A REMARK ON PARTIAL DATA INVERSE PROBLEMS FOR SEMILINEAR ELLIPTIC EQUATIONS

KATYA KRUPCHYK AND GUNTHER UHLMANN

ABSTRACT. We show that the knowledge of the Dirichlet–to–Neumann map on an arbitrary open portion of the boundary of a domain in $\mathbb{R}^n$, $n \geq 2$, for a class of semilinear elliptic equations, determines the nonlinearity uniquely.

1. INTRODUCTION AND STATEMENT OF RESULTS

The work [11] discovered that nonlinearity can be helpful in solving inverse problems for hyperbolic PDE, see also [2], [14], and the references given there. Similar phenomena for inverse problems for semilinear elliptic PDE have been revealed in the recent works [4], [12]. A common feature of all of the aforementioned works is that the presence of a nonlinearity allows one to solve inverse problems for non-linear equations in cases where the corresponding inverse problem in the linear setting is open.

The purpose of this note is to point out that the same phenomenon remains valid for partial data inverse boundary problems for a class of semilinear elliptic PDE. To state our result, let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary. Following [4], we consider a function $V : \overline{\Omega} \times \mathbb{C} \to \mathbb{C}$ satisfying the conditions:

(i) the map $\mathbb{C} \ni z \mapsto V(\cdot, z)$ is holomorphic with values in the Hölder space $C^\alpha(\overline{\Omega})$, for some $0 < \alpha < 1$,
(ii) $V(x, 0) = \partial_z V(x, 0) = 0$, for all $x \in \overline{\Omega}$.

It follows from (i) and (ii) that V can be expanded into a power series

$$V(x, z) = \sum_{k=2}^{\infty} V_k(x) \frac{z^k}{k!}, \quad V_k(x) := \partial_z^k V(x, 0) \in C^\alpha(\overline{\Omega}), \qquad (1.1)$$

converging in the $C^\alpha(\overline{\Omega})$ topology.

Let us consider the Dirichlet problem for the following semilinear elliptic equation,

$$\begin{cases} -\Delta u + V(x, u) = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \qquad (1.2)$$

1

It was shown in [4], [12], see also [5], that there exist $r_0, r_1 > 0$ sufficiently small such that when $f \in B_{r_0}^\alpha(\partial\Omega) := \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} \leq r_0\}$, the problem (1.2) has a unique solution $u = u_f \in B_{r_1}^\alpha(\overline\Omega) := \{u \in C^{2,\alpha}(\overline\Omega) : \|u\|_{C^{2,\alpha}(\overline\Omega)} \leq r_1\}$. Moreover, there is a constant $C > 0$ depending on r_0, r_1 only such that

$$\|u\|_{C^{2,\alpha}(\overline\Omega)} \leq C\|f\|_{C^{2,\alpha}(\partial\Omega)},$$

for all $f \in B_{r_0}^\alpha(\partial\Omega)$.

Let $\Gamma \subset \partial\Omega$ be an arbitrary non-empty open subset of the boundary $\partial\Omega$. Associated to the problem (1.2), we define the partial Dirichlet–to–Neumann map $\Lambda_V^\Gamma f = \partial_\nu u|_\Gamma$, where $f \in B_{r_0}^\alpha(\partial\Omega)$, supp $(f) \subset \Gamma$. Here $u \in B_{r_1}^\alpha(\overline\Omega)$ is the unique solution of (1.2) and ν is the unit outer normal to the boundary.

