x_i \in \mathcal{N}_h^I : \ k\sigma \leq |s_h(x_i)| < (k+1)\sigma \right\}.$$

Then we clearly have

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |s_h(x_i)| = \sum_{k=0}^{N} \sum_{x_i \in P_k} f_i |s_h(x_i)| \leq \sum_{k=0}^{N} (k+1)\sigma \mu(P_k).$$

We also have

$$A_k = \bigcup_{m \geq k}^{N} P_m,$$

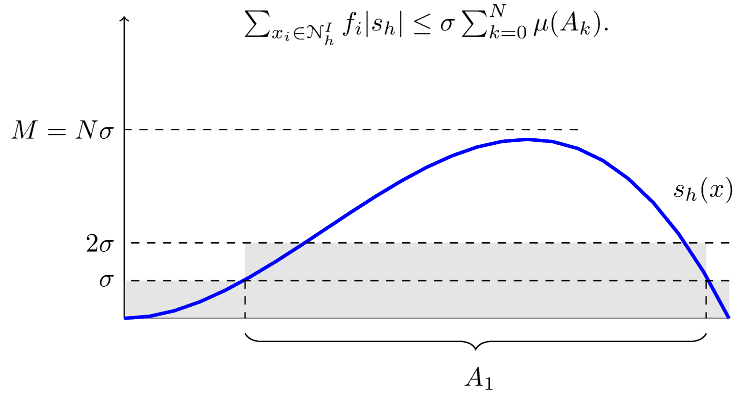

FIG. 3. *A pictorial illustration of Lemma 5.1. Here, the measure $\mu(A_k) := \sum_{x_i\in A_k} f_i$. The summation $\sum_{x_i\in\mathcal{N}_h^I} f_i|s_h|$ can be viewed as the integral of $|s_h|$ with respect to measure the $\sum f_i\delta_{x_i}$ (or the area under the blue curve), and $\sigma\mu(A_k)$ can be viewed as the area of rectangle with bases $\mu(A_k)$ and height σ. The rectangles with base $\mu(A_0)$ and $\mu(A_1)$ are plotted in gray.*

and so, since the sets $\{P_k\}$ are disjoint,

$$\mu(A_k) = \sum_{m=k}^{N} \mu(P_m).$$

Therefore

$$\sigma\sum_{k=0}^{N}\mu(A_k) = \sigma\sum_{k=0}^{N}\sum_{m=k}^{N}\mu(P_m) = \sigma\sum_{k=0}^{N}(k+1)\mu(P_k) \ge \sum_{x_i\in\mathcal{N}_h^I} f_i|s_h(x_i)|. \qquad \square$$

5.1. Ideal case. Now we are ready to prove the estimate in the case that the consistency error (3.1) holds for all interior grid points.

THEOREM 5.2. *Let u be the solution of the Monge–Ampère equation* (1.1). *Assume that*

$$\big||\partial N_h u(x_i)| - f_i\big| \le Ch^{2+d} \quad \text{for every node } x_i \in \mathcal{N}_h^I, \tag{5.2}$$

where $N_h u$ is the interpolation of u on the nodal set $\mathcal{N}_h$. Assume further that f is uniformly positive on Ω. Then the error in the weighted W_p^2-norm satisfies

$$\|N_h u - u_h\|_{W^2_{f,p}(\mathcal{N}_h^I)} \le C \begin{cases} h^2|\ln h| & \text{if } p = 1,\\ h^{2/p} & \text{if } p > 1\end{cases}$$

provided that h is sufficiently small.

Proof. We start by setting $v_h = (1 - Ch^2)^{1/d} N_h u$, where the constant $C > 0$ is large enough, but independent of h, to ensure that (cf. (5.2))

$$g_i := |\partial v_h(x_i)| = \left(1 - Ch^2\right)|\partial N_h u(x_i)| \le f_i.$$

By a comparison principle (cf. Lemma 2.4), we have $u_h \le v_h$ on $\mathcal{N}_h^I$, and we see that

$$|f_i - g_i| \le Ch^{2+d} \qquad \forall x_i \in \mathcal{N}_h^I \tag{5.3}$$

due to the assumption (5.2). We also have $g_i \geq Ch^d$ provided h is sufficiently small, and $|(v_h - N_h u)(x_i)| \leq Ch^2$.

