Linear Time-Periodic Systems with Exceptional Points of Degeneracy

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***Abstract*—We show how exceptional points of degeneracy (EPDs), which a coalescence of multiple eigenmodes, emerge in a linear time-periodic (LTP) systems. We establish the necessary conditions that yield an EPD in a single LTP LC resonator, however, the presented theory can be generalized to any kind of resonator with a time varying element. Furthermore, we propose an application of the EPD in a LTP LC resonator as a sensing device and show the ultra-sensitivity of such system to external perturbations.**

Keywords—Exceptional point, LC resonator, Linear time variant

I. Introduction

Recent developments on the concept of eexceptional points of degeneracy (EPDs) have attracted a surge of interest due to their applications in the microwave and optical regimes. An EPD is defined as a special point in the parameter space of a system at which multiple eigenmodes of the system coalesce in both their eigenvalues and eigenvectors. The EPDs are found in different structures such as systems with loss and/or gain under parity-time symmetry [1]-[4], and lossless spatially periodic structures [5]-[8]. In general, such points normally are not found in devices and systems, however, systems can be engineered to show EPDs which may be suitable in variety of applications. In this paper, we show the emergence of EPDs in a linear time periodic (LTP) system where we show that EPDs are induced in such system systems by simply temporally modulating one or more of the system elements. The EPDs found based on the concept of periodic time variation are analogous to EPDs found in spatially periodic waveguides [6], [7]. At an EPD, the eigenmode of the system coalesce, so that the system, when described with the evaluation of a state vector via a system matrix multiplication, exhibits a Jordan block degeneracy. Hence, the eigenstates of the system are described using generalized eigenvectors rather than regular eigenvectors, and this leads to an algebraic growth in the system eigenstates [5], [7]. Unique features that are associated with the coalescence of eigenmodes may contribute to many potential applications, such as enhancing the gain of active systems, highly directive antennas, and enhanced sensors [9]-[11].

II. EPDs in Linear Ttime Periodic systems

Generally, a linear time-periodic system may comprise of one or more periodic, time-varying elements. Here, we specifically consider an example of an LC resonator with a time-periodic capacitance $C(t)$ as the schematic of the circuit is shown in Fig. 1(a), though the presented formalism is general for any

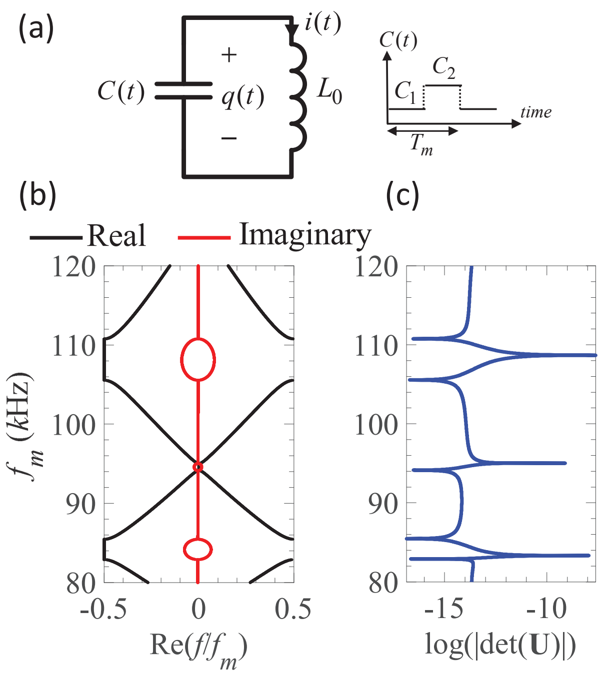

Fig. 1. (a) A single LC resonator with periodically (piece wise) time varying capacitance exhibiting EPDs. (b) The resonant frequency dispersion relation of the time-periodic varying LC resonator is plotted varying modulation frequency. The real and imaginary parts of the resonance frequency f are denoted by black and red colors respectively. (c) determinant of the similarity matrix $\mathbf{U}$ where the nulls indicate second order EPDs.

other time-periodic resonator. The periodic time-varying capacitor of the LC resonator is assumed, for simplicity, to be modulated by a piece-wise constant time-periodic function. We express the time domain behavior of time-varying LC resonator by multidimensional differential equations with $\mathbf{\Psi}(t)=[q(t),i(t)]^T$ being the state vector, where T represents the transpose operation. The governing multidimensional first-order differential equations which describes the system dynamics read as

$$\frac{d}{dt}\mathbf{\Psi}(t)=\begin{bmatrix} 0 & -1 \\ 1/(L_0C(t)) & 0 \end{bmatrix}\mathbf{\Psi}(t) \qquad (1)$$

