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Review

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Author for correspondence:
Fabio Semperlotti
e-mail: fsemperl@purdue.edu

Applications of variable-order fractional operators: a review

Sansit Patnaik, John P. Hollkamp and Fabio Semperlotti

School of Mechanical Engineering, Ray W. Herrick Laboratories, Purdue University, West Lafayette, IN 47907, USA

SP, 0000-0001-8975-130X; JPH, 0000-0001-8133-7083; FS, 0000-0002-8143-3902

Variable-order fractional operators were conceived and mathematically formalized only in recent years. The possibility of formulating evolutionary governing equations has led to the successful application of these operators to the modelling of complex real-world problems ranging from mechanics, to transport processes, to control theory, to biology. Variable-order fractional calculus (VO-FC) is a relatively less known branch of calculus that offers remarkable opportunities to simulate interdisciplinary processes. Recognizing this untapped potential, the scientific community has been intensively exploring applications of VO-FC to the modelling of engineering and physical systems. This review is intended to serve as a starting point for the reader interested in approaching this fascinating field. We provide a concise and comprehensive summary of the progress made in the development of VO-FC analytical and computational methods with application to the simulation of complex physical systems. More specifically, following a short introduction of the fundamental mathematical concepts, we present the topic of VO-FC from the point of view of practical applications in the context of scientific modelling.

1. Introduction

Fractional calculus refers to the study of differential and integral operators of either real or complex order. The first documented discussion on fractional-order differentials are letters exchanged between Leibniz and de l'Hôpital in 1695, in which they discussed the meaning and the interpretation of $\mathrm{d}^{\alpha}f(t)/\mathrm{d}t^{\alpha}$ when α is a non-integer. Many great mathematicians later contributed

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to the development of fractional calculus, including Liouville, Riemann, Abel, Riesz, Weyl and Caputo.

Although the branch of fractional calculus started almost simultaneously to its integer-order counterpart, the mathematics and especially its applications are considerably less developed. Many factors have contributed to this result, including the lack of methodologies to link both the geometrical and physical properties of a system to the corresponding order of the fractional operator. An intriguing aspect of these operators is their intrinsic multiscale nature. As a result, time-fractional operators enable memory effects (i.e. the response of a system is a function of its past history) while space-fractional operators enable non-local and scale effects.

The many characteristics of fractional operators have sparked, in recent years, much interest in fractional calculus and produced a plethora of applications with particular attention to the simulation of physical problems. Areas that have seen the largest number of applications include the formulation of constitutive equations for viscoelastic materials [1–4], transport processes in complex media [4–11], mechanics [12–15], non-local elasticity [16–19], plasticity [20–22], model-order reduction of lumped parameter systems [23] and biomedical engineering [24–26]. These studies have typically used constant-order (CO) fractional operators. Machado *et al*. [27] provided a historical perspective on the major developments in fractional calculus since the 1970s. Detailed reviews covering the fundamentals of CO fractional calculus (CO-FC) can be found in [27,28].

Although the CO-FC formalism is capable of addressing some very relevant physical problems, it cannot capture important classes of physical phenomena where the order itself is a function of either dependent or independent variables. For example, the reaction kinetics of proteins has been found to exhibit relaxation mechanisms that are properly described by a temperature-dependent fractional order [29]. Thus, the underlying physics of the reaction kinetics (captured by the order of the relaxation mechanism) changes with temperature. Hence, it is reasonable to think that a differential equation with operators that update their order as a function of temperature will better describe the protein kinetics. This simple example suggests that there exist classes of physical problems that would be better described by variable-order (VO) operators.

VO operators can be seen as a natural analytical extension of CO operators. In VO operators, the order can vary continuously as a function of either dependent or independent variables of integration (or differentiation), such as time, space or even of an independent external variable (e.g. temperature or applied loads). Although the extension from CO operators to VO operators may seem somewhat natural, the first definitions of these operators were given by Samko & Ross [30] in 1993. Recently, Lorenzo & Hartley [31,32] and Coimbra [33] put the mathematics of VO fractional calculus (VO-FC) in perspective by discussing possible applications of VO-FC in mechanics. These works marked the starting point for applications of VO operators to the analysis of different complex physical problems. The first few examples focused on modelling anomalous diffusion in complex structures [34,35] and mechanics [33], hence extending much of the previous work based on CO operators. All the studies on VO-FC recognized and took advantage of the potential of VO operators, and how they could be used to describe more accurately behaviour of systems with time and spatially varying properties. During the last decade, VO-FC has seen a surge in interest and in the number of applications to the modelling of scientific and engineering systems. The most direct result was a drastic increase in the number of technical publications, as summarized in figure 1. During the last decade, an average of 12 papers per year were published. This measure is comparable to the one for FC since 1974, as reported in [27].

The main goal of this paper is to provide a detailed review, to date, of the available applications of VO-FC in the general area of scientific and engineering modelling. While a few excellent review papers on VO-FC have already been presented by other authors [36–39], these works were mostly tailored towards the mathematics community and focused on the main mathematical properties of both VO operators and VO equations. On the contrary, this paper will present a perspective of VO-FC from the point of view of applications to scientific modelling; this paper is aimed at the broader scientific and engineering community with the intention of attracting the attention of this community towards this specific area of calculus. It should be pointed out

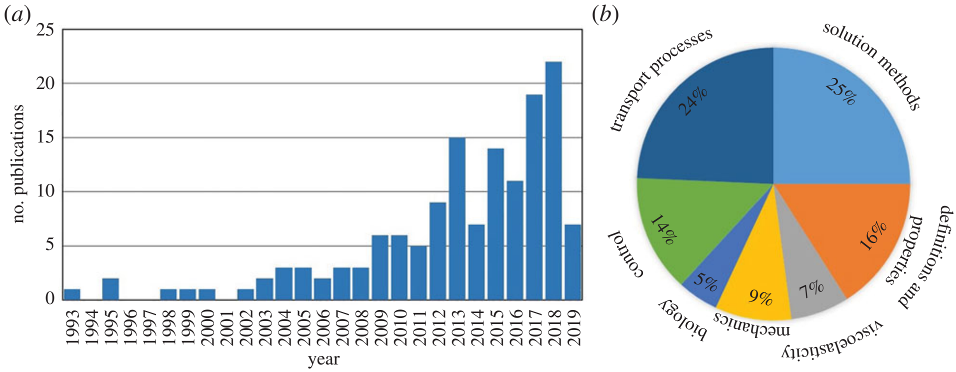

Figure 1. Overview of the historical development of the VO-FC field presented in terms of the number of publications. (*a*) Bar chart showing the number of publications per year since 1993. (*b*) Percentage distribution of VO-FC in different fields of application. (Online version in colour.)

that, while [39] did include a section of applications of VO-FC, the chief focus of the survey was still on presenting the relevant aspects of the mathematics and of the corresponding numerical methods. On the contrary, the present review will focus on the physical relevance and the practical applications of VO-FC, including additional fields not fully examined in [39]. Engineers, physicists, biologists and financial analysts are only some of the communities that will find several points of interest and material for further considerations in this work. More specifically, this review will serve a twofold objective: (i) it will help to disseminate to the broader scientific community a relatively less known mathematical tool that offers remarkable opportunities for interdisciplinary applications and (ii) it will provide a concise and comprehensive summary of VO-FC for modelling and simulation, hence serving as a starting point for the reader interested in approaching this fascinating field.

The remainder of this paper is structured as follows: the opening section will provide a description of the basic definitions and properties of VO operators. The main intent is to define the basic nomenclature and mathematics of VO-FC. Then, a brief discussion on the relevance and needs of VO operators for scientific modelling will be presented. The following sections of the paper will present a summary of the many applications of VO-FC, starting with applications to mechanics. The review will continue by considering applications of VO-FC to viscoelasticity, which is arguably the engineering field where fractional calculus has garnered the most attention. Then, we discuss applications to the modelling of different transport processes and to control theory. Finally, we conclude our review with a section on miscellaneous applications.

2. Review of fundamental concepts

We briefly review the main properties required for a proper definition of a fractional differ-integral derivative. In the following, $f(t)$ is a continuous function of the generalized variable t on the interval $[a, t]$. Ross [40] established a minimal set of properties to define a differ-integral operator ${}_aD_t^{\alpha}(\cdot)$ of order α on the interval $[a, t]$ as a fractional derivative. These are:

(i) *Analyticity*: ${}_aD_z^{\alpha}f(z)$ is an analytic function of α and z if $f(z)$ is analytic.
(ii) *Identity*: zero-order operation of a function returns the function itself, i.e. ${}_aD_t^{0}f(t) = f(t)$.
(iii) *Linearity*: the operator must be linear, i.e. ${}_aD_t^{\alpha}(Af(t) + Bg(t)) = A_aD_t^{\alpha}f(t) + B_aD_t^{\alpha}g(t)$.
(iv) *Backward compatibility*: when the order of the operator is an integer, the fractional operator must return the same result as the corresponding integer-order operator.
(v) *Law of exponents*: index law must be satisfied, i.e. ${}_aD_t^{\alpha}\,{}_aD_t^{\beta}f(t) = {}_aD_t^{\alpha+\beta}f(t)$. This leads to the existence of the left inverse of the operator, i.e. ${}_aD_t^{\alpha}\,{}_aD_t^{-\alpha}f(t) = f(t)$.

Note that the operator ${}_aD_t^{\alpha}(\cdot)$ is a fractional integral operator when $\alpha < 0$ and a fractional differential operator otherwise. Analyticity is not considered by most authors [31,38,41] as the majority of the formulations are based on real functions and real orders. As discussed in [42], CO fractional derivatives do not satisfy the classical Leibniz rule, hence a generalized Leibniz rule is derived in [38,41–43]. A fractional operator is required to satisfy such a generalized rule in order to qualify as a fractional derivative. We emphasize that several authors [31,38,40–44] have discussed the fundamental set of properties that a fractional operator should satisfy in order to be defined as a fractional derivative. As discussed in [41], this search for a criterion, especially the satisfaction of the law of exponents, is 'philosophically controversial' and amounts to a discussion on whether these properties should be defined for the operator or for the functions which are operated upon.

The above five properties were initially formulated for CO operators and were later evaluated for VO operators in [30,38]. Depending on the definitions (discussed in detail in §2a), different VO operators violate different above-stated properties. In this review, we refer to VO differ-integral operators as VO differential (or integral) operators, and not as VO derivatives (or integrals).

