FUNDAMENTAL MATRIX FACTORIZATION IN THE FJRW-THEORY REVISITED

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Abstract. We present an improved construction of the fundamental matrix factorization in the FJRW-theory given in [7]. The revised construction makes the independence on choices more apparent and works for a possibly nonabelian finite group of symmetries. One of the new ingredients is the category of dg-matrix factorizations over a dg-scheme.

Introduction

This short note is supposed to clarify the construction of the cohomological field theory associated with a quasihomogeneous polynomial W and its finite group of symmetries G. Such a cohomological field theory, called the *FJRW-theory* was first proposed in [4]. Then, in [7] a different construction, based on categories of matrix factorizations, was given (conjecturally, the two constructions give the same cohomological field theory).

The approach of [7] is based on constructing certain *fundamental matrix factorizations* which live over the product of certain finite coverings of $\overline{M}_{g,n}$ (the moduli of Γ-spin structures) with affine spaces. It is this construction that we aim to clarify. More precisely, we would like to present the construction in such a way that it would be analogous to the construction of Ciocan-Fontanine and Kapranov of the virtual fundamental class in Gromov-Witten theory via dg-manifolds (see [1]). The second goal that we achieve is to present the construction without using coordinates on the vector space V on which W lives. This has an additional bonus that we can handle the case when the group G is not necessarily commutative (but still finite).

The construction of [7] of the fundamental matrix factorization over $\mathcal{S}\times\prod_i V^{\gamma_i}$, where $\mathcal{S}$ is the moduli space of (rigidified) Γ-spin structures with some markings (see Sec. 3.1 for details) roughly has the following two steps. In Step 1 one considers the object $R\pi_*(\mathcal{V})$ in the derived category $D(\mathcal{S})$, where $\pi:\mathcal{C}\to\mathcal{S}$ is the universal curve, $\mathcal{V}$ is the underlying vector bundle of the universal Γ-spin structure, and then equips it with some additional structure. In Step 2 one realizes $R\pi_*(\mathcal{V})$ by a 2-term complex $[A\to B]$, where A and B a vector bundles over $\mathcal{S}$, such that there is a morphism

$$Z : X = \mathrm{tot}(A) \to \prod_i V^{\gamma_i}$$

and a Koszul matrix factorization of $Z^*(\sum W_i)$, where $W_i = W|_{V^{\gamma_i}}$. Then the fundamental matrix factorization is obtained by taking its push-forward with respect to the

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morphism $(p, Z) : X \to \mathcal{S} \times \prod_i V^{\gamma_i}$, where $p : X \to \mathcal{S}$ is the projection. Note that here the space X is non-canonical, so one has to check independence on the choices made.

The main idea of the present paper is to change the conceptual framework slightly by observing that in fact one gets a *dg-matrix factorization* on a *dg-scheme* over $\mathcal{S} \times \prod_i V^{\gamma_i}$ (the terminology is explained in Sec. 1). Namely, for a non-negatively graded complex of vector bundles $C^\bullet$ over $\mathcal{S}$, one can define the corresponding dg-scheme over $\mathcal{S}$,

$$[C^\bullet] := \mathrm{Spec}(S^\bullet(C^\bullet)^\vee).$$

In our case we consider the dg-scheme

$$\mathcal{X} := [R\pi_*(\mathcal{V})].$$

More concretely, if we realize $\mathcal{V}$ by a 2-term complex $\mathcal{V} = [A \to B]$ then our dg-scheme is realized by the sheaf of dg-algebras

$$\mathcal{O}_{\mathcal{X},[A\to B]} := S^\bullet[B^\vee \to A^\vee],$$

where the complex $[B^\vee \to A^\vee]$ is concentrated in degrees -1 and 0. Then we interpret the additional structure on $R\pi_*(\mathcal{V})$ coming from the universal Γ-spin structure as a structure of a dg-matrix factorization on the structure sheaf of $\mathcal{X}$. More precisely, we get a morphism

$$Z_{\mathcal{X}} : \mathcal{X} \to \prod_i V^{\gamma_i}$$

and a function of degree -1, $f_{-1} \in \mathcal{O}^{-1}_{\mathcal{X},[A\to B]}$, such that

$$d(f_{-1}) = -Z^*_{\mathcal{X}}(\sum W_i).$$

Now the fundamental matrix factorization is obtained as the push-forward of $(\mathcal{O}_{\mathcal{X}}, d + f_{-1} \cdot \mathrm{id})$ with respect to the morphism $\mathcal{X} \to \mathcal{S} \times \prod_i V^{\gamma_i}$.

