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Ferromagnetic to Paramagnetic Transition in Spherical Spin Glass

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Abstract
We consider the spherical spin glass model defined by a combination of the pure 2-spin spherical Sherrington–Kirkpatrick Hamiltonian and the ferromagnetic Curie–Weiss Hamiltonian. In the large system limit, there is a two-dimensional phase diagram with respect to the temperature and the coupling strength. The phase diagram is divided into three regimes; ferromagnetic, paramagnetic, and spin glass regimes. The fluctuations of the free energy are known in each regime. In this paper, we study the transition between the ferromagnetic regime and the paramagnetic regime in a critical scale.

Keywords Free energy · Spherical SK model · Phase transition · Spiked random matrices

1 Introduction

We consider a disordered system defined by random Gibbs measures whose Hamiltonian is the sum of a spin glass Hamiltonian and a ferromagnetic Hamiltonian. Depending on the strength of the coupling constant and the temperature, the system may exhibit several phases in the large system limit. The paper is concerned with the fluctuations of the free energy near the boundary between two phases known as ferromagnetic and paramagnetic regimes.

Consider the sum of the pure 2-spin spherical Sherrington–Kirkpatrick (SSK) Hamiltonian and the Curie–Weiss (CW) Hamiltonian. We call this sum the SSK + CW Hamiltonian. We denote the coupling constant by J and the inverse temperature by β. We consider the random

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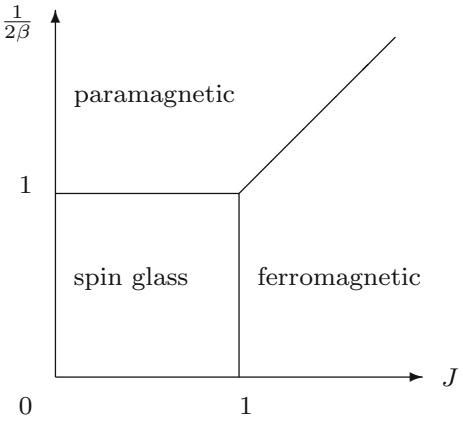

Fig. 1 Phase diagram for SSK + CW model. Here, β is the inverse temperature and J is the coupling constant

Gibbs measure with the SSK + CW Hamiltonian. The focus of this paper is on the free energy.

The limiting free energy was obtained non-rigorously by Kosterlitz et al. [21] in 1976. When $J = 0$, this formula is the explicit evaluation of the Crisanti–Sommers formula [15] (which was proved rigorously by Talagrand [26]) in the case of the pure 2-spin SSK. The Crisanti–Sommers formula is the spherical version of the Parisi formula [24,27]. The formula of Kosterlitz, Thouless, and Jones shows a two-dimensional phase transition: see Fig. 1. The three regimes are determined by the condition that $\max\{1, \frac{1}{2\beta}, J\}$ is equal to 1 (spin glass regime), $\frac{1}{2\beta}$ (paramagnetic regime) or J (ferromagnetic regime). The limiting free energy is analytic with respect to both β and J in each regime, but not on the boundary.

Recently, the authors of [5] showed that the result of Kosterlitz, Thouless, and Jones is rigorous. Furthermore, the authors also evaluated the distribution of the fluctuations of the free energy in each regime. (The case when $J = 0$ was obtained earlier in [3].) The order of the fluctuations are $N^{-2/3}$, N^{-1}, $N^{-1/2}$ and the limiting distributions are Tracy–Widom, Gaussian, and Gaussian in the spin glass, paramagnetic regime, ferromagnetic regime, respectively. In the same paper, the transition between the spin glass regime and the ferromagnetic regime was also studied. However, the other two transitions and the triple point were left open. The goal of this paper is to describe the transition between the paramagnetic regime and and the ferromagnetic regime.

Another system which combines a spin glass and a ferromagnetic model is the SSK with an external field. The difference between the CW Hamiltonian and an external field is that one is a quadratic function and the other is a linear function of the spin variables. These two models are related; see [12] for a one-sided inequality. For the spin glass with external field, the fluctuations of the free energy were computed recently in [13,14] when the coupling constant is positive (for both SSK and SK (Sherrington–Kirkpatrick) cases with general spin interactions). However, the transitions are not obtained except for certain large deviation results [16,18]. One of the interests of the SSK + CW model is that it is an easier model which can be analyzed in detail in the transitional regimes.

1.1 Model

Let

$$S_{N-1} = \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \sigma_1^2 + \cdots + \sigma_N^2 = N\} \tag{1.1}$$

be a sphere in $\mathbb{R}^N$ of radius $\sqrt{N}$. Define the SSK + CW Hamiltonian by

$$H_N(\boldsymbol{\sigma}) = H_N^{\mathrm{SSK}}(\boldsymbol{\sigma}) + H_N^{\mathrm{CW}}(\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} \in S_{N-1} \tag{1.2}$$

where

$$H_N^{\mathrm{SSK}}(\boldsymbol{\sigma}) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} A_{ij}\sigma_i\sigma_j, \qquad H_N^{\mathrm{CW}}(\boldsymbol{\sigma}) = \frac{J}{N} \sum_{i,j=1}^{N} \sigma_i\sigma_j = \frac{J}{N} \left(\sum_{i=1}^{N} \sigma_i \right)^2. \tag{1.3}$$

Here J is the coupling constant. The random coefficients A_{ij} satisfy $A_{ij} = A_{ji}$ and A_{ij}, $i \le j$, are independent centered random variables. We call A_{ij} disorder variables. The precise conditions are given in Definition 1.1 below. Note that as a function of $\boldsymbol{\sigma}$, $H_N^{\mathrm{CW}}(\boldsymbol{\sigma})$ is large when the coordinates of $\boldsymbol{\sigma}$ have same sign. On the other hand, the maximizers $\boldsymbol{\sigma}$ of $H_N^{\mathrm{SSK}}(\boldsymbol{\sigma})$ depend highly on $\{A_{ij}\}$.

With $\beta > 0$ representing the inverse temperature, the free energy and the partition function are defined by

$$F_N = \frac{1}{N} \log Z_N, \qquad Z_N = \int_{S_{N-1}} e^{\beta H_N(\sigma)} \mathrm{d}\omega_N(\boldsymbol{\sigma}) \tag{1.4}$$

where ω_N is the normalized uniform measure on S_{N-1}. Note that F_N and Z_N are random variables since they depend on the disorder variables A_{ij}. The free energy and the partition function depend on the parameters β and J,

$$F_N = F_N(\beta, J), \qquad Z_N = Z_N(\beta, J). \tag{1.5}$$

Since the Curie–Weiss Hamiltonian is a quadratic function of the spin variable, we can write the SSK + CW Hamiltonian as $H_N(\boldsymbol{\sigma}) = \sum_{i,j=1}^{N} M_{ij}\sigma_i\sigma_j$ where $M_{ij} = \frac{1}{\sqrt{N}}A_{ij} + \frac{J}{N}$ are non-centered random variables. In terms of matrix notations,

$$H_N(\boldsymbol{\sigma}) = \boldsymbol{\sigma}^T M \boldsymbol{\sigma}, \qquad M = \frac{1}{\sqrt{N}} A + \frac{J}{N} \mathbf{1}\mathbf{1}^T \tag{1.6}$$

with $A = (A_{ij})_{1\le i,j\le N}$, $\mathbf{1} = (1, \dots, 1)^T$, $M = (M_{ij})_{1\le i,j\le N}$, and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)^T$. The non-centered random symmetric matrix M is an example of a real Wigner matrix perturbed by a deterministic finite rank matrix. Such matrices are often called spiked random matrices. We will use the eigenvalues of spiked random matrices in our analysis of the free energy.

We assume the following conditions on the disorder variables.

Definition 1.1 (*Assumptions on disorder variables*) Let A_{ij}, $i \le j$, be independent real random variables satisfying the following conditions:

- All moments of A_{ij} are finite and $\mathbb{E}[A_{ij}] = 0$ for all $i \le j$.
- For all $i < j$, $\mathbb{E}[A_{ij}^2] = 1$, $\mathbb{E}[A_{ij}^3] = W_3$, and $\mathbb{E}[A_{ij}^4] = W_4$ for some constants $W_3 \in \mathbb{R}$ and $W_4 \ge 0$.
- For all i, $\mathbb{E}[A_{ii}^2] = w_2$ for a constant $w_2 \ge 0$.

Set $A_{ij} = A_{ji}$ for $i > j$. Let $A = (A_{ij})_{i,j=1}^N$ and we call it a Wigner matrix (of zero mean).

Definition 1.2 (*Eigenvalues of non-zero mean Wigner matrices*) Let M be the $N \times N$ symmetric matrix defined in (1.6). We call it a Wigner matrix of non-zero mean [1]. Its eigenvalues are denoted by

[1] In [5], we consider the case when the diagonal entries of M have mean $\frac{J'}{N}$ and the off-diagonal entries have mean $\frac{J}{N}$ where J and J' are allowed to be different. However, in this case, $M = \frac{1}{\sqrt{N}} + \frac{J}{N}\mathbf{1}\mathbf{1}^T + \frac{J'-J}{N} I$ where I is the identity matrix. This only shifts all eigenvalues by a deterministic small number. As we will see in Remark 2.1, it is not more general than the case with $J' = J$.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N. \tag{1.7}$$

We introduce the following terminology.

Definition 1.3 (*High probability event*) We say that an N-dependent event Ω_N holds with high probability if, for any given $D > 0$, there exists $N_0 > 0$ such that

$$\mathbb{P}(\Omega_N^c) \leq N^{-D}$$

for any $N \geq N_0$.

1.2 Previous Results in Each Regime

We review the results on the fluctuations in each regime obtained in [5]. We state two types of results: one in terms of the eigenvalues of M and the other in terms of limiting distributions.

Set

$$\tilde{J} := \max\{J, 1\}. \tag{1.8}$$

It was shown in [5] that the following holds with high probability. In both ferromagnetic and the spin glass regimes (given by $\tilde{J} > \frac{1}{2\beta}$), with any $\epsilon > 0$,

$$F_N = \tilde{F}_N + \left(\beta - \frac{1}{2\tilde{J}}\right)\left(\lambda_1 - \tilde{J} - \frac{1}{\tilde{J}}\right) + o\left(N^{-1+\epsilon}\right). \tag{1.9}$$

In the paramagnetic regime (given by $\tilde{J} < \frac{1}{2\beta}$),

$$F_N = \tilde{F}_N - \frac{1}{2N}\sum_{i=1}^{N} \log\left(2\beta + \frac{1}{2\beta} - \lambda_i\right) + o(N^{-1}). \tag{1.10}$$

Here, $\tilde{F}_N$ is a deterministic function of N, β, J. The above results show that the fluctuations of F_N are determined, to the leading order, by the top eigenvalue λ_1 in the ferromagnetic and spin glass regimes, while they are determined by all eigenvalues in the paramagnetic regime.

A limit theorem for F_N follows if we use limit theorems for the eigenvalues of random matrices. The relevant random matrices are Wigner matrices of non-zero mean in (1.6). For such random matrices, the following is known [11,25] (see [6] for complex matrices):

$$\begin{cases} N^{2/3}\,(\lambda_1 - 2) \Rightarrow \mathrm{TW}_1 & \text{if } J < 1, \\ N^{1/2}\left(\lambda_1 - J - \frac{1}{J}\right) \Rightarrow \mathcal{N}(W_3(J^{-2} - J^{-4}), 2(1 - J^{-2})) & \text{if } J > 1, \end{cases} \tag{1.11}$$

where the convergences are in distribution. Here TW_1 denotes the GOE Tracy–Widom distribution and $\mathcal{N}(a, b)$ denotes the Gaussian distribution of mean a and variance b. The dichotomy is due to the effect of the non-zero mean; if J is not large enough (i.e. $J < 1$), then the influence of the non-zero mean is negligible to contribute to the fluctuations of the top eigenvalue. For $J < 1$, the top eigenvalue is close to the second eigenvalue with order $O(N^{-2/3+\epsilon})$. But for $J > 1$, the difference of the top eigenvalue and the second eigenvalue is of order $O(1)$.

On the other hand, the following is also known (see [5, Theorem 1.6]): if a function φ is smooth in an open interval containing the interval $[-2, \tilde{J} + \tilde{J}^{-1}]$, then

$$\sum_{i=1}^{N} \varphi(\lambda_i) - N\int_{-2}^{2} \varphi(x)\mathrm{d}\sigma_{scl}(x) \Rightarrow \mathcal{N}(f, a), \qquad \mathrm{d}\sigma_{scl}(x) := \frac{\sqrt{4 - x^2}}{2\pi}\mathrm{d}x, \tag{1.12}$$

for some explicit constants f, a. This result is applicable to the paramagnetic regime.

Together, we have the following asymptotic results obtained in Theorem 1.4 of [5] (with a small correction in [4]):

(i) (Spin glass regime) If $\beta > \frac{1}{2}$ and $J < 1$, then

$$\frac{1}{\beta - \frac{1}{2}} N^{2/3} (F_N - F) \Rightarrow \mathrm{TW}_1 . \tag{1.13}$$

(ii) (Paramagnetic regime) If $\beta < \frac{1}{2}$ and $\beta < \frac{1}{2J}$, then

$$N (F_N - F) \Rightarrow \mathcal{N} (f_1, \alpha_1) . \tag{1.14}$$

(iii) (Ferromagnetic regime) If $J > 1$ and $\beta > \frac{1}{2J}$, then

$$\sqrt{N} (F_N - F) \Rightarrow \mathcal{N} \left(f_2', \alpha_2'\right) . \tag{1.15}$$

for some deterministic function $F = F(\beta, J)$ and some explicit constants f_1, α_1, f_2' and α_2' depending on β and J.

1.3 Results

We state the results on the transition between the paramagnetic regime and the ferromagnetic regime. The boundary between these two regimes is given by the equation $\frac{1}{2\beta} = J$ with $J > 1$. In the transitional regime, the correct scaling turns out to be the following: let $J > 1$ be fixed and let $\beta = \beta_N$ be given by

$$2\beta = \frac{1}{J} + \frac{B}{\sqrt{N}} \tag{1.16}$$

with fixed $B \in \mathbb{R}$. The following is the first main result of this paper. This relates the free energy with the eigenvalues of M.

Theorem 1.1 *Let β be given by (1.16). Then, for every $0 < \epsilon < \frac{1}{8}$,*

$$F_N = \tilde{F}_N - \frac{1}{2N} \sum_{i=2}^{N} g(\lambda_i) + \frac{1}{N} Q(\chi_N) + O\left(N^{-3/2+4\epsilon}\right), \tag{1.17}$$

with high probability as $N \to \infty$, where

$$\tilde{F}_N = \beta(J + J^{-1}) - \frac{1}{2} - \frac{1}{2} \log(2\beta) + \frac{1}{N} \left(\frac{1}{4} \log N + \log \frac{\beta}{\sqrt{\pi}} \right), \tag{1.18}$$

$\chi_N := \sqrt{N}(\lambda_1 - J + J^{-1})$ *and* $g(z) := \log(J + J^{-1} - z)$. *Also,*

$$Q(x) = \frac{s(x)}{2(s(x) - x)} - \frac{s(x)^2}{4(J^2 - 1)} + \frac{\log(s(x) - x)}{2} + \log \mathbf{I} \left(\frac{(s(x) - x)^2}{J^2 - 1} \right) \tag{1.19}$$

with

$$s(x) = \frac{x - B(J^2 - 1) + \sqrt{(x + B(J^2 - 1))^2 + 4(J^2 - 1)}}{2} \tag{1.20}$$

and

$$\mathbf{I}(\alpha) = \int_{-\infty}^{\infty} \frac{e^{-\frac{\alpha}{4} t^2 + \frac{it}{2}}}{\sqrt{1 + \mathrm{i}t}} \mathrm{d}t, \tag{1.21}$$

where the square root denotes the principal branch.