(a) Definitions of variable-order differ-integral operators

In this section, we review the definitions for VO operators proposed to date and, for the sake of brevity, we present only their left-handed version (the right-handed operators are a straightforward extension of the left-handed version). The notation ${}_a^{\square}D_t^{\alpha(t)}(\cdot)$ indicates an operator having order $\alpha(t)$ and operating on the interval $[a,t]$. Further $\alpha(t)$ is a continuous function on the interval $[a,t]$. In 1993, Samko & Ross [30] first proposed the differentiation of functions to a VO using two separate approaches: (i) a direct approach and (ii) a Fourier transform-based approach. The key difference between the two methods lies in the ability to satisfy the law of exponents. In the direct approach, Samko & Ross [30] extended the CO Riemann–Liouville (CO-RL) integration to VO as

$${}_aI_t^{\alpha(t)}f(t) = \frac{1}{\Gamma[\alpha(t)]}\int_a^t (t-\tau)^{\alpha(t)-1}f(\tau)\,\mathrm{d}\tau, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function and τ is a dummy variable. The VO integral was then used to define the VO Riemann–Liouville (VO-RL) differential operator of VO $\alpha(t) \in (0,1)$ as

$${}_a^{RL}D_t^{\alpha(t)}f(t) = \frac{1}{\Gamma[1-\alpha(t)]}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\tau)^{-\alpha(t)}f(\tau)\,\mathrm{d}\tau. \tag{2.2}$$

One of the most important findings in [30,44] was the proof of the violation of the law of exponents by the VO operators defined above, i.e. ${}_aI_t^{\alpha(t)}\,{}_aI_t^{\beta(t)}f(t) \neq {}_aI_t^{\alpha(t)+\beta(t)}f(t)$. This leads to the fact that the VO-RL differential and integral operators are not inverse to each other, which is in contrast to the case of CO operators. Further it was shown in [30] that the symmetry on power functions still holds for the VO integral operator, but it is violated for the VO-RL differential operator.

In the Fourier transforms approach [30], the VO integral operator is defined as

$$I^{\alpha(t)}f(t) = F^{-1}\frac{1}{(-\mathrm{i}\tau)^{\alpha(\tau)}}F[f(t)], \tag{2.3}$$

where F is the Fourier transform operator and F^{-1} is its inverse. The VO integral operator in equation (2.3) satisfies the law of exponents. These first steps to the definition of VO differ-integral operators, as Samko [44] notes, 'was out of mathematical curiosity as well as the fact that the spaces of functions with variable smoothness, for example, the spaces of $L_p^{\alpha(x)}$-type, can be characterized using these VO operators'.

In 1998, Hartley & Lorenzo [31] first presented the physical motivation towards VO operators. They presented the following definition for the initialized VO integral operator on the interval

$[c, t]$:

$$ {}_{c}I_t^{\alpha(t)} f(t) = \int_c^t \frac{(t-\tau)^{\alpha(t,\tau)-1}}{\Gamma[\alpha(t,\tau)]} f(\tau)\,\mathrm{d}\tau + \int_a^c \frac{(t-\tau)^{\alpha(t,\tau)-1}}{\Gamma[\alpha(t,\tau)]} f(\tau)\,\mathrm{d}\tau, \tag{2.4}$$

where the second integral is the initialization function denoted as $\psi(f, \alpha(t,\tau), a, c, t)$. The VO operator is initialized at $a \leq c \leq t$ such that $f(t) = 0 \ \forall \ t \leq a$. The most general case for the VO is $\alpha(t,\tau) \triangleq \alpha(At + B\tau)$ [32]. In [32], three cases are discussed—case 1: $\alpha(t,\tau) \triangleq \alpha(t)$, case 2: $\alpha(t,\tau) \triangleq \alpha(\tau)$, and case 3: $\alpha(t,\tau) \triangleq \alpha(t-\tau)$. Properties such as linearity, time invariance, memory, Laplace transforms and physical realization using switches were presented for these cases. Using the VO integral operator, the VO-RL differential operator is defined in [31,32] as

$$ {}_{c}^{RL}D_t^{\alpha(t)} f(t) = {}_{c}^{RL}D_t^{\lceil \alpha(t) \rceil}\, {}_{c}I_t^{\lceil \alpha(t) \rceil - \alpha(t)} f(t), \tag{2.5}$$

where $\lceil \alpha(t) \rceil$ is the upper integer bound on $\alpha(t)$. In general, the use of RL operators in fractional differential equations (FDEs) requires fractional-order boundary conditions whose physical interpretation is more elusive than their integer-order counterparts [28]. The function $\psi(t)$ shifts (from c to a) the initial time instant of the interval over which the fractional operator is defined. As $f(t) = 0 \ \forall \ t \leq a \leq c$, all the fractional-order derivatives of $f(t)$ at a are zero. Thus, we do not require fractional boundary conditions while solving FDEs with the initialized RL operator. Details on the initialization procedure and its importance in RL operators can be found in [31,45].

In 2003, Coimbra [33] proposed the following VO Caputo differential operator for $0 < \alpha(t) \leq 1$:

$$ {}_{a}^{C}D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma[1-\alpha(t)]} \int_a^t (t-\tau)^{-\alpha(t)} D^{(1)} f(\tau)\,\mathrm{d}\tau + \frac{(f(a^+) - f(a^-))t^{-\alpha(t)}}{\Gamma[1-\alpha(t)]}, \tag{2.6}$$

where $D^{(k)}$ is the k^{th} integer-order derivative and $f(a^+)$ and $f(a^-)$ are the right-hand and left-hand limits of $f(t)$ at $t = a$, respectively. The non-differ-integral term on the r.h.s. accounts for a discontinuous function behaviour at $t = a$. Later, VO definitions for Caputo operators based on the general-order variation $\alpha(t,\tau)$ were introduced in [36,46–48], where the VO $\alpha(t)$ in equation (2.6) is replaced by $\alpha(t,\tau)$. In [49], the relations between the CO-RL and Caputo derivatives have also been used to propose variations to the definition in equation (2.6) as well as to propose new definitions.

Several authors [36,47,48,50–55] have defined the VO Grünwald–Letnikov (VO-GL) operator corresponding to the general-order variation $\alpha(t,\tau)$ as

$$ {}_{a}^{GL}D_t^{\alpha(t)} f(t) = \lim_{h \to 0} \sum_{r=0}^{n} \frac{(-1)^r}{h^{\alpha(t,rh)}} \binom{\alpha(t,rh)}{r} f(t-rh), \tag{2.7}$$

where $h > 0$ is a time step and $n = \lfloor (t-a)/h \rfloor$ is the lower integer bound on $(t-a)/h$. Further, a recursive relation has been used in [51–53,56] to define the following type of VO-GL operator:

$$ {}_{a}^{GL}D_t^{\alpha(t)} f(t) = \lim_{h \to 0} \left[\frac{f(t)}{h^{\alpha(t)}} - \sum_{r=0}^{n} (-1)^r \binom{-\alpha(t)}{r} {}_{a}^{GL}D_{t-rh}^{\alpha(t)} f(t) \right]. \tag{2.8}$$

The above recursive definition for the VO-GL operator results in a difference equation and its equivalent matrix form has been derived in [51–53]. Further, it is shown in [38] that setting the upper limit of the summation in equation (2.7) to ∞ preserves an important property of the CO Grünwald–Letnikov derivative, that is,

$$ {}_{a}^{GL}D_t^{\alpha(t)}\, \mathrm{e}^{st} = s^{\alpha(t)}\, \mathrm{e}^{st} \ \forall\ \Re(s) > 0. \tag{2.9}$$

This is used in [38] to analyse VO linear systems and derive corresponding Mittag–Leffler functions.

Note that the above definitions for VO fractional operators can be readily extended to VO partial differential operators. Several works reviewed in the following sections make use of VO partial differential operators to model physical processes in a multi-dimensional space. Hence, for the sake of brevity, we do not provide their definitions here. We emphasize that the above-reviewed VO operators fail to satisfy the law of exponents property for the most general

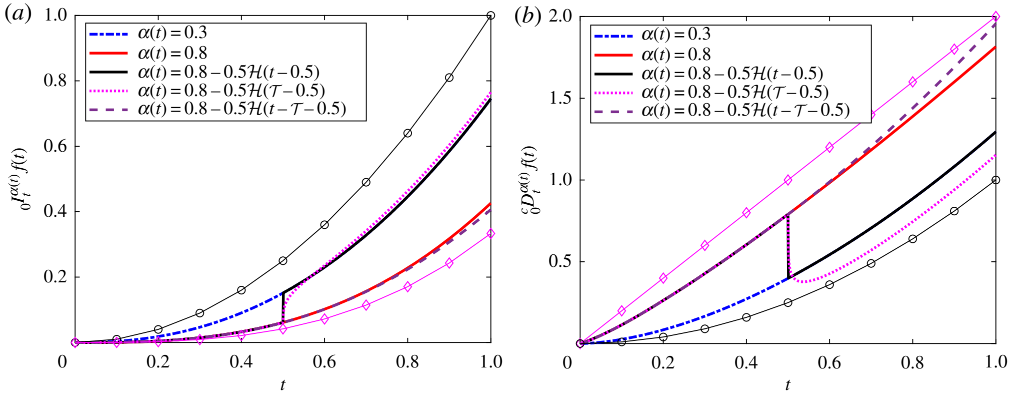

Figure 2. Variable-order (*a*) integration and (*b*) differentiation of $f(t)=t^2$. We have also plotted $f(t)$ (indicated by ○), and (*a*) $I^1f(t)$ and (*b*) $D^1f(t)$ (indicated by ◇) for reference. $\mathcal{H}(\cdot)$ denotes the Heaviside operator. Note that the VO $\alpha(t,\tau)\triangleq\alpha(t)$ has no order memory, while $\alpha(t,\tau)\triangleq\alpha(t-\tau)$ has a strong memory and $\alpha(t,\tau)\triangleq\alpha(\tau)$ has a weak memory of the order history. (Online version in colour.)

definitions of $f(t)$ and $\alpha(t)$, except for the Fourier transform-based approach given in [30,44]. A critical analysis of VO operators based on their dimensional inconsistency is presented in [57].

(b) Memory characteristics of variable-order operators

It is well known that the integral nature of the fractional operators is associated with the ability to account for the history (i.e. not only the instantaneous value) of the dependent variable. When time is chosen as the independent variable, this property is referred to as memory. When space is chosen as the independent variable, non-local properties (effectively a spatial memory) are obtained [28]. This memory is associated with both the CO and VO operators, and is sometimes called the fading memory [32]. Additionally, VO operators also enable memory of their order, which is sometimes called the order memory [32,58]. Lorenzo & Hartley [32] defined measures for the fading memory ($m_1(t)$) and order memory ($m_2(t)$) of the operator ${}_0I_t^{\alpha(t)}(\cdot)$ as

$$m_1(t)\triangleq\frac{1}{t}\int_0^t\frac{(t-\tau)^{\alpha(t,\tau)-1}}{\Gamma[\alpha(t,\tau)]}\,d\tau,\qquad m_2(t)\triangleq\frac{\int_{\tau_1}^{\tau_1}(((t-\tau)^{\alpha_0-1})/(\Gamma[\alpha_0]))\,d\tau}{\int_0^t(((t-\tau)^{\alpha(t,\tau)-1})/(\Gamma[\alpha(t,\tau)]))\,d\tau},\tag{2.10}$$

where a non-constant step-function-type order variation $\alpha(t)$ is assumed in the definition of $m_2(t)$. $m_2(t)$ measures the memory retentiveness of the order α_0 by the VO operator, and τ_1 and τ_2 are the lower and upper bounds of the interval for which the VO $\alpha(t)=\alpha_0$. It is shown in [32,58] that the response rate of the VO operator to changes in order is inversely related to its order memory. Evaluation of $m_2(t)$ for the three possible cases of order variation show that: case 1: $\alpha(t,\tau)\triangleq\alpha(t)$ has no memory of its past order, case 2: $\alpha(t,\tau)\triangleq\alpha(\tau)$ has a weak memory of its past order, and case 3: $\alpha(t,\tau)\triangleq\alpha(t-\tau)$ strongly remembers its order history [32,58]. Figure 2 depicts this behaviour of different VO operators for a step-function-type order variation.

