

Theory of Periodically Time-Varying Induced Exceptional Points of Degeneracy

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Abstract – We present the novel concept of exceptional points of degeneracy (EPDs), which denote a coalescence of multiple eigenmodes, that directly emerge in systems when a linear time-periodic (LTP) variation is applied. Though the presented theory is general, as an example we establish the general conditions that yield an EPD in a single LTP LC resonator with a capacitance that varies periodically in time. We show a potential application of the proposed LTP system in making sensors to exploit the ultra-sensitivity associated with operating at an EPD.

I. INTRODUCTION

Exceptional points of degeneracy (EPDs) recently have received a surge of interest due to their various applications in the microwave and optical regimes. An EPD is a point in the parameter space of a system at which multiple eigenmodes of the system coalesce in both their eigenvalues and eigenvectors. The concept of EPD has been vastly investigated in lossless spatially periodic structures [1]–[4], and systems with loss and/or gain under parity-time (PT-) symmetry [5]–[8]. In general, EPDs rarely occur unintentionally in devices and systems, yet different systems can be engineered to intentionally exhibit EPDs which may be useful in variety of applications. However, in this paper we show a new way to obtain EPDs together with the associated theory. Specifically we show that EPDs are induced in systems by simply applying time periodicity in one or more of its elements. We demonstrate the occurrence of EPDs induced by time-periodic variation in a system through a rigorous formalism mathematical formalism [9] and provide an experimental proof as well. The presented concept based on periodic time variation systems is analogous to that of EPDs found in spatially periodic waveguides [10], [11]. At an EPD, the eigenstates of the system are described using the generalized eigenvectors rather than the regular eigenvectors, which, in turn, lead to a Jordan block degeneracy and an algebraic growth in the system eigenstates [1], [3]. Unique and different properties associated with the emergence of EPDs may lead to many potential applications, such as enhancing the gain of active systems [2], [12], highly directive antennas, and enhanced sensors [13].

II. MATHEMATICAL DESCRIPTION OF EPDs IN LTP SYSTEMS

We consider a linear time-periodic system which, in general, may be comprised of one or more periodic, time-varying elements. Though the presented theory is general, here we specifically consider an example of an LC resonator with a time-periodic capacitance, $C(t)$. Fig. 1(a) shows the schematic of an LC resonator which has a periodically time-varying capacitor that is, for simplicity, assumed to be modulated by a piece-wise constant, time-periodic function, however other time-periodic functions may also be used to modulate the capacitance. The time domain behavior of such a system is captured using a multidimensional differential equation with a state vector $\Psi(t) = [v(t) \ i(t)]^T$, where T denotes the transpose operation. The multidimensional first-order differential equation that describes the system dynamics is described in matrix form as

$$\frac{d}{dt} \Psi(t) = \underline{\mathbf{M}}(t) \Psi(t), \quad (1)$$

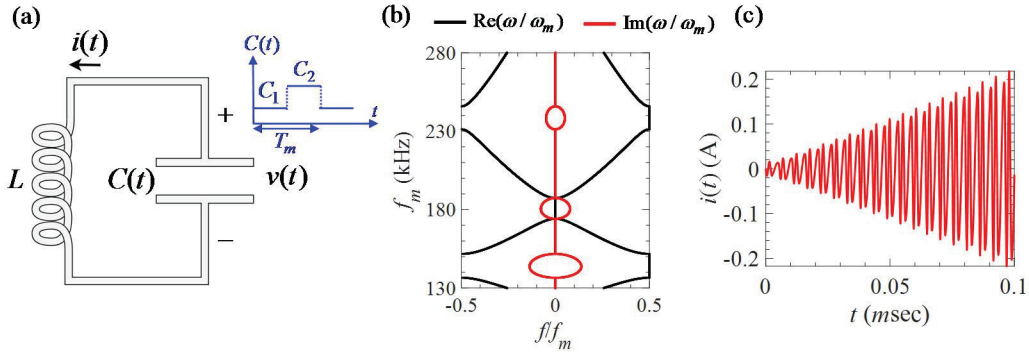


Fig. 1 (a) A single LC resonator with periodically (piece wise) time varying capacitance exhibiting EPDs. (b) The resonant frequency dispersion relation of the time-periodic variant LC circuit is plotted varying modulation frequency. The real and imaginary parts of the resonance frequency f are denoted by black and red colors respectively. (c) Time domain behavior of the inductor current $i(t)$ at an EPD ($f_m = 187.3\text{kHz}$) where a linear growth is observed.

where $\underline{\mathbf{M}}(t) = \begin{bmatrix} 0 & -C(t) \\ 1/L & 0 \end{bmatrix}$ is the *system matrix* of the LC resonator shown in Fig. 1(a). For such an LTP system

the state transition matrix, $\underline{\Phi}$, that translates the state vector from the time instant t to $t+T_m$ is defined as $\Psi(t+T_m) = \underline{\Phi}\Psi(t)$, where $T_m = 1/f_m$ is the period of modulation applied to the time-periodic elements. Assuming a two-level, piece-wise constant, time periodic variation in the capacitance as shown in Fig. 1(a), the state transition matrix can be related to the system matrix $\underline{\mathbf{M}}_j$ during the time interval T_j as $\underline{\Phi} = \prod_{j=1}^2 e^{\underline{\mathbf{M}}_j T_j}$.