Note that

$$\|N_h u - u_h\|_{W^2_{f,p}(\mathcal{N}_h^I)} \leq \|v_h - u_h\|_{W^2_{f,p}(\mathcal{N}_h^I)} + Ch^2 \|N_h u\|_{W^2_p(\mathcal{N}_h^I)}.$$

Thus, to prove the theorem, it suffices to show that

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e (v_h - u_h)(x_i)|^p \leq C \begin{cases} h^2 |\ln h| & \text{if } p = 1, \\ h^2 & \text{if } p > 1. \end{cases}$$

Define the positive and negative parts of $\delta_e(v_h - u_h)(x_i)$, respectively, as

$$\delta_e^+(v_h - u_h)(x_i) = \max\{\delta_e(v_h - u_h)(x_i), 0\},$$
$$\delta_e^-(v_h - u_h)(x_i) = \max\{-\delta_e(v_h - u_h)(x_i), 0\}.$$

We shall prove

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e^+ (v_h - u_h)(x_i)|^p \leq C \begin{cases} h^2 |\ln h| & \text{if } p = 1, \\ h^2 & \text{if } p > 1. \end{cases}$$

The estimate for the negative part can be proved in a similar fashion.

Due to the regularity assumption of u, a Taylor expansion shows that $|\delta_e v_h(x_i)| \leq C_2$ for all $x_i \in \mathcal{N}_h^I$, where $C_2 > 0$ depends on $\|u\|_{C^{1,1}(\bar{\Omega})}$. Moreover, from the L_∞ error estimate, Proposition 3.2, and the assumption (2.1) that interior nodes are at least $h/2$ away from the boundary, we deduce that

$$\delta_e^+(v_h - u_h)(x_i) \leq C_\infty \qquad \forall x_i \in \mathcal{N}_h^I,$$

where the constant $C_\infty > 0$ depends on $\|u\|_{C^{3,1}(\bar{\Omega})}$.

Let $\tau_k = C_2 k^{1/p} h^2$, and define the set

$$A_k := \left\{x_i \in \mathcal{N}_h^I, \quad \delta_e^+(v_h - u_h)(x_i) \geq \tau_k\right\}.$$

By Lemma 5.1 with $s_h(x_i) = |\delta_e^+(v_h - u_h)(x_i)|^p$, $\sigma = C_2^p h^{2p}$, and $M = C_\infty^p$, we obtain

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e^+ (v_h - u_h)(x_i)|^p \leq Ch^{2p} \left(\mu\left(\mathcal{N}_h^I\right) + \sum_{k=1}^{Ch^{-2p}} \mu(A_k) \right). \tag{5.4}$$

We aim to estimate the measure of set $\mu(A_k)$. Due to the relations of the second-order difference and contact set given in Remark 4.1, we have $A_k \subset S_{\epsilon_k} = \mathcal{N}_h^I \setminus \mathcal{C}_{\epsilon_k}$ with $\epsilon_k \in (0,1)$ satisfying $\tau_k = \epsilon_k/(1-\epsilon_k)$. Therefore, by the estimate (4.11) given in Proposition 4.1,

$$\mu(A_k) \leq \mu(S_{\epsilon_k}) \leq \frac{C_f}{\tau_k} \left\| g^{1/d} - f^{1/d} \right\|_{\ell^d(\mathcal{C}_{\epsilon_k})} = \frac{C_f}{k^{1/p} h^2} \left\| g^{1/d} - f^{1/d} \right\|_{\ell^d(\mathcal{C}_{\epsilon_k})}.$$

From the concavity of $t \to t^{1/d}$, we have $(t+\epsilon)^{1/d} - t^{1/d} \leq d^{-1} t^{1/d-1} \epsilon$. Setting $t = g_i$ and $\epsilon = f_i - g_i \geq 0$, we get

$$\left| f_i^{1/d} - g_i^{1/d} \right| = f_i^{1/d} - g_i^{1/d} \leq d^{-1} g_i^{1/d-1} (f_i - g_i) \leq Ch^3$$

due to the consistency error (5.3) and the lower bound $g_i \geq Ch^d$. Consequently, we find that

$$\left\| f^{1/d} - g^{1/d} \right\|_{\ell^d(\mathcal{C}_{\epsilon_k})} \leq Ch^2,$$

and therefore $\mu(A_k) \leq \frac{C}{k^{1/p}}$. Applying this bound in (5.4), we derive the estimate

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e^+ (u_h - v_h)(x_i)|^p \leq Ch^{2p} \sum_{k=1}^{Ch^{-2p}} \frac{1}{k^{1/p}} \leq C \begin{cases} h^2 |\ln h| & \text{if } p = 1, \\ h^2 & \text{if } p > 1. \end{cases}$$

This completes the proof. ∎

Remark 5.2. It is worth mentioning that the assumption on the consistency error (5.2) holds for nodes bounded away from the boundary $\partial\Omega$ provided that $u \in C^{3,1}(\bar{\Omega})$. However, for nodes close to the boundary $\partial\Omega$, such an estimate holds only for a structured domain, such as a rectangle domain; see Remark 3.2 and the first numerical experiment in section 6. In general, this estimate may not be true. In fact, Lemma 3.1 shows that the (relative) consistency error, $\mathcal{O}(h)$ away from the boundary, is of order $\mathcal{O}(1)$. In the following subsection, we take into account the lack of consistency in the boundary layer.