The Formula (1.17) shows a combined contribution from $\lambda_2, \ldots, \lambda_N$ and a distinguished contribution from λ_1. Compare the formula with (1.9) and (1.10).

Now we state a result analogous to (1.14) and (1.15). This follows if we have limit theorems for $Q(\chi_N)$ and $\sum_{i=2}^N g(\lambda_i)$. From the second part of (1.11), $Q(\chi_N)$ converges to an explicit function of a Gaussian random variable. On the other hand, $\sum_{i=2}^N g(\lambda_i)$ is different from $\sum_{i=1}^N g(\lambda_i)$ by one term. It is not difficult to show that removing one term does not affect the fluctuations much and the fluctuations are still given by a Gaussian random variable similar to (1.12); see Theorem 2.1 in the next section. In random matrix theory, these sums are known as partial linear statistic and linear statistic, respectively. The main technical part of this paper is to evaluate the joint distribution of $Q(\chi_N)$ and $\sum_{i=2}^N g(\lambda_i)$. We show that jointly they converge in distribution to a bivariate Gaussian variable with an explicit covariance. See the next section for the precise statement. These results are interesting on their own in random matrix theory. Putting together, we obtain the following result.

Theorem 1.2 *We have*

$$N\left(F_N - \frac{1}{4J^2} - \frac{B}{2J\sqrt{N}} - \frac{\log N}{4N} - \frac{B^2J^2}{4N}\right) \Rightarrow \mathcal{G}_1 + Q(\mathcal{G}_2) \tag{1.22}$$

in distribution as $N \to \infty$ where $\mathcal{G}_1$ and $\mathcal{G}_2$ are bivariate Gaussian random variables with

$$\mathbb{E}[\mathcal{G}_1] = \frac{1}{4}\log(J^2-1) + \frac{w_2-2}{4J^2} + \frac{W_4-3}{8J^4} + \log\frac{1}{2\sqrt{\pi}J}, \tag{1.23}$$

$$\mathrm{Var}[\mathcal{G}_1] = -\frac{1}{2}\log(1-J^{-2}) + \frac{w_2-2}{4J^2} + \frac{W_4-3}{8J^4}, \tag{1.24}$$

$$\mathbb{E}[\mathcal{G}_2] = W_3(J^{-2}-J^{-4}), \qquad \mathrm{Var}[\mathcal{G}_2] = 2(1-J^{-2}), \tag{1.25}$$

and

$$\mathrm{Cov}(\mathcal{G}_1, \mathcal{G}_2) = \frac{W_3(J^{-2}-J^{-4})}{2}. \tag{1.26}$$

Note that $\mathcal{G}_1$ and $\mathcal{G}_2$ do not depend on B. The function Q is defined in (1.19).

Note that if the third moment W_3 of A_{ij} with $i \neq j$ is zero, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent Gaussians.

The above result is consistent with the results on ferromagnetic and paramagnetic regimes if we let formally $B \to +\infty$ and $B \to -\infty$, respectively. One can show that when $B \to +\infty$, $Q(\mathcal{G}_2)$ dominates $\mathcal{G}_1$. Furthermore, while $Q(\mathcal{G}_2)$ is not Gaussian, upon proper normalization, it converges to a Gaussian as $B \to +\infty$. See Fig. 2. On the other hand, when $B \to -\infty$, the leading two terms of $Q(\mathcal{G}_2)$ are constants and the random part is smaller than $\mathcal{G}_1$. See Sect. 6 for details.

Let us comment on the other transitions in the phase digram in Fig. 1. As mentioned before, the transition between the spin glass and ferromagnetic regimes was discussed in [5]. Note that (1.9) is valid in both regimes. It was shown that if we let $\beta > 1/2$ be fixed and consider N-dependent $J = 1 + wN^{-1/3}$, then for each $w \in \mathbb{R}$, (1.9) still holds. Now, for such J, it was shown in [9] that $N^{2/3}(\lambda_1 - 2) \Rightarrow \mathrm{TW}_{1,w}$ where $\mathrm{TW}_{1,w}$ is a one-parameter family of random variables interpolating TW and Gaussian distributions. Hence, we obtain the fluctuations for the transitional regime.

On the other hand, the transition between the spin glass and paramagnetic regimes is an open question. By matching the fluctuation scales in both regimes, we expect that the critical scale is $\beta = \frac{1}{2} + O(\frac{\sqrt{\log N}}{N^{1/3}})$.

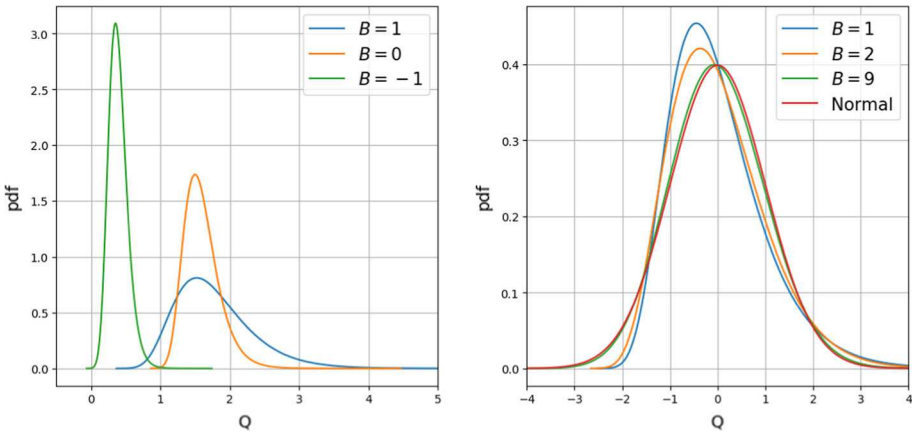

Fig. 2 **a** Probability density function of $Q(\mathcal{G}_2)$ for $B = -1, 0, 1$, **b** probability density function of normalized $Q(\mathcal{G}_2)$ resembles a Gaussian density as $B \to +\infty$

1.4 Organization

The rest of the paper is organized as follows. In Sect. 2, we first state new results on random matrices. They are given in Theorem 2.1 (partial linear statistics) and Theorem 2.2 (joint convergence). Using them, we derive Theorem 1.2 from Theorem 1.1. In Sect. 3, we prove Theorem 1.1. In the next two sections, we prove the random matrix results stated in Sect. 2; Theorem 2.1 in Sect. 4 and Theorem 2.2 in Sect. 5. In Sect. 6, we show that Theorem 1.2 is consistent with the previous results on ferromagnetic and paramagnetic regimes.

2 Results on Wigner Matrices with Non-zero Mean

In order to prove Theorem 1.2 from Theorem 1.1, we need some new results on random matrices. We need (i) a limit theorem for partial linear statistics $\sum_{i=2}^{N} g(\lambda_i)$ and (ii) a joint convergence of the large eigenvalue and partial linear statistics. These results are interesting on their own in random matrix theory. We state them here and prove them in Sects. 4 and 5 below. Using these results, we prove Theorem 1.2 in Sect. 2.3.

Recall that the $N \times N$ symmetric matrix M is given by $M = \frac{1}{\sqrt{N}} A + \frac{J}{N} \mathbf{1}\mathbf{1}^T$ where $A = (A_{ij})$ is a symmetric matrix with independent entries for $i \le j$ satisfying the conditions given in Definition 1.1 and $\mathbf{1} = (1, \dots, 1)^T$. The matrix M is called a Wigner matrix with a non-zero mean $\frac{J}{N}$. Recall that we assume

$$J > 1. \tag{2.1}$$

The eigenvalues of M are denoted by $\lambda_1 \ge \cdots \ge \lambda_N$.

It is known that λ_1 is close to $J + J^{-1}$ with high probability and $\lambda_2, \dots, \lambda_N$ are in a neighborhood of $[-2, 2]$ with high probability. See Lemma 3.2 below for the precise statement.

2.1 Partial Linear Statistics

A linear statistic is the sum of a function of the eigenvalues. The fluctuations of linear statistics for Wigner matrices and other random matrix ensembles are of central interest in the random matrix theory; see, for example, [2,19,22]. For Wigner matrices with non-zero mean, the following result was obtained in Theorem 1.6 and Remark 1.7 of [5]. Set

$$\hat{J} = J + J^{-1}. \tag{2.2}$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function which is analytic in an open neighborhood of $[-2, \hat{J}]$ and has compact support. Then, as $N \to \infty$, the random variable

$$\mathcal{N}_N(\varphi) := \sum_{i=1}^{N} \varphi(\lambda_i) - N \int_{-2}^{2} \varphi(x) \mathrm{d}\sigma_{scl}(x) \Rightarrow \mathcal{N}(M(\varphi), V(\varphi)) \tag{2.3}$$

where

$$\begin{aligned} M(\varphi) &= \frac{1}{4}(\varphi(2) + \varphi(-2)) - \frac{3}{2}\tau_0(\varphi) - J^{-1}\tau_1(\varphi) + (w_2 - 2)\tau_2(\varphi) \\ &\quad + (W_4 - 3)\tau_4(\varphi) + \varphi(\hat{J}) - \sum_{\ell=2}^{\infty} J^{-\ell}\tau_\ell(\varphi), \\ V(\varphi) &= (w_2 - 2)\tau_1(\varphi)^2 + (W_4 - 3)\tau_2(\varphi)^2 + 2\sum_{\ell=1}^{\infty} \ell\tau_\ell(\varphi)^2. \end{aligned} \tag{2.4}$$

Here, $W_4 = \mathbb{E}[A_{12}^4]$, $w_2 = \mathbb{E}[A_{11}^2]$, and

$$\tau_\ell(\varphi) = \frac{1}{\pi}\int_{-2}^{2} \varphi(x)\frac{T_\ell(x/2)}{\sqrt{4 - x^2}}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(2\cos(\theta))\cos(\ell\theta)\mathrm{d}\theta, \tag{2.5}$$

where $T_\ell(t)$ are the Chebyshev polynomials of the first kind.

We are interested in a partial linear statistic, $\sum_{i=2}^{N} \varphi(\lambda_i)$. See [7,23] for other types of partial linear statistics. The partial linear static $\sum_{i=2}^{N} \varphi(\lambda_i)$ is the linear statistic minus one term $\varphi(\lambda_1)$. Since $\lambda_1 \to \hat{J}$ in probability (see the second part of (1.11)), by (2.3), Slutsky's theorem implies that

$$\sum_{i=2}^{N} \varphi(\lambda_i) - N \int_{-2}^{2} \varphi(x) \mathrm{d}\sigma_{scl}(x) \Rightarrow \mathcal{N}(M(\varphi) - \varphi(\hat{J}), V(\varphi)).$$

Since this follows from (2.3), this is true assuming that φ is analytic in an open neighborhood of $[-2, \hat{J}]$. However, we are interested in the test function $\varphi(x) = g(x) = \log(\hat{J} - x)$ (see (1.17)). Since this function is not analytic at $x = \hat{J}$, the above simple argument does not apply. Nonetheless, if we adapt the proof of (2.3), one can show that it is enough to assume that the test function is analytic in a neighborhood of the interval $[-2, 2]$, not of $[-2, \hat{J}]$.

Theorem 2.1 *Let $J > 1$. Then for every test function φ which is analytic in a neighborhood of $[-2, 2]$,*

$$\mathcal{N}_N^{(2)}(\varphi) := \sum_{i=2}^{N} \varphi(\lambda_i) - N \int_{-2}^{2} \varphi(x) \mathrm{d}\sigma_{scl}(x) \Rightarrow \mathcal{N}(M^{(2)}(\varphi), V^{(2)}(\varphi)) \tag{2.6}$$

as $N \to \infty$ *with*

$$M^{(2)}(\varphi) = \frac{1}{4}(\varphi(2)+\varphi(-2)) - \frac{3}{2}\tau_0(\varphi) - J^{-1}\tau_1(\varphi) + (w_2-2)\tau_2(\varphi) + (W_4-3)\tau_4(\varphi) - \sum_{\ell=2}^{\infty} J^{-\ell}\tau_\ell(\varphi), \tag{2.7}$$

and $V^{(2)}(\varphi) = V(\varphi)$ *where* $V(\varphi)$ *is defined in* (2.4).

Note that

$$M^{(2)}(\varphi) = M(\varphi) - \varphi(\hat{J}) \tag{2.8}$$

for φ analytic in a neighborhood of $[-2, \hat{J}]$.

Remark 2.1 We comment on a case when the test function depends on N. Consider the function φ_N defined by

$$\varphi_N(x) = \varphi(x) + \frac{\phi(x)}{N} + O(N^{-2})$$

uniformly for x in a neighborhood of $[-2, 2]$ for analytic functions φ and ϕ. Define the corresponding linear statistic

$$\begin{aligned}\mathcal{N}_N^{(2)}(\varphi_N) :=& \sum_{i=2}^{N} \varphi_N(\lambda_i) - N\int_{-2}^{2} \varphi_N(x)\mathrm{d}\sigma_{scl}(x) \\ =& \mathcal{N}_N^{(2)}(\varphi) + \frac{1}{N}\left(\sum_{i=2}^{N}\phi(\lambda_i) - N\int \phi(x)\mathrm{d}\sigma_{scl}(x)\right) + O\left(\frac{1}{N}\right).\end{aligned} \tag{2.9}$$

By Theorem 2.1, the second order term converges to zero in probability. Thus, $\mathcal{N}_N^{(2)}(\varphi_N)$ and $\mathcal{N}_N^{(2)}(\varphi)$ converge to the same Gaussian distribution. The same argument also applies to full linear statistics; this is used in Remark 5.1 below. Now, the claim in footnote[1] is verified by noting that $\varphi(x + \frac{J'-J}{N}) = \varphi(x) + \frac{\varphi'(x)(J'-J)}{N} + O(N^{-2})$.

2.2 Joint Convergence of the Largest Eigenvalue and Linear Statistics

By Theorem 2.1 and the second part of (1.11), the partial linear statistic and the largest eigenvalue each converge to Gaussian distributions individually. The following theorem shows that they converge jointly to a bivariate Gaussian with an explicit covariance.

Theorem 2.2 *Let* $J > 1$. *Then for* $\varphi(x)$ *which is analytic in a neighborhood of* $[-2, 2]$, $\mathcal{N}_N^{(2)}(\varphi) := \sum_{i=2}^{N}\varphi(\lambda_i) - N\int_{-2}^{2}\varphi(x)\mathrm{d}\sigma_{scl}(x)$ *and* $\chi_N := \sqrt{N}(\lambda_1 - \hat{J})$ *converges jointly in distribution to a bivariate Gaussian variable with mean*

$$(M^{(2)}(\varphi),\ W_3(J^{-2}-J^{-4})) \tag{2.10}$$

and covariance

$$\begin{pmatrix} V^{(2)}(\varphi) & 2W_3\tau_2(\varphi)(1-J^{-2}) \\ 2W_3\tau_2(\varphi)(1-J^{-2}) & 2(1-J^{-2}) \end{pmatrix}. \tag{2.11}$$

The proof of this theorem, given in Sect. 5, is the main technical part of this paper. We prove the theorem first for the Gaussian case, and then use an interpolation argument.