(c) Solution methods for variable-order fractional differential equations

The theory of generalized VO operators and the corresponding VO equations has been addressed in [59–61]. It is shown in [62] that the solutions to VO fractional differential equations (VO-FDEs) are defined in fractional Besov spaces of VO on $\mathbb{R}^n$. Since the kernel of VO operators has a variable exponent, it is difficult to obtain closed-form solutions to VO-FDEs. Various authors have attempted to prove the existence and uniqueness of the solutions to VO-FDEs. Malesza *et al.* [63] presented an approach based on switching schemes that realize different types of VO operators to obtain closed-form solutions to specific VO-FDEs. The existence and uniqueness of the solution of a generalized VO-FDE has been addressed in [64–67] using standard techniques in analysis and, particularly, the Arzela–Ascoli theorem. Since closed-form solutions to VO-FDEs are even more difficult to obtain in real applications, numerical solutions have become the key to solving VO-FDEs. Alikhanov [68] obtained *a priori* estimates for the solutions of boundary value problems using the method of energy inequalities. Many of the numerical methods developed for CO-FC have been extended to VO-FC including finite difference (FD), spectral and meshless methods.

Finite difference methods have been widely used to approximate CO operators and find solutions to CO fractional differential equations (CO-FDEs), and hence are fairly well developed. A detailed review of numerical methods for CO-FC can be found in [69]. Here, we discuss the numerical schemes developed for solving VO-FDEs. We start by first reviewing first-order accurate FD schemes, and then move to higher-order accurate FD schemes as well as meshless, and spectral methods, before finally reviewing perturbation solutions to VO-FDEs. We emphasize that, in all the following sections, we have used $\mathbf{x}$ and t to denote the space and time variables, respectively, for a given process.

In 2009, Lin *et al*. [70] presented a conditionally stable explicit FD scheme for a one-dimensional VO diffusion equation. In the same year, Zhuang *et al*. [71] constructed both a conditionally stable explicit scheme and an unconditionally stable implicit scheme for a VO fractional advection–diffusion equation (VO-FADE). Both these works used space-fractional Riesz operators with VO $\alpha(t,x) \in (1,2]$ and approximated them using shifted GL expressions. Following these works, several researchers developed explicit, implicit and Crank–Nicolson schemes to simulate a variety of VO-FDEs in [72–77], and analysed the stability and convergence of the methods. Chen *et al*. [78] constructed a stable alternating directions implicit scheme for the two-dimensional VO percolation equation. The Adams–Bashforth–Moulton predictor–corrector (ABM-PC) scheme was used in [79,80] to simulate VO-FDEs with time delays. All the above-discussed schemes are convergent to first order, i.e. $O(\Delta t + \Delta x)$.

Chen *et al*. [81] first proposed a highly accurate unconditionally stable FD scheme based on GL expansions, which is convergent to $O(\Delta t^2 + \Delta x^4)$, for the solution of VO sub-diffusion equations (VO-SDEs). Following the work in [81], several researchers [82–84] have proposed second-order spatially accurate schemes to study transport processes. Sun *et al*. [85] constructed explicit, implicit and Crank–Nicolson schemes for VO-SDEs, which are convergent to $O(\Delta t + \Delta x^2)$, and established the conditions of convergence and stability. An implicit scheme was presented in [86] to solve two-dimensional VO-SDEs, convergent to $O(\Delta t + \Delta x^2 + \Delta y^2)$. Cao & Qiu [87] used shifted GL approximations to construct a second-order accurate scheme for a VO-FDE of the type ${}^{RL}_{-\infty}D_t^{\alpha(t)}y(t) = f(t)$. Ma *et al*. [88] proposed a second-order accurate ABM-PC method to investigate the stable and unstable equilibrium points of a three-dimensional VO financial system.

Second-order accurate schemes have been developed in [89] to study VO anomalous diffusion and wave propagation. The scheme presented in [89] used Lagrange second-degree interpolation polynomials for VO operators with order $\alpha(t) \in (0,1]$ and cubic interpolation polynomials in Hermite form for VO operators with $\alpha(t) \in (1,2]$. Moghaddam & Machado [90] developed an explicit B-spline approximation for VO Caputo operators to obtain numerical solutions for nonlinear time-fractional VO-FDEs. The scheme presented in [90] is convergent with $O(\Delta t^2 + \Delta x^2 + ((\Delta t)/(\Delta x))^2)$. Lagrange polynomial interpolation-based approximations were also used in [91], while the Adam's method along with standard FD schemes were used in [92] for solving VO-FDEs.

Highly accurate FD schemes based on spline interpolations were proposed in [90,93]. A piecewise integro-quadratic cubic spline interpolation was used in [90], with convergence at least to $O(\Delta t^4)$, to simulate the VO Bagley–Torvik and Basset equations. An explicit predictor–corrector scheme based on cubic spline interpolation, with convergence to $O(\Delta t^4)$, was developed in [93] to study VO-FDEs with delays. Further, highly accurate meshless methods based on Bernstein polynomials were constructed in [94,95] to solve VO-FDEs. Operational matrices were derived for the VO operators and used to transform the VO-FDEs into a system of algebraic equations. Similarly, a non-standard FD scheme was developed in [96] to solve VO optimal control problems by reducing the VO-FDEs into a set of linear equations. A meshless method based on a moving least squares algorithm was developed in [97] to solve a two-dimensional VO-FADE, while the method of approximate particular solutions was used in [98] to solve VO diffusion equations.

Spectral methods have also been applied to VO-FDEs which have smooth exact solutions and the classical Jacobi polynomials (typically Legendre or Chebyshev polynomials) were used as approximation bases [99–107]. Weighted Jacobi polynomials of the form $(1 \pm x)^{\mu} P_j^{a,b}(x)$ and $(1+x)^{\mu_1}(1-x)^{\mu_2} P_j^{a,b}(x)$ with $(a, b, \mu, \mu_1, \mu_2 > -1)$ were used to construct collocation methods to solve VO boundary value problems with endpoint singularities in [108–110]. It is shown that the singular basis greatly enhances the accuracy of the numerical solution through proper tuning of the parameter μ. In a series of works, Bhrawy *et al.* [111–116] used spectral collocation methods based on shifted Jacobi polynomials to study a variety of VO-FDEs. Numerical VO differentiation of noisy signals by wavelet denoising was presented in [117]. Recently, Chebyshev wavelets were used in [118,119] to solve VO-FDEs. The VO equation was reduced to a set of algebraic equations using collocation methods. The proposed scheme had the advantage of working with different kinds of initial and Dirichlet boundary conditions. All these works showed that spectral methods are highly accurate when compared with standard FD schemes.

Very recently, Patnaik & Semperlotti [120] proposed a perturbation solution to study the dynamics of a nonlinear oscillator system with VO damping. Using the method of averaging, the expressions for the amplitude and phase of the oscillator were derived, and a close match between the derived perturbation solutions and the numerical solution of the oscillator equation was demonstrated.

3. Relevance and need of variable-order operators

Many physical processes can often be mathematically described by differential or integral models. The order of the model is typically indicative of the underlying physics dominating the process. As an example, it was mentioned in the introduction that, in the reaction kinetics of proteins, the order is indicative of the nature of the relaxation mechanism [29]. Similarly, different types of frictional damping are modelled using different-order derivatives of displacement with respect to time. For example, viscoelastic damping is typically accounted for by using a half-order derivative in time, while viscous damping is modelled using a first-order derivative. In real-world applications, such as in shock absorbers for automotive systems or in dampers used in structural elements for seismic energy dissipation [121,122], the nature of the damping may depend on the instantaneous position of the damper or on other variables characteristic of the process, such as temperature. When this situation occurs, the underlying physical process and, subsequently, the order of the differential model changes as a function of either external or internal variables. This is precisely where the use of VO-FC can be advantageous.

Other examples of such an evolutionary nature of certain physical problems include the transition in the behaviour of materials from a linear to nonlinear response [123–126], the transition from sub-diffusive to super-diffusive flows [34,127,128], and the development of VO controllers and filters in control systems [129–133]. Currently available modelling approaches to these problems typically rely on nonlinear CO differential equations (CO-DEs) whose coefficients are functions of the process variables. Although CO models are invaluable tools for the analysis of complex engineering systems, they are unable to evolve between different governing equations and their corresponding physical behaviours. The modelling of these transitions using CO derivatives would require a continuous update of the underlying governing equations. Further, the ability of CO-DEs to account for nonlinearities must be integrated in the model *a priori*, often requiring somewhat arbitrary assumptions on what elements will experience nonlinear behaviour. Several applications in biology, field transport theory and other complex dynamical processes support this observation since their responses and behaviour are not well captured by CO (either integer or fractional) models. Examples include, but are not limited to, the modelling of changing climate patterns; ocean surface temperatures and salinity, which affect marine life and economies; the spread of pollutants and disease-causing pathogens through air; and the growth of bone tumours and cancerous tissues [134–143].

The definitions of VO operators and their ability to update the system's order depending on its instantaneous or, even, its historical response allow VO-FC-based models to describe widely dissimilar dynamics without the need for changing the underlying governing equations. This remarkable property of VO-FC has led to the development of evolutionary models capable of describing numerous complex physical processes. Most of the work to date has concentrated on mechanics, viscoelasticity, transport processes and control, as well as the biological interactions seen in nature. In the following, we report the different applications of VO-FC to the simulation of several physical processes in the above-mentioned fields.

4. Application of VO-FC to mechanics

As discussed above, real-world applications in mechanics involve transitions across widely dissimilar nonlinear physical phenomena. These transitions have been modelled by exploiting the evolutionary nature of VO-FC-based physical models, thus leading to several applications of VO-FC in mechanics and, particularly, nonlinear dynamics. The theoretical foundation was laid by Atanackovic & Pilipović [144], who derived FDEs from a variational dynamics perspective. A generalized Hamilton's principle was formulated by introducing VO operators into the Lagrangian. The functional was minimized with respect to the generalized coordinates as well as the order of the VO operators. The minimization of the Lagrangian-based functional is

$$I = \int_0^T L(t, y(t), D^{\alpha(t)} y(t), \alpha(t))\, \mathrm{d}t, \tag{4.1}$$

where L is the Lagrangian. Equation (4.1) was applied to the following four cases:

(i) The function for the order $\alpha(t)$ is known.
(ii) The function for the order $\alpha(t)$ is unknown, but is constant.
(iii) The function for the VO $\alpha(t)$ is unknown.
(iv) The function for the order $\alpha(t)$ is given in terms of an additional differential equation.

