Corruption Detection in Networks of Bi-directional Dynamical Systems

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***Abstract*— Modeling complex networked systems as graphs is prevalent, with nodes representing the agents and the links describing a notion of dynamic coupling between them. Passive methods to identify such influence pathways from data are central to many applications. However, dynamically related data-streams originating at different sources are prone to corruption caused by asynchronous time-stamps of different streams, packet drops and noise. Earlier results have shown that spurious links are inferred in the graph structure identified using corrupt data-streams. In this article, we provide a novel approach to detect the location of corrupt agents in the network solely by observing the inferred directed graph. Here, the generative system that yields the data admits bidirectionally coupled nonlinear dynamic influences between agents. A simple, but novel and effective approach, using graph theory tools is presented to arrive at the results.**

I. Introduction

For an effective abstraction, many complex systems are modeled as networks of interacting components. This is prevalent across several application domains such as geo-science [1], finance [2] neuroscience [3] and in engineered networked systems such as internet-of-things [4]. Identification of influence pathways is a primary objective in such complex systems whose data are dynamically related as dictated by the physics of the interacting agents.

In scenarios such as the power grid [5] and financial markets, it is impossible or impermissible to actively inject signals to influence the system. Here, network structure identification must be achieved via passive means. With advancements in information measurement, data processing and communication systems, passive identification of networks has become more tenable.

Often, the measurements in such large systems are subjected to effects of noise [6], asynchronous sensor clocks [7] and packet drops [8]. When dealing with problems of identifying structural and functional connectivity of a large network, there is a pressing need to rigorously study such uncertainties and address the problem of locating corrupt agents and removing spurious links for performing accurate system identification on networked systems.

Network identification for linear systems using instrument variables has been studied in [9]. However, the effects of data corruption are not studied in this work. Authors in [10] leveraged multivariate Wiener filters to reconstruct the undirected topology of the generative network model. With assumptions of perfect measurements, and linear time invariant interactions, it is established that the multivariate Wiener filter can recover the *kin graph*. In other words, for each node, its parents, children and spouses are detected.

For a network of interacting agents with nonlinear and strictly causal interactions, the authors in [11] proposed the use of directed information to determine the directed structure of the network. Here too, it is assumed that the data-streams are ideal with no distortions.

The effects of data corruption in network reconstruction was studied in [12] and [13] wherein the spurious links in the inferred network structure was characterized. In this work, we extend the analysis to identify the location of corrupt nodes in a network. We consider causal and non-linear dynamical systems. Here, every coupling is assumed to be a bi-directional. Such a framework is applicable in many domains such as power networks [14], thermal monitoring [15], networks of oscillators [16] and consensus networks [17].

In this article, directed information method is first employed to infer the *corrupt* graph from uncertain data-streams. We then use graph theory tools to isolate the corrupt nodes by observing the directed graph inferred. We remark here, that the solution methodology provides an effective method to detect sources of corruption that only involve examining paths in the constructed graph.

The paper is organized as follows. Graph theory preliminaries and the generative model that generates the measured data is described in Section II. Corruption models are highlighted in Section III. Section IV describes the method of network inference using Directed Information. The main result to detect the corrupt nodes is presented in Section V.

II. Preliminaries

Notations:
$z[\cdot]$ denotes a sequence and $z^{(t)}$ denotes the sequence $z[0], z[1], \dots z[t]$.
P_X represents the probability density function of a random variable X.
$i \to j$ indicates an arc or edge from node i to node j in a directed graph.
$i \leftrightarrow j$ denotes $i \to j$ and $i \leftarrow j$.
$i - j$ denotes an undirected edge or link between nodes i and j in an undirected graph. If the graph is directed, then $i - j$ denotes at least one of $i \to j$ or $i \leftarrow j$.
$\mathbb{E}[\cdot]$ denotes the expectation operator.

A. Definitions

In this subsection, some graph theory notions that will be used in the article are presented. See [18] for more details.