5.2. Estimate on general domain. We define the barrier nodal function

$$b_h(x_i) = \begin{cases} h^2 & \text{if } x_i \in \mathcal{N}_h^I, \\ 0 & \text{if } x_i \in \mathcal{N}_h^B, \end{cases} \tag{5.5}$$

which will be used to "push down" the graph of the nodal interpolant of u and as such, develop error estimates in a general setting.

THEOREM 5.3. *Let $u \in C^{3,1}(\bar{\Omega})$ be the solution of the Monge–Ampère equation (1.1) with $0 < \lambda I \leq D^2 u \leq \Lambda I$, and assume that the nodal set $\mathcal{N}_h^I$ is translation invariant and that f is uniformly positive on Ω. Then the error in the weighted $W^{2,p}$-norm satisfies*

$$\|N_h u - u_h\|_{W^2_{f,p}(\mathcal{N}_h^I)} \leq C \begin{cases} h^{1/p} & \text{if } p > d, \\ h^{1/d} \big(\ln\left(\frac{1}{h}\right)\big)^{1/d} & \text{if } p \leq d, \end{cases}$$

where $N_h u$ is the interpolation of u on the nodal set $\mathcal{N}_h$ and the constant C depends on $\|u\|_{C^{3,1}(\bar{\Omega})}$, the dimension d, and the constant p.

Proof. We define the boundary layer:

$$\Omega_h := \left\{ x_i \in \mathcal{N}_h^I, \operatorname{dist}(x_i, \partial\Omega) \leq Rh \right\}, \tag{5.6}$$

where the constant R is the constant in the consistency error, Lemma 3.1, which depends on the ellipticity constants λ and Λ of $D^2 u$. We set

$$v_h = N_h u - C b_h, \qquad g_i = |\partial v_h(x_i)|, \tag{5.7}$$

where the constant $C > 0$ is sufficiently large so that $u_h \leq v_h$; see Proposition 3.2. It is clear from the definition of b_h that

$$|\partial v_h(x_i)| = |\partial N_h u(x_i)| \quad \text{for any } x_i \in \mathcal{N}_h^I \setminus \Omega_h \tag{5.8a}$$

and

$$|\partial N_h u(x_i)| \geq |\partial v_h(x_i)| \geq 0 \quad \text{for any } x_i \in \Omega_h. \tag{5.8b}$$

This implies that $|f_i - g_i| \leq Ch^{2+d}$ in $\mathcal{N}_h^I \setminus \Omega_h$ and $|f_i - g_i| \leq Ch^d$ in Ω_h. We have that $|\delta_e v_h(x_i)| \leq C_2$ and $|\delta_e(v_h - u_h)(x_i)| \leq C_\infty$ for all $x_i \in \mathcal{N}_h^I$. As in Theorem 5.2, we shall prove the estimate for the positive part:

$$\sum_{x_i \in \mathcal{N}_h^I} f_i \left(\delta_e^+(v_h - u_h)(x_i)\right)^p \leq \begin{cases} Ch & \text{if } p > d, \\ Ch \ln\left(\frac{1}{h}\right) & \text{if } p = d. \end{cases}$$

The estimate for the negative part can be proved in a similar fashion. Also note that the estimate for $p < d$ follows from the estimate of $p = d$ and Hölder's inequality:

$$\|N_h u - u_h\|_{W^2_{f,p}(\mathcal{N}_h^I)} \leq C_\mu \|N_h u - u_h\|_{W^2_d(\mathcal{N}_h^I)}, \quad \text{where } C_\mu := \mu\left(\mathcal{N}_h^I\right)^{1/p - 1/d}.$$

We set $\tau_k = C_2 k^{1/p} h$ and define the set

$$A_k := \left\{x_i \in \mathcal{N}_h^I, \quad \left(\delta_e^+(v_h - u_h)(x_i)\right) \geq \tau_k\right\}. \tag{5.9}$$

Then, by similar arguments as in Theorem 5.2, we find by Lemma 5.1 that

$$\sum_{x_i \in \mathcal{N}_h^I} f_i \left(\delta_e^+(v_h - u_h)(x_i)\right)^p \leq C_2 h^p \left(\mu\left(\mathcal{N}_h^I\right) + \sum_{k=1}^{h^{-p}} \mu(A_k)\right). \tag{5.10}$$