The connection with the original approach is the following: for each presentation $\mathcal{V} = [A \to B]$, for which the first construction works, there is a morphism $q : \mathcal{X} \to X = \mathrm{tot}(A)$, such that $Z \circ q = Z_{\mathcal{X}}$, and an isomorphism of the push-forward $q_*(\mathcal{O}_{\mathcal{X}}, d + f_{-1} \cdot \mathrm{id})$ with the Koszul matrix factorization of $Z^*(\sum W_i)$ constructed through the first approach.

The second technical improvement we present is in the construction of f_{-1}. The idea is to work systematically with the categories of sheaves over pairs (scheme, closed subscheme) to deal with non-functoriality of the cone construction (such categories fit into the framework of Lunts's poset schemes in [6]) . Namely, we work with the enhancement of the usual push-forward with respect to the projection $\pi : \mathcal{C} \to \mathcal{S}$ to a morphism of pairs $(\mathcal{C}, \Sigma) \to (\mathcal{S}, \mathcal{S})$, where $\Sigma \subset \mathcal{C}$ is the union of the images of the universal marked points (see Sec. 2).

Recall that in [7], we used the fundamental matrix factorizations to construct cohomological field theories associated with (W, G) by viewing them as kernels for Fourier-Mukai functors and passing to Hochschild homology. It seems that the approach via dg-matrix factorizations presented here could also be useful in the development of a more general construction in Gauged Linear Sigma Model, see [5], [2].

Throughout this work the ground field is $\mathbb{C}$.

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1. Matrix factorizations over dg-schemes

1.1. **Definition.** We consider dg-schemes in the spirit of [1]. We fix a space S (a scheme or a stack), and consider the structure sheaf of a dg-scheme over S to be a sheaf $(\mathcal{O}_X^\bullet, d)$ of $\mathbb{Z}_-$-graded commutative dg-algebras over $\mathcal{O}_S$ (one can make a restriction $\mathcal{O}_X^0 = \mathcal{O}_S$, but it is not really necessary).

Given a function $f_0 \in \mathcal{O}_X^0$ we can consider the category of (quasicoherent) *dg-matrix factorizations* of f_0. By definition, these are $\mathbb{Z}/2$-graded complexes of sheaves $P = P^{\overline{0}} \oplus P^{\overline{1}}$ together with a (quasicoherent) $\mathcal{O}_X^\sharp$-module structure, such that $\mathcal{O}_X^i \cdot P^{\overline{a}} \subset P^{\overline{i+a}}$. In addition P is equipped with an odd differential δ satisfying the Leibnitz identity

$$\delta(\phi \cdot p) = d(\phi) \cdot p + (-1)^k \phi \delta(p),$$

for $\phi \in \mathcal{O}_X^k$, $p \in P$, and the equation $\delta^2 = f_0 \cdot \mathrm{id}_P$.

Example 1.1.1. Given an element $f_{-1} \in \mathcal{O}_X^{-1}$, such that $d(f_{-1}) = f_0$, we get a structure of a dg-matrix factorization on $\mathcal{O}_X^\bullet$ by setting

$$\delta(\phi) = d(\phi) + f_{-1} \cdot \phi.$$

(In checking that $\delta^2 = 0$ one has to use the fact that $f_{-1}^2 = 0$.)

The above example can be obtained from the following more general operation. Suppose we are given a function $f_0 \in \mathcal{O}_X^0$ and a dg-matrix factorization (P, δ) of f_0. Then for any $f_{-1} \in \mathcal{O}_X^0$ we can change the differential δ to $\delta + f_{-1} \cdot \mathrm{id}_P$. Then $(P, \delta + f_{-1} \cdot \mathrm{id}_P)$ will be a dg-matrix factorization of $f_0 + d(f_{-1})$.

1.2. **Positselski's framework of quasicoherent CDG-algebras.** More generally, we can assume that f_0 a section in $\mathcal{O}_X^0 \otimes L$, where L is a locally free $\mathcal{O}_X^0$-module of rank 1. The theory of the corresponding categories of dg-matrix factorizations fits into the framework of quasicoherent CDG-algebras developed by Positselski (see [3, Sec. 1]).

With the data $(\mathcal{O}_X^\bullet, L, f_0)$ as above we can associate a quasicoherent CDG-algebra

$$\mathcal{B} := \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X^\bullet \otimes_{\mathcal{O}_X^0} L^{\otimes n}[-2n],$$

with the natural structure of a complex of sheaves (i.e., the $\mathbb{Z}$-grading and the differential d), the natural product and the global curvature element given by $f_0 \in \mathcal{O}^0 \otimes L \subset \mathcal{B}_2$.

Now a quasicoherent dg-matrix factorization is a quasicoherent DG-module over $\mathcal{B}$, i.e., a graded $\mathcal{B}$-module $M = \bigoplus_n M_n$, equipped with a differential $\delta = \delta_M$ such that $\delta^2 = f_0 \cdot \mathrm{id}_M$ and δ satisfies the Leibnitz identity with respect to the $\mathcal{B}$-action. Note that such a DG-module necessarily has

$$M_{n+2} \simeq M_n \otimes L,$$

so it is determined by the components M_0 and M_1, and we get the structure of a dg-matrix factorization on $M_0 \oplus M_1$.