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(a) Generative Graph G (b) Topology G^{τ}

Fig. 1: This figure shows 1a generative graph and its topology 1b

Definition 1 (Directed and Undirected Graphs): A *directed graph* G is a pair (V, A) where V is a set of vertices or nodes and A is a set of edges or links given by ordered pairs (i, j) where $i, j \in V$. If $(i, j) \in A$, then we say that there is an edge from i to j which is also denoted as $i \to j \in A$. (V, A) forms an *undirected graph* if V is a set of nodes or vertices and A is a set of the un-ordered pairs $\{i, j\}$.

Definition 2 (Topology): Suppose $G = (V, A)$ is a directed graph. The *topology* of the graph G is an undirected graph $G^{\tau} = (V, A^{\tau})$, where $A^{\tau} = \{i - j \mid i \to j \in A\} \cup \{i - j \mid i \leftarrow j \in A\}$.

Definition 3 (Trail/Path): Nodes $v_1, v_2, \dots, v_k \in V$ forms a *trail/path* in a graph G (directed or undirected) if for every $i = 1, 2, \dots, k-1$ we have $v_i - v_{i+1}$. We will denote the path connecting v_1 and v_k by $v_1 - v_2 - \dots - v_{k-1} - v_k$.

Definition 4 (Collider): A node v_k is a *collider* in a directed graph G, if there are two other nodes v_i, v_j such that $v_i \to v_k \leftarrow v_i$ holds.

Definition 5 (Tree): An undirected graph $G = (V, A)$ is called a *tree* if there is a unique path connecting any two nodes in V.

B. Generative Model

In this subsection, the *generative model* that is assumed to generate the measured data is described. Consider N agents that interact over a network. Let Y denote the set of all random process $\{y_1, \dots, y_N\}$ with a parent set $\mathcal{P}(i)$ defined for $i = 1, \dots, N$. The *generative model* for y_i is described by the structural relationship:

$$y_i[t] = f_i\left(y_i^{(t-1)}, \bigcup_{j \in \mathcal{P}(i)} y_j^{(t-1)}, e_i[t]\right), \qquad (1)$$

where f_i's are arbitrary functions.

Here, to each agent we associate a discrete time sequence $y_i[\cdot]$ and another sequence $e_i[\cdot]$. The process $e_i[\cdot]$ is considered innate to agent i and thus e_i and e_j are independent for $i \neq j$. It is also assumed that $e_i[\cdot]$ is independent across time. All discrete time sequences have a finite horizon assumed to be T. The structural description of (1) induces a *generative graph* $G = (V, A)$ formed by identifying the set of vertices, V, with random processes y_i and the set of directed links, A, obtained by introducing a directed link from every element in the parent set $\mathcal{P}(i)$ of agent i to i.

In this article, we consider *bi-directional generative models* whose associated generative graph $G = (V, A)$, is bi-directional. That is for all $i \to j \in A$, we also have $j \to i \in A$.

For an illustration, consider the dynamics of a generative model described by:

$$\begin{aligned}
y_1[t] &= f_1(y_1^{(t-1)}, y_2^{(t-1)}, y_3^{(t-1)}, e_1[t]),\\
y_2[t] &= f_2(y_1^{(t-1)}, y_2^{(t-1)}, y_4^{(t-1)}, e_2[t]),\\
y_3[t] &= f_3(y_1^{(t-1)}, y_3^{(t-1)}, y_4^{(t-1)}, e_3[t]),\\
y_4[t] &= f_4(y_2^{(t-1)}, y_3^{(t-1)}, y_4^{(t-1)}, e_4[t]),
\end{aligned}$$

Its associated generative graph is shown in Fig. 1(a). Note that for all i in $\{1, 2, \dots, 4\}$, $i \to i$ is not shown. Figure 1(b) shows the topology.

III. Uncertainty Description

In this section we provide a description for how uncertainty affects the time-series y_i. We interchangeably use corruption or perturbation to denote uncertainties in measurement.

A. General Perturbation Models

Consider i^{th} node in a generative graph and it's associated unperturbed time-series y_i. The corrupt data-stream u_i associated with i is assumed to follow:

$$u_i[t] = g_i(y_i^{(t)}, u_i^{(t-1)}, \zeta_i[t]), \qquad (2)$$

where u_i can depend dynamically on y_i till time t, its own values in the strict past, and $\zeta_i[t]$ which represents a stochastic process that is independent across time. We highlight a few important perturbation models that are practically relevant. See [13] for more details.