For an LTP system, the translation of the state vector from the time instant t to $t+T_m$ is carried out using the state transition

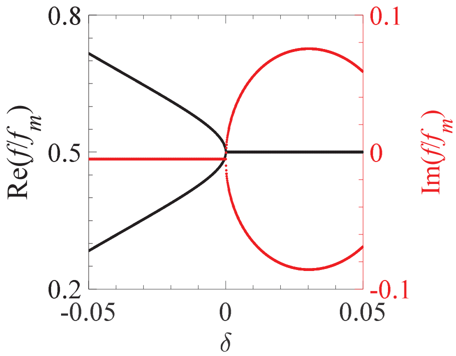

Fig. 2. Complex resonance frequencies of a lossy time vaying LC resonator slightly away from the EPD. Large variations of resonance frequency occur even for small perturbations δ.

matrix $\underline{\mathbf{\Phi}}$ that is defined by $\mathbf{\Psi}(t+T_m) = \underline{\mathbf{\Phi}}\, \mathbf{\Psi}(t)$, where $T_m = 1/f_m$ is the modulation period. Moreover, by enforcing the Floquet periodicity as $\mathbf{\Psi}(t+T_m) = \exp(-i\omega T_m)\mathbf{\Psi}(t)$, the eigenvalue problem is constructed as $(\underline{\mathbf{\Phi}} - \lambda\, \underline{\mathbf{I}})\, \mathbf{\Psi}(t) = \mathbf{0}$ with $\lambda = \exp(-i\omega T_m)$ as the system eigenvalue. The dispersion plot of the eigenfrequencies versus the modulation frequency f_m is illustrated in Fig. 1(b) when the parameters are set as $L_0 = 22$ μH, and $C_1 = 5$ nF and $C_2 = 15$ nF. Note that the eigenfrequency f in this figure, corresponds to harmonics $f \pm nf_m$, where n is an integer. Furthermore, in this system without gain or loss element, the imaginary parts of the eigenvalue solutions are symmetric with respect to the center of the Brillouin zone and the resonance frequencies are purely real at an EPD (e.g., f_m = 105.5 kHz).

EPDs occur in this system when the two eigenvalues and their corresponding eigenvector solutions coalesce at a specific modulation frequency f_m and become exactly equal. The determinant of the similarity matrix $\underline{\mathbf{U}} = [\mathbf{\Psi}_1, \mathbf{\Psi}_2]$, which contains the eigenvectors corresponding to each eigenvalue, can be employed as measure of the coalescence of the eigenvectors. For instance, in the proximity of an EPD and since the eigenvectors of the system are similar and the determinant of $\underline{\mathbf{U}}$ approaches zero. Figure 1(c) shows the determinant of the similarity matrix where it can be observed that the local minimums of the determinant coincide with points in the dispersion diagram where two eigenfrequencies coalesce. It is worth to note that one can adjust the EPD frequency to any desired operating frequency through tuning the system parameters such as capacitances, inductor or modulation frequency.

III. Ultra-sensitivity to perturbations

The existence of EPDs in LTP systems is associated with unique characteristics due to the extreme sensitivity to system perturbations. This extreme sensitivity makes such an LTP LC resonator promising candidate to conceive ultra-sensitive sensor and biosensor. The enormous and desirable sensitivity enhancement at EPDs in LPT systems is due to the coalescence of eigenvectors of the system. For instance, let us consider a small perturbation δ introduced in the time varying capacitance as $C_p(t) = (1+\delta)C(t)$. This perturbation leads to a perturbed state transition matrix $\underline{\mathbf{\Phi}}(\delta)$, which consequently results in perturbed system eigenvalues. For an LTP resonator near an EPD, the perturbed eigenvalues are proportional to $\lambda_p \propto \delta^{1/2}$. Hence, when $\delta \ll 1$ then $\delta^{1/2} \gg \delta$, which implies much higher sensitivity to a small variation δ than that of regular sensors where the measurable quantities are linearly proportional to the perturbation [12], [13]. Such a small perturbation changes the complex resonance frequency of the LTP LC resonator drastically as shown in Fig. 2.

IV. Conclusion

We have explored the general and novel principle that a linear time-periodic variation in a system induces exceptional points of degeneracy (EPDs). As a simple example, we have investigated the linear time-periodic LC resonator which exhibits EPDs without any time-invariant gain- and loss-circuit elements (though time variation can inject energy in the system). We also explore how such an EPD concept is used to conceive extremely sensitive sensors.

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