2.3 Proof of Theorem 1.2

We now derive Theorem 1.2 from Theorem 1.1 using the results on the eigenvalues stated in the previous two subsections. The term $Q(\chi_N)$ converges to $Q(\mathcal{G}_2)$ in distribution from Theorem 2.2. Consider the rest. It was shown in (A.5) of [3] that for $g(z) = \log(J + J^{-1} - z)$,

$$\int g(z)\mathrm{d}\sigma_{scl}(x) = \frac{1}{2J^2} + \log J. \tag{2.12}$$

Inserting $2\beta = J^{-1} + BN^{-1/2}$ and using the Taylor expansion $\log(1+\frac{BJ}{\sqrt{N}}) = \frac{BJ}{\sqrt{N}} - \frac{B^2J^2}{2N} + O(N^{-3/2})$,

$$\begin{aligned}\tilde{F}_N - \frac{1}{2}\int g(z)\mathrm{d}\sigma_{scl}(x) =& \frac{1}{4J^2} + \frac{B}{2J\sqrt{N}} + \frac{\log N}{4N} \\ &+ \frac{1}{N}\left[\frac{B^2J^2}{4} + \log\frac{1}{2\sqrt{\pi}J}\right] + O(N^{-3/2}).\end{aligned} \tag{2.13}$$

We can evaluate $M^{(2)}(g)$ using (2.7) of [5] which evaluated the $M(h)$ with $h(x) = \log(2\beta + \frac{1}{2\beta} - x)$: (note that $J' = J$ here)

$$\begin{aligned}M^{(2)}(g) &= \lim_{\beta \to \frac{1}{2J}} \left(M(h) - \log\left(2\beta + \frac{1}{2\beta} - J - J^{-1}\right)\right) \\ &= -\frac{1}{2}\log(J^2 - 1) - \frac{w_2 - 2}{2J^2} - \frac{W_4 - 3}{4J^4}.\end{aligned} \tag{2.14}$$

The variance $V^{(2)}(g) = V(g)$, which is independent of J, is given by 4 times (3.13) of [3] if we replace 2β by J^{-1}:

$$V^{(2)}(g) = -2\log(1 - J^{-2}) + \frac{1}{J^2}(w_2 - 2) + \frac{1}{2J^4}(W_4 - 3). \tag{2.15}$$

For the covariance term, we have $\tau_2(g) = -\frac{1}{2J^2}$ from (A.17) of [3]. Hence, from Theorem 2.1 and 2.2, we obtain the result.

3 Proof of Theorem 1.1

The proof follows the steps for the proof of the Theorem 1.5 of [5] for paramagnetic and ferromagnetic regimes with necessary adjustments. The analysis is based on applying a method of steepest-descent to a random integral. The location of the critical point is important. In the transitional regime, the critical point is close to the largest eigenvalue but not as close as the ferromagnetic case. On the other hand, the critical point is away from the largest eigenvalue in the paramagnetic case. See Sect. 3.2 below for details.

3.1 Preliminaries

The following formula is a simple result in [21].

Lemma 3.1 ([21]; also Lemma 1.3 of [3]) *Let M be a real $N \times N$ symmetric matrix with eigenvalue $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Then for fixed $\beta > 0$,*

$$\int_{S_{N-1}} e^{\beta \boldsymbol{\sigma}^T M \boldsymbol{\sigma}} \mathrm{d}w_N(\boldsymbol{\sigma}) = C_N \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2} G(z)} \mathrm{d}z, \tag{3.1}$$

where

$$G(z) = 2\beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i) \tag{3.2}$$

and γ *is any constant satisfying* $\gamma > \lambda_1$, *the integration contour is the vertical line from* $\gamma - \mathrm{i}\infty$ *to* $\gamma + \mathrm{i}\infty$, *the* log *function is defined in the principal branch, and*

$$C_N = \frac{\Gamma(N/2)}{2\pi \mathrm{i} (N\beta)^{N/2-1}}. \tag{3.3}$$

Here $\Gamma(z)$ *denotes the Gamma function.*

Let M be a Wigner matrix with non-zero mean as in (1.6). Then its eigenvalues λ_i are random variables, and hence the above result gives a random integral representation of the partition function. In [3,5], the above random integral was evaluated using the method of steepest-descent for different choices of random matrices. The key ingredient in controlling the error term is a precise estimate for the eigenvalues which are obtained in the random matrix theory.

Lemma 3.2 (Rigidity of eigenvalues: Theorem 2.13 of [17] and Theorem 6.3 of [20]) *For each positive integer* $k \in [1, N]$, *set* $\hat{k} := \min\{k, N + 1 - k\}$. *Let* γ_k *be the classical location defined by*

$$\int_{\gamma_k}^{\infty} \mathrm{d}\sigma_{scl}(x) = \frac{1}{N} \left(k - \frac{1}{2} \right). \tag{3.4}$$

Then, for every $0 < \epsilon < \frac{1}{2}$,

$$|\lambda_k - \gamma_k| \le \hat{k}^{-1/3} N^{-2/3+\epsilon} \tag{3.5}$$

for all $k = 2, 3, \ldots, N$ *with high probability. Furthermore, for fixed* $J > 1$, *recall* $\hat{J} = J + J^{-1}$,

$$|\lambda_1 - \hat{J}| \le N^{-1/2+\epsilon} \tag{3.6}$$

holds with high probability.

From the rigidity, it is easy to obtain the following law of large numbers for eigenvalues.

Corollary 3.1 (c.f. Lemma 5.1 of [3]) *Fix* $\delta > 0$, *let* $\{f_\alpha\}_{\alpha \in I} \subset C^1[-2-\delta, 2+\delta]$ *be a family of monotonic increasing functions satisfying* $\sup_{\alpha \in I} \max_x |f_\alpha(x)| \le C_0$ *and* $\sup_{\alpha \in I} \max_x |f'_\alpha(x)| \le C_1$. *Then, for every* $0 < \epsilon < 1$,

$$\sup_{\alpha \in I} \left| \frac{1}{N} \sum_{i=2}^{N} f_\alpha(\lambda_i) - \int_{-2}^{2} f_\alpha(x) \mathrm{d}\sigma_{scl}(x) \right| = O\left(N^{-1+\epsilon} \right) \tag{3.7}$$

with high probability.

Proof Let $f = f_\alpha$ for some $\alpha \in I$. The absolute value on the left hand-side is bounded above by

$$\left| \frac{1}{N} \sum_{i=2}^{N} f(\lambda_i) - \frac{1}{N} \sum_{i=2}^{N} f(\gamma_i) \right| + \left| \frac{1}{N} \sum_{i=2}^{N} f(\gamma_i) - \int_{-2}^{2} f(x) \mathrm{d}\sigma_{scl}(x) \right|. \tag{3.8}$$

By Lemma 3.2,

$$\left|\frac{1}{N}\sum_{i=2}^{N}(f(\lambda_i)-f(\gamma_i))\right| \le \frac{\max|f'(x)|}{N}\sum_{i=2}^{N}|\lambda_i-\gamma_i| \le \frac{C_0}{N^{1-\epsilon}} \tag{3.9}$$

with high probability. On the other hand, set $\hat{\gamma}_j$ by

$$\int_{\hat{\gamma}_j}^{2} \mathrm{d}\sigma_{scl}(x) = \frac{j}{N}, \qquad j = 1, 2, \ldots, N, \tag{3.10}$$

and by convention $\hat{\gamma}_0 = 2$. As $f(x)$ is a monotonic increasing function, for $i = 2, 3, \ldots, N-1$,

$$\int_{\hat{\gamma}_{i+1}}^{\hat{\gamma}_i} f(x)\mathrm{d}\sigma_{scl}(x) \le \frac{1}{N}f(\gamma_i) \le \int_{\hat{\gamma}_{i-1}}^{\hat{\gamma}_{i-2}} f(x)\mathrm{d}\sigma_{scl}(x). \tag{3.11}$$

Thus,

$$\left|\frac{1}{N}\sum_{i=2}^{N} f(\gamma_i) - \int_{-2}^{2} f(x)\mathrm{d}\sigma_{scl}(x)\right| \le \frac{3\max|f(x)|}{N} \le \frac{3C_1}{N}. \tag{3.12}$$

Since the upper bounds are independent of f, we obtain the result. □

3.2 Steepest-Descent Analysis

We now apply steepest descent analysis to the integral in Lemma 3.1. We deform the contour to pass a critical point and show that the main contribution to the integral comes from a small neighborhood of the critical point. For $G(z)$ given in (3.2), it is easy to check that all solutions of $G'(z) = 0$ are real-valued, and there is a unique critical point γ which lies in the interval (λ_1, ∞) (see [5, Lemma 4.1]).

Note that since G is random, the critical point is also random. For the paramagnetic regime, it was shown in [5] that $\gamma - \lambda_1 = O(1)$ with high probability. In the same paper, it was also shown that in the ferromagnetic regime, $\gamma - \lambda_1 = O(N^{-1+\epsilon})$ with high probability. The following lemma establishes a corresponding result for the transitional regime; it shows that $\gamma - \lambda_1 = O(N^{-\frac{1}{2}+\epsilon})$ with high probability.

Lemma 3.3 (Critical point) *Recall that* (*see* (1.16)) $J > 1$ *is fixed and* $2\beta = 2\beta_N = \frac{1}{J} + \frac{B}{\sqrt{N}}$ *with fixed* $B \in \mathbb{R}$. *Then, for every* $0 < \epsilon < \frac{1}{4}$,

$$\gamma = \lambda_1 + \frac{-\chi_N - B(J^2-1) + \sqrt{(\chi_N + (J^2-1)B)^2 + 4(J^2-1)}}{2\sqrt{N}} + O\left(N^{-1+\epsilon}\right) \tag{3.13}$$

with high probability, where we set $\chi_N := \sqrt{N}(\lambda_1 - \hat{J})$.

Note that γ given above is larger than λ_1 with high probability since the term in the big parenthesis is positive.

Proof Set

$$\theta := \frac{-\chi_N - B(J^2-1) + \sqrt{(\chi_N + (J^2-1)B)^2 + 4(J^2-1)}}{2}. \tag{3.14}$$

Note that $\theta > 0$. By the rigidity of λ_1, we have $|\chi_N| \le N^{\frac{\epsilon}{4}}$ and hence, $\theta \le N^{\frac{\epsilon}{3}}$ with high probability. On the other hand, using $-a + \sqrt{a^2+b^2} = \frac{b^2}{\sqrt{a^2+b^2}+a}$,

$$\theta = \frac{2(J^2-1)}{\sqrt{(J^2-1)B + \chi_N)^2 + 4(J^2-1)} + ((J^2-1)B + \chi_N)},$$

and hence $\theta \ge CN^{-\frac{\epsilon}{4}}$ for some constant $C > 0$ with high probability. Hence,

$$N^{-\frac{\epsilon}{3}} \le \theta \le N^{\frac{\epsilon}{3}} \tag{3.15}$$

with high probability. Set

$$\gamma_\pm := \lambda_1 + \frac{\theta}{\sqrt{N}} \pm N^{-1+\epsilon}. \tag{3.16}$$

By the above properties of θ, we have $\gamma_\pm > \lambda_1$ with high probability. We will show that $G'(\gamma_-) < 0$ and $G'(\gamma_+) > 0$ with high probability. Since $G'(z)$ is a monotone increasing function for real z in the interval (λ_1, ∞), this shows that $\gamma_- < \gamma < \gamma_+$ with high probability, proving the lemma.

Recall that $\lambda_1 \to \hat{J}$ in probability. Let us write

$$\gamma_\pm = J + \frac{1}{J} + \frac{\phi}{\sqrt{N}} \pm N^{-1+\epsilon}, \qquad \phi := \theta + \chi_N \tag{3.17}$$

where $\chi_N = \sqrt{N}(\lambda_1 - \hat{J})$. Note that $\phi = O(N^{\frac{\epsilon}{3}})$ with high probability. Now, notice that

$$G'(z) = 2\beta - \frac{1}{N}\sum_{i=2}^{N} \frac{1}{z-\lambda_i} - \frac{1}{N(z-\lambda_1)}. \tag{3.18}$$

We apply Corollary 3.1 to the family of the function $\{\frac{1}{z-x}\}_{z>2+c}$ for some constant $c > 0$ and obtain

$$G'(\gamma_\pm) = 2\beta - \frac{\gamma_\pm - \sqrt{\gamma_\pm^2 - 4}}{2} + O(N^{-1+\frac{\epsilon}{3}}) - \frac{1}{N(\gamma_\pm - \lambda_1)}$$

with high probability. By (3.17),

$$\frac{\gamma_\pm - \sqrt{\gamma_\pm^2 - 4}}{2} = \frac{1}{J} - \frac{1}{J^2-1}\left(\frac{\phi}{\sqrt{N}} \pm N^{-1+\epsilon}\right) + O(N^{-1+\frac{2\epsilon}{3}}).$$

By (3.16),

$$\frac{1}{N(\gamma_\pm - \lambda_1)} = \frac{1}{\theta\sqrt{N}}\left(1 \mp \frac{N^{-\frac{1}{2}+\epsilon}}{\theta} + O\left(\frac{N^{-1+2\epsilon}}{\theta^2}\right)\right).$$

Using the formula of 2β and the estimate (3.15) for $\frac{1}{\theta}$, we find that

$$G'(\gamma_\pm) = \frac{1}{\sqrt{N}}\left(B + \frac{\phi}{J^2-1} - \frac{1}{\theta}\right) \pm \left(\frac{1}{J^2-1} + \frac{1}{\theta^2}\right) N^{-1+\epsilon} + O(N^{-1+\frac{2\epsilon}{3}}) \tag{3.19}$$

with high probability since $0 < \epsilon < \frac{1}{4}$. By the definition of θ, the leading term is zero. The coefficient of the second term is positive. Hence we find that $G'(\gamma_-) < 0$ and $G'(\gamma_+) > 0$, and we obtain the lemma. □

Then we have the following lemma.

Lemma 3.4 *Set*

$$s = s_N := \sqrt{N}(\gamma - J - J^{-1}) \text{ and } \Delta = \Delta_N := \sqrt{N}(\gamma - \lambda_1) = s_N - \chi_N. \tag{3.20}$$

Then, for every $\epsilon > 0$,

$$s = \frac{\chi_N - B(J^2-1) + \sqrt{(\chi_N + (J^2-1)B)^2 + 4(J^2-1)}}{2} + O(N^{-\frac{1}{2}+\epsilon}) \tag{3.21}$$

with high probability. We also have

$$|s| \le N^{\epsilon} \text{ and } N^{-\epsilon} \le \Delta \le N^{\epsilon} \tag{3.22}$$

with high probability.

Proof The previous lemma implies (3.21). The first part of (3.22) follows from the fact that $\chi_N = O(N^{\epsilon})$ with high probability. The second part is the estimate (3.15) in the proof of the previous lemma. □

We also need the following lemma.

Lemma 3.5 *For every* $0 < \epsilon < 1$,

$$\frac{1}{N}\sum_{i=2}^{N} \frac{1}{(\gamma - \lambda_i)^2} = \frac{1}{J^2-1} + O(N^{-1+\epsilon}) \tag{3.23}$$

with high probability.