To estimate the measure of set $\mu(A_k)$, we note that $A_k \subset S_{\epsilon_k} = \mathcal{N}_h^I \setminus \mathcal{C}_{\epsilon_k}$ with $\tau_k = \epsilon_k/(1-\epsilon_k)$. Invoking the estimate of the measure of the noncontact set S_ϵ stated in Proposition 4.1, we obtain

$$\mu(A_k) \leq \mu(S_{\epsilon_k}) \leq \nu_{\tau_k}(\mathcal{C}_{\epsilon_k}) - \mu(\mathcal{C}_{\epsilon_k}).$$

We then divide the estimate of $\nu_{\tau_k}(\mathcal{C}_{\epsilon_k}) - \mu(\mathcal{C}_{\epsilon_k})$ into two parts:

$$\begin{aligned} \nu_{\tau_k}(\mathcal{C}_{\epsilon_k}) - \mu(\mathcal{C}_{\epsilon_k}) &= \sum_{x_i \in \mathcal{C}_{\epsilon_k}} \left[\left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d}\right)^d - f_i\right] \\ &= \left(\sum_{x_i \in \mathcal{C}_{\epsilon_k} \cap \Omega_h} + \sum_{x_i \in \mathcal{C}_{\epsilon_k} \setminus \Omega_h}\right) \left[\left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d}\right)^d - f_i\right], \end{aligned}$$

where we recall that $e_i^{1/d} = f_i^{1/d} - g_i^{1/d}$. Since $f_i^{1/d} = \mathcal{O}(h)$ and $g_i^{1/d} = \mathcal{O}(h)$, we have

$$\begin{aligned} \left|\left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d}\right)^d - f_i\right| &\leq \frac{d}{\tau_k} \max\left\{\left|f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d}\right|, f_i^{1/d}\right\}^{d-1} \left|e_i^{1/d}\right| \\ &\leq \frac{Ch^{d-1}}{\tau_k^d} \left|e_i^{1/d}\right|. \end{aligned}$$

In the set $\mathcal{C}_{\epsilon_k} \cap \Omega_h$, the consistency error satisfies $|e_i^{1/d}| = \mathcal{O}(h)$; see Lemma 3.1. Therefore, we have

$$\left|\left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d}\right)^d - f_i\right| \leq \frac{Ch^d}{\tau_k^d} \qquad \forall x_i \in \mathcal{C}_{\epsilon_k} \cap \Omega_h.$$

On the other hand, in the set $\mathcal{C}_{\epsilon_k} \setminus \Omega_h$, we conclude as in Theorem 5.2 that $|e_i^{1/d}| = \mathcal{O}(h^3)$, and

$$\left| \left(f_i^{1/d} + \frac{1}{\tau_k} e_i^{1/d} \right)^d - f_i \right| \le \frac{Ch^{2+d}}{\tau_k^d}.$$

Combining both estimates and applying the fact that $\#(\mathcal{C}_{\epsilon_k} \cap \Omega_h) \le Ch^{1-d}$ and $\#(\mathcal{C}_{\epsilon_k} \setminus \Omega_h) \le Ch^{-d}$, we obtain

$$\nu_{\tau_k}(\mathcal{C}_\epsilon) - \mu(\mathcal{C}_\epsilon) \le \frac{Ch}{\tau_k^d} + \frac{Ch^2}{\tau_k^d} \le \frac{Ch}{\tau_k^d}$$

because $h \le 1$. Hence, we conclude that

$$\mu(A_k) \le \frac{Ch}{\tau_k^d}.$$

Applying this estimate to (5.10), we arrive at

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e^+ (v_h - u_h)(x_i)|^p \le C_2 h^p \sum_{k=1}^{h^{-p}} \frac{h}{h^d k^{d/p}}.$$

Since

$$\sum_{k=1}^{h^{-p}} \frac{1}{k^{d/p}} \le \begin{cases} C(d,p) h^{d-p} & \text{if } p > d, \\ C \ln\left(\frac{1}{h}\right) & \text{if } p = d, \end{cases}$$

we conclude that

$$\sum_{x_i \in \mathcal{N}_h^I} f_i |\delta_e^+ (v_h - u_h)(x_i)|^p \le \begin{cases} Ch & \text{if } p > d, \\ Ch \ln\left(\frac{1}{h}\right) & \text{if } p = d. \end{cases}$$

This completes the proof. ∎

5.3. Estimate for solutions with less regularity. In this subsection, we exploit our stability estimate established in section 4 and show that it may be possible to apply the arguments given in the previous sections to solutions with low regularity, in particular, with regularity lower than $C^{3,1}(\bar{\Omega})$. We show this by means of an example, which is a modification of the test problem in the numerical experiments below.