The main result of this note is as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary, and let $\Gamma \subset \partial\Omega$ be an arbitrary open non-empty subset of the boundary $\partial\Omega$. Let $V^{(1)}, V^{(2)} : \overline\Omega \times \mathbb{C} \to \mathbb{C}$ satisfy the assumptions (i) and (ii). Assume that $\Lambda_{V^{(1)}}^\Gamma f = \Lambda_{V^{(2)}}^\Gamma f$ for all $f \in B_{r_0}^\alpha(\partial\Omega)$ with supp $(f) \subset \Gamma$. Then $V^{(1)} = V^{(2)}$ in $\overline\Omega \times \mathbb{C}$.*

Remark 1.2. *In particular Theorem 1.1 shows that the potential $q \in C^\alpha(\overline\Omega)$ in the semilinear Schrödinger equation with a power nonlinearity,*

$$-\Delta u + q(x)u^m = 0 \quad in \quad \Omega,$$

is uniquely determined by the partial Dirichlet–to–Neumann map for $m \geq 2$, when both the data and the measurements are confined to an arbitrary open portion $\Gamma \subset \partial\Omega$. It may be interesting to note that the corresponding partial data inverse problem for the linear Schrödinger equation, i.e. $m = 1$, is still open in dimensions $n \geq 3$. We refer to [6] and [7] for the study of the partial data inverse problem in the linear and non-linear settings, respectively, in dimension $n = 2$.

Remark 1.3. *Theorem 1.1 is an immediate consequence of the main result of [3] combined with the higher order linearization procedure introduced in [4], [12].*

Let us remark that inverse problems for nonlinear elliptic PDE have been studied extensively, both in the semilinear setting, see [7], [8], [9], [19], as well as the quasilinear one, see [1], [10], [16], [17], [18]. In particular, the second order linearization of the nonlinear Dirichlet–to–Neumann map has already been used in the works [1], [10], [17], and [18].

Finally, we mention that after this note appeared on arXiv, we were informed by Mikko Salo that in his joint work with Matti Lassas, Tony Liimatainen, and Yi–Hsuan Lin [13], the same result was obtained. Their paper was posted on arXiv shortly afterwards.

2. Proof of Theorem 1.1

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{C}^m$, $m \geq 2$, and consider the Dirichlet problem (1.2) with

$$f = \sum_{k=1}^{m} \varepsilon_k f_k, \quad f_k \in C^\infty(\partial\Omega), \quad \text{supp } (f_k) \subset \Gamma, \quad k = 1, \ldots, m.$$

Then for all $|\varepsilon|$ sufficiently small, the problem (1.2) has a unique solution $u(\cdot, \varepsilon) \in B_{r_1}^\alpha(\overline{\Omega})$ which is holomorphic in a neighborhood of $\varepsilon = 0$ in the $C^{2,\alpha}(\overline{\Omega})$ topology, see [4].

Following [4], we use an induction argument on $m \geq 2$ to prove that all the coefficients $V_m(x)$ of V in (1.1) can be determined from the partial Dirichlet–to–Neumann map. The computations below related to the higher order linearization procedure reproduce those of [4], [12], the only minor difference being that we consider the case of partial measurements.

First let $m = 2$, and following [4], [12], we shall proceed to carry out a second order linearization of the partial Dirichlet–to–Neumann map. Let $u_j = u_j(x; \varepsilon) \in B_{r_1}^\alpha(\overline{\Omega})$ be the unique small solution of the Dirichlet problem

$$\begin{cases} -\Delta u_j + \sum_{k=2}^{\infty} V_k^{(j)}(x)\frac{u_j^k}{k!} = 0 & \text{in} \quad \Omega, \\ u_j = \varepsilon_1 f_1 + \varepsilon_2 f_2 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

for $j = 1, 2$. Differentiating (2.1) with respect to ε_l, $l = 1, 2$, and using that $u_j(x, 0) = 0$, we get

$$\begin{cases} \Delta v_j^{(l)} = 0 & \text{in} \quad \Omega, \\ v_j^{(l)} = f_l & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where $v_j^{(l)} = \partial_{\varepsilon_l} u_j|_{\varepsilon=0}$. By the uniqueness and the elliptic regularity for the Dirichlet problem (2.2), we see that $v^{(l)} := v_1^{(l)} = v_2^{(l)} \in C^\infty(\overline{\Omega})$, $l = 1, 2$.