Proof This follows from Corollary 3.1 applied to $f(x) = \frac{1}{(\gamma - x)^2}$. □

The following auxiliary lemma is used to estimate an error in the steepest descent analysis.

Lemma 3.6 *Define*

$$\mathbf{I}_m(\alpha) := \int_{-\infty}^{\infty} \frac{t^m}{\sqrt{1+\mathrm{i}t}} e^{-\frac{\alpha}{4}t^2 + \frac{\mathrm{i}t}{2}} \mathrm{d}t \tag{3.24}$$

for non-negative integers m *and* $\alpha > 0$, *where the square root is the defined on the principal branch. We set* $\mathbf{I}(\alpha) := \mathbf{I}_0(\alpha)$*; see* (1.21). *Then,*

$$\mathbf{I}(\alpha) = \sqrt{\frac{4\pi}{\alpha}}(1 + O(\alpha^{-1})) \quad \text{as } \alpha \to +\infty, \tag{3.25}$$

$$\mathbf{I}(\alpha) = \sqrt{\frac{8\pi}{e}}(1 + O(\alpha)) \quad \text{as } \alpha \to 0_+, \tag{3.26}$$

and for every $m \ge 0$,

$$\mathbf{I}_m(\alpha) \text{ is uniformly bounded for } \alpha \in (0, \infty). \tag{3.27}$$

A particular consequence is that the derivative $\mathbf{I}'(\alpha) = -\frac{1}{4}\mathbf{I}_2(\alpha)$ *is uniformly bounded for* $\alpha > 0$. *Furthermore,* $\mathbf{I}(\alpha) > 0$ *for all* $\alpha > 0$.

Proof Consider (3.25). Applying the method of steepest-descent to $\mathbf{I}(\alpha) = \int_{-\infty}^{\infty} g(t) e^{\alpha h(t)} \mathrm{d}t$ with $h(z) = -\frac{z^2}{4}$ and $g(z) = \frac{1}{\sqrt{1+\mathrm{i}z}} e^{\frac{\mathrm{i}z}{2}}$, we find that

$$\mathbf{I}(\alpha) = \frac{e^{\alpha h(z_c)}}{\sqrt{\alpha}} \left[\sqrt{\frac{2\pi}{|h''(z_c)|}} g(z_c) + O(\alpha^{-1}) \right] = \sqrt{\frac{4\pi}{\alpha}} (1 + O(\alpha^{-1})) \tag{3.28}$$

as $\alpha \to +\infty$. For $\mathbf{I}_m(\alpha)$, using $\int_{-\infty}^{\infty} y^m e^{-\alpha y^2} \mathrm{d}y = O(\alpha^{-(m+1)/2})$, we find that

$$\mathbf{I}_m(\alpha) = O\left(\alpha^{-\frac{m+1}{2}}\right) \quad \text{as } \alpha \to +\infty. \tag{3.29}$$

Consider the limit $\alpha \to 0_+$. After the change of the variables $t = z/\alpha$,

$$\mathbf{I}(\alpha) = \frac{e^{-\frac{1}{4\alpha}}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(z-\mathrm{i})^2}{4\alpha}}}{\sqrt{\alpha + \mathrm{i}z}} \mathrm{d}z. \tag{3.30}$$

The integrand is analytic in the complex plane minus the vertical line from $\mathrm{i}\alpha$ to $\mathrm{i}\infty$. Note that the saddle point is i and it is on the branch cut. We show that the main contribution to the integral comes from the branch point $z = \mathrm{i}\alpha$. We deform the contour so that it consists of the following four line segments: L_1 from $\mathrm{i} - \infty$ to i on the left half-plane, L_2 from i to $\mathrm{i}\alpha$ lying on the left of the branch cut, L_3 from $\mathrm{i}\alpha$ to i lying on the right of the branch cut, and L_4 from i to $\mathrm{i} + \infty$ lying on the right-half plane. On L_4, setting $z = \mathrm{i} + \sqrt{\alpha}x$,

$$\int_{L_4} \frac{e^{-\frac{(z-\mathrm{i})^2}{4\alpha}}}{\sqrt{\alpha + \mathrm{i}z}} \mathrm{d}z = \sqrt{\alpha} \int_0^{\infty} \frac{e^{-\frac{x^2}{4}}}{\sqrt{\alpha - 1 + \mathrm{i}\sqrt{\alpha}x}} \mathrm{d}x = O(\sqrt{\alpha}) \tag{3.31}$$

as $\alpha \to 0$. Similarly, the integral over L_1 is also of the same order. On the other hand, setting $z = \mathrm{i}\alpha + \mathrm{i}y$,

$$\int_{L_2 \cup L_3} \frac{e^{-\frac{(z-\mathrm{i})^2}{4\alpha}}}{\sqrt{\alpha + \mathrm{i}z}} \mathrm{d}z = 2 \int_0^{1-\alpha} \frac{e^{\frac{(\alpha+y-1)^2}{4\alpha}}}{\sqrt{y}} \mathrm{d}y = 2e^{\frac{(\alpha-1)^2}{4\alpha}} \int_0^{1-\alpha} \frac{e^{\frac{y}{2} + \frac{y^2-2y}{4\alpha}}}{\sqrt{y}} \mathrm{d}y. \tag{3.32}$$

The function $y^2 - 2y$ decreases as y increases from $y = 0$ to $y = 1$. Hence the main contribution to the integral comes near the point $y = 0$. Using Watson's lemma,

$$\int_0^{1-\alpha} \frac{e^{\frac{y}{2} + \frac{y^2-2y}{4\alpha}}}{\sqrt{y}} \mathrm{d}y = \Gamma(1/2)\sqrt{2\alpha}(1 + O(\alpha)). \tag{3.33}$$

Combining together and using $\Gamma(1/2) = \sqrt{\pi}$, we obtain (3.26). For $\mathbf{I}_m(\alpha)$, the analysis is same except that we use

$$\int_0^{1-\alpha} (\mathrm{i}\alpha + \mathrm{i}y)^m \frac{e^{\frac{y}{2} + \frac{y^2-2y}{4\alpha}}}{\sqrt{y}} \mathrm{d}y = O(\alpha^{m+1/2}). \tag{3.34}$$

Hence, we find that for $m \geq 0$, $\mathbf{I}_m(\alpha) = O(1)$ as $\alpha \to 0_+$. Together with (3.29), this implies the uniform boundness of $\mathbf{I}_m(\alpha)$.

For the positiveness of $\mathbf{I}(\alpha)$, we first write it as

$$\mathbf{I}(\alpha) = \int_{-\infty}^{\infty} \frac{e^{-\frac{\alpha}{4}t^2 + \frac{\mathrm{i}}{2}(t - \arctan t)}}{(1+t^2)^{1/4}} \mathrm{d}t = \int_0^{\infty} \frac{2e^{-\frac{\alpha}{4}t^2}}{(1+t^2)^{1/4}} \cos\left(\frac{1}{2}(t - \arctan t)\right) \mathrm{d}t. \tag{3.35}$$

The function $\theta(t) = t - \arctan t$ is monotone increasing. We use the inverse function, $t = t(\theta)$, to change the variables and find that

$$\mathbf{I}(\alpha) = 2\int_0^\infty e^{-\frac{\alpha}{4}t^2}\frac{(1+t^2)^{3/4}}{t^2}\cos\left(\frac{\theta}{2}\right)\mathrm{d}\theta, \qquad t = t(\theta). \tag{3.36}$$

Since $e^{-\frac{\alpha}{4}t(\theta)^2}$ is positive and monotone decreasing in θ, we obtain $I(\alpha) > 0$ for every $\alpha > 0$ if we show that (i)

$$\int_0^\pi \frac{(1+t^2)^{3/4}}{t^2}\cos\left(\frac{\theta}{2}\right)\mathrm{d}\theta \geq -\int_\pi^{3\pi}\frac{(1+t^2)^{3/4}}{t^2}\cos\left(\frac{\theta}{2}\right)\mathrm{d}\theta, \tag{3.37}$$

and (ii)

$$(-1)^k\int_{(2k-1)\pi}^{(2k+1)\pi}\frac{(1+t^2)^{3/4}}{t^2}\cos\left(\frac{\theta}{2}\right)\mathrm{d}\theta, \qquad k = 1, 2, 3, \ldots, \tag{3.38}$$

is decreasing in k. (i) can be verified numerically. On the other hand, (ii) follows immediately from the fact $(1+t^2)^{3/4}/t^2$ is a decreasing function of t. This completes the proof. □

We now evaluate the integral in (3.1) using the steepest descent analysis.

Lemma 3.7 *Fix $J > 1$ and let $2\beta = J^{-1} + BN^{-1/2}$. Consider $G(z)$ defined in (3.2). Then, for every $0 < \epsilon < \frac{1}{8}$,*

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)}\mathrm{d}z = \frac{\mathrm{i}\Delta e^{\frac{N}{2}G(\gamma)}}{\sqrt{N}}\mathbf{I}(F''(\gamma)\Delta^2)\left(1 + O(N^{-\frac{1}{2}+4\epsilon})\right) \tag{3.39}$$

with high probability, where

$$F(z) = 2\beta z - \frac{1}{N}\sum_{i=2}^N \log(z-\lambda_i) - \frac{1}{N}\log(\gamma-\lambda_1) - \frac{z-\gamma}{N(\gamma-\lambda_1)} \tag{3.40}$$

and $\mathbf{I}(\alpha)$ is defined in (1.21). Recall that $\Delta = \sqrt{N}(\gamma-\lambda_1)$ (see Lemma 3.4.)

Proof We choose the γ, which defines the contour, as the critical point of $G(z)$. The path of steepest-descent is locally a vertical line near the critical point. It turns out that, instead of using the path of steepest-descent, it is enough to proceed the analysis using the straight line $\gamma + \mathrm{i}\mathbb{R}$ globally. This choice was also made for the analysis in the paramagnetic regime in [5].

We first write, using the function $F(z)$,

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)}\mathrm{d}z = e^{\frac{N}{2}G(\gamma)}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(G(z)-F(z))+\frac{N}{2}(F(z)-G(\gamma))}\mathrm{d}z. \tag{3.41}$$

From the definitions of $G(z)$ and $F(z)$,

$$e^{\frac{N}{2}(G(z)-F(z))} = \sqrt{\frac{\gamma-\lambda_1}{z-\lambda_1}}e^{\frac{z-\gamma}{2(\gamma-\lambda_1)}}. \tag{3.42}$$

Changing the variables $z = \gamma + \mathrm{i}tN^{-1/2}$ and using the notation $\Delta = \sqrt{N}(\gamma-\lambda_1)$,

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)}\mathrm{d}z = \frac{\mathrm{i}e^{\frac{N}{2}G(\gamma)}}{\sqrt{N}}\int_{-\infty}^{\infty}\frac{e^{\frac{\mathrm{i}t}{2\Delta}}}{\sqrt{1+\frac{\mathrm{i}t}{\Delta}}}e^{\frac{N}{2}(F(\gamma+\mathrm{i}tN^{-1/2})-G(\gamma))}\mathrm{d}t. \tag{3.43}$$

It is easy to check that the part of the integral with $|t| \geq N^{\epsilon}$ is small. To show this, we first note that

$$\Re\left(N\left(F\left(\gamma+\frac{\mathrm{i}t}{\sqrt{N}}\right)-G(\gamma)\right)\right)=-\Re\sum_{i=2}^{N}\log\left(\frac{\gamma-\lambda_i+\mathrm{i}tN^{-1/2}}{\gamma-\lambda_i}\right)$$
$$\leq-\frac{N-1}{2}\log\left(1+\frac{c^2t^2}{N}\right)$$

with high probability for some constant $c > 0$, since there is a constant $c > 0$ such that $c \leq \gamma - \lambda_i \leq \frac{1}{c}$ for all $i = 2, \ldots, N$, with high probability. Hence,

$$\left|\int_{N^{\epsilon}}^{\infty}\frac{e^{\frac{\mathrm{i}t}{2\Delta}}}{\sqrt{1+\frac{\mathrm{i}t}{\Delta}}}e^{\frac{N}{2}(F(\gamma+\mathrm{i}tN^{-1/2})-G(\gamma)}\mathrm{d}t\right|\leq\int_{N^{\epsilon}}^{\infty}e^{-\frac{N-1}{2}\log\left(1+\frac{c^2t^2}{N}\right)}\mathrm{d}t$$
$$\leq\int_{N^{\epsilon}}^{N}e^{-\frac{c^2}{8}N^{2\epsilon}}\mathrm{d}t+\int_{N}^{\infty}\frac{1}{(c^2N^{-1}t^2)^{N/4}}\mathrm{d}t=O\left(e^{-N^{\epsilon}}\right)+O\left(N^{-N/8}\right) \tag{3.44}$$

with high probability.

Consider the part $|t| \leq N^{\epsilon}$. Note that $F(z)$ satisfies $F(\gamma) = G(\gamma)$, $F'(\gamma) = G'(\gamma) = 0$, and for each $m \geq 2$, $F^{(m)}(z) = O(1)$ uniformly for z in a small neighborhood of γ (by Corollary 3.1). For $m = 2$, by Lemma 3.5,

$$c_1 \leq F''(\gamma) \leq c_2 \tag{3.45}$$

for some constants $0 < c_1 < c_2$, uniformly in N. By Taylor expansion, for $|t| \leq N^{\epsilon}$,

$$F(\gamma+\mathrm{i}tN^{-1/2})-G(\gamma)=-\frac{F''(\gamma)t^2}{2N}-\frac{\mathrm{i}F'''(\gamma)t^3}{6N^{3/2}}+O\left(N^{-2+4\epsilon}\right) \tag{3.46}$$

and hence,

$$e^{\frac{N}{2}(F(\gamma+\mathrm{i}tN^{-1/2})-G(\gamma))}=e^{-\frac{F''(\gamma)t^2}{4}}\left(1-\frac{\mathrm{i}F'''(\gamma)t^3}{12N^{1/2}}+O\left(N^{-1+6\epsilon}\right)\right). \tag{3.47}$$

Therefore,

$$\int_{-N^{\epsilon}}^{N^{\epsilon}}\frac{e^{\frac{\mathrm{i}t}{2\Delta}}}{\sqrt{1+\frac{\mathrm{i}t}{\Delta}}}e^{\frac{N}{2}\left(F(\gamma+\mathrm{i}tN^{-1/2})-G(\gamma)\right)}\mathrm{d}t$$
$$=\int_{-\infty}^{\infty}\frac{e^{\frac{\mathrm{i}t}{2\Delta}-\frac{F''(\gamma)}{4}t^2}}{\sqrt{1+\frac{\mathrm{i}t}{\Delta}}}\mathrm{d}t-\frac{\mathrm{i}F'''(\gamma)}{12N^{1/2}}\int_{-\infty}^{\infty}\frac{t^3e^{\frac{\mathrm{i}t}{2\Delta}-\frac{F''(\gamma)}{4}t^2}}{\sqrt{1+\frac{\mathrm{i}t}{\Delta}}}\mathrm{d}t+O\left(N^{-1+6\epsilon}\right)$$
$$=\Delta\,\mathbf{I}(F''(\gamma)\Delta^2)-\frac{\mathrm{i}F'''(\gamma)\Delta^4}{12N^{1/2}}\mathbf{I}_3(F''(\gamma)\Delta^2)+O\left(N^{-1+6\epsilon}\right). \tag{3.48}$$