Set the domain Ω to be a unit ball centered at 0 in $\mathbb{R}^2$, and define

$$p(x) = (|x| - 1/2)^+, \quad u(x) = \frac{1}{2} p(x)^2 + \frac{1}{2}|x|^2 - \frac{5}{8}. \tag{5.11}$$

It is easy to see that $u \in W^2_\infty(\Omega)$, but $u \notin C^2(\Omega)$, and therefore the hypotheses of Theorem 5.3 do not hold. Nonetheless, we are still able to prove a error estimate with the same rate established in the theorem.

THEOREM 5.4. *Let $\Omega \subset \mathbb{R}^2$ be the unit ball, and let $u \in W^2_\infty(\Omega)$, defined by* (5.11), *be the solution to the Monge–Ampère problem. Let u_h be the solution of the Oliker–Prussner method. Then there holds*

$$\|N_h u - u_h\|_{W^2_{f,2}(\mathcal{N}_h^I)} \le Ch^{1/2} \left(\ln\left(\frac{1}{h}\right) \right)^{1/2}.$$

Proof. Our goal is to find a nodal function v_h satisfying the following three conditions:

(5.12a) $$u_h(x_i) \le v_h(x_i) \quad \text{for all nodes } x_i \in \mathcal{N}_h,$$

(5.12b) $$\|e\|^d_{\ell^d(\mathcal{N}_h^I)} = \mathcal{O}(h), \quad \text{where } e_i^{1/2} := \left(|\partial v_h(x_i)|^{1/2} - |\partial u_h(x_i)|^{1/2}\right)^+,$$

(5.12c) $$\|v_h - N_h u\|^2_{W^2_{f,2}(\mathcal{N}_h^I)} = \mathcal{O}(h).$$

With such a function v_h, we can show by the same arguments as in the previous theorem that

$$\sum_{x_i \in \mathcal{N}_h^I} f_i \left(\delta_e^+ (v_h - u_h)(x_i)\right)^2 \le Ch \left(\ln\left(\frac{1}{h}\right)\right).$$

The bound for the negative component of the error can be proved in a similar way. We construct the nodal function v_h in three steps.

Step one. We define

$$p^h(x) = \left(|x| - 1/2 - 2Rh\right)^+,$$

$$\text{and set} \quad u^h(x) = \frac{1}{2} p^h(x)^2 + \frac{1}{2}|x|^2 - \left(\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - 2Rh\right)^2\right),$$

where $R > 0$ is defined in Lemma 3.1. We assume that h is sufficiently small so that $1/2 - 2Rh \ge 0$, implying that $p^h = 1/2 - 2Rh$ on $\partial\Omega$. It is then easy to check that $u = u^h = 0$ on $\partial\Omega$.

We first show that $D^2 u^h \le D^2 u$ for all $x \in \Omega$. To do so, we divide the unit ball into three regions

$$\underbrace{\{|x| \le 1/2\}}_{=:D_1} \cup \underbrace{\{1/2 \le |x| \le 1/2 + 2Rh\}}_{=:D_2} \cup \underbrace{\{1/2 + 2Rh \le |x| \le 1\}}_{=:D_3}.$$

By direct calculation, we immediately have

$$D^2 u = I \quad \text{in } D_1, \quad \text{and} \quad D^2 u = I + D^2\left(p^2\right) \quad \text{in } D_2 \cup D_3,$$

while

$$D^2 u^h = I \quad \text{in } D_1 \cup D_2, \quad \text{and} \quad D^2 u^h = I + D^2\left(\left(p^h\right)^2\right) \quad \text{in } D_3.$$

Since $p(x)^2$ is a convex function in Ω and $D^2(p^2) \ge 0$, we obtain

$$D^2 u^h \le D^2 u \quad \text{in } D_1 \cup D_2.$$

Next, we show that $D^2 u^h \le D^2 u$ in D_3. Since $\boldsymbol{\nabla}|x| = \frac{x}{|x|}$, we obtain for all $|x| \ge 1/2 + 2Rh$,

$$\begin{aligned} \boldsymbol{\nabla} u(x) &= x + (|x| - 1/2)\frac{x}{|x|} = 2x - \frac{1}{2}\frac{x}{|x|}, \\ D^2 u(x) &= 2I - \frac{|x|^2 I - x \otimes x}{2|x|^3}. \end{aligned}$$

Similarly,

$$D^2u^h(x) = 2I - \left(\frac{1}{2} + Rh\right)\frac{|x|^2 I - x\otimes x}{|x|^3}.$$

Hence, we get

$$D^2u - D^2u^h = 2Rh \underbrace{\frac{|x|^2 I - x\otimes x}{|x|^3}}_{=:A(x)} \geq 0, \quad \text{and} \quad A(x) \leq 2|x|^{-1}RhI \leq 4RhI \quad \text{in } D_3,$$

and thus, $D^2u \geq D^2u^h$ in D_3 as desired.