Applying $\partial_{\varepsilon_1}\partial_{\varepsilon_2}|_{\varepsilon=0}$ to (2.1), we get

$$\begin{cases} -\Delta(\partial_{\varepsilon_1}\partial_{\varepsilon_2} u_j|_{\varepsilon=0}) + V_2^{(j)}(x)\partial_{\varepsilon_1} u_j|_{\varepsilon=0}\partial_{\varepsilon_2} u_j|_{\varepsilon=0} = 0 & \text{in} \quad \Omega, \\ \partial_{\varepsilon_1}\partial_{\varepsilon_2} u_j|_{\varepsilon=0} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3}$$

since $\partial_{\varepsilon_1}\partial_{\varepsilon_2}(\sum_{k=3}^{\infty} V_k^{(j)}(x)\frac{u_j^k}{k!})$ is a sum of terms each of them containing positive powers of u_j, which vanish when $\varepsilon = 0$, and the only term in $\partial_{\varepsilon_1}\partial_{\varepsilon_2}(V_2^{(j)}(x)\frac{u_j^2}{2!})$ which does not contain a positive power of u_j is $V_2^{(j)}(x)\partial_{\varepsilon_1} u_j\partial_{\varepsilon_2} u_j$. Setting $w_j = \partial_{\varepsilon_1}\partial_{\varepsilon_2} u_j|_{\varepsilon=0}$, we get from (2.3) that

$$\begin{cases} -\Delta w_j + V_2^{(j)}(x)v^{(1)}v^{(2)} = 0 & \text{in} \quad \Omega, \\ w_j = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.4}$$

The fact that $\Lambda^\Gamma_{V^{(1)}}(\varepsilon_1 f_1 + \varepsilon_2 f_2) = \Lambda^\Gamma_{V^{(2)}}(\varepsilon_1 f_1 + \varepsilon_2 f_2)$ for all small $\varepsilon_1, \varepsilon_2$ and all $f_1, f_2 \in C^\infty(\partial\Omega)$ with $\mathrm{supp}\,(f_1), \mathrm{supp}\,(f_2) \subset \Gamma$ implies that $\partial_\nu u_1|_\Gamma = \partial_\nu u_2|_\Gamma$. Therefore, an application of $\partial_{\varepsilon_1}\partial_{\varepsilon_2}|_{\varepsilon=0}$ gives $\partial_\nu w_1|_\Gamma = \partial_\nu w_2|_\Gamma$. Multiplying (2.4) by $v^{(3)} \in C^\infty(\overline\Omega)$ harmonic in Ω and applying Green's formula, we get

$$\int_\Omega (V_2^{(1)} - V_2^{(2)}) v^{(1)} v^{(2)} v^{(3)} dx = \int_{\partial\Omega\setminus\Gamma} (\partial_\nu w_1 - \partial_\nu w_2) v^{(3)} dS = 0,$$

provided that $\mathrm{supp}\,(v^{(3)}|_{\partial\Omega}) \subset \Gamma$. Hence, we obtain that

$$\int_\Omega (V_2^{(1)} - V_2^{(2)}) v^{(1)} v^{(2)} v^{(3)} dx = 0$$

for any $v^{(l)} \in C^\infty(\overline\Omega)$ harmonic in Ω, such that $\mathrm{supp}\,(v^{(l)}|_{\partial\Omega}) \subset \Gamma$, $l = 1, 2, 3$.

Now taking $v^{(3)} \not\equiv 0$ and using the result of [3] which says that the set of products of two harmonic functions in $C^\infty(\overline\Omega)$ which vanish on the closed proper subset $\partial\Omega \setminus \Gamma$ of the boundary $\partial\Omega$ is dense in $L^1(\Omega)$, we conclude that

$$(V_2^{(1)} - V_2^{(2)}) v^{(3)} = 0 \quad \text{in} \quad \Omega.$$

Now $v^{(3)}$ is harmonic and therefore, the set $(v^{(3)})^{-1}(0)$ is of measure zero, see [15]. Hence $V_2^{(1)} = V_2^{(2)}$ in Ω.