Minimization of the functional in case (i) yielded a generalization of the Euler–Lagrange equations for VO, and in cases (ii) and (iii) led to the determination of the order of the FDEs describing various processes. Lastly, case (iv) was solved in the form of an optimization problem where the VO $\alpha(t)$ was treated as an internal variable of the system and the minimization of the Lagrangian was achieved using Lagrange multipliers imposing a differential-type constraint for the order. The concept of the order being an internal system variable is particularly striking because the order can now evolve according to an appropriate physical law (in [144] that law being the differential equation for the order) driving the system response through widely dissimilar dynamics. We now review various applications where this evolution in the system order has been exploited to understand and model complex time-varying problems in mechanics.

Experiments have shown that the nature of the viscous drag force experienced by a particle due to the oscillatory flow of a viscous fluid is dependent on the value of the Reynolds number (*Re*) and the Reynolds–Strouhal number (*SlRe*). The motion of the particle in the viscous flow in the limit of infinitesimal *Re* is given by Tchen's equation,

$$m_p \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = (m_p - m_f)\mathbf{g} - 6\pi\mu a \mathbf{V} - \frac{m_f}{2}\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} - \frac{6\pi\mu a^2}{(\pi\nu)^{(1/2)}} \int_{-\infty}^{t} (t-\tau)^{-(1/2)} \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau}, \tag{4.2}$$

where $\mathbf{V}$, m_p and a are the velocity, mass and effective radius of the particle respectively. Also, ν is the kinematic viscosity of the fluid, and $m_f = ((4\pi a^3)/3)\rho$, where ρ is the fluid density. In equation (4.2) the second term on the right-hand side is the quasi-steady linear Stokes drag, and the fourth term is the history of the drag force which takes into account the weighted history of the local acceleration acting on the particle [145]. It is observed that, for particles at small but finite

Re, the decay of the history force changes to t^{-2} at long times. This evolution in the order of decay of the drag force has been modelled using VO-FC in [145,146]. In [146], the two drag force terms in equation (4.2) have been combined and the drag force on the particle is modelled using a VO Caputo operator (equation (2.6)) as

$$f_D = F(SlRe, Re)\, {}_0^C D_t^{\alpha(SlRe,Re,|\mathbf{w}|)} \mathbf{w}, \tag{4.3}$$

where F is a dimensionless function, $0 < \alpha < 1$ is the VO and $\mathbf{w}$ is the displacement field of the particle. Ramirez & Coimbra [145] isolated the order behaviour of the drag history not by combining the viscous drag terms of equation (4.2) as in [146], but instead by modelling the drag history term as

$$f_h = F(Re(t), Re_\tau, \beta) 6\pi \mu a \left(\frac{a^2}{\nu}\right)^{\alpha(Re(t),Re_\tau)} {}_0^C D_t^{\alpha(Re(t),Re_\tau)} \mathbf{V}, \tag{4.4}$$

where Re_τ is the Reynolds number based on the terminal velocity and β is the ratio of particle-to-fluid density. The values of F and α in equations (4.3) and (4.4) were obtained using a least squares method against numerically obtained data of flow past a sphere. As shown in [145,146], the VO drag force models could accurately and robustly reflect complicated fluid flow, particularly for several ranges of Re when the local perturbations produced an asymmetric flow field where memory of the wake influenced the flow. It was established that the VO captures the transition of the order of the decay of drag history over the entire time of the motion of the particle. Thus VO-FC models are an effective way to examine the effects of the wake on the drag forces acting on the particle.

The dynamics of complex non-local media evolve across viscoelastic and viscoinertial regimes depending on the forcing frequencies. In [147], Orosco & Coimbra developed a framework to model these complex transitions in the spectral dynamics of non-local media. First, a Laplace transform was used to convert a CO-DE to either the frequency or wavenumber domain depending on whether the independent variable is time or space, respectively. This generalized CO-DE is given as

$$\sum_{n=0}^{N} \gamma_n D^{\mu_n} x(t) = F(t), \tag{4.5}$$

where F is the forcing, γ_n is a generalized coefficient and μ_n is the CO. Note that the case $N = 2$, where $\mu_n \in \mathbb{Z}$, corresponds to the classical mass–spring–damper problem where γ_0 and γ_1 are the elastic and damping coefficients respectively, and γ_2 is the mass. The region where $\mu \in (0, 1)$ is viscoelastic, while $\mu \in (1, 2)$ is viscoinertial. The transformed version of equation (4.5) is

$$\frac{X(s)}{F(s)} = G^{-1}(s) = \sum_{n=0}^{N} \gamma_n s^{\mu_n}, \tag{4.6}$$

where $G(s)$ is the transfer function and s is the transformed variable; $s = \mathrm{i}\nu$, where $\mathrm{i} = \sqrt{-1}$ and ν is frequency. A matching procedure, similar to the methodologies in [8,23], was then performed in the transformed domain between a VO framework and the CO system. The VO framework replaces the CO parameters γ_n and μ_n with a VO q_m and a generalized coefficient ζ_m. Consider a frequency domain set of M data points (such as experimental data of a non-local medium). Using the VO model to describe the data rather than CO model reduces the linear system of $2N + 2$ unknowns in two equations (real and imaginary parts of transfer function) at each of M different frequency data points into a nonlinear system of two equations in two unknowns at each of M data points [147]. This is a model order reduction technique similar to the fractional model order reduction technique in [23]. The VO parameters at each frequency datum point m are then given as

$$q_m = \frac{\pi}{2} \tan^{-1}(Z_m), \qquad \zeta_m = \frac{1}{\nu^{q_m}} |Z_m|, \tag{4.7}$$

where $Z_m (= 1/G(s_m))$ is the set of frequency domain data.

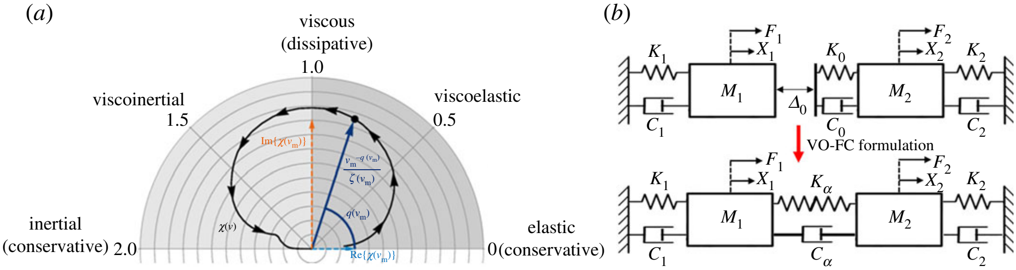

Figure 3. (*a*) Frequency contour plot from [147] for amorphous quartz silica depicting the process to obtain the VO parameters q_m and ζ_m. Note that in this figure $\chi(s) = G(s)$. (Reprinted with permission from [147]. Copyright © 2018 by the American Physical Society.) (*b*) The VO-FC model in [120,148] allows simulation of the coupled oscillator system through an evolutionary VO-FDE (equation (4.9)) irrespective of the status of contact between the masses. The nonlinearity in the contact problem is captured in the VO of the terms K_α and C_α. The schematic (*b*) is adapted from [148]. (Online version in colour.)

Perhaps this VO framework can best be understood from the contour plot of the real and imaginary parts of the system response in the frequency domain as shown in figure 3*a* (the contour for amorphous quartz silica given in [147]). It is evident from figure 3*a* that, when the transfer function of the dynamics of a complex medium is plotted as a contour in a frequency domain, the VO parameters can be easily calculated as a function of frequency, similar to what has been presented in [8,23]. Noteworthy is the ability of the VO operator in capturing the evolution of the system dynamics across the very different viscoelastic and viscoinertial regimes. Thus the authors conclude that the VO is a homogenized differential order of the system that interpolates the conservative and dissipative dynamics directly rather than by the superposition performed using only integer-order operators [147]. Further, the authors used the VO parameters in equation (4.7) to analyse a set of spectroscopic data for the high-frequency dielectric response of a nanofluidic graphene dispersion and the mid-infrared optical response of amorphous quartz silica. The results, when compared against other integer-order models whose parameters were derived through optimization, indicated that the VO model accurately reflected non-locality that could not be well represented using integer-order models.

Similar to the work presented above, VO-FC has also been used to model the evolution of material properties with time or external loads. Experiments have shown that properties of polymers, ductile metals and rocks evolve across strain hardening and softening regimes depending on their internal microstructure and applied strain rates. In a series of papers, Meng *et al*. [123–125] have shown that VO models can accurately capture these transitions in the response of polymers and metals. The VO in [123–125] is obtained by fitting the VO model against experimental data. VO-FC has also been used in the modelling of creep in rocks [126] and the dynamics of shape-memory polymers [149]. In all these works, it is shown that VO-FC models admit fewer parameters than the existing models, and the evolution of the mechanical property is well captured by the VO.

Patnaik & Semperlotti [120,148] have also modelled these transitions in material response using a specific simulation strategy that leverages the peculiar properties of the VO-RL operation of a constant. The VO-RL operation over a constant (A_0) was shown to rapidly change its value between 0 and A_0 for the VO $\alpha(t) = \exp(-\kappa_0\kappa(t))$ for an appropriate choice of κ_0. Mathematically, ${}_0^{RL}D_t^{\alpha(t)}A_0 = 0$ for $\kappa(t) \le 0$, and ${}_0^{RL}D_t^{\alpha(t)}A_0 = A_0$ for $\kappa(t) > 0$. This switch-type behaviour allows the governing equations to describe systems whose dynamics evolve from linear to nonlinear without requiring a modification of the fundamental governing equations. The VO is crafted such that the VO-RL term captures the evolution of a system's stiffness from linear to nonlinear either for reversible or irreversible (e.g. hysteresis) problems depending on the total elongation in the system.