There are several exotic derived categories associated to a quasicoherent CDG-algebra. The one that is most relevant for the theory of dg-matrix factorizations is the category

$$\mathrm{qcoh}-\mathrm{MF}_{ffd}(f_0) := D^{co}(\mathcal{B}-\mathrm{qcoh}_{ffd}) \simeq D^{co}(\mathcal{B}-\mathrm{qcoh}_{fl}) \simeq D^{abs}(\mathcal{B}-\mathrm{qcoh}_{fl}),$$

where the superscripts "abs" and "co" refer to "absolute" and "coderived", while the subscripts "fl" and "ffd" mean "flat" and "finite flat dimension" (see [3, Sec. 1]).

Assume that $f : (X, \mathcal{O}_X^\bullet) \to (Y, \mathcal{O}_Y^\bullet)$ is a morphism of finite flat dimension, L is a locally free $\mathcal{O}_Y^0$-module of rank 1, W_0 is a section of L. Then we have the induced section f^*W_0 of f^*L. In this situation we have the push-forward functor (see [3, Prop. 1.9])

$$Rf_* : \mathrm{qcoh}-\mathrm{MF}_{ffd}(f^*W_0) \to \mathrm{qcoh}-\mathrm{MF}_{ffd}(W_0).$$

1.3. **Koszul matrix factorizations as push-forwards.** Let V be a vector bundle over a scheme X, and suppose we have sections $\alpha \in H^0(X, V^\vee)$, $\beta \in H^0(X, V)$. With these data one associates a Koszul matrix factorization $\{\alpha, \beta\}$ of $W = \langle \alpha, \beta \rangle$, whose underlying super-vector bundle is $\bigwedge^\bullet(V)$. On the other hand, we have the derived zero locus of β, $\mathcal{Z}(\beta) \to X$, which corresponds to the dg-algebra given by the Koszul complex of β:

$$\mathcal{O}_{\mathcal{Z}(\beta)} = (\bigwedge^\bullet(V), d = \iota_\beta).$$

Now we can view α as a function of degree -1 on $\mathcal{Z}(\beta)$ such that $d(\alpha)$ is the pull-back of W. Thus, by definition, $\{\alpha, \beta\}$ is the push-forward of the dg-matrix factorization $(\mathcal{O}_{\mathcal{Z}(\beta)}, d + \alpha \cdot \mathrm{id})$ by the morphism $\mathcal{Z}(\beta) \to X$.

This explains why in the case when β is a regular section of V, the Koszul matrix factorization $\{\alpha, \beta\}$ is equivalent to the push-forward of the structure sheaf on the usual zero locus of β.

2. Trace maps via morphisms of pairs

2.1. **Sheaves on pairs.** Let $\iota : Y \to X$ be a closed embedding.

We consider a very simple poset scheme in the sense of [6] for the poset consisting of two elements $\alpha > \beta$, so that $X_\alpha = Y$ and $X_\beta = X$. Then a quasicoherent sheaf on this poset scheme is a triple $(\mathcal{F}_\alpha, \mathcal{F}_\beta, \phi)$, with $\mathcal{F}_\alpha \in \mathrm{Qcoh}(Y)$, $\mathcal{F}_\beta \in \mathrm{Qcoh}(X)$ and $\phi : \mathcal{F}_\beta \to \iota_*\mathcal{F}_\alpha$ is a morphism. We denote by $\mathrm{Qcoh}(X, Y)$ this abelian category, and by $\mathrm{Coh}(X, Y)$ its subcategory corresponding to $\mathcal{F}_\alpha \in \mathrm{Coh}(Y)$, $\mathcal{F}_\beta \in \mathrm{Coh}(X)$. Furthermore, we have a subcategory of locally free coherent sheaves (those with $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ locally free).

The perfect derived category $\mathrm{Perf}(X, Y)$ of bounded complexes of locally free sheaves on (X, Y) has a natural monoidal structure given by the tensor product, so we can also define symmetric powers of objects in $\mathrm{Perf}(X, Y)$.

Given a morphism of pairs $f : (X, Y) \to (X', Y')$ we have a natural derived push-forward morphism

$$Rf_* : D^+ \mathrm{Qcoh}(X, Y) \to D^+ \mathrm{Qcoh}(X', Y'),$$

where D^+ denotes the derived category of bounded below complexes.