Moreover, a direct calculation shows that

$$\left\|D^2\left(u-u^h\right)\right\|^2_{L^2(\Omega)} = \int_{D_2}\left|D^2\left(p^2\right)\right|^2 dx + \int_{D_3}|(2Rh)A(x)|^2 dx = \mathcal{O}(h). \tag{5.13}$$

Step two. Let b_h be defined by (5.5), and set $w_h = N_h u^h - Cb_h$. Since u^h is a quadratic polynomial in $D_1\cup D_2$, and the adjacent set of $x_i \in D_1\cap \mathcal{N}_h^I$ is contained in $D_1\cup D_2$ (see [26, Lemma 5.3] for details), and $|\partial u_h(x_i)| = f_i = \int_\Omega f(x)\phi_i(x)dx = \int_\Omega \det(D^2u(x))\phi_i(x)dx$, we have for any $x_i\in D_1\cap\mathcal{N}_h$

$$|\partial w_h(x_i)| = \int_\Omega \det\left(D^2u^h\right)(x)\phi_i(x)dx = \int_\Omega f(x)\phi_i(x)dx = |\partial u_h(x_i)|.$$

For nodes $x_i\in(D_2\cup D_3)\cap\mathcal{N}_h$, we have

$$|\partial w_h(x_i)| = \left|\partial N_h u^h(x_i)\right| \leq |\partial N_h u(x_i)|$$

because $D^2u^h(x)\leq D^2u(x)$ for all $x\in B_{Rh}(x_i)$. By Lemma 3.1 (consistency error), we obtain

$$|\partial N_h u(x_i)| = h^2\det\left(D^2u\right)(x_i) \pm \mathcal{O}\left(h^4\right) = |\partial u_h(x_i)| \pm \mathcal{O}\left(h^4\right)$$

for all nodes $x_i\in(D_2\cup D_3)\cap\mathcal{N}_h$ and $\operatorname{dist}(x_i,\partial\Omega)\geq Rh$. To deal with the consistency error at nodes $\operatorname{dist}(x_i,\partial\Omega)\leq Rh$, we note that for the constant C large enough, the second difference

$$\delta_e w_h(x_i) = \delta_e u^h(x_i) - C\delta_e b_h(x_i) \leq 0,$$

where $e = \boldsymbol{\nabla} d(x_i)$ and $d(x) = \operatorname{dist}(x,\partial\Omega)$. This implies that $|\partial w_h(x_i)| = 0$ at x_i close to the boundary ($\operatorname{dist}(x_i,\partial\Omega)\geq Rh$) with sufficiently large constant C.

Combining all these estimates, we get

$$r_i^{1/2} := \left(|\partial w_h(x_i)|^{1/2} - |\partial u_h(x_i)|^{1/2}\right)^+ = 0 \quad \forall x_i\in D_1\cap\mathcal{N}_h$$

and

$$r_i^{1/2} = \frac{|\partial w_h(x_i)| - |\partial u_h(x_i)|}{|\partial w_h(x_i)|^{1/2} + |\partial u_h(x_i)|^{1/2}} \leq O\left(h^3\right) \quad \forall x_i\in(D_2\cup D_3)\cap\mathcal{N}_h.$$

By a discrete Alexandroff estimate [26, Proposition 4.3], we have

$$\sup(w_h-u_h)^- \leq C\|r\|_{\ell^2(\mathcal{N}_h)} \leq Ch^2. \tag{5.14}$$

Let $v_h = w_h - Cb_h$. From the estimate (5.14), there holds $u_h \le v_h$ for all $x_i \in \mathcal{N}_h$, i.e., the first condition (5.12a) is satisfied.

Step three. To verify (5.12b), we set

$$f_i = \int_\Omega \det\left(D^2 u\right)(x)\phi_i(x)dx \quad \text{and} \quad f_i^h = \int_\Omega \det\left(D^2 u^h\right)(x)\phi_i(x)dx.$$

Since $\|D^2(u-u^h)\|^2_{L^2(\Omega)} = \mathcal{O}(h)$, we have $\|f_i^{1/2} - (f_i^h)^{1/2}\|^2_{\ell^2(\mathcal{N}_h^I)} = \mathcal{O}(h)$. On the other hand, we have

$$\left\| |\partial w_h(x_i)|^{1/2} - \left(f_i^h\right)^{1/2} \right\|^2_{\ell^2(\mathcal{N}_h^I)} = \mathcal{O}(h)$$

by Lemma 3.1 (consistency error). It then follows that $\|e\|^2_{\ell^2(\mathcal{N}_h^I)} = \mathcal{O}(h)$, i.e., (5.12b) is satisfied.