The VO-RL framework was further used in [120,148] to model contact dynamics in a system of coupled masses (figure 3*b*). The transition in the status of contact between the two masses was modelled using VO-RL terms K_α and C_α, which are the VO-RL operations over the contact stiffness and damping, K_0 and C_0, to the VO $\alpha(t) = \exp(-\kappa_0(X_1(t) - X_2(t) - \Delta_0))$. They simplify as

$$\phi_\alpha = {}^{RL}_{0}D_t^{\alpha(t)}\phi_0 = \begin{cases} 0 & X_1 - X_2 \leq \Delta_0 \\ \phi_0 & X_1 - X_2 > \Delta_0, \end{cases} \tag{4.8}$$

where ϕ is either K or C. Now, K_α and C_α evolve according to equation (4.8) and the nonlinearity associated with the contact between the two masses is fully captured in $\alpha(t)$. By using K_α and C_α, the equation of motion of the coupled masses was given by the single system of equations,

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} C_\alpha + C_1 & -C_\alpha \\ -C_\alpha & C_\alpha + C_2 \end{bmatrix} \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{Bmatrix} + \begin{bmatrix} K_\alpha + K_1 & -K_\alpha \\ -K_\alpha & K_\alpha + K_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}. \tag{4.9}$$

The above equation of motion of the coupled masses can evolve from linear to nonlinear behaviour based on the system response. This again highlights the very unique feature of VO-FC for the simulation of dynamical systems with time-varying properties.

5. Application of VO-FC to viscoelasticity

Time-varying properties are a characteristic feature of viscoelastic media. This reason led to the development of several applications of VO-FC to viscoelasticity. Here, we review in detail the progress in the modelling of viscoelastic media and their dynamics. In viscoelastic applications of fractional calculus, the fractional derivative of a quantity such as displacement or strain is taken with respect to time. The intrinsic damping of a fractional operator [7,23] grants time-fractional derivatives the ability to accurately model dissipation in viscoelastic or lossy materials.

Many authors have applied CO fractional derivatives to viscoelastic models [1–4]. From a physical perspective, the use of fractional derivatives to describe the viscoelastic behaviour is fairly logical since the overall response of such systems is simultaneously elastic and viscous. Recall that the generalized one-dimensional relationship between stress σ and strain ϵ of a purely elastic solid is given by Hooke's law $\sigma = E\epsilon$, where E is Young's modulus. On the other hand, the stress–strain relationship of a viscous medium is given by Newton's law $\sigma = \eta((\mathrm{d}\epsilon)/(\mathrm{d}t))$, where η is the damping coefficient. Note that in Hooke's law the order of the derivative of strain with respect to time is zero, while in Newton's law the order of the derivative of strain with respect to time is 1. Thus, from an empirical standpoint, the stress–strain relationship of a viscoelastic material would be $\sigma = c((\mathrm{d}^\alpha\epsilon)/(\mathrm{d}t^\alpha))$, where $(\mathrm{d}^\alpha\epsilon)/(\mathrm{d}t^\alpha)$ is the fractional derivative of the strain and c is a generalized coefficient (whose units have the proper corresponding dimensionality). The value of the fractional derivative α is between 0 and 1 (corresponding to the purely elastic and viscous limit cases). In this context, we note that VO differential operators have been used mostly to model systems with time-dependent variations of the elastic and viscous behaviour.

(a) Variable-order constitutive relationship

Ingman *et al*. [150] were the first to introduce a VO stress–strain constitutive relationship, and used it to study nonlinear contact phenomena. Owing to the state-dependent dynamics and the inability of previously developed models to accurately represent wide ranges of stresses and strains over a loading history of an indentor test, Ingman *et al*. [150] developed a stress–strain model of the form

$$\sigma(t) = \frac{\mathrm{d}^{\alpha(S(t))}\epsilon(t)}{\mathrm{d}t^{\alpha(S(t))}}, \tag{5.1}$$

where $\alpha(S(t))$ is the VO that is dependent on the continuous variation of the state $S(t)$. In [150], the VO operator was defined according to the RL definition. Ingman *et al*. [150] implemented equation (5.1) to model both the viscoelastic and the elastoplastic deformation of a material induced by a spherical indentor. For the viscoelastic deformation, the order varies from 0 to 1 as the process changes from elastic to viscous. For the elastoplastic deformation, Ingman *et al.* [150] developed a fractional model using the well-known Hertz contact formulae to model the deformation as evolving from elastic to semi-plastic to totally plastic. By using experimental data from a spherical indentation test, they were able to calculate the VO as a function of load. The order function served to better understand the evolving dynamics of the nonlinear contact processes.

The development of equation (5.1) in [150] paved the way for other authors [151,152] to use the VO stress–strain relationship. In [151], Ingman & Suzdalnitsky used a slight variation of equation (5.1) to model the response of a viscoelastic plate to impact. A VO operator was necessary 'due to the intrinsic dynamic nature of the phenomenon, reflecting the dominance of elasticity or viscosity in the course of the same loading history' [151]. In the models, as stress increased, the contact dynamics shifted from an elastic to a viscous behaviour. The fractional operator reflected the rate of the damping process that immediately takes place following the termination of contact interaction.

Ramirez & Coimbra [152] further analysed the advantages of using a VO stress–strain relationship. It is well known that the stress in viscoelastic materials is dependent on the strain history of the material, with strains in the recent past contributing a larger influence than the strain in the more distant past. Ramirez & Coimbra [152] recalled the Maxwell and Voigt models, the two most common ways to model viscoelasticity. The Maxwell model consists of a spring and a dashpot connected in series, while in the Voigt model the same elements are connected in parallel. Although being well known and widely used, these models are mere approximations of the actual phenomenon and yield a limited accuracy. In order to improve the accuracy of the model, more spring and dashpot elements should be added, hence leading to a constitutive relationship of the form

$$\sum_{n=1}^{N} a_n \frac{\mathrm{d}^n \sigma}{\mathrm{d}t^n} = \sum_{m=1}^{M} b_m \frac{\mathrm{d}^m \epsilon}{\mathrm{d}t^m}, \tag{5.2}$$

where a_n and b_m are constants related to material properties or parameters [152]. Even with the increased accuracy of equation (5.2), its usefulness is still limited to various ranges of stress or strain before requiring further tuning (i.e. adjustments in the number of springs and dashpots, values of parameters, etc). To contrast the trend of these models, Ramirez & Coimbra [152] examined the constitutive relation

$$\sigma = E\, {}_0^C D_t^{\alpha(t)} \epsilon(t), \tag{5.3}$$

where E is a material property with units of stress. Equation (5.3) is the same as equation (5.1) except for the fact that, in equation (5.3), the VO Caputo operator in equation (2.6) is used. This definition is used for physical modelling since, as argued in [33], the operator satisfies backward compatibility. Ramirez & Coimbra [152] asserted that the two main advantages of equation (5.3) were that it required fewer terms and material constants and that the VO model is still accurate over a wide range of strain rates. In practical terms, this means that it is not necessary to add more elements in series or parallel since the variation of $\alpha(t)$ can accurately capture the varying dynamics.

(b) The viscoelastic oscillator

Strictly related to the VO viscoelastic stress–strain constitutive relationship is the viscoelastic oscillator model. This model was first introduced by Coimbra [33] and has been studied in a multitude of papers ever since [151,153–157]. In [33], Coimbra considered a mass–spring model that oscillates over a surface guide with non-uniform frictional forces. In this model, as depicted in figure 4*a*, the order of damping varies continuously according to the position of the mass. For example, when the mass translates from the viscous to viscoelastic region of the guide, the order smoothly changes from a value of 1 to 1/2. Coimbra [33] termed the model 'viscoelastic–viscous' when the friction behaviour is purely viscoelastic at the centre of the guide while it is viscous at

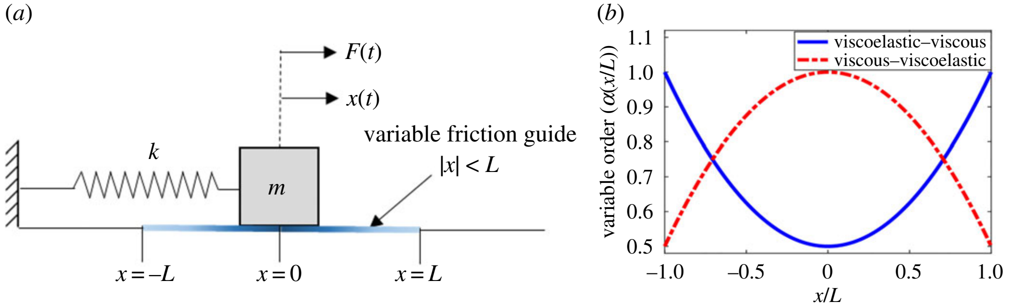

Figure 4. (*a*) Schematic of the mass–spring system described in [33,155,156] moving along a variable friction guide. In the 'viscoelastic–viscous' model, the guide is viscoelastic at $x = 0$ while it is viscous at $|x| = L$. The opposite behaviour leads to the 'viscous–viscoelastic' model. The shading of the guide symbolizes the continuous nature of the VO damping. (*b*) Plot of the VO α as a function of location for the 'viscoelastic–viscous' model and the 'viscous–viscoelastic' model. (Online version in colour.)

the ends of the guide. The opposite frictional behaviour is called 'viscous–viscoelastic'. The model assumed a smooth transition between the viscous and viscoelastic sections of the guide.

Coimbra [33] formulated the following mass-normalized equation of motion for the oscillator with non-uniform frictional forces:

$$D^2\xi(\tau) + c_0\, {}_0^C D_\tau^{\alpha(\xi(\tau))}\xi(\tau) + k_0 D^0 \xi(\tau) = F_0^*(\tau), \tag{5.4}$$

where $\xi \in [-1, 1]$ is the non-dimensionalized position of the mass, τ is the non-dimensionalized time, $\alpha(\xi(\tau))$ is the spatially varying VO, c_0 is a generalized damping coefficient with appropriate dimensions, k_0 is a mass-normalized spring constant and $F_0^*(\tau)$ is the mass-normalized forcing function. For the viscoelastic–viscous oscillator, $\alpha(\xi(\tau)) = (1 + \xi(\tau)^2)/2$, while $\alpha(\xi(\tau)) = (1 - \xi(\tau)^2)/2$ for the viscous–viscoelastic oscillator (figure 4*b*). The friction force, $f(\tau) = c_0 D^{\alpha(\xi(\tau))}\xi(\tau)$, changes its behaviour between purely viscoelastic and viscous types of damping.

For comparison, Coimbra [33] also formulated the equivalent CO model given by

$$D^2\xi(\tau) + c_0 f(\xi) D^1 \xi(\tau) + c_0(1 - f(\xi))\, {}_0^C D_\tau^{1/2}\xi(\tau) + k_0 D^0 \xi(\tau) = F_0^*(\tau), \tag{5.5}$$

where $f(\xi) = \xi^2$ for the viscous–viscoelastic oscillator and $f(\xi) = 1 - \xi^2$ for the viscoelastic–viscous oscillator. Using FD methods, Coimbra [33] numerically simulated both the VO and the CO model and analysed the phase diagrams of the system. It was concluded that the numerical solutions to equations (5.4) and (5.5) possess the same overall behaviour for the oscillator. However, the VO exhibited an advantage when the mass transitioned between the viscous and viscoelastic portions of the guide since it better captured the quickly changing dynamics. This work paved the path to better model practical problems whose dynamics are poorly represented by current mathematical tools (such as purely integer-order models) that face strong constraints and limitations.