Let $m \geq 3$ and assume that $V_k := V_k^{(1)} = V_k^{(2)}$ in Ω for all $k = 2, \ldots, m-1$. To show that $V_m^{(1)} = V_m^{(2)}$, following [4], [12], we shall perform the mth order linearization of the partial Dirichlet–to–Neumann map. To that end, let $u_j = u_j(x; \varepsilon) \in B_{r_1}^\alpha(\overline\Omega)$ be the unique small solution of the Dirichlet problem

$$\begin{cases} -\Delta u_j + \sum_{k=2}^\infty V_k^{(j)}(x) \frac{u_j^k}{k!} = 0 & \text{in} \quad \Omega, \\ u_j = \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m & \text{on} \quad \partial\Omega, \end{cases} \tag{2.5}$$

for $j = 1, 2$. Next we would like to apply $\partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m}|_{\varepsilon=0}$ to (2.5). We first observe that $\partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m}(\sum_{k=m+1}^\infty V_k^{(j)}(x) \frac{u_j^k}{k!})$ is a sum of terms each of them containing positive powers of u_j, which vanish when $\varepsilon = 0$. Moreover, the only term in $\partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m}(V_m^{(j)}(x) \frac{u_j^m}{m!})$ which does not contain a positive power of u_j is $V_m^{(j)}(x) \partial_{\varepsilon_1} u_j \ldots \partial_{\varepsilon_m} u_j$. Finally, the expression $\partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m}(\sum_{k=2}^{m-1} V_k(x) \frac{u_j^k}{k!})|_{\varepsilon=0}$ contains only derivatives of u_j of the form $\partial^k_{\varepsilon_{l_1}, \ldots, \varepsilon_{l_k}} u_j|_{\varepsilon=0}$ with $k = 1, \ldots, m-1$, $\varepsilon_{l_1}, \ldots, \varepsilon_{l_k} \in \{\varepsilon_1, \ldots, \varepsilon_m\}$. We claim that $\partial^k_{\varepsilon_{l_1}, \ldots, \varepsilon_{l_k}} u_1|_{\varepsilon=0} = \partial^k_{\varepsilon_{l_1}, \ldots, \varepsilon_{l_k}} u_2|_{\varepsilon=0}$ for $k = 1, \ldots, m-1$, $\varepsilon_{l_1}, \ldots, \varepsilon_{l_k} \in \{\varepsilon_1, \ldots, \varepsilon_m\}$. This follows by applying the operators $\partial^k_{\varepsilon_{l_1}, \ldots, \varepsilon_{l_k}}|_{\varepsilon=0}$ to (2.5), using the fact that $V_k^{(1)} = V_k^{(2)}$, $k = 2, \ldots, m-1$, and the unique solvability of the Dirichlet problem for the Laplacian. Hence, $H_m(x) := \partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m}(\sum_{k=2}^{m-1} V_k(x) \frac{u_j^k}{k!})|_{\varepsilon=0}$ is independent of $j = 1, 2$.

Therefore, we get

$$\begin{cases} -\Delta(\partial_{\varepsilon_1}\ldots\partial_{\varepsilon_m}u_j|_{\varepsilon=0}) + V_m^{(j)}(x)\partial_{\varepsilon_1}u_j|_{\varepsilon=0}\ldots\partial_{\varepsilon_m}u_j|_{\varepsilon=0} = -H_m \quad \text{in} \quad \Omega, \\ \partial_{\varepsilon_1}\ldots\partial_{\varepsilon_m}u_j|_{\varepsilon=0} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

(2.6)

Proceeding as in the case $m = 2$, we see that

$$\int_\Omega (V_m^{(1)} - V_m^{(2)})v^{(1)}\cdots v^{(m+1)}dx = 0$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma$, $l = 1,\ldots,m+1$. Arguing as in the case $m = 2$, we complete the proof of Theorem 1.1.