Temporal Uncertainty: Consider a node i in a generative graph. Suppose t is the true clock index but the node i measures a noisy clock index which is given by a random process, $\zeta_i[t]$. One such probabilistic model is given by the following IID Bernoulli process:

$$\zeta_i[t] = \begin{cases} d_1, & \text{with probability } p_i \\ d_2, & \text{with probability } (1-p_i), \end{cases}$$

where d_1 and d_2 are any non-positive integers such that at least one of d_1 and d_2 are not equal to 0. Randomized delays in information transmission can be modeled as a convolution operation with the impulse function $\delta[t]$ shifted by $\zeta_i[t]$ as follows :

$$u_i[t] = \delta[t + \zeta_i[t]] * y_i[t]. \qquad (3)$$

Noisy Filtering: Given a node i in a generative graph, the data-stream y_i is causally filtered by a stable filter L_i and corrupted with independent measurement noise $\zeta_i[\cdot]$. This perturbation model is described by:

$$u_i[t] = (L_i * y_i)[t] + \zeta_i[t]. \qquad (4)$$

Packet Drops: The measurement $u_i[t]$ corresponding to a ideal measurement $y_i[t]$ packet reception at time t can be stochastically modeled as:

$$u_i[t] = \begin{cases} y_i[t], & \text{with probability } p_i \\ u_i[t-1], & \text{with probability } (1-p_i). \end{cases} \quad (5)$$

Consider an IID Bernoulli process ζ_i described by,

$$\zeta_i[t] = \begin{cases} 1, & \text{with probability } p_i \\ 0, & \text{with probability } (1-p_i). \end{cases}$$

The corruption model in (2) takes the form:

$$u_i[t] = \zeta_i[t]y_i[t] + (1-\zeta_i[t])u_i[t-1]. \quad (6)$$

IV. Network Inference using Directed Information

In this section, we will recall how to infer directed graphs for networks from corrupt data-streams [13].

Definition 6 (Directed Information): Denote the measured data-streams by $U = \{u_1, \ldots, u_N\}$. The directed information (DI) from data stream u_j to u_i is given by:

$$I(u_j \to u_i \parallel U_{\bar{i}\bar{j}}) = \mathbb{E}\left[\log \frac{P_{u_i \| u_j, U_{\bar{i}\bar{j}}}}{P_{u_i \| U_{\bar{i}\bar{j}}}}\right], \quad (7)$$

where $P_{u_i \| u_j, U_{\bar{i}\bar{j}}} = \prod_{t=1}^{T} P_{u_i[t] | u_i^{(t-1)}, u_j^{(t-1)}, U_{\bar{i}\bar{j}}^{(t-1)}}$, $P_{u_i \| U_{\bar{i}\bar{j}}} = \prod_{t=1}^{T} P_{u_i[t] | u_i^{(t-1)}, U_{\bar{i}\bar{j}}^{(t-1)}}$ and $U_{\bar{i}\bar{j}} = U \setminus \{u_i, u_j\}$.

Definition 7 (Perturbed Graph): Let $G = (V, A)$ be a generative graph. Suppose $Z \subset V$ is the set of perturbed nodes where each perturbation is described by (2). The perturbed graph, $G_Z = (V, A_Z)$, is a directed graph where there is an edge $i \to j \in A_Z$ if and only if there is a trail $i = v_1 - v_2 - \cdots - v_{k-1} - v_k = j$ in G such that the following conditions hold:

P1) If $j \notin Z$, then $v_{k-1} \to j \in A$.

P2) For $m \in \{2, 3, \ldots, k-1\}$, if $v_{m-1} \to v_m \leftarrow v_{m+1}$, and $v_m \notin Z$, then $v_{m+1} \in Z$.

P3) If v_m is a node such that $v_{m-1} - v_m - v_{m+1}$ is a subpath of the path $v_1 - \ldots - v_k$ and v_m is not a collider, then $v_m \in Z$.