The push-forward is compatible with the tensor products in the usual way: we have natural morphisms

$$Rf_*(F) \otimes Rf_*(G) \to Rf_*(F \otimes G), \quad S^\bullet Rf_*(F) \to Rf_*S^\bullet(F). \tag{2.1.1}$$

We have a fully faithful exact embedding $j_! : D\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(X,Y)$ sending $\mathcal{G}$ to $\mathcal{F}_\alpha = 0$, $\mathcal{F}_\beta = \mathcal{G}$. There is a right adjoint functor to it (see [6]),

$$Rj^! : D^+\,\mathrm{Qcoh}(X,Y) \to D^+\,\mathrm{Qcoh}(X),$$

which is defined as the right derived functor of the functor

$$j^! : \mathrm{Qcoh}(X,Y) \to \mathrm{Qcoh}(X) : \mathcal{F}_\bullet \mapsto \ker(\mathcal{F}_\beta \to \iota_*\mathcal{F}_\alpha).$$

Note that objects $\mathcal{F}_\bullet \in \mathrm{Qcoh}(X,Y)$, such that $\mathcal{F}_\beta \to \iota_*\mathcal{F}_\alpha$ is surjective, are acyclic with respect to $j^!$. Furthermore, every object of $\mathrm{Qcoh}(X,Y)$ has a canonical resolutions by such acyclic objects:

$$0 \to (\mathcal{F}_\alpha, \mathcal{F}_\beta) \to (\mathcal{F}_\alpha, \mathcal{F}_\beta \oplus \iota_*\mathcal{F}_\alpha) \to (0, \iota_*\mathcal{F}_\alpha) \to 0$$

Computing $Rj^!$ using these resolutions has a very simple interpretation: given a complex $(\mathcal{F}_\alpha^\bullet, \mathcal{F}_\beta^\bullet)$ over $\mathrm{Qcoh}(X)$, the functor $Rj^!$ sends it to the complex

$$\mathrm{Cone}(\mathcal{F}_\beta^\bullet \to \iota_*\mathcal{F}_\alpha^\bullet)[-1].$$

In particular, there is a natural exact triangle

$$Rj^!(\mathcal{F}_\alpha^\bullet, \mathcal{F}_\beta^\bullet) \to \mathcal{F}_\beta^\bullet \to \iota_*\mathcal{F}_\alpha^\bullet \to \dots$$

We also have the following compatibility between $Rj^!$ and the push-forward.

Lemma 2.1.1. *Let $f : (X,Y) \to (X',Y')$ be a morphism of pairs. Assume that there exists a finite open covering of X, affine over X'. Then for $\mathcal{F} \in D^+\,\mathrm{Qcoh}(X,Y)$ we have a natural isomorphism*

$$Rj^! Rf_*(\mathcal{F}) \simeq Rf_* Rj^!(\mathcal{F}) \tag{2.1.2}$$

in $D^+\,\mathrm{Qcoh}(X')$.

Proof. Let us choose a quasi-isomorphism $\mathcal{F} \to \widetilde{\mathcal{F}}$, such that all $\widetilde{\mathcal{F}}_\alpha^i$ and $\widetilde{\mathcal{F}}_\beta^i$ are f_*-acyclic (this can be done using Cech resolutions). Then the left-hand side of (2.1.2) is represented by the complex

$$\mathrm{Cone}(f_*\widetilde{\mathcal{F}}_\beta \to \iota_* f_*\widetilde{\mathcal{F}}_\alpha)[-1].$$

On the other hand, the terms of $\mathrm{Cone}(\widetilde{\mathcal{F}}_\beta \to \iota_*\widetilde{\mathcal{F}}_\alpha)[-1]$ are also f_*-acyclic, so the right-hand side of (2.1.2) is represented by the complex

$$f_*\,\mathrm{Cone}(\widetilde{\mathcal{F}}_\beta \to \iota_*\widetilde{\mathcal{F}}_\alpha)[-1],$$

which is isomorphic to the one above. □

2.2. **Differentials on curves.** Let $\pi : \mathcal{C} \to \mathcal{S}$ be a family of stable curves, $p_i : \mathcal{S} \to \mathcal{C}$, $i = 1, \dots, r$, be sections of π, such that π is smooth along their images, and let $\Sigma = \sqcup_i p_i(\mathcal{S})$. We view $(\mathcal{C}, \Sigma)$ as a poset scheme and consider the corresponding category $\mathrm{Coh}(\mathcal{C}, \Sigma)$ whose objects are collections $(F, (F_i), (f_i))$, where F is a coherent sheaf on $\mathcal{C}$, F_i is a coherent sheaf on $\mathcal{S}$ and $f_i : F \to p_{i*}F_i$ is a morphism. Sometimes we will omit the morphisms (f_i) from the notation and just write $(F, (F_i))$.