By (3.45) and Lemma 3.4, $c_1N^{-\epsilon} \leq F''(\gamma)\Delta^2 \leq c_2N^{\epsilon}$. Hence, Lemma 3.6 implies that

$$\mathbf{I}(F''(\gamma)\Delta^2) \geq cN^{-\epsilon} \tag{3.49}$$

for some constant $c > 0$. Hence, using Lemma 3.4, Lemma 3.6, and the uniform boundedness of $F'''(\gamma)$, we find that (3.48) is equal to

$$\Delta\,\mathbf{I}(F''(\gamma)\Delta^2)\left(1+O\left(N^{-\frac{1}{2}+4\epsilon}\right)\right) \tag{3.50}$$

if $0 < \epsilon < \frac{1}{8}$. Thus, using (3.49) and Lemma 3.4 again, we conclude that

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)}\mathrm{d}z = \frac{\mathrm{i}\Delta e^{\frac{N}{2}G(\gamma)}}{\sqrt{N}}\mathbf{I}(F''(\gamma)\Delta^2)\Big(1+O\Big(N^{-1/2+4\epsilon}\Big)\Big). \tag{3.51}$$

□

3.3 Proof of Theorem 1.1

Proof of Theorem 1.1 From Lemma 3.1 and Lemma 3.7, for every $0 < \epsilon < \frac{1}{8}$,

$$Z_N = C_N\frac{\mathrm{i}\Delta e^{\frac{N}{2}G(\gamma)}}{\sqrt{N}}\mathbf{I}(F''(\gamma)\Delta^2)\Big(1+O\Big(N^{-\frac{1}{2}+4\epsilon}\Big)\Big) \tag{3.52}$$

with high probability. Using Stirling's formula,

$$C_N = \frac{\Gamma(N/2)}{2\pi\mathrm{i}(N\beta)^{N/2-1}} = \frac{\sqrt{N}\beta}{\mathrm{i}\sqrt{\pi}(2\beta e)^{N/2}}\Big(1+O\Big(N^{-1}\Big)\Big), \tag{3.53}$$

thus we find that $F_N = \frac{1}{N}\log Z_N$ satisfies

$$F_N = \frac{1}{2}(G(\gamma)-1-\log(2\beta)) + \frac{1}{N}\left(\log\left(\frac{\beta\Delta}{\sqrt{\pi}}\right)+\log\mathbf{I}(F''(\gamma)\Delta^2)\right)+O\Big(N^{-\frac{3}{2}+4\epsilon}\Big) \tag{3.54}$$

with high probability.

Let us consider $G(\gamma)$. Since γ and $\hat{J} = J + J^{-1}$ are away from $\lambda_2, \ldots, \lambda_N$ with high probability,

$$\begin{aligned}\log(\gamma-\lambda_i) &= \log(\hat{J}-\lambda_i) - \log\left(1-\frac{\gamma-\hat{J}}{\gamma-\lambda_i}\right)\\ &= \log(\hat{J}-\lambda_i) + \frac{\gamma-\hat{J}}{\gamma-\lambda_i} + \frac{(\gamma-\hat{J})^2}{2(\gamma-\lambda_i)^2} + O(|\gamma-\hat{J}|^3)\end{aligned} \tag{3.55}$$

for $i = 2, \ldots, N$, where we also use that $\gamma - \hat{J} = O(N^{-\frac{1}{2}+\epsilon})$ with high probability (see Lemma 3.4). Then, using Lemma 3.5 and the fact that $G'(\gamma) = 2\beta - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma-\lambda_i} = 0$,

$$\begin{aligned}\frac{1}{N}\sum_{i=2}^{N}\log(\gamma-\lambda_i) =& \frac{1}{N}\sum_{i=2}^{N}\log(\hat{J}-\lambda_i) + 2\beta(\gamma-\hat{J})\\ &- \frac{\gamma-\hat{J}}{N(\gamma-\lambda_1)} + \frac{(\gamma-\hat{J})^2}{2(J^2-1)} + O\Big(N^{-\frac{3}{2}+3\epsilon}\Big)\end{aligned}$$

with high probability. Hence, from the formula of $G(z)$ in (3.2),

$$\begin{aligned}G(\gamma) = 2\beta\hat{J} &- \frac{1}{N}\sum_{i=2}^{N}\log(\hat{J}-\lambda_i)\\ &- \frac{1}{N}\left(\log\left(\frac{\Delta}{\sqrt{N}}\right)+\frac{s_N}{\Delta}-\frac{s_N^2}{2(J^2-1)}\right)+O\Big(N^{-\frac{3}{2}+3\epsilon}\Big)\end{aligned}$$

using the notations $s_N = \sqrt{N}(\gamma - \hat{J})$ and $\Delta = \sqrt{N}(\gamma - \lambda_1)$ in Lemma 3.4. Thus,

$$F_N = \beta \hat{J} - \frac{1}{2} - \frac{1}{2}\log(2\beta) - \frac{1}{2N}\sum_{i=2}^{N}\log(\hat{J} - \lambda_i) + \frac{1}{4N}\log N + \frac{1}{N}\left(\frac{s_N}{2\Delta} - \frac{s_N^2}{4(J^2-1)} + \log\frac{\sqrt{\Delta}\beta}{\sqrt{\pi}} + \log \mathbf{I}(F''(\gamma)\Delta^2)\right) + O\left(N^{-\frac{3}{2}+4\epsilon}\right). \tag{3.56}$$

To conclude Theorem 1.1, we use (i) the fact that $\Delta = s_N - \chi_N$, (ii) the asymptotic (3.21) of s_N in terms of χ_N, (iii) the fact that $F''(\gamma) = \frac{1}{J^2-1} + O(N^{-1+\epsilon})$ which follows from Lemma 3.5, and (iv) the fact that $\mathbf{I}'(\alpha)$ is uniformly bounded for $\alpha > 0$ (see Lemma 3.6). □

4 Partial Linear Statistics

This section is devoted to a proof of Theorem 2.1 on partial linear statistics. The proof is a simple modification of [5] for the linear statistics of all eigenvalues, which, in turn, follows the proof of [1,2] for the case when the random matrix has zero mean.

4.1 Proof of Theorem 2.1

Recall $\hat{J} := J + J^{-1}$ denotes the classical location of the largest eigenvalue of a Wigner matrix of non-zero mean. Fix (N-independent) constants $a_- < -2$ and $2 < a_+ < \hat{J}$. Let Γ be the rectangular contour whose vertices are $(a_- \pm \mathrm{i}v_0)$ and $(a_+ \pm \mathrm{i}v_0)$ for some $v_0 \in (0, 1]$. The contour is oriented counter-clockwise. For a test function $\varphi(x)$ which is analytic in a neighborhood of $[-2, 2]$, we consider

$$\begin{aligned}\mathcal{N}_N^{(2)}(\varphi) &:= \sum_{i=2}^{N}\varphi(\lambda_i) - N\int_{\mathbb{R}}\varphi(x)\mathrm{d}\sigma_{scl}(x) \\ &= \oint_{\Gamma}\varphi(z)\left[\sum_{i=2}^{N}\frac{1}{z-\lambda_i} - N\int_{\mathbb{R}}\frac{\mathrm{d}\sigma_{scl}(x)}{z-x}\right]\mathrm{d}z = -\frac{1}{2\pi i}\oint_{\Gamma}\varphi(z)\xi_N^{(2)}\mathrm{d}z\end{aligned} \tag{4.1}$$

where

$$\xi_N^{(2)}(z) := \sum_{i=2}^{N}\frac{1}{\lambda_i - z} - N\int_{\mathbb{R}}\frac{1}{x-z}\mathrm{d}\sigma_{scl}(x). \tag{4.2}$$

Decompose Γ into $\Gamma_u \cup \Gamma_d \cup \Gamma_l \cup \Gamma_r \cup \Gamma_0$, where

$$\Gamma_u = \{z = x + \mathrm{i}v_0 : a_- \le x \le a_+\}, \tag{4.3}$$
$$\Gamma_d = \{z = x - \mathrm{i}v_0 : a_- \le x \le a_+\}, \tag{4.4}$$
$$\Gamma_l = \{z = a_- + \mathrm{i}y : N^{-\delta} \le |y| \le v_0\}, \tag{4.5}$$
$$\Gamma_r = \{z = a_+ + \mathrm{i}y : N^{-\delta} \le |y| \le v_0\}, \tag{4.6}$$
$$\Gamma_0 = \{z = a_- + \mathrm{i}y : |y| < N^{-\delta}\} \cup \{z = a_+ + \mathrm{i}y : |y| < N^{-\delta}\} \tag{4.7}$$

for some $\delta > 0$. In the proof of Theorem 1.6 in [5], the authors showed that

$$\xi_N(z) := \sum_{i=1}^{N} \frac{1}{\lambda_i - z} - N \int_{\mathbb{R}} \frac{1}{x - z} \mathrm{d}\sigma_{scl}(x) = \xi_N^{(2)}(z) + \frac{1}{\lambda_1 - z} \tag{4.8}$$

converges weakly to a Gaussian process with mean $b(z) = b^{(2)}(z) + \frac{1}{\hat{J}-z}$ and covariance $\Gamma(z_i, z_j) = \Gamma^{(2)}(z_i, z_j)$ where $b^{(2)}(z)$ and $\Gamma^{(2)}(z_i, z_j)$ are given in the proposition below. Since for each fixed $z \in \mathbb{C}_+$, $\frac{1}{\lambda_1 - z} \to \frac{1}{\hat{J}-z}$ in probability (by Lemma 3.2), it is natural to expect the following result for a partial sum.

Proposition 4.1 *Let*

$$s(z) = \int \frac{1}{x - z} \mathrm{d}\sigma_{scl}(x) = \frac{-z + \sqrt{z^2 - 4}}{2} \tag{4.9}$$

be the Stieltjes transform of the semicircle measure. Fix a constant $c > 0$ and a path $\mathcal{K} \subset \mathbb{C}_+$ such that $\Im z > c$ for $z \in \mathcal{K}$. Then the process $\{\xi_N^{(2)}(z) : z \in \mathcal{K}\}$ converges weakly to a Gaussian process with the mean

$$b^{(2)}(z) = \frac{s(z)^2}{1 - s(z)^2} \left(\frac{-J}{1 + Js(z)} + (w_2 - 1)s(z) + s'(z)s(z) + (W_4 - 3)s(z)^3 \right) - \frac{1}{\hat{J} - z} \tag{4.10}$$

and the covariance matrix

$$\Gamma^{(2)}(z_i, z_j) = s'(z_i)s'(z_j) \left((w_2 - 2) + 2(W_4 - 3)s(z_i)s(z_j) + \frac{2}{(1 - s(z_i)s(z_j))^2} \right). \tag{4.11}$$

Remark 4.1 Note that as $z \to \hat{J}$,

$$\frac{s(z)^2}{1 - s(z)^2} \frac{J}{1 + Js(z)} = \frac{s'(z)}{\frac{1}{J} + s(z)} = \frac{1}{z - \hat{J}} + \frac{s''(\hat{J})}{s'(\hat{J})} + O(z - \hat{J}). \tag{4.12}$$

Hence, $b^{(2)}(z)$ is analytic near $\hat{J}$ and thus analytic for $z \in \mathbb{C} \setminus [-2, 2]$.

In order to complete the proof of Theorem 2.1, we will prove the following lemma.

Lemma 4.1 *Define the events*

$$\Omega_N := \{\lambda_1 \geq \hat{J} - N^{-1/3}, \lambda_2 \leq 2 + N^{-1/3}\} \tag{4.13}$$

which satisfies $\mathbb{P}(\Omega_N^c) < N^{-D}$ for any fixed (large) $D > 0$. Then for some $\delta > 0$,

$$\lim_{v_0 \to 0^+} \limsup_{N \to \infty} \int_{\Gamma_\#} \mathbb{E} \, |\xi_N^{(2)}(z) \mathbb{1}_{\Omega_N}|^2 \mathrm{d}z = 0, \tag{4.14}$$

where $\Gamma_\#$ can be Γ_r, Γ_l or Γ_0.

From the explicit formulas (4.10) and (4.11), it is easy to check that

$$\lim_{v_0 \to 0^+} \int_{\Gamma_\#} \mathbb{E} \, |\xi^{(2)}(z)|^2 \mathrm{d}z = 0. \tag{4.15}$$

Proposition 4.1, Lemma 4.1 and (4.15) imply that $\mathcal{N}_N^{(2)}(\varphi)$ converges in distribution to a Gaussian random variable with the following mean and variance:

$$-\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \varphi(z) b^{(2)}(z) \mathrm{d}z, \qquad \frac{1}{(2\pi \mathrm{i})^2} \oint_\Gamma \oint_\Gamma \varphi(z_1)\varphi(z_2)\Gamma(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2. \tag{4.16}$$

It is direct to check that these are equal to $M^{(2)}(\varphi)$ and $V^{(2)}(\varphi)$ (see [5, Sect. 4.2]). We thus obtain Theorem 2.1.

4.2 Proof of Proposition 4.1

From Theorem 7.1 of [8], we need to show (i) the finite-dimensional convergence of $\xi_N^{(2)}(z)$ to a Gaussian vector with desired mean and variance, and (ii) the tightness of $\xi_N^{(2)}(z)$. We will base our proof on the corresponding properties of $\xi_N(z)$ obtained in [5]. Let us first recall the limit theorem for $\xi_N(z)$.

Lemma 4.2 ([5, Proposition 4.1]) *Let $s(z)$ and $\mathcal{K}$ defined in the same way as in Proposition 4.1. Then, the process $\{\xi_N(z) : z \in \mathcal{K}\}$ converges weakly to a Gaussian process $\{\xi(z) : z \in \mathcal{K}\}$ with the mean*

$$b(z) = \frac{s(z)^2}{1 - s(z)^2}\left(-\frac{J}{1 + Js(z)} + (w_2 - 1)s(z) + s'(z)s(z) + (W_4 - 3)s(z)^3\right) \tag{4.17}$$

and the covariance matrix

$$\Gamma(z_i, z_j) = s'(z_i)s'(z_j)\left((w_2 - 2) + 2(W_4 - 3)s(z_i)s(z_j) + \frac{2}{(1 - s(z_i)s(z_j))^2}\right). \tag{4.18}$$

Let $z_1, z_2, \ldots, z_p$ are p distinct points in $\mathcal{K}$. The above lemma implies that the random vector $(\xi_N(z_i))_{i=1}^p$ converges weakly to a p-dimensional Gaussian distribution with the mean $(b(z_i))_{i=1}^p$ and the covariance matrix $\Gamma(z_i, z_j)$. Since the distance between $\mathcal{K}$ and λ_1 is bounded below, $\frac{1}{\lambda_1 - z_i} \to \frac{1}{\hat{J} - z_i}$ in probability for $i = 1, \ldots, p$. Hence, by Slutsky's theorem, $(\xi_N^{(2)}(z_i))_{i=1}^p$ converges weakly to a p-dimensional Gaussian distribution vector with the mean $(b^{(2)}(z_i))_{i=1}^p$ and the covariance matrix $\Gamma^{(2)}(z_i, z_j)$, where

$$b^{(2)}(z) = b(z) - \frac{1}{\hat{J} - z}, \tag{4.19}$$

and $\Gamma^{(2)}(z_i, z_j) = \Gamma(z_i, z_j)$.