***Remark** 1:* Note that the existence of any trail that does not meet the ‘if’ conditions in P1), P2) and P3) guarantees that $i \to j \in A_Z$. For example, if $i \to j \in A$ then $i \to j \in A_Z$. Indeed, if $j \notin Z$ then $i \to j \in A_Z$ by condition P1). Conditions P2) and P3) are not applicable. On the other hand, if $j \in Z$, then none of the conditions P1), P2) or P3) are applicable to the trail $i \to j$. So, $i \to j \in A_Z$.

Definition 8 (Spurious Links): Let $G = (V, A)$ be a generative graph, $Z \subset V$ be the set of perturbed nodes and $G_Z = (V, A_Z)$ be the perturbed graph. Spurious links are those links $i \to j \in A_Z$ that do not belong to A.

The following theorem from [13] states that the the perturbed graph can be determined using directed information.

***Theorem** 1: Consider a generative graph $G = (V, A)$ consisting of N nodes. Let $Z = \{v_1, \ldots, v_n\} \subset V$ be the set of n perturbed nodes where each perturbation is described by* (2)*. Let $U = \{u_1, \ldots, u_N\}$ be the measured data-streams. There is a directed edge from i to j in the perturbed graph, $G_Z = (V, A_Z)$, if and only if $I(u_i \to u_j \parallel U_{\bar{j}\bar{i}}) > 0$.*

***Remark** 2:* We consider dynamical interactions such that they are *faithful*. (See [13].) This condition means that every directed edge in G can be detected via directed information using measured data-streams.

A. Perturbed Graphs in Bi-directional Networks

The following proposition provides a precise and simplified characterization for perturbed graphs for networks whose generative graphs are bi-directional.

***Proposition** 1: Let $G = (V, A)$ be a bi-directional generative graph. Let $Z \subset V$ be set of perturbed nodes and let $G_Z = (V, A_Z)$ be the corresponding perturbed graph. Then, $i \to j$ in A_Z if and only if either one of the following condition holds:*

B1) $i \leftrightarrow j$ in G or

B2) There is a trail of length at least 3, $i = v_1 \leftrightarrow v_2 \leftrightarrow v_3 \leftrightarrow \cdots \leftrightarrow v_k = j$, such that for every pair of consecutive nodes $v_m,\ v_{m+1}$ with $m \geq 2$ at least one of v_m or v_{m+1} is in Z.

Proof: Let $\hat{A}_Z$ be the edge set described in the proposition. We will show that $\hat{A}_Z = A_Z$.

First, we show that $\hat{A}_Z \subset A_Z$. Suppose, $i \to j \in \hat{A}_Z$. If $i \leftrightarrow j$ was an edge of G, then $i \to j \in A_Z$.

Now, consider the case of a trail of length at least 3 with $i = v_1 \leftrightarrow v_2 \leftrightarrow v_3 \leftrightarrow \cdots \leftrightarrow v_k = j$.

Since the network is bi-directional, we can choose the directionality of the edges. We will show that by suitable choice of directionality, we can retrieve a directed path between i and j that satisfy all the conditions P1), P2) and P3) for the link $i \to j$ to be in A_Z. So, for each pair of nodes along the trail, set the directionality as follows:

- For $m \geq 2$, if $v_m \notin Z$, set $v_{m-1} \to v_m \leftarrow v_{m+1}$.
- Set all other edges in the $\to$ direction.

Since no two consecutive nodes, v_m and v_{m+1} with $m \geq 2$ are unperturbed, this construction is feasible.

P1) If $v_k = j \notin Z$, then, since the trail has $k \geq 3$, we must have $k - 1 \geq 2$. So, we must have that $v_{k-1} \in Z$. Thus, by our convention for choosing directions, we have $v_{k-1} \to v_k$. Thus, P1) holds.

P2) If v_m is a collider, then we have $v_m \leftarrow v_{m+1}$. This directionality is chosen only when when $v_m \notin Z$. Furthermore, since v_m is a collider, we must have $m \geq 2$. Thus, we have $v_{m+1} \in Z$. Therefore, P2) holds.

P3) If v_m is an intermediate node which is not a collider, then by construction, it cannot be unperturbed. Thus, P3) holds.