Set $\omega^{\log}_{\mathcal{C}/\mathcal{S}} = \omega_{\mathcal{C}/\mathcal{S}}(\Sigma)$. Recall that we have natural residue maps

$$\mathrm{Res}_\Sigma : \omega^{\log}_{\mathcal{C}/\mathcal{S}}|_\Sigma \xrightarrow{\sim} \mathcal{O}_\Sigma,$$

so that $\ker(\mathrm{Res}_\Sigma)$ is identified with $\omega_{\mathcal{C}/\mathcal{S}}$. Thus, we can view the triple

$$[\omega^{\log}_{\mathcal{C}/\mathcal{S}}, \Sigma] := (\omega^{\log}_{\mathcal{C}/\mathcal{S}}, \mathcal{O}_\Sigma, \mathrm{Res}_\Sigma)$$

as an object of the category $\mathrm{Coh}(\mathcal{C}, \Sigma)$. Furthermore, we have

$$Rj^![\omega^{\log}_{\mathcal{C}/\mathcal{S}}, \Sigma] \simeq \omega_{\mathcal{C}/\mathcal{S}}.$$

Note that we have a morphism of pairs

$$\pi : (\mathcal{C}, \Sigma) \to (\mathcal{S}, \mathcal{S}). \tag{2.2.1}$$

By Lemma 2.1.1, the object $R\pi_*[\omega^{\log}_{\mathcal{C}/\mathcal{S}}, \Sigma]$ satisfies

$$Rj^! R\pi_*[\omega^{\log}_{\mathcal{C}/\mathcal{S}}, \Sigma] \simeq R\pi_*\omega_{\mathcal{C}/\mathcal{S}}. \tag{2.2.2}$$

Note also that we have a morphism of exact triangles (which will be used later)

$$\begin{array}{ccccccc}
\oplus_{i=1}^r \mathcal{O}_\mathcal{S}[-1] & \longrightarrow & R\pi_*(\omega_{\mathcal{C}/\mathcal{S}}) & \longrightarrow & R\pi_*(\omega^{\log}_{\mathcal{C}/\mathcal{S}}) & \longrightarrow & \oplus_{i=1}^r \mathcal{O}_\mathcal{S} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow t \\
\mathcal{O}_\mathcal{S}[-1] & \xrightarrow{\mathrm{id}} & \mathcal{O}_\mathcal{S}[-1] & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_\mathcal{S}
\end{array} \tag{2.2.3}$$

where t is given by the summation.

The above constructions also work in the case of a family of orbicurves with stable coarse moduli spaces.

3. Fundamental matrix factorization

3.1. **Setup and the moduli spaces of Γ-spin structures.** Let us recall the setup of the FJRW theory (see [4], [7]), or rather its slight generalization to noncommutative finite groups of symmetries (as in [5]).

We start with a finite-dimensional vector space V equipped with an effective $\mathbb{G}_m$-action called the *R-charge*, such that all the weights of this action on V are positive. We denote the corresponding subgroup in $\mathrm{GL}(V)$ by $\mathbb{G}_{m,R}$. and let W be a function of weight d on V. Also, we fix a finite subgroup $G \subset \mathrm{GL}(V)$ such that W is G-invariant, G commutes with $\mathbb{G}_{m,R}$ and G contains a fixed element $J \in \mathbb{G}_{m,R}$ of order d.

We define $\Gamma \subset \mathrm{GL}(V)$ to be the algebraic subgroup generated by G and by $\mathbb{G}_{m,R}$. There is a canonical exact sequence

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathbb{G}_m \to 1,$$

where χ restricts to the subgroup $\mathbb{G}_{m,R}$ as $\lambda \mapsto \lambda^d$.

As in [7], we consider the moduli space of Γ-spin structures: it classifies stable orbicurves $(C, p_1, \dots, p_n)$ equipped with Γ-principal bundle P (our convention is that we have a right action of Γ on P), together with an isomorphism $\chi_* P \xrightarrow{\sim} \omega^{\log}_C \setminus 0$. We can think of the latter isomorphism as a morphism $\chi_P : P \to \omega^{\log}_C \setminus 0$ satisfying

$$\chi_P(x\gamma) = \chi(\gamma) \cdot \chi_P(x)$$

for $\gamma \in \Gamma$.

In addition to requiring the coarse moduli of C to be Deligne-Mumford stable, we require that for each marked point p_i the morphism $B\operatorname{Aut}(p_i) \to B\Gamma$ induced by P is representable. By looking at the corresponding embedding $\operatorname{Aut}(p_i) \simeq \mathbb{Z}/m_i \to \Gamma$ defined up to a conjugacy, we get a conjugacy class γ_i in Γ. Thus, we get a decomposition of our moduli stack into a disjoint union of open and closed substacks $\mathcal{S}_g(\gamma_1, \dots, \gamma_n)$. As in [4, Sec. 2.2], one shows that these are smooth and proper DM stacks with projective coarse moduli.