Acknowledgements

References

[1] Cârstea, C., Nakamura, G., Vashisth, M., *Reconstruction for the coefficients of a quasilinear elliptic partial differential equation*, preprint https://arxiv.org/abs/1903.07034.

[2] Chen, X., Lassas, M., Oksanen, L., Paternain, G., *Detection of Hermitian connections in wave equations with cubic non-linearity*, preprint https://arxiv.org/abs/1902.05711.

[3] Dos Santos Ferreira, D., Kenig, C., Sjöstrand, J., Uhlmann, G., *On the linearized local Calderón problem*, Math. Res. Lett. **16** (2009), no. 6, 955–970.

[4] Feizmohammadi, A., Oksanen, L., *An inverse problem for a semi-linear elliptic equation in Riemannian geometries*, preprint https://arxiv.org/abs/1904.00608.

[5] Gilbarg, D., Trudinger, N., *Elliptic partial differential equations of second order*, Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[6] Imanuvilov, O., Uhlmann, G., Yamamoto, M., *The Calderón problem with partial data in two dimensions*, J. Amer. Math. Soc. **23** (2010), no. 3, 655–691.

[7] Imanuvilov, O., Yamamoto, M., *Unique determination of potentials and semilinear terms of semilinear elliptic equations from partial Cauchy data*, J. Inverse Ill-Posed Probl. **21** (2013), no. 1, 85–108.

[8] Isakov, V., Nachman, A., *Global uniqueness for a two-dimensional semilinear elliptic inverse problem*, Trans. Amer. Math. Soc. **347** (1995), no. 9, 3375–3390.

[9] Isakov, V., Sylvester, J., *Global uniqueness for a semilinear elliptic inverse problem*, Comm. Pure Appl. Math. **47** (1994), no. 10, 1403–1410.

[10] Kang, K., Nakamura, G., *Identification of nonlinearity in a conductivity equation via the Dirichlet–to–Neumann map*, Inverse Problems **18** (2002), no. 4, 1079–1088.

[11] Kurylev, Y., Lassas, M., Uhlmann, G., *Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations*, Invent. Math. 212 (2018), no. 3, 781–857.

[12] Lassas, M., Liimatainen, T., Lin, Y-H., Salo, M., *Inverse problems for elliptic equations with power type nonlinearities*, preprint https://arxiv.org/abs/1903.12562.

[13] Lassas, M., Liimatainen, T., Lin, Y-H., Salo, M., *Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations*, preprint https://arxiv.org/abs/1905.02764.

[14] Lassas, M., Uhlmann, G., Wang, Y., *Inverse problems for semilinear wave equations on Lorentzian manifolds*, Comm. Math. Phys. **360** (2018), no. 2, 555–609.

[15] Mityagin, B., *The zero set of a real analytic function*, preprint https://arxiv.org/abs/1512.07276v1.

[16] Muñoz, C., Uhlmann, G., *The Calderon problem for quasilinear elliptic equations*, preprint, arXiv:1806.09586.

[17] Sun, Z., *On a quasilinear inverse boundary value problem*, Math. Z. **221** (1996), no. 2, 293–305. .

[18] Sun, Z., Uhlmann, G., *Inverse problems in quasilinear anisotropic media*, Amer. J. Math. **119** (1997), no. 4, 771–797.

[19] Sun, Z., *An inverse boundary-value problem for semilinear elliptic equations*, Electron. J. Differential Equations 2010, No. 37, 5 pp.

K. Krupchyk, Department of Mathematics, University of California, Irvine, CA 92697-3875, USA

E-mail address: katya.krupchyk@uci.edu

G. Uhlmann, Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA, and Institute for Advanced Study of the Hong Kong University of Science and Technology

E-mail address: gunther@math.washington.edu