From Theorem 12.3 of [8], in order to show the tightness of a random process $(\zeta_N(z))_{z \in \mathcal{K}}$, it is sufficient to show that (i) $(\zeta_N(z))_N$ is tight for a fixed z, and (ii) the following Hölder condition holds: for some N-independent constant $K > 0$,

$$\mathbb{E}\,|\zeta_N(z_1) - \zeta_N(z_2)|^2 \le K|z_1 - z_2|^2, \qquad z_1, z_2 \in \mathcal{K}. \tag{4.20}$$

In [5], the authors considered the random process $\zeta_N(z) := \xi_N(z) - \mathbb{E}[\xi_N(z)]$, and proved that it satisfies conditions (i) and (ii). Now, we consider $\xi_N^{(2)}(z) := \zeta_N^{(2)} + \mathbb{E}[\xi_N(z)]$, where $\zeta_N^{(2)}(z) := \zeta_N(z) - \frac{1}{\lambda_1 - z}$. Since $\mathbb{E}[\xi_N(z)]$ converges, it is enough to check that $(\zeta_N^{(2)}(z))_N$ satisfies conditions (i) and (ii). Now for a fixed z, the tightness of $(\zeta_N(z))_N$ and the boundedness of $\frac{1}{\lambda_1 - z}$ imply that $(\zeta_N^{(2)}(z))_N$ is tight. On the other hand, since $\zeta_N(z)$ satisfies the Hölder condition and $\Im z \ge c$ for $z \in \mathcal{K}$,

$$\begin{aligned}\mathbb{E}\,|\zeta_N^{(2)}(z_1) - \zeta_N^{(2)}(z_2)|^2 &\le 2\,\mathbb{E}\,|\zeta_N(z_1) - \zeta_N(z_2)|^2 + 2\,\mathbb{E}\left|\frac{1}{\lambda_1 - z_1} - \frac{1}{\lambda_1 - z_2}\right|^2 \\ &\le 2K|z_1 - z_2|^2 + \frac{2|z_1 - z_2|^2}{c^4} = \left(K + \frac{2}{c^4}\right)|z_1 - z_2|^2.\end{aligned} \tag{4.21}$$

Thus $\{\xi_N^{(2)}(z), z \in \mathcal{K}\}$ is tight. This completes the proof of Proposition 4.1.

4.3 Proof of Lemma 4.1

For $z \in \Gamma_0$, we notice that $|\xi_N^{(2)} \mathbb{1}_{\Omega_N}| \le CN$ and then

$$\int_{\Gamma_0} \mathbb{E}\,|\xi_N^{(2)} \mathbb{1}_{\Omega_N}|^2 \le CN^{2-\delta}. \tag{4.22}$$

Thus (4.14) holds for Γ_0 with $\delta > 2$. For Γ_r and Γ_l, it is sufficient to show $\mathbb{E}\,|\xi_N^{(2)}|^2 < K$ for some N-independent constant $K > 0$. The authors in [5] showed[2] that $\mathbb{E}\,|\xi_N(z)|^2 < K$. Hence, for $z \in \Gamma_r$,

$$|\xi_N^{(2)}(z)\mathbb{1}_{\Omega_N}|^2 \le 2|\xi_N(z)\mathbb{1}_{\Omega_N}|^2 + 2\left|\frac{1}{\lambda_1 - z}\mathbb{1}_{\Omega_N}\right|^2. \tag{4.23}$$

The lemma then follows from the fact that $|\frac{1}{\lambda_1 - z}\mathbb{1}_{\Omega_N}|$ is bounded.

5 Joint Distribution of χ_N and $\mathcal{N}_N^{(2)}(\varphi)$

As before, let A be a random symmetric matrix of size N whose entries are (up to the symmetry condition) independent centered random variables satisfying Definition 1.1. Let $M = \frac{1}{\sqrt{N}}A + \frac{J}{N}\mathbf{1}\mathbf{1}^T$ where $J > 1$. Let $\lambda_1 \ge \cdots \ge \lambda_N$ be the eigenvalues of M.

Let $\chi_N = \sqrt{N}(\lambda_1 - \hat{J})$ denoting the rescaled largest eigenvalue. Given an analytic function $\varphi(x)$, recall the partial linear statistics $\mathcal{N}_N^{(2)}(\varphi) = \sum_{i=2}^N \varphi(\lambda_i) - N\int_{-2}^2 \varphi(x)\mathrm{d}\sigma_{scl}(x)$. We saw in the previous sections that χ_N and $\mathcal{N}_N^{(2)}(\varphi)$ converge individually to Gaussian random variables. In this section, we consider the joint distribution and prove Theorem 2.2. In Sect. 5.1, we first prove Theorem 2.2 assuming that the disorder variables are Gaussian random variables. In Sect. 5.2, the general disorder variables are considered using an interpolation trick.

5.1 Asymptotic Independence for the GOE Case

Let the off-diagonal entries of A be Gaussian random variables of variance 1 and the diagonal entries be Gaussian random variables of variance 2. In random matrix theory, the random symmetric matrix $H = \frac{1}{\sqrt{N}}A$ is said to belong to the Gaussian orthogonal ensemble (GOE). A special property of GOE, compared with general random symmetric matrices, is that the probability measure of GOE is invariant under orthogonal conjugations.

The following result is basically in [10].

Lemma 5.1 *Let $(\frac{1}{\sqrt{2}}A_{ii}, A_{ij}, y_i)_{1\le i<j\le N}$ be i.i.d. standard Gaussian random variables. Let $H = \frac{1}{\sqrt{N}}A$ with $A = (A_{ij})_{1\le i,j\le N}$ and let $Y = \frac{1}{\sqrt{N}}(y_1, \ldots, y_N)^T$. Define $G(z) = (H - zI)^{-1}$ for $z \in \mathbb{C}\backslash[-2-\delta, 2+\delta]$, which is well defined with high probability for fixed $\delta > 0$.*

[2] Even though it is stated in Lemma 4.2 of [5] that the lemma holds for sufficiently small $\delta > 0$, the proof of it is valid for any $\delta > 0$, and we use $\delta > 2$ for our purpose.

Then, for $z \in \mathbb{R}\backslash[-2-\delta, 2+\delta]$,

$$n_N(z) := \sqrt{N}(Y^* G(z) Y - \frac{1}{N}\mathrm{Tr}(G(z))) \Rightarrow n(z) \tag{5.1}$$

where $n = n(z) := \mathcal{N}\left(0, 2\int \frac{\mathrm{d}\sigma_{scl}(x)}{(x-z)^2}\right)$ *is a Gaussian random variable.*

Proof of Lemma 5.1 We follow the idea presented in [10]. By Theorem 5.2 of [10], it is enough to check the following three conditions for G: (i) There exists an N-independent constant a such that $\|G\| \le a$ with high probability, (ii) $\frac{1}{N}$ Tr G^2 converges to a constant in probability, and (iii) $\frac{1}{N}\sum_{i=1}^N G_{ii}^2$ converges to a constant in probability. They follow from rigidity of eigenvalue (Lemma 3.2), law of large numbers (Corollary 3.1), and local law ([17, Theorem 2.9]), respectively. □

We are now ready to prove the following property of GOE matrices.

Proposition 5.1 *For H defined in Lemma 5.1, denote its eigenvalues by $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_N$. For fixed k, consider a random vector $(X_N^1, X_N^2, \ldots, X_N^k)$ whose entries are real measurable functions of those eigenvalues, i.e., $X_N^i = X_N^i(\rho_1, \rho_2, \ldots, \rho_N)$ for $i = 1, 2, \ldots, k$. Suppose there is a random vector $(X^i)_{i=1}^k$ such that $(X_N^i)_{i=1}^k \Rightarrow (X^i)_{i=1}^k$ as $N \to \infty$. Then for n_N and n defined as in (5.1), $(X_N^1, X_N^2, \ldots, X_N^k, n_N) \Rightarrow (X^1, X^2 \ldots, X^k, n)$, where n is independent from $(X^1, X^2, \ldots, X^k)$.*

Proof For the convergence, it is enough to show (i) $(X_N^1, X_N^2 \ldots, X_N^k, n_N)$ is tight, and (ii) convergence of characteristic function. The tightness follows from the tightness of individual random vector (variable), which is a consequence of individual convergence.

For (ii), consider the eigenvalue decomposition $H = OPO^T$, where $P = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_N)$ and O is an orthogonal matrix. Since the H is orthogonal invariant, P and O are independent. Set $X = O^T Y$. Then $X = \frac{1}{\sqrt{N}}(x_1, \ldots, x_N)$ where $x_1, \ldots, x_N$ are i.i.d standard Gaussian (X is also independent with P).

Now, $n_N = Y^* G(z) Y - \frac{1}{N}\mathrm{Tr} G(z) = \frac{1}{N}\sum_{i=1}^N \frac{x_i^2-1}{\rho_i - z}$. Since $\mathbb{E}[e^{tx_1^2}] = \frac{1}{\sqrt{1-2t}}$, we find that for any $t \in \mathrm{i}\mathbb{R}$, the conditional expectation over X given P satisfies

$$\mathbb{E}_X\left[e^{tn_N}\middle|P\right] = \mathbb{E}_X\left[e^{\frac{t}{\sqrt{N}}\sum_{i=1}^N \frac{x_i^2-1}{\rho_i - z}}\middle|P\right] = \prod_{i=1}^N e^{-\frac{1}{2}\log(1-\frac{2t}{\sqrt{N}(\rho_i - z)}) - \frac{t}{\sqrt{N}(\rho_i - z)}}.$$

Note that $(X_N^1, X_N^2, \ldots, X_N^k)$ only depends on the eigenvalues, and hence it is independent of X. Thus, for any $u_1, u_2, \ldots, u_k, t \in \mathrm{i}\mathbb{R}$,

$$\mathbb{E}\left[e^{\sum_{j=1}^k u_j X_N^j + tn_N}\right] = \mathbb{E}\left[e^{\sum_{j=1}^k u_j X_N^j} \prod_{i=1}^N e^{-\frac{1}{2}\log(1-\frac{2t}{\sqrt{N}(\rho_i - z)}) - \frac{t}{\sqrt{N}(\rho_i - z_2)}}\right]. \tag{5.2}$$

Since $-\frac{1}{2}\log(1-2z) - z = z^2 + O(z^3)$ as $z \to 0$, using Corollary 3.1,

$$\begin{aligned}&\prod_{i=1}^N e^{-\frac{1}{2}\log(1-\frac{2t}{\sqrt{N}(\rho_i - z)}) - \frac{t}{\sqrt{N}(\rho_i - z)}} \\ &= e^{\frac{1}{N}\sum_{i=1}^N \frac{t^2}{(\rho_i - z)^2} + O(N^{-\frac{1}{2}})} = e^{t^2 \int \frac{1}{(x-z)^2}\mathrm{d}\sigma_{scl}(x) + O(N^{-\frac{1}{2}})} = \mathbb{E}\left[e^{tn(z)}\right] e^{O(N^{-\frac{1}{2}})}\end{aligned} \tag{5.3}$$

with high probability. Denote this high probability event by Ω_N. Then,

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[e^{\sum_{j=1}^k u_j X_N^j + t n_N}\right] \\
&= \lim_{N\to\infty} \left(\mathbb{E}\left[e^{\sum_{j=1}^k u_j X_N^j + t n_N} \big| \Omega_N\right] \mathbb{P}(\Omega_N) + \mathbb{E}\left[e^{\sum_{j=1}^k u_j X_N^j + t n_N} \big| \Omega_N^c\right] \mathbb{P}(\Omega_N^c)\right) \\
&= \mathbb{E}\left[e^{\sum_{j=1}^k u_j X^j}\right] \mathbb{E}\left[e^{t n(z)}\right],
\end{aligned} \tag{5.4}$$

since $t, u_1, u_2, \ldots, u_k \in \mathrm{i}\mathbb{R}$ and hence all exponents are pure imaginary. Note that the characteristic function of $(X^1, \ldots, X^k, n)$ is equal to the product of the characteristic functions of individual random vector (variable). Thus $n(z)$ is independent from $(X^1, \ldots, X^k)$. This completes the proof. □

Corollary 5.1 *Fix $\delta > 0$, consider $z_1 \in \mathbb{C}\backslash\mathbb{R}$ and $z_2 \in \mathbb{R}\backslash[-2-\delta, 2+\delta]$. Recall $s(z)$ defined in (4.9). Then $(\mathrm{Tr}(G(z_1)) - Ns(z_1), n_N(z_2))$ converges in distribution to independent Gaussian random variables.*

Proof Note that $\mathrm{Tr}(G(z_1)) - Ns(z_1)$ is complex, we consider the random vector $(\Re(\mathrm{Tr}(G(z_1)) - Ns(z_1)), \Im(\mathrm{Tr}(G(z_1)) - Ns(z_1)))$. By Proposition 5.1, it is enough to show that $(\Re(\mathrm{Tr}(G(z_1)) - N(s_1)), \Im(\mathrm{Tr}(G(z_1)) - Ns(z_1))$ converges to a Gaussian random vector. Consider the expression $z_1 = E + i\eta$ for $\epsilon, \eta \in \mathbb{R}$ and $\eta \neq 0$. Recalling the definition of linear statistics $\mathcal{N}_N(\varphi)$ defined in (2.3), we have

$$\Re(\mathrm{Tr}(G(z_1) - Ns(z_1))) = \mathcal{N}_N(\varphi_r), \quad \varphi_r(x) = \frac{x - E}{(x-E)^2 + \eta^2},$$

and

$$\Im(\mathrm{Tr}(G(z_1) - Ns(z_1)) = \mathcal{N}_N(\varphi_i), \quad \varphi_i(x) = \frac{\eta}{(x-E)^2 + \eta^2}.$$

That is, they are both linear statistics. Then Corollary then follows from Theorem 1.1 of [2]. □

Remark 5.1 When we prove Theorem 2.2 for GOE, we use Proposition 5.1 and Corollary 5.1 with N-dependent z_i. First, for a fixed $z_2 \in \mathbb{R}\backslash[-2-\delta, 2+\delta]$ for some $\delta > 0$, let $\tilde{z}_2 = \tilde{z}_2(N) := \sqrt{\frac{N+1}{N}} z_2$. Using the exactly same argument in the proof of Lemma 5.1, one can show $n_N(\tilde{z}_2) \Rightarrow n(z_2)$. Since the (5.3) still holds for $\tilde{z}_2$ and $n(z_2)$, the asymptotic independence in Proposition 5.1 is still valid, i.e.

$$(X_N^1, X_N^2, \ldots, X_N^k, n_N(\tilde{z}_2)) \Rightarrow (X^1, X^2 \ldots, X^k, n(z_2)),$$

where $n(z_2)$ is independent from $(X^1, X^2, \ldots, X^k)$. Second, for $z_1 \in \mathbb{C}\backslash\mathbb{R}$, consider $\tilde{z}_1 = \tilde{z}_1(N) := \sqrt{\frac{N+1}{N}} z_1$. Notice that

$$\frac{1}{x - \tilde{z}_1} = \frac{1}{x - z_1} + \frac{z_1}{2N(x - z_1)^2} + O(N^{-2}).$$

Then, by the discussion in Remark 2.1, $\mathrm{Tr}(G(\tilde{z}_1)) - Ns(\tilde{z}_1) = \mathcal{N}_N(\frac{1}{x - \tilde{z}_1})$ converges to a Gaussian random variable. Now, putting together, for $\tilde{z}_1$ and $\tilde{z}_2$ defined as above, $(\mathrm{Tr}(G(\tilde{z}_1)) - Ns(\tilde{z}_1), n_N(\tilde{z}_2))$ converge jointly to independent Gaussian random variables.