Let $\pi : \mathcal{C} \to \mathcal{S}_g(\gamma_1, \dots, \gamma_n)$ be the universal curve over $\mathcal{S}_g(\gamma_1, \dots, \gamma_n)$, and let $\mathcal{V} = \mathcal{P} \times_\Gamma V$ be the vector bundle over $\mathcal{C}$ associated with the universal Γ-spin structure $\mathcal{P}$ via the embedding $\Gamma \subset \operatorname{GL}(V)$. Note that $\mathcal{V}$ is equipped with a $\mathbb{G}_{m,R}$-action (through its action on V).

As in [7], we also consider a Galois covering $\mathcal{S}_g^{\mathrm{rig}}(\gamma_1, \dots, \gamma_n) \to \mathcal{S}_g(\gamma_1, \dots, \gamma_n)$ corresponding to choices of a rigidification at every marked point. A *rigidification* is an isomorphism of the restriction of P to $p_i/\operatorname{Aut}(p_i) \simeq B\langle\gamma_i\rangle$ with $\Gamma/\langle\gamma_i\rangle$ (viewed as a bundle over $B\langle\gamma_i\rangle$. There is a natural simply transitive action of the group $\prod_i C_G(\gamma_i)$ on the set of rigidifications at $p_1, \dots, p_n$, where $C_G(\gamma) \subset G$ is the centralizer of $\gamma \in G$.

3.2. **Construction.** Let us set for now $\mathcal{S} = \mathcal{S}_g^{\mathrm{rig}}(\gamma_1, \dots, \gamma_n)$ and consider the pull-back of all the objects to $\mathcal{S}$ (denoting them by the same symbols).

Note that we have a natural projection $V/\langle\gamma_i\rangle \to V^{\gamma_i}$. Thus, from rigidification structures we get morphisms

$$Z_i : p_i^*\mathcal{V} \to V^{\gamma_i} \otimes \mathcal{O}_\mathcal{S}. \tag{3.2.1}$$

Hence, by adjunction we can extend $\mathcal{V}$ to an object

$$[\mathcal{V}, \Sigma] := (\mathcal{V}, (V^{\gamma_i} \otimes \mathcal{O}_\mathcal{S}), (Z_i))$$

of $\operatorname{Coh}(\mathcal{C}, \Sigma)$.

On the other hand, we can combine χ_P with W into a polynomial morphism

$$W_\mathcal{V} : \mathcal{V} = \mathcal{P} \times_\Gamma V \to \omega_{\mathcal{C}/\mathcal{S}}^{\log} : (x, v) \mapsto W(v) \cdot \chi_P(x).$$

We can view it as a linear morphism of vector bundles on $\mathcal{C}$,

$$W_\mathcal{V} : S^\bullet(\mathcal{V})_d \to \omega_{\mathcal{C}/\mathcal{S}}^{\log},$$

where we grade the symmetric algebra of $\mathcal{V}$ using the $\mathbb{G}_{m,R}$-action on $\mathcal{V}$. Furthermore, this morphism is compatible with the morphisms (3.2.1), so that the following diagram is commutative

$$\begin{array}{ccc} p_i^* S^\bullet(\mathcal{V})_d & \xrightarrow{p_i^* W_\mathcal{V}} & p_i^*\omega_{\mathcal{C}/\mathcal{S}}^{\log} \\ {\scriptstyle S^\bullet(Z_i)}\downarrow & & \downarrow \\ S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_\mathcal{S} & \xrightarrow{W_i} & \mathcal{O}_\mathcal{S} \end{array}$$

where $W_i = W|_{V^{\gamma_i}}$. This means that we have a morphism

$$(W_{\mathcal{V}}, (W_i)) : S^\bullet[\mathcal{V}, \Sigma]_d \to [\omega^{\log}_{\mathcal{C}/\mathcal{S}}, \Sigma] \tag{3.2.2}$$

in the category $\mathrm{Qcoh}(\mathcal{C}, \Sigma)$ (where again we take the part of weight d with respect to $\mathbb{G}_{m,R}$). Next, we can take the derived push-forward with respect to the morphism of pairs (2.2.1). Together with (2.1.1) this gives us a morphism

$$S^\bullet(R\pi_*[\mathcal{V}, \Sigma])_d \to R\pi_* S^\bullet[\mathcal{V}, \Sigma]_d \to R\pi_*[\omega^{\log}_{\mathcal{C}/\mathcal{S}}, \Sigma] \tag{3.2.3}$$

in $D\,\mathrm{Qcoh}(\mathcal{S}, \mathcal{S})$.