Approximately at the same time Coimbra was developing the viscoelastic oscillator, Ingman & Suzdalnitsky [153] produced and studied an equation that is essentially the same as equation (5.4). Using the VO constitutive stress–strain equation, Ingman & Suzdalnitsky [153] considered the dynamics of a single degree of freedom (SDOF) oscillator consisting of a mass, an elastic spring and a 'viscoelastic' spring. First, Ingman & Suzdalnitsky [151,153] rewrote the equation of motion in the following standard form:

$$D^2 x(t) + 2\eta\omega_0^{2-\alpha(t)}\, {}_0^{RL} D_t^{\alpha(t)} x(t) + \omega_0^2 D^0 x(t) = f(t), \tag{5.6}$$

where ω_0 is the natural frequency of the undamped system and η is the damping coefficient. The VO $\alpha(t)$ characterizes the changing physical properties. Further, Ingman & Suzdalnitsky developed a method to determine the natural frequency (first eigenvalue) of the system in [153], and then expanded on it in [151] to present three total approaches. After obtaining the eigenvalues, Ingman & Suzdalnitsky [153] formulated a numerical iterative approach to solve the VO-FDE. Using the work in [153], Ingman & Suzdalnitsky [154] determined the dependence of the VO on the strain and the strain rate from experimental data of deformable polymers. An expression for the VO was obtained for the viscoelastic deformation by fitting the experimental data to a fractional constitutive relationship at each measured instant. Using the iterative technique developed in [153], Ingman & Suzdalnitsky numerically obtained the oscillation of the SDOF oscillator in [154]. Comparing the results with a CO model used to represent the experimental data, Ingman & Suzdalnitsky concluded that the CO models can be used as initial approximations in oscillatory systems. The approximations can then be improved by tuning the value of the order.

Two years after [33], Soon *et al*. [156] extended the work of [33] by considering a viscoelastic–viscous oscillator. The work was similar to that of Ingman & Suzdalnitsky [153,154], but used the VO Caputo operator from [33] (equation (2.6)). The damping force in this oscillator varied continuously between purely elastic and purely viscous depending on the location of the mass, therefore justifying the use of VO operators. Soon *et al*. [156] compared the solution of the VO-FDE with a multi-weighted nonlinear CO approximation of the motion of the mass, which was given as

$$D^2\xi(\tau) + c_0 \sum_{n=1}^{N} f_n(\xi(\tau)) D^{\alpha_n}\xi(\tau) + kD^0\xi(\tau) = F_0^*(\tau), \tag{5.7}$$

where the variables have the same meaning as in equation (5.4) and f_n are weighting functions of the CO derivatives. The numerical solutions of the VO-FDE and the CO-DE indicated that the multi-weighted CO model was able to provide a reasonable approximation. Unsurprisingly, the accuracy of the CO models was directly dependent on factors such as the number of CO derivatives in the summation, the weighting functions and the forcing frequency [156]. In order to reach an accuracy comparable to that of the VO model, the CO model needed at least five weighted CO terms. Although the numerical computational expenses of a VO operator are greater than those of a CO operator, the need for multiple CO terms in order to produce an accurate model of the viscoelastic oscillator rendered the CO approach just as expensive as the VO model, if not more so. Thus, because of the enhanced accuracy of the VO model, Soon *et al*. [156] deduced that it is more prudent to use a VO operator to model the complex dynamics of this viscoelastic oscillator.

More recently, Sahoo *et al*. [155] also used VO models to study viscoelastic oscillator systems. Although Sahoo *et al*.'s study was simply a review of the model presented in [33,156], novel contributions included a study of the Laplace transform of the VO integral operator as well as a numerical scheme based on the GL definition. On the other hand, Morales-Delgado *et al*. [157] were concerned with obtaining an analytical solution to the VO oscillator problem. To do so, they implemented the VO Atangana–Koca–Caputo (AKC) operator [158,159]. By using the Laplace transform of the VO-AKC operator, they rigorously worked through the Laplace domain before transforming back to obtain analytical solutions. Note that the VO-FDEs that they considered were fundamentally different from equation (5.4). Thus, the work in [157] is not applicable to the main model of the mass–spring–viscodamper considered throughout this section.

6. Application of VO-FC to the modelling of transport processes

Recent theoretical and experimental studies have shown that transport processes in complex media are often characterized by either hybrid or anomalous mechanisms. Further, the nature of the transport processes transitions across very different underlying physical phenomena such as transitions from sub-diffusive flow to diffusive flow, and from diffusive flow to super-diffusive flow [34,127,128,160–162]. These complex transport processes have been observed experimentally in various fields, including fluid flow through porous media [5,134–136,163,164], reaction–diffusion interactions between chemical substances leading to pattern formations in nature [165–167], diffusion of ions in human neurons [168], analysis of financial data [169], advection–diffusion of groundwater [134–136] and elastography [7]. Given the complex nature of transport in the above cases, CO differential models have failed to model many characteristics of the above transport processes. We start by reviewing the need behind VO fractional models in describing complex diffusive transport. Then, we review the applications of VO-FC to the modelling of complex advection–diffusion, reaction–diffusion and hybrid propagating systems.

(a) Anomalous diffusion

The mean square displacement (MSD), which is used to characterize diffusion processes, scales as a constant power of time for classical diffusion models, and as a fractional power of time for CO fractional diffusion models. However, experiments have shown that in several complex diffusive processes the scaling power varies as a function of time and/or spatial location, i.e. $\langle x^2(t)\rangle \propto t^{\mu(t,x)}$. Clearly CO (fractional or classical) diffusion equations are unable to model these complex processes. This variation in the scaling law stems from the medium heterogeneity, memory effects, long-range interactions and heavy tail characteristics seen in anomalous diffusion [34,127,160]. Hence, VO models have been used to model these complex processes as the MSD for VO diffusion equations has been shown to have a spatially and/or temporally varying scaling exponent.

In their seminal study on VO diffusion models, Chechkin *et al*. [34] in 2005 derived a VO diffusion equation as the continuum limit of a continuous time random walk (CTRW) model for a spatially heterogeneous system. The particles in this heterogeneous system experience a spatially varying power-law waiting-time $\psi(t,x) \propto t^{-1-\alpha(x)}$. The VO diffusion equation is given as

$$\frac{\partial}{\partial t}c(t,x) = \frac{\partial^2}{\partial x^2}\left(K(x)\,{}_0^{RL}D_t^{1-\alpha(x)}c(t,x)\right), \tag{6.1}$$

where $c(t,x)$ is the particle concentration, $0<\alpha(x)<1$ and $K(x)$ is the diffusion constant. Since closed-form solutions are difficult to obtain for equation (6.1), the authors have analysed the VO diffusion equation by taking the example of a composite medium consisting of two semi-infinite sub-diffusive systems with different sub-diffusion exponents. The solution to this simplified set-up shows the appearance of drift, which at small times is in the direction of the region with a smaller diffusion exponent and at large times is in the direction of a larger diffusion exponent. Further, the time dependence for drift and diffusion spreading also changes in the course of time. Similarly, a VO Fokker–Planck equation was also derived in [170] as the continuum limit of a CTRW where the spatially variable time-scaling behaviour of the MSD is modelled by a spatially varying $\beta(x)$-stable Lévy noise in the waiting time probability density function.

Diffusion processes are also characterized by the Hurst exponent, which is expressed relative to the MSD as $\langle x^2(t)\rangle \propto t^{2H}$. Clearly, the classical diffusion process corresponds to $H=0.5$. Sun *et al*. [128] derived the MSD law starting from a different VO diffusion equation given as

$${}_0^{C}D_t^{\alpha(t)}c(t,x) = K\frac{\partial^2 c(t,x)}{\partial x^2}, \tag{6.2}$$

where $0<\alpha(t)<1$ and K is the diffusion coefficient. The MSD for equation (6.2) is found as

$$\langle x^2(t)\rangle = \frac{2Kt^{\alpha(t)}}{\Gamma(\alpha(t)+1)}. \tag{6.3}$$

Clearly, the Hurst exponent is a function of time. The time-dependent Hurst exponent has been used in [164,169,171,172] to analyse multi-fractional Brownian processes. Further, anomalous diffusion in fractal structures has been analysed in [168,173] by constructing an MSD which is a function of the fractal dimension and time. Particularly, anomalous diffusion of chemical ions in Purkinje cell dendrites has been modelled in [168] by considering the complex disordered neuronal dendrites as a fractal body, the dimension of which is a function of space (and time) (figure 5*a*).

Sun *et al*. [35] proposed a more generic VO diffusion equation where the order of the time derivative is a function of time, space, concentration or other process variables, i.e. $\alpha \triangleq \alpha(t,x,c)$.

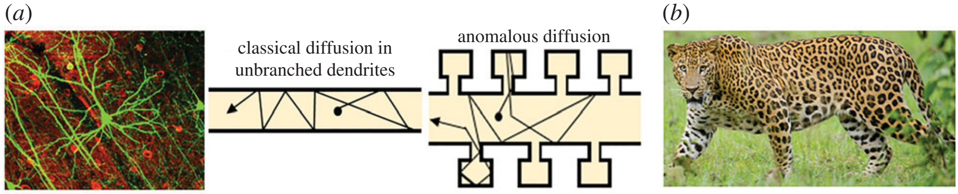

Figure 5. (*a*) Diffusion of ions in neuronal dendrites is accurately modelled by VO models where the Hurst exponent is a function of the fractal dimension of the dendrites, which varies in both space and time [168]. (*b*) Spatial patterns occurring in nature are modelled by the VO Gray–Scott model in [92], where it is argued that medium heterogeneity leads to anomalous diffusion of the reacting substances. The images of the dendrites and the leopard are taken from Wikipedia, while the image of the particle diffusing in the dendrites is adapted from [168]. (Online version in colour.)

For example, when $\alpha \triangleq \alpha(t)$, the model could more accurately depict diffusion processes with a time-variable Hurst exponent. The VO $\alpha(x)$ was used to represent diffusion in heterogeneous or anisotropic media, such as porous materials [174]. Sun *et al*. [174] used the VO diffusion equation to describe transient dispersion observed in case (i): uranine transport at the small-scale Grimsel test site where the transport transitions from strong sub-diffusion to Fickian dispersion and case (ii): the transport of tritium at the regional-scale macrodispersion experimental site which transitions from near-Fickian dispersion to strong super-dispersion. Further, they used the VO-FC model to study conservative particle transport through a regional-scale discrete fracture network which transitions from super-dispersion to Fickian dispersion. As shown in [174], the VO model can accurately capture these transitions with the VO $\alpha(t,x)$ transitioning in $(0,2)$ depending on the temporal or spatial state. Furthermore, the time-derivative order α can also be a function of the concentration, or some other system or independent variable [35]. The particular form of the VO would dictate the functional form of the Hurst exponent of the diffusion process.