Now, we show that $A_Z \subset \hat{A}_Z$. Say that $i \to j \in A_Z$. Let $i = v_1 - \cdots - v_k = j$ be trail in G that satisfies conditions P1), P2), and P3). If the trail is $i - j$, then from B1) we have $i \to j \in \hat{A}_Z$.

Now consider the case that the trail has length of at least 3. We must show that for any pair v_m, v_{m+1} with $m \geq 2$, at least one of the nodes is in Z. Since $m \geq 2$, we must have $v_{m-1} - v_m - v_{m+1}$ on the trail. If m is a collider, then P2) implies that $v_{m+1} \in Z$. If m is not a collider, then P3) implies that $v_m \in Z$. Thus, at least one of v_m or v_{m+1} is perturbed. Thus, $i \to j \in \hat{A}_Z$. ∎

V. Identification of Corrupt Nodes

In this section, we present the main result of the paper to detect the location of corrupt nodes in a network of bi-directional systems. For the rest of the article we have the following assumption on the perturbations.

***Assumption** 1:* Let $G = (V, A)$ be a bi-directional generative graph. Let $Z \subset V$ be the set of perturbed nodes satisfying (2). We consider perturbations that satisfy the following: for every unperturbed node $i \in V$ there exists at least one more unperturbed node $j \in V$ such that $i \leftrightarrow j$ holds in G.

***Remark** 3:* The above assumption states that we consider perturbations such that every unperturbed node in the generative graph is connected to at least one other unperturbed node. However, any node(corrupt/unperturbed) can be can be connected to multiple perturbed nodes.

A. Main result: Corruption Identification

Theorem 2 is the main result which detects the exact location of all the corrupt nodes in the network. To this, we will require the following definitions.

Definition 9 (Bi-directional Clique): Suppose, $G = (V, A)$ is a directed graph. A subset of nodes $S \subset V$ forms a *bi-directional clique* in G if $i \to j \in A$ and $j \to i \in A$ for all $i, j \in S$.

Definition 10 (Bi-directional Neighbors): Suppose, $G = (V, A)$ is a directed graph. The *bi-directional neighbors* of a node $i \in V$, bidNr(i), is given by: bidNr(i) $= \{j \mid i \leftrightarrow j \text{ holds in } G\}$.

***Theorem** 2: Suppose $G = (V, A)$ is a bi-directional generative graph with the topology, $G^\tau = (V, A^\tau)$, a tree. Let $Z \subset V$ be the set of perturbed nodes satisfying* (2) *and Assumption 1. Let $U = \{u_1, \ldots, u_N\}$ be the measured data streams. Let $G_Z = (V, A_Z)$ be the perturbed graph. Consider the following statements:*

T1) $i \to j \in A_Z$

T2) $j \to i \notin A_Z$.

T3) j and $bidNr(j)$ form a bi-directional clique in G_Z.

If statements T1) and T2) hold, then j is corrupted if and only if statement T3) holds.

Proof: Let statements T1) and T2) be true. Now, we will show that j is corrupted if and only if statement T3) holds.

($\Rightarrow$) Say that j is corrupted. Let p and q be bi-directional neighbors of j in G_Z. We will show that $p \to q$ in G_Z as well. Since p and q are arbitrary, switching their roles shows that we must have $p \leftrightarrow q$. Furthermore, since p and q are arbitrary bidirectional neighbors of j, $\{j\} \cup$ bidNr(j) must form a bidirectional clique in G_Z.

Now, we will show that $p \to q \in A_Z$. Since $p \leftrightarrow j \in A_Z$ and $j \leftrightarrow q \in A_Z$, by Proposition 1, there must be a trail in G of the form

$$p = v_1 \leftrightarrow \cdots \leftrightarrow v_k = j = w_1 \leftrightarrow \cdots \leftrightarrow w_l = q \tag{8}$$

such that the following conditions hold:

- If $v_m \leftrightarrow v_{m+1}$ with $m \geq 2$, then at least one of v_m or v_{m+1} must be perturbed.
- If $w_m \leftrightarrow w_{m+1}$ with $m \geq 2$, then at least one of w_m or w_{m+1} must be perturbed.

Note that these conditions automatically hold if the paths have length 2.