Now let us set

$$E := Rj^! S^\bullet(R\pi_*[\mathcal{V}, \Sigma])_d.$$

Applying $Rj^!$ to morphism (3.2.3), we obtain a morphism

$$E = Rj^! S^\bullet(R\pi_*[\mathcal{V}, \Sigma])_d \to Rj^! R\pi_*[\omega^{\log}_{\mathcal{C}/\mathcal{S}}, \Sigma] \simeq R\pi_*\omega_{\mathcal{C}/\mathcal{S}},$$

where the last isomorphism is (2.2.2). It is easy to see that it fits into a morphism of exact triangles

$$\begin{array}{ccccccc} E & \longrightarrow & S^\bullet(R\pi_*(\mathcal{V}))_d & \longrightarrow & \oplus_{i=1}^r S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_{\mathcal{S}} & \longrightarrow & E[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle (W_i)} & & \downarrow \\ R\pi_*(\omega_{\mathcal{C}/\mathcal{S}}) & \longrightarrow & R\pi_*(\omega^{\log}_{\mathcal{C}/\mathcal{S}}) & \longrightarrow & \oplus_{i=1}^r \mathcal{O}_{\mathcal{S}} & \longrightarrow & R\pi_*(\omega_{\mathcal{C}/\mathcal{S}})[1] \end{array} \tag{3.2.4}$$

Combining it with the morphism of triangles (2.2.3), we get a commutative diagram with the exact triangle in the first row

$$\begin{array}{ccccc} S^\bullet(R\pi_*(\mathcal{V}))_d & \longrightarrow & \oplus_{i=1}^r S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_{\mathcal{S}} & \longrightarrow & E[1] \\ & & \downarrow{\scriptstyle \sum W_i} & & \downarrow{\scriptstyle \tau} \\ & & \mathcal{O}_{\mathcal{S}} & \xrightarrow{\mathrm{id}} & \mathcal{O}_{\mathcal{S}} \end{array}$$

Dualizing we get a commutative diagram

$$\begin{array}{ccccc} E^\vee[-1] & \longrightarrow & \oplus_{i=1}^r S^\bullet(V^{\gamma_i})_d^\vee \otimes \mathcal{O}_{\mathcal{S}} & \longrightarrow & S^\bullet(R\pi_*(\mathcal{V}))_d^\vee \\ \uparrow{\scriptstyle \tau^\vee} & & \uparrow{\scriptstyle \sum W_i} & & \\ \mathcal{O}_{\mathcal{S}} & \xrightarrow{\mathrm{id}} & \mathcal{O}_{\mathcal{S}} & & \end{array}$$

This implies that the pull-back $Z^*(\bigoplus_i W_i)$ with respect to the morphism

$$Z : [R\pi_*(\mathcal{V})] \to \prod_i V^{\gamma_i} \tag{3.2.5}$$

induced by (3.2.1), gives the zero morphism from the structure sheaf to itself in the derived category of quasicoherent sheaves on $[R\pi_*(\mathcal{V})]$.

In fact, we can realize this functon by an explicit coboundary. For this we need a realization of the above diagram in the homotopy category of complexes. As in [7, Sec. 4.2], the starting point is that $R\pi_*(\mathcal{V})$ can realized ($\mathbb{G}_{m,R}$-equivariantly) by a complex of the form $[A \to B]$ in such a way that the morphism (3.2.5) is realized by a surjective morphism $A \to \bigoplus_{i=1}^r V^{\gamma_i} \otimes \mathcal{O}_S$. Then the first line of the diagram (3.2.4) can be realized by a short exact sequence of complexes

$$0 \to \ker(S^\bullet(Z)_d) \to S^\bullet(A \to B)_d \xrightarrow{S^\bullet(Z)_d} \bigoplus_{i=1}^r S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_S \to 0$$

where the complex $S^\bullet(A \to B)_d$, concentrated in degrees $[0, \mathrm{rk}(B)]$, has form

$$S^\bullet(A)_d \to (S^\bullet(A) \otimes B)_d \to (S^\bullet(A) \otimes \wedge^2 B)_d \to \dots$$

Using this we get a canonical quasi-isomorphism of E with the bounded complex of vector bundles

$$K^\bullet := \mathrm{Cone}(S^\bullet(R\pi_*(\mathcal{V}))_d \to \oplus_{i=1}^r S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_{\mathcal{S}})[-1].$$

Now we want to realize the morphism $\tau : E \to \mathcal{O}_S[-1]$ in the derived category by a morphism $K^\bullet \to \mathcal{O}_S[-1]$ in the homotopy category of complexes.

By changing $[A \to B]$ to a quasi-isomorphic complex $[\overline{A} \to \overline{B}]$ one can achieve that for $i \geq 1$ the terms K^i satisfy $\mathrm{Ext}^{>0}(K^i, \mathcal{O}_S) = 0$ (see [7, Lem. 4.2.5]). This implies that morphisms $K \to \mathcal{O}_S[-1]$ in the homotopy category of complexes and in the derived category are the same.