Umarov & Steinberg [175] presented a thorough mathematical analysis of VO diffusion equations. They studied diffusion processes with changing modes where the memory of the diffusion process alters between a 'long-term' memory and a 'short-term' memory. The main findings in [175] focus on theorems, corollaries and proofs of the mathematics of the VO diffusion equations.

Heat transfer through complex structures has also been shown to be anomalously diffusive in nature. Recently, a VO diffusive heat transfer equation was used to model heat transfer through a structure with holes, where the grid-hole geometry changed in time in [176]. The grid-hole structure was a copper plate with time-varying insulated areas that are arranged according to a pattern akin to the first tier of a Sierpinski carpet fractal design. This resulted in changing diffusivity coefficients and the diffusive phenomenon was modelled using a VO operator. From the fractional diffusion equation for temperature (equation (6.2), where temperature $T(t,x)$ replaces $c(t,x)$), the relationship between the temperature and the heat flux $H(t,x)$ was found to be

$$T(t,x) = \frac{1}{\sqrt{\lambda_\alpha}} \frac{\partial^{-\alpha/2} H(t,x)}{\partial t^{-\alpha/2}}, \tag{6.4}$$

where λ_α is a homogenized diffusion coefficient. Through an optimization tool, the values of α and λ_α were obtained for two different ratios of the length of the insulators to the length of the copper plate. The VO model was developed from CO models where the size ratio between the length of the insulators to the length of the copper plate suddenly changed at a specific time instant. The results from this VO model by Sakrajda & Sierociuk [176] were shown to be in good agreement (error less than about 3%) with the results obtained via a finite-element model.

A year later, Sakrajda & Wiraszka [177] extended the work in [176] to include the iteration tier of the Sierpinski fractal geometry. The relationship between the fractional-order α and the iteration level n_f of the Sierpinski carpet design was obtained by fitting parameters of the fractional equation to results obtained by a finite-element method for two sets of simulations. In the first study, the CO was calculated for different fractal levels n_f of the structure. This yielded a plot of the fractional CO as a function of n_f. In the second set of simulations, the structure's geometry could vary in time. The geometry would immediately change from a homogeneous structure (where α=1) to a fractal structure (where the order α is obtained through the first set of CO simulations) at a specific time. Results of the VO model compared with the results from a finite-element analysis showed that a definition of a VO operator called the '$\mathscr{D}$-type' by [177] could accurately represent the diffusion process in a heterogeneous, spatially varying, fractal geometry.

(b) Advection–diffusion systems

The VO diffusion equation formed the basis of several interesting investigations involving anomalous advection–diffusion systems, especially those seen in nature. For example, the flow of groundwater through underground aquifers has been modelled in [134–136] through VO time-fractional operators. Groundwater diffuses through porous, fractured, layered and heterogeneous aquifers, whose structure changes with space as well as time, leading to anomalous diffusion and a VO scaling of the MSD with time. Hence, this flow process cannot be modelled accurately through CO models. Thus, in [134,135] this anomalous flow is modelled through a VO version of the groundwater flow equation (called the Theis equation) given as

$$S{}_0^C D_t^{\alpha(\mathbf{r},t)} \Phi(\mathbf{r},t) = T\nabla_r^2 \Phi(\mathbf{r},t) + \frac{1}{r}\nabla_r \Phi(\mathbf{r},t), \tag{6.5}$$

where $\mathbf{r}$ is the radial location, $\alpha(\mathbf{r},t) \in (0,1)$, Φ is the flow head, S is the specific storativity and T is the transmissivity of the aquifer. Further, the spread of groundwater pollutants through these aquifers has also been modelled using VO advection–diffusion equations in [136].

(c) Reaction–diffusion systems

Another application of VO-FC involves modelling of reaction–diffusion systems. These systems correspond to a change in the concentration of interacting chemical substances in both space and time, i.e. they involve local chemical reactions, in which the substances react with each other, and diffusion, which causes the chemicals to spread out in space creating rich patterns. Reaction–diffusion processes have been linked to spots on deer and patterns in giraffe, zebra, leopards and butterfly wings [165–167]. The Gray–Scott model is often used to understand reaction–diffusion systems as patterns created by this model very closely resemble many patterns seen in living things (figure 5*b*). These living systems exhibit rich dynamics due to heterogeneity of the diffusing media, which causes the diffusion reactions to exhibit transience between sub-diffusion to Fickian diffusion to super-diffusion. Coronel-Escamilla *et al.* [92] proposed a VO Gray–Scott model where they analysed the effect of several VOs on the patterns created by two interacting chemicals.

(d) Wave propagation

A very interesting application of the VO-FC in the modelling of transport processes was presented by Zhao & Karniadakis [89], who used VO differential operators to control errors in the solutions of classical integer-order partial differential equations arising from truncated domains and erroneous boundary conditions or from the loss of monotonicity of the numerical solution either because of under-resolution or because of the presence of discontinuities. VO operators are used to control erroneous wave reflections in truncated computational domains of the integer-order wave equation by switching from a wave to a diffusion-dominated equation at the boundaries.

Using VO-FC the wave equation in [89] for a bounded domain Ω is modified as

$$ {}_0^C D_t^{\alpha(t,x)} u = u_{xx}, \; x \in \Omega, \; t \in (0, T], \tag{6.6}$$

where $u(x,t)$ denotes the particle displacement in the x direction. The initial conditions are taken as $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, and $u(x,t) = 0 \; \forall \; x \in \partial\Omega$ is imposed as the boundary condition. The VO $\alpha(t,x)$ is crafted such that $\alpha(t,x) = 2$ in the inner part of the domain and $\alpha(t,x) \leq 2$ near the boundaries. The results show that maximum errors are less and decay in time for the VO wave equation, unlike the classical wave equation for which the errors are order 1 and which increase with time. Similarly, the VO viscous Burgers' equation is

$$ {}_0^C D_t^{\alpha(t,x)} u + \frac{1}{2} \frac{\partial u^2}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathcal{R}, \; t > 0, \tag{6.7}$$

with the initial condition $u(x,0) = \phi(x)$. It was shown that $0 < \alpha(t,x) \leq 1$ suppresses the loss of monotonicity owing to the formation of wiggles as $\mu \to 0$. Clearly, the VO-FC model allows the use of super-diffusion/sub-diffusion in controlling the above numerical artefacts and is particularly useful in large-scale practical simulations, where all fine scales of a heterogeneous field in a complex geometry are difficult to capture or outflow boundary conditions are not readily available [89].

7. Application of VO-FC to control theory

This section analyses applications of VO-FC in control theory. There is a multitude of research on the control and stability of CO fractional systems that essentially serves as the foundation of VO-FC-based formulations for control applications. As noted by Orosco & Coimbra [178], the main advantage of introducing fractional calculus into control theory is to improve robustness and dynamic characteristics. CO fractional controllers include, but are not limited to, the fractional proportional–integral–derivative (PID) controller, fractional lead–lag controllers and fractional adaptive controllers. The interested reader is referred to [179] for a thorough review of the control of CO systems, as well as practical applications and implementations. In this section, we focus on the extension to VO control by reviewing the mathematics and applications of the VO PID controllers, as well as other control and stability studies that involve systems with VO dynamics. We conclude with a brief discussion of VO filters.

(a) Variable-order fractional PID control

Podlubny [180] generalized the well-known PID controller to a fractional PID controller where the integral control is a fractional integral while the derivative control is accomplished using a fractional derivative. The fractional PID controller is more commonly referred to as $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ control, where λ is the order of the fractional integral and μ is the order of the fractional derivative. Liu *et al*. [181] discussed the main advantages of CO $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ control, such as the addition of two more parameters (the orders λ and μ) to the usual PID gains k_P, k_I and k_D. The additional parameters λ and μ enhance the flexibility and robustness of the controllers. The value of the order λ in $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ control affects the slope of the low-frequency range of the system as well as the peak value of the system. On the other hand, the value of the derivative μ affects the accuracy of the dynamic closed-loop response, system overshoot and stability as well. For a more detailed discussion of the roles of λ and μ, the interested reader is referred to §3 of [181].

The first extensions of $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controllers to VO occurred in the early 2010s. In [182], Sheng *et al*. fit experimental data of the temperature of an electrical element (called a 'fractor') to a VO system. Previous work by the authors showed that the order of a transfer function of a circuit containing the fractor changed over time as a function of temperature. Sheng *et al*. [182] proceeded to propose

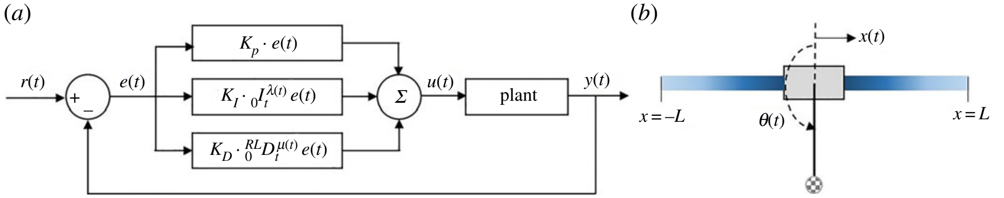

Figure 6. (*a*) Block diagram illustrating the feedback VO PI$^\lambda$D$^\mu$ controller based on [182]. The orders of the VO integral and derivative are both $0 \le \lambda, \mu \le 1$. The definition of the VO integral and derivative are from equation (2.1) and equation (2.2), respectively, although any form presented in §2 can be used. (*b*) Schematic of the inverted pendulum attached to a block sliding along a VO damped guide, based on the problem presented in [178]. The shaded guide depicts the smooth evolution of the damping from purely viscous at the ends to viscoelastic at the centre. (Online version in colour.)

a VO PI$^\lambda$D$^\mu$ controller for the fractor system described by

$$u(t) = k_P e(t) + k_I D_t^{-\lambda(t)} e(t) + k_D D_t^{\mu(t)} e(t), \tag{7.1}$$

where $u(t)$ is the input voltage and $e(t)$ is the error between the output voltage $y(t)$ and the reference signal $r(t)$. The controller is depicted in figure 6*a*. Sheng *et al*. [182] do not include any simulations or results of this PI$^{\lambda(t)}$D$^{\mu(t)}$ controller, but merely proposed the concept.

In a series of three papers, Ostalczyk *et al*. [129–131] explored the development of VO PI$^\lambda$D$^\mu$ controllers. In [129], Ostalczyk presented a VO finite backward difference equation (VO-FBDE) and focused on the stability of a closed-loop single input–single output (SISO) system with a controller described by the VO-FBDE. They determined the optimal parameters of the VO controller that gave a stable transient closed-loop dynamic response. Then, in [130] Ostalczyk examined the VO PI$^\lambda$D$^\mu$ controller and explained that the variability of the orders led to 'new possibilities of shaping the transient characteristics of a closed-loop system that are unattainable in classic PID control' [130]. While exact analytical expressions for the VO λ and μ are still open questions, in [130,131] Ostalczyk *et al*. proposed that values of the order change according to the present and past values of the error function and the output signal of the controller. This in turn significantly affected the transient properties and stability of a closed-loop response. In [131], the VO controller was also applied to a microprocessor. According to Ostalczyk *et al.* [131], the main advantages of the VO controller include the fact that it provides a larger scope to design the dynamics of the closed-loop system than classical PID control and that smoother transient states result from continuous VO functions [131].