Step four. It remains to verify (5.12c). Since $v_h = N_h u^h - Cb_h$ by definition,

$$\begin{aligned}
\|v_h - N_h u\|_{W^2_{f,2}(\mathcal{N}_h^I)} &\le C\|b_h\|_{W_{f,2}(\mathcal{N}_h^I)} + \left\|N_h u^h - N_h u\right\|_{W^2_{f,2}(\mathcal{N}_h^I)} \\
&\le C\|b_h\|_{W^2_{f,2}(\mathcal{N}_h^I)} + C\left\|D^2\left(u^h - u\right)\right\|_{L^2(\Omega)} \\
&\le O\left(h^{1/2}\right) + O\left(h^{1/2}\right),
\end{aligned}$$

where the estimate of $\|D^2(u^h - u)\|_{L^2(\Omega)}$ follows from (5.13). This completes the proof. ∎

6. Numerical experiments.

In this section, we perform numerical examples to illustrate the accuracy of the method, and to compare the results with the theory. In the tests, we replace the homogeneous boundary condition (1.1b) with $u = g$ on $\partial\Omega$. For simplicity, we carry out numerical experiments on a box, instead of a strictly convex domain. The theoretical results developed in the previous sections can be applied to this slightly more general problem with minor modifications.

We consider three different test problems, each reflecting different scenarios of regularity. Each set of problems is performed in two dimensions ($d = 2$), and errors are reported in the (discrete) L_∞, H^1, W_1^2, and W_2^2 norms. Here, a nine-point stencil is used in the definition of the W_p^2 norms with $e_1 = (1, 0)$, $e_2 = (0, 1)$, $e_3 = (1, 1)$, and $e_4 = (1, -1)$. That is, with an abuse of notation, we set

$$\|v\|^p_{W_p^2(\mathcal{N}_h^I)} = \sum_{j=1}^4 \sum_{x_i \in \mathcal{N}_h^I} |\delta_{e_j} v(x_i)|^p.$$

As explained in [26] and in section 2.2, a convex nodal function induces a triangulation of Ω whose set of vertices corresponds to $\mathcal{N}_h$. For a computed solution u_h, we associate with it a piecewise linear polynomial on the induced mesh, which we still denote by u_h, and use the quantity $\|u - u_h\|_{H^1(\Omega)}$ to denote the H^1 error in the experiments below.

A summary of the theoretical results in sections 2.3 and 5 when $d = 2$ is

$$\|N_h u - u_h\|_{L_\infty(\mathcal{N}_h^I)} = \mathcal{O}\left(h^2\right), \qquad \|N_h u - u_h\|_{W_p^2(\mathcal{N}_h^I)} = \mathcal{O}\left(h^{1/2-\epsilon}\right),\ p = 1, 2,$$

for any $\epsilon > 0$, provided that $u \in C^{3,1}(\bar\Omega)$.

TABLE 1
Rate of convergence for a smooth solution (Example I).

h	L_∞	Rate	H^1	Rate	W_1^2	Rate	W_2^2	Rate
1	1.12e-01	0.00	2.24e-01		4.49e-01		1.44e+01	
1/2	4.78e-02	1.23	1.35e-01	0.73	6.02e-01	-0.42	4.24e-01	5.08
1/4	1.37e-02	1.80	4.35e-02	1.63	2.94e-01	1.03	1.93e-01	1.13
1/8	3.55e-03	1.95	1.16e-02	1.91	9.93e-02	1.57	6.34e-02	1.61
1/16	8.96e-04	1.99	2.94e-03	1.98	2.86e-02	1.80	1.80e-02	1.82
1/32	2.24e-04	2.00	7.39e-04	1.99	7.66e-03	1.90	4.79e-03	1.91
1/64	5.61e-05	2.00	1.85e-04	2.00	1.98e-03	1.95	1.24e-03	1.95

Example I: Smooth solution $u \in C^\infty(\bar{\Omega})$. We consider the example

$$u(x,y) = e^{\frac{x^2+y^2}{2}}, \quad f(x,y) = \left(1 + x^2 + y^2\right) e^{x^2+y^2}, \quad \text{and } \Omega = (-1,1)^2, \tag{6.1}$$

and list the resulting errors and rates of the scheme in Table 1. The table clearly shows that the errors decay with rate $\mathcal{O}(h^2)$ in all norms. This behavior matches the theoretical results of Proposition 3.2 but indicates that the W_2^2 estimates stated in Theorem 5.2 may not be sharp. This numerical experiment is done on a laptop with a single core processor of 2.90 GHZ. To compute on the finest mesh in this table with approximately 16,000 degrees of freedom, it takes approximately 85 seconds.