Since $j \in Z$, at least one node is perturbed on each of $v_{k-1} \leftrightarrow j$ and $j \leftrightarrow w_2$. Thus, there are no consecutive unperturbed nodes along the subtrail:

$$v_2 \leftrightarrow \cdots \leftrightarrow v_{k-1} \leftrightarrow j \leftrightarrow w_2 \cdots \leftrightarrow w_l = q \tag{9}$$

Thus, $p \to q$.

($\Leftarrow$) As $i \to j \in A_Z$ and $j \to i \notin A_Z$, $i \to j$ is a spurious link. By Proposition 1, there exists a trail in G, $trl_G : i = v_1 \leftrightarrow v_2 \leftrightarrow v_3 \leftrightarrow \cdots \leftrightarrow v_{k-1} \leftrightarrow v_k = j$, such that there no two consecutive unperturbed nodes along the subtrail $v_3 \leftrightarrow v_4 \leftrightarrow \cdots \leftrightarrow v_{k-1} \leftrightarrow v_k = j$. Suppose, $j \notin Z$ for the sake of contradiction. We will show that T3) does not hold, leading to a contradiction. That is, we will show that there exists at least one subset of nodes $S \subset$ bidNr(j), such that S does not form a bi-directional clique with j in G_Z. Since $j \notin Z$, by Assumption 1, there is at least one unperturbed node l such that $j \leftrightarrow l$ holds in G. Therefore, $v_{k-1} \leftrightarrow j \leftrightarrow l$ exists in G. That is, v_{k-1} and l are bi-directional neighbors of j. We will now prove that $\{v_{k-1}, l\}$ does not form a bi-directional clique with j in G_Z. Particularly, we will show that $v_{k-1} \to l \notin A_Z$.

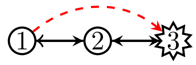
To show that $v_{k-1} \to l \notin A_Z$, we will show that there exists no trail connecting v_{k-1} and l in G satisfying conditions P1), P2) and P3). Suppose, there exists a trail different from $v_{k-1} \leftrightarrow j \leftrightarrow l$ in G. Such a trail will imply two paths connecting v_{k-1} and l in the topology, G^τ, of G. This contradicts the assumption that G^τ is a tree. Therefore, $v_{k-1} \leftrightarrow j \leftrightarrow l$ is the only trail in G that connects v_{k-1} and l. Now, we will show that all directed paths in this trail violate at least one of P1), P2) and P3). Consider the directed path $v_{k-1} \to j \leftarrow l$ where j is a collider. As $j \notin Z$ and $l \notin Z$, condition P2) is not met. Now consider $v_{k-1} \to j \to l$. Then, as $j \notin Z$, P3) is violated. Consider the directed path $v_{k-1} \leftarrow j \to l$. As $j \notin Z$, P3) is violated. Finally consider $v_{k-1} \leftarrow j \leftarrow l$. Again, as $j \notin Z$, P3) is violated. Thus all possible directed paths in $v_{k-1} \leftrightarrow j \leftrightarrow l$ violate atleast one of P1), P2) and P3). Thus $v_{k-1} \to l$ cannot be in the perturbed graph. Therefore, $\{v_{k-1}, l, j\}$ does not form a bi-directional clique in G_Z. This completes the proof that j must be a perturbed node. ∎

B. Illustrative Example

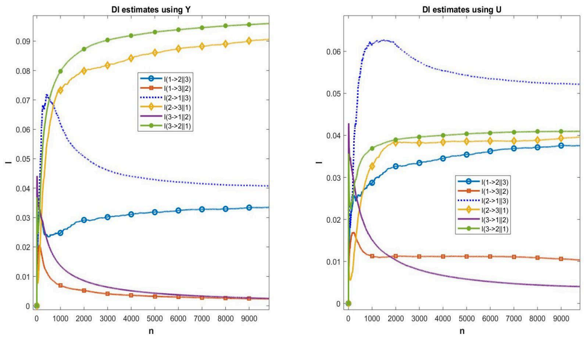
In this subsection we provide examples and discuss the significance of the detection procedure described above.