In [50], a VO PID controller was used in conjunction with a switching objective function. A switching objective function could arise from manual user input of the reference signal. The changing objective function was robustly captured by changing the order of the controller, which was accomplished by using a VO PID controller. Sierociuk & Macias [50] implemented the designed VO controller on a heat transfer process in a steel beam with a changing reference signal. Results exemplified the utility of VO for controlling a system with rapidly varying dynamics in the form of changing objective functions.

In addition to a review of the CO PI$^\lambda$D$^\mu$ control, Liu *et al*. [181] extended their work to VO PI$^\lambda$D$^\mu$ control combined with fuzzy logic. They developed a control algorithm where a fuzzy logic controller used the value of the error and the time derivative of the error to determine the changes in the tuning parameters k_P, k_I and k_D as well as the fractional orders λ and μ used in the VO PID controller. The analysis then compared the control of an example plant using the VO fuzzy logic PID controller, an optimal CO controller and an optimal classical PID controller. The comparison between the three controllers was conducted for both fractional-order and integer-order plants. The fractional-order controllers performed better than the classical PID controller when applied to a fractional-order plant. While both the VO and CO controllers performed well, the VO fuzzy PI$^\lambda$D$^\mu$ controller had a smaller overshoot and smaller rise time. Even for controlling the integer-order plants, the VO fuzzy $PI^{\lambda}D^{\mu}$ controller outperformed the optimized classical PID controller, again with a smaller rise time and less overshoot. Thus, the VO fuzzy controller exhibited enhanced system performance, robustness and adaptability to VO plants [181].

Most recently, Dabiri *et al*. [183] also designed a VO $PI^{\lambda}D^{\mu}$ controller for a SISO linear dynamical system and addressed its stability. To accomplish this, Dabiri *et al*. implemented spline and backward FD approximations to discretize the governing equations. Furthermore, a particle swarm optimization algorithm determined the optimal and stable values of the orders and gains. The algorithm was then applied to multiple dynamical examples, including the control of a typical mass–spring system and a Duffing oscillator. In general, the VO $PI^{\lambda}D^{\mu}$ control was shown to perform slightly better owing to its increased flexibility and robustness.

(b) Other variable-order fractional control and stability studies

Diaz & Coimbra [184] and Orosco & Coimbra [178] focused on developing VO controllers for Coimbra's viscoelastic oscillator [33] using a strategy different from $PI^{\lambda}D^{\mu}$ control. In [184], two different controllers were designed to track a reference function for the nonlinear VO viscoelastic–viscous oscillator given in equation (5.4). The VO oscillator equation was converted to a state-space form in the form of a matrix equation. After finding the locations of the eigenvalues of the linear portion of the matrix system, a feed-forward controller was created according to $u = -K\mathbf{x} + Gr$, where $K = [k_0, k_1]$, k_0 and k_1 are constants corresponding to the location of the preferred values of the closed-loop eigenvalues and G is the feed-forward gain for the reference r [184]. Results of the performance of this controller were provided for a Heaviside and oscillatory reference signal. The second controller developed in [184] was an optimal tracking controller which functioned by minimizing a defined performance index. Finally, Diaz & Coimbra [184] developed a method to rewrite the Van der Pol equation in the form of a VO-FDE. Recall the Van der Pol equation

$$D^2y(t) + C(y^2 - 1)D^1y(t) + D^0y(t) = u(t), \tag{7.2}$$

where C is a coefficient. Equation (7.2) is matched to equation (5.4), resulting in

$$C(y^2 - 1)D^1y(t) = {}_0^C D_t^{\alpha(x(t))} x(t), \tag{7.3}$$

where $y(t)$ is the solution of the Van der Pol equation and $x(t)$ is the solution of the VO-FDE. Setting $y = x$, a minimization algorithm can be applied to calculate the VO such that equation (7.3) holds true. Once this method was applied to transform the Van der Pol equation to the VO-FDE, the developed VO control algorithms could be applied to control a Van der Pol oscillator.

Orosco & Coimbra [178] developed a controller for the classic inverted pendulum problem, depicted in figure 6*b*. The inverted pendulum is connected to a cart which moves along a track that is coated in a thin film with spatially varying order of damping. This damping film introduced VO Caputo operators (equation (2.6)) into the nonlinear equations of motion, similar to the viscoelastic damping in [33]. After non-dimensionalizing the equations of motion, two controller strategies were proposed. The first was a quasi-linearized state-space method where the VO term was treated as a nonlinear disturbance, while the second was a model-predictive controller (see [178] for more details). Simulations established that there was a critical value of the friction coefficient above which unstable oscillations persisted in the pendulum. A perturbation and eigenvalue analysis was performed using a half-order derivative to determine the critical friction coefficient value. For the VO system, this critical value depended on the initial perturbation of the system and increasing the value of the VO produced a continuous destabilizing effect.

Additionally, VO fractional controllers were designed and implemented using Crone approximations in [47] to control plants with time-varying poles and gain. The drift in the parameters was significantly slower than the plant dynamics and a constant phase margin was sought resulting in constant overshoot in step responses. Valério & Sá da Costa [47] varied the order of the fractional controller to analytically adapt to the time-varying plant in order to overcome the drift.

Lastly, some recent papers considered the VO control of chaotic systems. Ávalos-Ruiz *et al*. [185] implemented a controller on a field-programmable gate array for VO chaotic systems using four different types of chaotic attractors. Using an extensive LabVIEW model that was based on sliding mode control, the control of the VO chaos of the four different types of attractors was studied and stabilized according to a developed theorem. Coronel-Escamilla *et al*. [186] considered a state-observer-based approach for the synchronization of VO chaotic systems. The approach considered in [186] implemented a master–slave relationship where the slaves were VO observers whose input signal arose from the master dynamical system. This predictor–corrector numerically solved chaotic systems such as the Rössler oscillator, the Chua system and the multi-scrolls system using VO.

(c) Variable-order filters

VO digital filters have been very popular for signal processing as they allow for quick changes in frequency characteristics without needing to redesign a new filter. Conventional CO fractional differentiators have been widely used to design finite-impulse responses (FIRs) and have been extended to design VO FIRs in [132]. VO digital filters were also designed in [133] using Taylor series expansion, and in [187] using an iterative method. Charef & Idiou [188] developed a design technique for VO analogue filters based on a polynomial interpolation of the residues of the analogue rational function approximations of the filters. VO filters were also used for adaptive-order and parameter estimation in [189], where a gradient-based algorithm was used to identify the VO. The VO of the system was estimated on the basis of known input and output signals measured in real time. An adaptive estimation law based on the minimization of the next instantaneous cost function was used subject to a constraint on the upper bound of the order.

8. Miscellaneous applications

Complex competitive interactions are commonly seen in nature; for example, in ecological models including food chains of species linked by trophic interactions, diffusion or spread of nutrients or species in different states, competition between healthy and disease-causing cells. These biological systems exhibit long-range temporal memory or long-range spatial interactions and the strength of such interactions varies with space and time. Thus, the use of VO fractional operators can handle efficiently the dynamics of these interactions that often change with space and time. On these lines, Ghanbari & Gómez-Aguilar [137] have modelled the competitive dynamics in a nutrient–phytoplankton–zooplankton interaction model using VO operators. It is shown that the VO model leads to a change in the memory effects of the system, wherein the temporal memory of the interactions is affected by the relative populations of the nutrient–phytoplankton–zooplankton system, as well as the particular order variation. Further, a VO growth model was used in [138] to study the population history of several countries, and it was shown that the VO model was highly accurate when compared with the existing CO models.

A similar competitive dynamics involves interaction between people who are affected by three different strains of tuberculosis (TB), namely drug sensitive, emerging multi-drug resistant and extensively drug resistant, and people unaffected by TB. Sweilam & Al-Mekhlafi [139] have numerically modelled this complex dynamics using VO-FDEs with GL operators. Further, using data provided by the World Health Organization they estimate the required rate of treatment to achieve control over the spread of TB in Egypt.

Another biological example which has been modelled via VO fractional operators involves the competitive dynamics between healthy and tumorous bone cells. Neto *et al*. [140,141] have shown that VO-FDEs provide results similar to those of original integer-order bone cells and tumour interaction models with fewer parameters. The VO-FDE results in a non-local approach with memory effects where the VO depends on time as well spatial location. The VO is influenced by the tumour dynamics and induces the effect of tumours in the original healthy bone model. The authors comment that comparison of the VO models with real experimental data will provide real insights into tumour growth and will form the basis of efficient and targeted tumour therapies.

A particularly interesting application of VO-FC pertains to modelling the impact of Twitter on the spread of alcoholism [142]. VO operators with time delays were added to existing CO models and it was shown that the VO model better captured the spread of alcoholism than the CO models. In all the above-reviewed works, VO-FC has been used to model complex competitive dynamics between various biological entities and it has been shown that the VO successfully captures areas of transition between dynamic regimes of the various biological phenomena.

VO-FC has also been applied to study random-order models [128,143,190]. However, to date, the work on random-order operators and their applications has been limited. Particularly lacking is a rigorous mathematical definition of these operators and their properties. Despite this issue, the concept of random-order fractional calculus is particularly exciting and can have important applications in modelling random and chaotic dynamics observed, as an example, in financial systems, turbulent dynamics, noise and vibration control. These models could potentially form the basis for the development of highly accurate risk analysis and control models.

9. Conclusion

This review paper provided a comprehensive overview of the remarkable progress made in the general area of scientific and engineering modelling based on VO-FC. The relatively recent development of VO-FC has sparked much interest across several scientific communities. This interest directly resulted in a fast-growing number of studies focusing on the many unique opportunities and modelling capabilities offered by this outstanding mathematical tool. Although a few excellent reviews and critical analyses of VO-FC have been presented to date, the existing literature mostly targeted the mathematics community and focused on presenting the many important mathematical aspects of such operators. The present review, instead, is intended for the broader scientific community and it provides an overview of the many areas of science and engineering that have already found much benefit in the use of VO-FC. Particular attention was given to the application of VO-FC to the numerical modelling and simulation of complex physical systems.

Although there are still significant mathematical difficulties preventing the widespread use of VO fractional operators, the development of dedicated numerical approaches has helped in uncovering and leveraging the hidden potential of these operators. A critical step to further promote the use of VO-FC models for numerical simulations is to establish the connection between the mathematical properties of VO operators (i.e. the order variation) and the physical properties and parameters of the system to be modelled. In other terms, closed-form relations allowing the selection of both the order and its functional relationship with the physical parameters are paramount to turning VO-FC into a mainstream modelling tool.

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