Example II: Piecewise smooth solution $u \in W_\infty^2$. In this example, the domain is $\Omega = (-1,1)^2$, and the exact solution and data are taken to be

$$u(x) = \begin{cases} 2|x|^2 & \text{in } |x| \le 1/2, \\ 2(|x| - 1/2)^2 + 2|x|^2 & \text{in } 1/2 \le |x|, \end{cases}$$
$$f(x) = \begin{cases} 16 & \text{in } |x| \le 1/2, \\ 64 - 16|x|^{-1} & \text{in } 1/2 \le |x|. \end{cases}$$

This is essentially the example we consider in Theorem 5.4. A simple calculation shows that $u \in C^{1,1}(\bar{\Omega})$ and $u \in C^4(\Omega \setminus \partial B_1)$, but $u \notin C^2(\bar{\Omega})$. The errors and rates of convergence are given in Table 2. The table shows that, while all errors tend to zero as the mesh is refined, the rates of convergence in the L_∞ and W_1^2 norms are less obvious than the previous set of experiments. Nonetheless, while Theorem 5.3 assumes more regularity of the exact solution, we do observe a convergence rate of approximately $\mathcal{O}(h^{1/2})$ in the W_2^2 as stated in Theorem 5.4. It takes approximately 150 seconds to compute the solution on the finest mesh with approximately 16,000 degrees of freedom.

Example III: Singular solution $u \in W_p^2$ with $p < 2$. In the last series of experiments, the domain is $\Omega = (-1,1)^2$, and the solution and data are

$$u(x) = \begin{cases} x^4 + \frac{3}{2} y^2/x^2 & \text{in } |y| \le |x|^3, \\ \frac{1}{2} x^2 y^{2/3} + 2y^{4/3} & \text{in } |y| \ge |x|^3, \end{cases}$$
$$f(x) = \begin{cases} 36 - 9y^2/x^6 & \text{in } |y| \le |x|^3, \\ \frac{8}{9} - \frac{5}{9} x^2/y^{2/3} & \text{in } |y| > |x|^3. \end{cases}$$

TABLE 2
Rate of convergence of piecewise smooth viscosity solution (Example II*).*

h	L_∞	Rate	H^1	Rate	W_1^2	Rate	W_2^2	Rate
1	4.02e-01	0.00	8.04e-01	0.00	1.61	0.00	1.61	0.00
1/2	4.19e-02	3.26	1.30e-01	2.63	6.08e-01	1.40	5.39e-01	1.58
1/4	2.89e-02	0.53	6.84e-02	0.92	6.46e-01	-0.09	5.54e-01	-0.04
1/8	1.27e-02	1.18	3.50e-02	0.97	5.14e-01	0.33	4.54e-01	0.29
1/16	4.58e-03	1.47	1.38e-02	1.34	2.76e-01	0.90	3.15e-01	0.53
1/32	8.02e-04	2.51	3.59e-03	1.94	1.08e-01	1.35	2.08e-01	0.60
1/64	4.33e-04	0.89	1.50e-03	1.26	6.36e-02	0.77	1.56e-01	0.42

TABLE 3
Rate of convergence of W_p^2 solution with $p < 2$ (Example III*).*

h	L_∞	Rate	H^1	Rate	W_1^2	Rate	W_2^2	Rate
1	8.36e-01	0.00	1.67	0.00	3.35	0.00	3.35	0.00
1/2	2.34e-01	1.84	9.11e-01	0.88	5.48	-0.71	3.94	-0.24
1/4	1.86e-01	0.33	4.80e-01	0.92	4.90	0.16	4.02	-0.03
1/8	8.52e-02	1.13	2.41e-01	1.00	4.00	0.29	3.94	0.03
1/16	3.41e-02	1.32	1.02e-01	1.24	2.38	0.75	3.33	0.24
1/32	1.35e-02	1.34	4.79e-02	1.09	1.59	0.58	3.17	0.07

This example is constructed in [30] to show that $D^2u(x)$ may not be in W_p^2 for large p for discontinuous f. The errors of the method for this problem are listed in Table 3. Because the exact solution does not enjoy W_2^2 regularity, it is not expected that the discrete solution will converge in the discrete W_2^2 norm, and this is observed in the table. However, we do observe convergence in the L_∞, H^1, and W_1^2 norms with approximate rates $\|N_h u - u_h\|_{L_\infty(\mathcal{N}_h^I)} = \mathcal{O}(h^{4/3})$, $\|N_h u - u_h\|_{H^1(\mathcal{N}_h^I)} = \mathcal{O}(h)$, and $\|N_h u - u_h\|_{W_{f,1}^2(\mathcal{N}_h^I)} = \mathcal{O}(h^{1/2})$. Finally, we would like to mention that it takes approximately 150 seconds to compute the solution on the finest mesh with approximately 4,000 degrees of freedom.

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