***Example** 1 (Non-Linear System):* Consider a network consisting of 3 nodes as shown in Figure 2a). The true

(a) Ideal Measurements Y



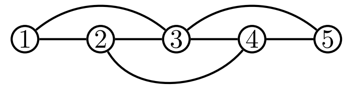
(b) Unreliable Measurements U.



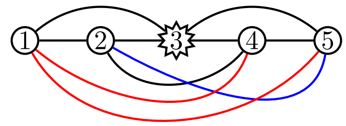
(c) Comparison of directed information estimates between perfect measurements and corrupted data-streams. DIR I is shown along X-axis and the sample length n is along Y-axis.

Fig. 2: This figure shows how node 3 can be inferred as a perturbed node. The only unidirectional link is $1 \to 3$ and 2 is the only bi-directional neighbor of 3. Using Theorem 2, node 3 is detected as a corrupt node.

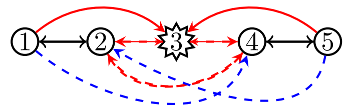
(a) True Generative Graph



(b) True kin graph



(c) Perturbed kin graph



(d) Directed perturbed graph

Fig. 3: This figure shows how graph theory notions can detect corrupt nodes from the inferred directed perturbed graph. In 3d), node 3 forms bi-directional clique with all its bi-directional neighbors as shown in dashed red arrows. This does not hold for unperturbed nodes.

generative model is described by:

$$y_1[t] = e_1[t].y_2[t-1], \tag{10a}$$
$$y_2[t] = y_1[t-1] + y_3[t-1] \cdot e_2[t], \tag{10b}$$
$$y_3[t] = y_2[t-1] + e_3[t] \tag{10c}$$

where $e_1[t] \sim$ Bernouilli(0.2), $e_2[t] \sim$ Bernouilli(0.2) and $e_3[t] \sim$ Bernouilli(0.35), and '+' is the logical 'OR' operation and '·' is the logical 'AND' operation. Each of $y_1[t], y_2[t]$ and $y_3[t]$ has a finite alphabet $\{0, 1\}$. The perturbation considered here is the time-origin uncertainty at node 3. The corruption model takes the form:

$$u_3[t] = \begin{cases} y_3[t-3], & \text{with probability } 0.33 \\ y_3[t], & \text{with probability } 0.67. \end{cases} \tag{11}$$

We used the methods proposed in [19] to compute directed information rate(DIR). The perturbed graph is shown in Figure 2b). Here, the only unidirectional spurious link introduced is $1 \to 3$. Also, bidNr(3) $= \{2\}$. Clearly, 2 forms a bi-directional clique with 3. Applying Theorem 2, therefore, we can conclude node 3 as a perturbed node just by observing the graph structure.

***Example** 2 (Linear System):* Let the generative graph, G, be as showin in Fig. 3a) with the following dynamics:

$$\begin{aligned} y_1[t] &= (H_{12} * y_2)[t] + e_1[t] \\ y_2[t] &= (H_{21} * y_1)[t] + (H_{23} * y_3)[t] + e_2[t] \\ y_3[t] &= (H_{32} * y_2)[t] + (H_{34} * y_4)[t] + e_3[t] \\ y_4[t] &= (H_{43} * y_3)[t] + (H_{45} * y_5)[t] + e_4[t] \\ y_5[t] &= (H_{54} * y_4)[t] + e_5[t] \end{aligned} \tag{12}$$

where H_{ij} are stable LTI filters for all $i, j \in \{1, 2, 3, 4, 5\}$. Suppose, node 3 is perturbed. That is, the measured data-stream for node 3, $u_3[t]$, is given by a noisy filter:

$$u_3[t] = (L * y_3)[t] + \zeta_3[t]$$

where, L is a stable first-order filter and $\zeta_3[t]$ is an IID Gaussian noise. Let $U[t] = \begin{bmatrix} y_1[t] & y_2[t] & u_3[t] & y_4[t] & y_5[t] \end{bmatrix}^\top$ be the vector of observed signals.