We now prove Theorem 2.2 for the case where the disorder belongs to GOE.

Proof (Proof of Theorem 2.2 when A belongs to GOE) Recall that λ_i are the eigenvalues of $M = \frac{1}{\sqrt{N}}A + \frac{J}{N}\mathbf{1}\mathbf{1}^T$ with A from the GOE. Since the means and variances follow from [10] and Theorem 2.1, it is enough to prove the asymptotic independence of χ_N and $\mathcal{N}_N^{(2)}(\varphi)$. (Notice that that $W_3 = 0$ for Gaussian A_{ij}.) Now, for any analytic test function φ, the partial linear statistics can be expressed as (see (4.2)) an integral of

$$\xi_N^{(2)}(z) = \sum_{i=2}^{N} \frac{1}{\lambda_i - z} - N \int_{\mathbb{R}} \frac{1}{x - z} \mathrm{d}\sigma_{scl}(x), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{5.5}$$

Then according to Lemma 4.1 and what follows, it is enough to prove that χ_N and $\xi_N^{(2)}(z)$ are asymptotically independent for fixed $z \in \mathbb{C} \setminus \mathbb{R}$. Let

$$\xi_N(z) = \xi_N^{(2)}(z) + \frac{1}{\lambda_1 - z} = \mathrm{Tr}(M - zI)^{-1} - Ns(z).$$

Since $\frac{1}{\lambda_1 - z} \to \frac{1}{\hat{J} - z}$ in probability, it is enough to prove that χ_N and $\xi_N(z)$ are asymptotically independent.

Since the GOE is orthogonal invariant, for every deterministic matrix U, the eigenvalues of $A + U$ have the same distribution as $A + OUO^T$ for any orthogonal matrix O. Thus, we may consider the following equivalent model:

$$M = \frac{1}{\sqrt{N}}A + \mathrm{diag}(J, 0, \ldots, 0). \tag{5.6}$$

Following the proof of Theorem 2.2 in [10], we write

$$M = \begin{bmatrix} \frac{A_{11}}{\sqrt{N}} + J & Y^* \\ Y & \hat{M} \end{bmatrix}. \tag{5.7}$$

Since $\det(M - zI) = \det(\hat{M} - zI)\left(\frac{A_{11}}{\sqrt{N}} + J - z - Y^*\hat{G}(z)Y\right)$ with

$$\hat{G}(z) := (\hat{M} - zI_{N-1})^{-1} = \left(\frac{1}{\sqrt{N}}\hat{A} - zI_{N-1}\right)^{-1}, \tag{5.8}$$

the largest eigenvalue of M satisfies

$$\lambda_1 = J + \frac{A_{11}}{\sqrt{N}} - Y^*\hat{G}(\lambda_1)Y \tag{5.9}$$

if λ_1 is not an eigenvalue of $\hat{M}$, which holds with high probability. Using the resolvent formula twice, we write

$$\begin{aligned} \hat{G}(\lambda_1) &= \hat{G}(\hat{J}) + (\hat{G}(\lambda_1) - \hat{G}(\hat{J})) = \hat{G}(\hat{J}) + (\lambda_1 - \hat{J})\hat{G}(\lambda_1)\hat{G}(\hat{J}) \\ &= \hat{G}(\hat{J}) + (\lambda_1 - \hat{J})\hat{G}(\hat{J})^2 + (\lambda_1 - \hat{J})^2\hat{G}(\lambda_1)\hat{G}(\hat{J})^2. \end{aligned}$$

Hence,

$$\begin{aligned} \lambda_1 - \hat{J} &= \frac{A_{11}}{\sqrt{N}} - \frac{1}{J} - Y^*\hat{G}(\lambda_1)Y \\ &= \frac{A_{11}}{\sqrt{N}} - \frac{1}{J} - Y^*\hat{G}(\hat{J})Y + (\lambda_1 - \hat{J})Y^*\hat{G}(\hat{J})^2Y + (\lambda_1 - \hat{J})^2Y^*\hat{G}(\lambda_1)\hat{G}(\hat{J})^2Y \end{aligned}$$

with high probability. Moving all terms with factor $\lambda_1 - \hat{J}$ to the left and taking it out as a common factor, we arrive at

$$\chi_N = \sqrt{N}(\lambda_1 - \hat{J}) = \frac{A_{11} - \sqrt{N}\left(\frac{1}{J} + Y^*\hat{G}(\hat{J})Y\right)}{1 + Y^*\hat{G}(\hat{J})^2Y + (\lambda_1 - \hat{J})Y^*\hat{G}(\lambda_1)\hat{G}(\hat{J})^2Y} \tag{5.10}$$

with high probability.

Note that $\hat{M}$ and Y satisfy the setting of Corollary 5.1 up to the scaling factor $\sqrt{\frac{N}{N-1}}$. Set

$$\tilde{Y} = \sqrt{\frac{N}{N-1}}Y, \qquad \tilde{G}(z) = \left(\sqrt{\frac{N}{N-1}}\hat{M} - zI_{N-1}\right)^{-1}. \tag{5.11}$$

Then, $\tilde{Y}$ and $\tilde{G}$ satisfy the setting of Corollary 5.1, and

$$Y^*\hat{G}(\hat{J})Y = \sqrt{\frac{N-1}{N}}\tilde{Y}^*\tilde{G}(\tilde{J})\tilde{Y}, \qquad \tilde{J} := \sqrt{\frac{N}{N-1}}\hat{J}. \tag{5.12}$$

Now, by Corollary 3.1,

$$\frac{1}{N-1}\operatorname{Tr}(\tilde{G}(\tilde{J})) = s(\hat{J}) + O(N^{-1+\epsilon}) = -\frac{1}{J} + O(N^{-1+\epsilon}) \tag{5.13}$$

with high probability. By Lemma 5.1, Corollary 3.1 and Lemma 3.2,

$$Y^*\hat{G}(\hat{J})^2Y \to \frac{1}{J^2-1}, \qquad (\lambda_1 - \hat{J})Y^*\hat{G}(\lambda_1)\hat{G}(\hat{J})^2Y \to 0 \tag{5.14}$$

in probability. Using (5.14), (5.13) and denoting the denominator in (5.10) by D_1, we write

$$\chi_N = D_1^{-1}\left(A_{11} - \tilde{n}_{N-1}(\tilde{J}) + O(N^{-\frac{1}{2}+\epsilon})\right), \tag{5.15}$$

where $n_{N-1}(\tilde{J}) = \sqrt{N-1}(\tilde{Y}^*\tilde{G}(\tilde{J})\tilde{Y} - \frac{1}{N-1}\operatorname{Tr}(\tilde{G}(\tilde{J})))$ (see (5.1)) and $D_1 \to \frac{J^2}{J^2-1}$ in probability. Note that A_{11} and $n_{N-1}(\tilde{J})$ are independent, the distribution of χ_N is governed by their convolution.

We now turn to the linear statistic $\xi_N(z)$. Using Schur complement of M with block structure in (5.7), for any $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\operatorname{Tr}(M - zI)^{-1} = \left(J + \frac{A_{11}}{\sqrt{N}} - z - Y^*\hat{G}(z)Y\right)^{-1}(1 + Y^*\hat{G}(z)^2Y) + \operatorname{Tr}(\hat{G}(z)). \tag{5.16}$$

Using Lemma 5.1 and Lemma 3.1,

$$D_2 = D_2(N) := \frac{1 + Y^*\hat{G}(z)^2Y}{J + \frac{A_{11}}{\sqrt{N}} - z - Y^*\hat{G}(z)Y} \to \frac{1 + s'(z)}{J - z - s(z)}$$

in probability. Then, by setting $\tilde{z} := \tilde{z}(N) = \sqrt{\frac{N}{N-1}}z$, we write

$$\begin{aligned}\xi_N(z) =& \operatorname{Tr}(M - zI)^{-1} - Ns(z) = D_2 + \operatorname{Tr}\hat{G}(z) - Ns(z) + O(N^{-\frac{1}{2}+\epsilon}) \\ =& D_2 - \frac{s(z)}{2} + \frac{zs'(z)}{2} + \sqrt{\frac{N}{N-1}}\left(\operatorname{Tr}\tilde{G}(\tilde{z}) - (N-1)s(\tilde{z})\right) + O(N^{-\frac{1}{2}+\epsilon}).\end{aligned} \tag{5.17}$$

That is, the fluctuation of $\xi_N(z)$ is govern by $\mathrm{Tr}\tilde{G}(\tilde{z}) - (N-1)s(\tilde{z})$. Now using Corollary 5.1 and Remark 5.1, one can conclude that $(\mathrm{Tr}\tilde{G}(\tilde{z}) - (N-1)s(\tilde{z}), n_{N-1}(\tilde{J}))$ converge to independent Gaussian random variables. Furthermore, A_{11} is independent of both Y and $\hat{M}$. Thus by (5.15) and (5.17), $(\xi_N(z), \chi_N)$ converge to independent random variables. Theorem 2.2 then follows. □

5.2 Proof of Theorem 2.2 for General Case

We prove Theorem 2.2 for general disorders, where the disorder matrix A is a Wigner matrix and satisfies Definition 1.1. Unlike the GOE, Wigner matrices are not orthogonal invariant, hence we cannot apply (5.6) where we replaced the rank-1 perturbation in M by a diagonal matrix. To overcome the difficulty, we use an interpolation method. It has been successfully applied in many works in random matrix theory, where a given matrix and a reference matrix such as GOE are interpolated. We refer to [22] for its application in the analysis of linear eigenvalue statistics.

Let $V = \frac{1}{\sqrt{N}}A$ be a (normalized) Wigner matrix and V^G be a (normalized) GOE matrix independent from V. Define

$$H(t) = V\cos t + V^G \sin t \tag{5.18}$$

so that $H(0) = V$ and $H(\frac{\pi}{2}) = V^G$. Note that $\mathbb{E}[H_{ij}^2] = \frac{1}{N}$ for $i \neq j$. Let

$$\boldsymbol{e} = \frac{1}{\sqrt{N}}\mathbf{1}^T = \frac{1}{\sqrt{N}}(1, 1, \ldots, 1)^T \in \mathbb{R}^N \tag{5.19}$$

and

$$M(t) = H(t) + J\boldsymbol{e}\boldsymbol{e}^T, \tag{5.20}$$

whose eigenvalues are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Define the resolvents

$$G(z) = (M - zI)^{-1}, \qquad \hat{G}(z) = (H - zI)^{-1}. \tag{5.21}$$

Here, we omit the dependence on t for the ease of notation. We note that G and $\hat{G}$ are symmetric (not Hermitian). For any (small) fixed $\delta > 0$, $\hat{G}(z)$ is well-defined for $z \in \mathbb{C} \setminus [-2-\delta, 2+\delta]$ with high probability.

For $\chi_N = \sqrt{N}(\lambda_1 - \hat{J})$, we notice that

$$\hat{G}_{\boldsymbol{e}\boldsymbol{e}}(\lambda_1) := \langle \boldsymbol{e}, \hat{G}(\lambda_1)\boldsymbol{e}\rangle = -\frac{1}{J} \tag{5.22}$$

with high probability. The claim holds since

$$\begin{aligned} 0 = \det(M - \lambda_1 I) &= \det(H - \lambda_1 I)\det(I + J\hat{G}(\lambda_1)\boldsymbol{e}\boldsymbol{e}^T) \\ &= \det(H - \lambda_1 I)\left(1 + J\hat{G}_{\boldsymbol{e}\boldsymbol{e}}(\lambda_1)\right) \end{aligned} \tag{5.23}$$

and λ_1 is not an eigenvalue of H with high probability (See [20, Lemma 6.1]). Furthermore, by Taylor expansion,

$$-\frac{1}{J} = \hat{G}_{\boldsymbol{e}\boldsymbol{e}}(\lambda_1) = \hat{G}_{\boldsymbol{e}\boldsymbol{e}}(\hat{J}) + \hat{G}'_{\boldsymbol{e}\boldsymbol{e}}(\hat{J})(\lambda_1 - \hat{J}) + O(N^{-1+\epsilon}) \tag{5.24}$$

with high probability, since $|\lambda_1 - \hat{J}| = O(N^{-\frac{1}{2}+\epsilon})$ and $\|\hat{G}''(z)\| = O(1)$ with high probability. From the isotropic local law, Theorem 2.2 of [20], we find that

$$\hat{G}_{\boldsymbol{ee}}(\hat{J}) = s(\hat{J}) + O(N^{-\frac{1}{2}+\epsilon}), \qquad \hat{G}'_{\boldsymbol{ee}}(\hat{J}) = s'(\hat{J}) + O(N^{-\frac{1}{2}+\epsilon}) \tag{5.25}$$

with high probability. Thus, using Lemma 5.1,

$$\chi_N = \sqrt{N}(\lambda_1 - \hat{J}) = -\frac{\sqrt{N}(J^{-1} + \hat{G}_{\boldsymbol{ee}}(\hat{J}))}{s'(\hat{J})} + O(N^{-\frac{1}{2}+2\epsilon}) \tag{5.26}$$

with high probability. That is, the behavior of χ_N is governed by the fluctuation of $\hat{G}_{\boldsymbol{ee}}(\hat{J})$.