Furthermore, the time-periodicity of the capacitance enforces a pseudo-periodicity of the state vector as $\Psi(t+T_m) = \exp(-i\omega T_m)\Psi(t)$, where $\omega/(2\pi)$ is the resonant frequency of the system. Using the state transition matrix to represent the time evolution of the state vector, we can form the eigenvalue problem as $(\underline{\Phi} - \lambda \underline{\mathbf{I}})\Psi(t) = \mathbf{0}$ where $\lambda = \exp(-i\omega T_m)$ are the eigenvalues of the system. Figure 1(b) illustrates the dispersion of the eigenfrequencies f of $\underline{\Phi}$, vs the modulation frequency f_m where the black and red curves denote the real and imaginary parts of the eigenfrequency f , respectively. The dispersion diagram is plotted for an LTP LC resonator with its parameters set as $L_0 = 22\ \mu\text{H}$, and $C_1 = 4.5\ \text{nF}$ and $C_2 = 22.5\ \text{nF}$, each over half of the period. The resonance frequencies are symmetrically located with respect to the center of the Brillouin zone. Note that due to the time periodicity of the system, the eigenfrequency f , corresponds to all the Floquet harmonics at $f \pm nf_m$, where n is an integer. In this system, EPDs occur when the two eigenvalue solutions of the system coalesce at a specific modulation frequency f_m and become exactly equal (though not shown here the eigenvector coalesce as well). One can note that in this particular system EPDs occur either at the center (i.e., $f = \pm nf_m$) or at the edge of the Brillouin zone (i.e., $f = 0.5f_m \pm nf_m$). In this example, we consider a lossless and gainless system, i.e., without any gain or loss element, therefore the imaginary parts of the eigenvalue solutions are symmetric with respect to the center of the Brillouin zone and the resonance frequencies are purely real at an EPD. The Fig. 1(c) shows the time domain behavior of the state vector element $i(t)$ at an EPD ($f_m = 187.3\text{kHz}$) where the linear growth of the state vector is observed even though there is no gain element. This is an important characteristic associated with the generalized eigenvector of a second-order EPD. The state vector growth is due to the reception of energy from the time variation mechanism.

The LC resonator system considered here has a 2×2 state transition matrix $\underline{\Phi}$ where at an EPD the transition matrix is similar to a Jordan block containing two degenerate eigenvalues that are associated with a regular eigenvector and a generalized eigenvector. At this point, the state transition matrix is non-diagonalizable and is similar to a matrix that contains a nontrivial Jordan block with two degenerate eigenvalues where the order of the Jordan Block defines the order of the degeneracy. For the 2×2 system in the example we provide, it is shown in [9] that the degeneracy of eigenvalues is a sufficient condition for the coalescence of the eigenvectors, unless the state transition matrix $\underline{\Phi}$ is a diagonal matrix. In general, to have a degenerate eigenvalue and as result an EPD in a second order LTP system the state transition matrix must satisfy $\text{trace}(\underline{\Phi})/2 = \pm \sqrt{\det(\underline{\Phi})}$. This is a sufficient

and general condition that describes the occurrence of EPDs in any system that can be described with the set of differential Eqs. (1).

The premise of EPDs may be beneficial toward various applications, such as low-threshold oscillators, highly directive antennas, and RF and microwave circuits. Furthermore, systems operating at or near EPDs are characterized by an exceptional sensitivity of their eigenvalues (resonance frequencies in our case) to external perturbations, which makes them potential candidates to be used to conceive a new class of highly sensitive sensors. For instance, in the presentation we will show theoretically and experimentally that the second order time-periodically induced EPD described in this paper shows an unprecedented sensitivity to external perturbations in the capacitance.

III. CONCLUSIONS

We have explored the general and novel principles that a linear time-periodic variation of a system induces exceptional points of degeneracy (EPDs) and we have provided the mathematical conditions necessary for such occurrence. The simple example of a linear time-periodic LC resonator is investigated, and we have illustrated that without any time-invariant gain- and loss-circuit elements, the resonance frequency of an EPD is purely real, yet the system state vector grows linearly in time. We also explore how such EPD concept is used to conceive extremely sensitive devices.

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