If the measurements were unperturbed, the true *kin graph* (undirected graph) can be recovered using multivariate Wiener filter [10]. See Fig. 3b). However, if the measurements of y_3 are perturbed, then the undirected network structure identified using Wiener filtering will contain spurious links as proved in [12]. In this case, the perturbed kin graph will be complete, as shown in Fig. 3c. (See [12] for more details on this construction). The Wiener filtering method implies that if $i - j$ is in the kin topology, then the corresponding entry in the inverse power spectrum satisfies $(\Phi_{UU}(e^{j\omega})^{-1})_{ij} \neq 0$ for all ω. From a trajectory of length 5×10^5, we estimated:

$$\Phi_{UU}(1)^{-1} = \begin{bmatrix} 1.052 & -1.233 & 0.007 & 0.242 & -0.073 \\ -1.233 & 1.621 & -0.036 & -0.538 & 0.227 \\ 0.007 & -0.036 & 0.028 & -0.034 & 0.005 \\ 0.242 & -0.538 & -0.034 & 1.515 & -1.146 \\ -0.073 & 0.227 & 0.005 & -1.146 & 0.983 \end{bmatrix}. \tag{13}$$

Thus, the Wiener filter method predicts a full graph. Due to high symmetry and completeness of the graph, it is not possible to detect the corrupt node purely by looking at the inferred graph structure. Moreover, separation techniques as described in [20] cannot be used to remove spurious edges.

However, directed perturbed graph yields more insightful results. This could be estimated from data using directed

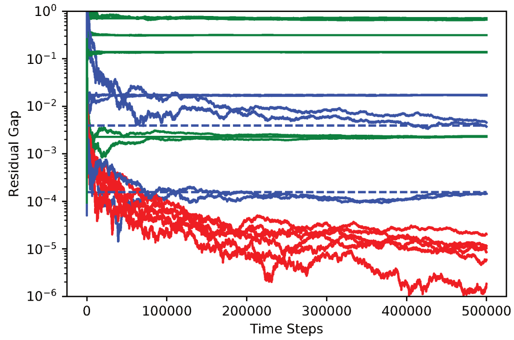


Fig. 4: **Residual Errors**. Each line depicts a running estimate of the residual gap, (14). The dashed lines correspond to theoretical predictions, while the solid lines are the estimates. The green lines correspond to true links $i \to j \in A$, the blue lines correspond to spurious links $i \to j \in A_Z \setminus A$, and the red lines correspond to pairs (i, j) with no predicted link $i \to j \notin A_Z$. As can be seen the green and blue lines plateau near predicted values while the red lines continue to decrease.

information [11], or in the case of linear systems, Granger causality [21]. As discussed in [21] Granger causality is equivalent to directed information in the case of linear Gaussian dynamic systems. Let $U_{\bar{j}}$ denote the entries of U other than j and let $U_{\overline{ij}}$ denote the entries of U other than i and j. For the Granger filter methods, we used recursive least-squares to estimate the difference in prediction error residuals:

$$r_{i,j} = \mathbb{E}\left[\left(u_j[t] - \mathbb{E}[u_j[t]|U_{\bar{j}}^{t-1}]\right)^2\right] - \mathbb{E}\left[\left(u_j[t] - \mathbb{E}[u_j[t]|U_{\overline{ij}}^{t-1}]\right)^2\right]. \quad (14)$$

It can be shown that for linear Gaussian systems that $r_{i,j} > 0$ if and only if $I(u_i \to u_j | U_{\overline{ij}}) > 0$. The results of the estimation are shown in Fig. 4. As can be seen, Fig. 3d is the inferred perturbed graph, G_Z. Applying Theorem 2, we can conclude node 3 as a perturbed node just by observing the graph structure.

VI. Conclusion

We studied the problem of identifying the location of corrupt nodes in a network. We described a method to detect all the corrupt nodes in a network admitting causal and non-linear dynamical interactions with bi-directional coupling between its agents. We showed that solely by examining the perturbed graph structure we can locate corrupt nodes. In particular, we showed that a corrupt node will always form a bi-directional clique with all its bi-directional neighbors. Immediate future extension entails removing spurious edges in the network reconstructed. Furthermore, determining sufficient conditions and detection algorithms to locate corrupt nodes in networks that admit unidirectional coupling between agents will be addressed in future.

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