The dual of this morphism can be interpreted as a canonical homotopy (up to a homotopy between homotopies) f_{-1} between the function $Z^*(\bigoplus_i W_i)$ on $[R\pi_*\mathcal{V}]$ and 0. As we have seen in Example 1.1.1, this corresponds to a structure $\delta = d - f_{-1} \cdot \mathrm{id}$ of a dg-matrix factorization of $-Z^*(\bigoplus_i W_i)$ on the structure sheaf of $[R\pi_*\mathcal{V}]$.

Furthermore, it carries an equivariant structure with respect to the action of the center $Z(\Gamma)$ of Γ (acting trivially on the base) and with respect to $\prod_i C_G(\gamma_i)$ (changing the rigidifications).

3.3. **Properties.** The first important property is that our dg-matrix factorization over $[R\pi_*\mathcal{V}]$ is supported on the zero section in $[R\pi_*\mathcal{V}]$. Indeed, first, we recall that any matrix factorization is supported on the critical locus of the potential. Since each W_i is non-degenerate, we get that the support belongs to the zero locus of $Z^*(\bigoplus_i W_i)$. Note also that the support can be calculated pointwise (see [7, Sec. 1.4]), so it is enough to deal with the case of a single curve with a Γ-spin structure. Thus, we are reduced to considering the following situation. Let C be a curve, and let $\mathcal{V}$ be a vector bundle over C. Assume also we have a polynomial morphism $W_{\mathcal{V}} : \mathcal{V} \to \omega_C$, such that over an open dense subset of C there exists a trivialization $\mathcal{V} \simeq V \otimes \mathcal{O}_C$ such that $W_{\mathcal{V}}$ is induced by our polynomial W on V. Then we have the induced polynomial function of degree -1 on the dg-affine space $[H^0(C, \mathcal{V}) \oplus H^1(C, \mathcal{V})[-1]]$ (recall that the base is now a point), induced by $W_{\mathcal{V}}$ and by the identification $H^1(C, \omega_C) \simeq \mathbb{C}$. We claim that it is supported at the origin. Indeed, we start by observing that the preimage of the origin under the gradient morphism $\Delta W : V \to V^\vee$ is still the origin (since W is non-degenerate). From this we get the similar assertion about the preimage of the zero section under the relative gradient morphism $\Delta W_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}^\vee \otimes \omega_C$. Finally, we note that the support of our function on $[H^0(C, \mathcal{V}) \oplus H^1(C, \mathcal{V})[-1]]$ coincides with the vanishing locus of the polynomial morphism

$$H^0(C, \mathcal{V}) \to H^0(\mathcal{V}^\vee \otimes \omega_C) \simeq H^1(C, \mathcal{V})^\vee$$

induced by the relative gradient map. This implies our claim.

Next, the key gluing property satisfied by the fundamental matrix factorizations (cf. [7, Sec. 5.2, 5.3]) holds in the situation when we consider two natural families of orbicurves $\widetilde{C} \xrightarrow{\widetilde{\pi}} S$, $C \xrightarrow{\pi} S$, over

$$S := S^{\mathrm{rig}}_{g_1}(\gamma_1, \dots, \gamma_{n_1}, \gamma) \times S^{\mathrm{rig}}_{g_2}(\gamma'_1, \dots, \gamma'_{n_2}, \gamma^{-1}),$$

where C is obtained by gluing two smooth points on $\widetilde{C}$ into a node. We denote by $f : \widetilde{C} \to C$ the gluing morphism.

In this setting there are natural Γ-spin structures $\widetilde{P}$ (resp., P) over $\widetilde{C}$ (resp., C), where P is obtained by gluing fibers of $\widetilde{P}$ over the two points that are glued into a node, using the rigidifications and the square root of J, $J^{1/2} \in \mathbb{G}_{m,R}$ such that $\chi(J^{1/2}) = -1$ (see [7, Sec. 5.2]). The main compatibility between the push-forwards of the corresponding vector bundles $\widetilde{\mathcal{V}}$ and $\mathcal{V}$ is given by the cartesian diagram

$$\begin{array}{ccc} [R\pi_*\mathcal{V}] & \longrightarrow & V^\gamma \\ \Big\downarrow & & \Big\downarrow \Delta^{J^{1/2}} \\ [R\widetilde{\pi}_*\widetilde{\mathcal{V}}] & \longrightarrow & V^\gamma \times V^{\gamma^{-1}} \end{array}$$

where $\Delta^{J^{1/2}} : V^\gamma \to V^\gamma \times V^{\gamma^{-1}}$ is the twisted diagonal map: $x \mapsto (x, J^{1/2}x)$. Furthermore, the analysis of [7, Sec. 5.2] shows that the natural dg-matrix factorization on $[R\pi_*\mathcal{V}]$ is identified with the pull-back of the one on $[R\widetilde{\pi}_*\widetilde{\mathcal{V}}]$.

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