To prove the Theorem 2.2, as in the Gaussian disorder case, it is enough to show the convergence of the joint distribution of χ_N and the full linear statistics $\xi_N(z) = \mathrm{Tr}(G(z)) - Ns(z)$ for fixed $z \in \mathbb{C} \setminus \mathbb{R}$. Under the light of (5.26), we set out to calculate the following characteristic function involving $\xi_N(z)$ and $\hat{G}_{\boldsymbol{ee}}(\hat{J})$. Explicitly, for $t_1, t_2, t_3 \in \mathrm{i}\mathbb{R}$ and $z = E + \mathrm{i}\eta$ with $E \in \mathbb{R}$ and $\eta > 0$, we define

$$\mathbb{E}\left[e^{P(t)}\right] := \mathbb{E}\left[e^{t_1 \Re \xi_N + t_2 \Im \xi_N + t_3 n_N}\right], \quad P(t) := t_1 \Re \xi_N(z) + t_2 \Im \xi_N(z) + t_3 n_N, \tag{5.27}$$

where

$$n_N = \sqrt{N}\left(\hat{G}_{\boldsymbol{ee}}(\hat{J}) + \frac{1}{J}\right). \tag{5.28}$$

Note that n_N is real, the exponent $P(t)$ is pure imaginary and thus $|e^{P(t)}| \leq 1$. For our purpose, it is desired to estimate $\mathbb{E}[e^{P(0)}]$. At $t = \frac{\pi}{2}$, the disorder $H(\frac{\pi}{2})$ reduces to the GOE case. From Sect. 5.1, χ_N and ξ_N are asymptotically independent in the GOE case, then

$$\lim_{N\to\infty} \mathbb{E}\left[e^{P(\frac{\pi}{2})}\right] = \mathbb{E}\left[e^{t_1 \Re \xi + t_2 \Im \xi}\right] \cdot \mathbb{E}\left[e^{t_3 n}\right] \tag{5.29}$$

for some Gaussian random variables ξ, n with known mean and variance. Thus, it only remains to estimate the t-derivative of $\mathbb{E}[e^{P(t)}]$. Here, we recall the following identity for the derivative of the resolvent G. For $i, j, a, b = 1, 2, \ldots, N$,

$$\frac{\partial}{\partial M_{ij}} G_{ab} = -\beta_{jk}(G_{aj}G_{kb} + G_{ak}G_{jb}) \tag{5.30}$$

with

$$\beta_{jk} = \begin{cases} 1 & j \neq k, \\ 1/2 & j = k. \end{cases} \tag{5.31}$$

We note that the above identity also holds if one replace G by $\hat{G}$. Thus for any fixed event Ω, $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[e^{P(t)}|\Omega\right]$ equals

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i\leq j} \frac{\mathrm{d}M_{ij}}{\mathrm{d}t} \frac{\partial}{\partial M_{ij}} e^{P(t)} \Big| \Omega\right] \\
&= \sin t \sum_{i,j} \mathbb{E}\left[V_{ij}\left(t_1 \Re\left(G^2\right)_{ij} + t_2 \Im\left(G^2\right)_{ij} + \frac{t_3}{\sqrt{N}} \sum_{p,q} \hat{G}_{pi}\hat{G}_{jq}\right) e^{P(t)} \Big| \Omega\right] \\
&\quad - \cos t \sum_{i,j} \mathbb{E}\left[V^G_{ij}\left(t_1 \Re\left(G^2\right)_{ij} + t_2 \Im\left(G^2\right)_{ij} + \frac{t_3}{\sqrt{N}} \sum_{p,q} \hat{G}_{pi}\hat{G}_{jq}\right) e^{P(t)} \Big| \Omega\right].
\end{aligned} \tag{5.32}$$

The reason for the introduction of Ω will be revealed in a minute. The right hand side of (5.32) motivates us to apply the generalized Stein's lemma. More precisely, we will use Proposition 3.1 of [22] with a small modification as follows:

Proposition 5.2 *Given an event Ω, let X be a random variable such that $\mathbb{E}[|X|^{p+2}|\Omega] < \infty$ for a certain non-negative integer p. Denote the conditional cumulants of X by $\kappa_l := \kappa_l(\Omega)$, $l = 1, \dots, p+1$. Then for any function $\Phi : \mathbb{R} \to \mathbb{C}$ of the class C^{p+1} with bounded derivatives $\Phi^{(l)}$, $l = 1, \dots, p+1$, we have*

$$\mathbb{E}[X\Phi(X)|\Omega] = \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!} \mathbb{E}[\Phi^{(l)}(X)|\Omega] + \epsilon_p, \tag{5.33}$$

where the remainder term ϵ_p admits the bound

$$|\epsilon_p| \le C_p \, \mathbb{E}\left[|X|^{p+2} \left(1 + \max_{1 \le j \le p+1} \left(\int_0^1 |\Phi^{(p+1)}(vX)| \mathrm{d}v \right)^{\frac{p+2}{j}} \right) \Big| \Omega \right] \tag{5.34}$$

for some constant C_p that depends only on p.

Proof We basically follow the proof of Proposition 3.1 of [22]. Let π_p be the degree p Taylor polynomial of Φ and let $r_p = \Phi - \pi_p$. Then, as in the proof of Proposition 3.1 of [22],

$$\mathbb{E}[X\pi_p(X)|\Omega] = \sum_{j=0}^{p} \frac{\kappa_{j+1}}{j!} \mathbb{E}[\pi_p^{(j)}(X)|\Omega]. \tag{5.35}$$

Thus

$$\left| \mathbb{E}[X\Phi|\Omega] - \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!} \mathbb{E}[\Phi^{(l)}|\Omega] \right| \le \left| \mathbb{E}[Xr_p|\Omega] \right| + \sum_{l=0}^{p} \frac{|\kappa_{l+1}|}{l!} \left| \mathbb{E}\left[r_p^{(l)} | \Omega \right] \right|. \tag{5.36}$$

Since

$$r_p(X) = \frac{X^{p+1}}{p!} \int_0^1 \Phi^{(p+1)}(vX)(1-v)^p \mathrm{d}v, \tag{5.37}$$

by the estimate $|\kappa_j| \le (2j)^j \, \mathbb{E}[|X|^j|\Omega]$ and Hölder's inequality,

$$\begin{aligned} &\sum_{l=0}^{p} \frac{|\kappa_{l+1}|}{l!} \left| \mathbb{E}\left[r_p^{(l)}(X) | \Omega \right] \right| \\ &\le \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!(p-l)!} \mathbb{E}\left[|X|^{p+1-l} \int_0^1 |\Phi^{(p+1)}(vX)| \mathrm{d}v \Big| \Omega \right] \\ &\le \sum_{l=0}^{p} \frac{(2l+2)^{l+1}}{l!(p-l)!} \mathbb{E}\left[|X|^{p+2} \left(1 + \left(\int_0^1 |\Phi^{(p+1)}(vX)| \mathrm{d}v \right)^{\frac{p+2}{p+1-l}} \right) \Big| \Omega \right]. \end{aligned} \tag{5.38}$$

As $\left| \mathbb{E}[Xr_p|\Omega] \right|$ can also be bounded by the right hand side of (5.38), the proof is complete. □

In order to apply Proposition 5.2 to (5.32), we need prior bounds of $P(t)$ and its derivatives to bound ϵ_p in (5.33). As we will see later, it is enough to bound G_{ij}, $(G^2)_{ij}$, $\hat{G}_{ij}$ and $\sum_p \hat{G}_{ip}$.

In the following, we are going to introduce a high probability event Ω, on which we have the desired bounds.

With the trivial bound $\|G\| \leq \frac{1}{\eta}$ (recall that $z = E + i\eta$), we have that $|G_{ij}| \leq \frac{1}{\eta}$ and $\left|(G^2)_{ij}\right| \leq \left\|G^2\right\| \leq \frac{1}{\eta^2}$. For $\hat{G}_{ij}$, we introduce the high probability event $\Omega_1 = \{\lambda_1 \leq (2 + \hat{J})/2\}$. It is easy to check that $\|\hat{G}\mathbb{1}_{\Omega_1} \leq \frac{1}{\hat{J}-2}$ and thus

$$|\hat{G}_{ij}\mathbb{1}_{\Omega_1}| \leq \frac{1}{\hat{J}-2}, \tag{5.39}$$

For $\sum_p \hat{G}_{ip}$, we recall the following concentration theorem for the quadratic function of $\hat{G}$:

Proposition 5.3 (Theorem 2.3 and Remark 2.4 of [20]) *Fix $\Sigma \geq 3$. Set $\varphi = (\log N)^{\log\log N}$. Then there exist constants C_1 and C_2 such that for any*

$$E \in \left[\Sigma, -2 - \varphi^{C_1} N^{-\frac{2}{3}}\right] \cup [2 + \varphi^{C_1} N^{-\frac{2}{3}}, \Sigma],$$

and any $\eta \in (0, \Sigma]$, and any deterministic $v, w \in \mathbb{C}^N$,

$$|\langle v, \hat{G}(z)w\rangle - s(z)\langle v, w\rangle| \leq \varphi^{C_2}\sqrt{\frac{\Im s(z)}{N\eta}}\,\|v\|\,\|w\| \tag{5.40}$$

with high probability, uniformly on $z = E + \mathrm{i}\eta$.

Let $\boldsymbol{e}_i := (0, \ldots, 1, \cdots, 0)$. Noting that $\sum_{p=1}^N \hat{G}_{pi} = \sqrt{N}\langle \boldsymbol{e}, \hat{G}\boldsymbol{e}_i\rangle$, we can derive a prior bound for $\sum_{p=1}^N \hat{G}_{pi}$, which is summarized in the following Corollary.

Corollary 5.2 *For any fixed $E \in \mathbb{R}\backslash[-2, 2]$, the tail bound*

$$\left|\sum_p (\hat{G}(E))_{pi}\right| \leq N^{\epsilon} \tag{5.41}$$

holds simultaneously for $i = 1, \ldots, N$ with high probability. We also have that

$$|\langle v, \hat{G}(E)w\rangle - s(E)\langle v, w\rangle| \leq \|v\|\,\|w\|\,N^{-\frac{1}{2}+\epsilon} \tag{5.42}$$

with high probability.

Proof We first prove (5.42). Consider $z = E + \mathrm{i}N^{-1/2}$. Using Proposition 5.3, we find there exists some $C > 0$ such that

$$\begin{aligned}
&\langle v, \hat{G}(E)w\rangle - s(E)\langle v, w\rangle| \\
&\quad \leq |\langle v, (\hat{G}(z) - \hat{G}(E)w\rangle| + |\langle v, \hat{G}(z)w\rangle - s(z)\langle v, w\rangle| + |s(z) - s(E)||\langle v, w\rangle| \\
&\quad \leq CN^{-1/2}\,\|v\|\,\|w\| + C\varphi^C N^{-\frac{1}{2}}\,\|v\|\,\|w\| + CN^{-1/2}.
\end{aligned} \tag{5.43}$$

Here we also use the fact that Ω_1 holds with high probability. Since $\varphi \ll N^{\epsilon}$, (5.42) then follows. The tail bound (5.41) can be obtained from (5.42) by setting $v = \sqrt{N}\boldsymbol{e}$ and $w = \boldsymbol{e}_i$. □

We are ready to introduce the high probability event as promised. Set $s_1 := s(z)$, $s_1' := s'(z)$ and $s_2 := s(\hat{J}) = -J^{-1}$, the desired high probability event Ω is the intersection of Ω_1

and the following events:

$$\Omega_2 = \left\{ \left| \sum_p (\hat{G}(\hat{J}))_{pi} \right| \le N^{\epsilon}, \quad \forall i = 1, \dots, N \right\} \cap \{ |\hat{G}_{\boldsymbol{ee}}(\hat{J}) - s_2| \le N^{-\frac{1}{2}+\epsilon} \}, \tag{5.44}$$

$$\Omega_3 = \{ |\hat{G}_{ij} - \delta_{ij} s_2| \le N^{-\frac{1}{2}+\epsilon}, \quad \forall i, j = 1, \dots, N \}, \tag{5.45}$$

$$\Omega_4 = \{ |G_{ij} - \delta_{ij} s_1|, |(G^2)_{ij} - \delta_{ij} s_1'| \le N^{-\frac{1}{2}+\epsilon}, \quad \forall i, j = 1, \dots, N \}, \tag{5.46}$$

$$\Omega_5 = \{ |V_{ij}|, |V_{ij}^G|, |M_{ij}| \le N^{-\frac{1}{2}+\epsilon}, \quad \forall i, j = 1, \dots, N \}. \tag{5.47}$$

Here, by Corollary 5.2, Ω_2 is a high probability event. The fact that Ω_3 and Ω_4 are high probability events can be checked from Theorem 2.8 and Theorem 2.9 of [17]. It is easy to check that Ω_5 is a high probability event from the existence of all moments. Furthermore, by the Lipshitz continuity of the resolvents, we also find that Ω holds uniformly on t with high probability.

Applying Proposition 5.2 to Equation (5.32) conditioning on Ω, we claim that for any $1 \le i, j \le N$,

$$\begin{aligned} &\mathbb{E}\left[V_{ij} \left(t_1 \Re (G^2)_{ij} + t_2 \Im (G^2)_{ij} + \frac{t_3}{\sqrt{N}} \sum_{p,q} \hat{G}_{pi} \hat{G}_{jq} \right) e^P \Big| \Omega \right] \\ &= \sum_{l=1}^{3} \frac{\kappa_{l+1}^{V_{ij}} \cos^l t}{l!} \mathbb{E}\left[\left(\frac{\partial}{\partial M_{ij}} \right)^l \left(\left(t_1 \Re (G^2)_{ij} + t_2 \Im (G^2)_{ij} \right.\right.\right. \\ &\quad \left.\left.\left. + \frac{t_3}{\sqrt{N}} \sum_{p,q} \hat{G}_{pi} \hat{G}_{jq} \right) e^P \right) \Big| \Omega \right] + O(N^{-\frac{5}{2}+\epsilon}) \end{aligned} \tag{5.48}$$

where $\kappa_l^{V_{ij}}$ denotes the lth cumulant of V_{ij}. Here, it is legal to replace the conditional cumulants by $\kappa_l^{V_{ij}}$, since Ω is a high probability event.

To prove the claim, we begin by controlling the remainder term ϵ_p in (5.33). On Ω, G_{ij}, $\hat{G}_{ij}$ and $(G^2)_{ij}$ are $O(1)$, and

$$N^{-\frac{1}{2}} \sum_{p,q} \hat{G}_{pi} \hat{G}_{jq} = N^{-\frac{1}{2}} \left(\sum_p \hat{G}_{pi} \right) \left(\sum_q \hat{G}_{qj} \right) = O\left(N^{-\frac{1}{2}+\epsilon} \right).$$

Thus, $\frac{\partial}{\partial M_{ij}} P = O(1)$ on Ω. From the resolvent identity and the definition of event Ω, we find $\| G(z; vV_{ij}) - G(z; V_{ij}) \| = O(N^{-\frac{1}{2}+\epsilon})$ for $0 \le v \le 1$. Thus on Ω, $\frac{\partial}{\partial M_{ij}} P(t; vV_{ij}) = O(1)$ for $0 \le v \le 1$. Furthermore, we notice that

$$\frac{\partial}{\partial M_{ij}} (G^2)_{ij} = \frac{\partial}{\partial M_{ij}} \sum_k G_{ki} G_{jk} = -\beta_{ij} \left(2G_{ij} (G^2)_{ij} + G_{ii} (G^2)_{jj} + G_{jj} (G^2)_{ii} \right), \tag{5.49}$$

and

$$\frac{\partial}{\partial M_{ij}} \sum_p \hat{G}_{pi} = -\beta_{ij} \left(\hat{G}_{ji} \sum_p \hat{G}_{pi} + \hat{G}_{ii} \sum_p \hat{G}_{pj} \right). \tag{5.50}$$

Thus we can obtain similar estimates for higher derivatives of P. Since $V_{ij}^5 = O(N^{-\frac{5}{2}+5\epsilon})$ on Ω_5, we find that

$$|V_{ij}|^5 \left(1 + \max_{1\le j\le 5}\left(\int_0^1 \left|\left(\frac{\partial}{\partial M_{ij}}\right)^5 P\right| \mathrm{d}v\right)^{\frac{5}{j}}\right) \le CN^{-\frac{5}{2}+C\epsilon} \tag{5.51}$$

on Ω. That is, $\epsilon_3 \le CN^{-\frac{5}{2}+C\epsilon}$, and after summing over i, j, the claim (5.48) is proved.

We next consider the term in (5.32) containing V^G. Noting that the cumulants of order higher than 2 vanish for Gaussian random variables, for any $1 \le i, j \le N$,

$$\begin{aligned}&\mathbb{E}\left[V_{ij}^G\left(t_1\Re(G^2)_{ij} + t_2\Im(G^2)_{ij} + \frac{t_3}{\sqrt{N}}\sum_{p,q}\hat{G}_{pi}\hat{G}_{jq}\right)e^P\Big|\Omega\right]\\ &= \kappa_2^{V_{ij}^G}\sin t\,\mathbb{E}\left[\frac{\partial}{\partial M_{ij}}\left(\left(t_1\Re(G^2)_{ij} + t_2\Im(G^2)_{ij} + \frac{t_3}{\sqrt{N}}\sum_{p,q}\hat{G}_{pi}\hat{G}_{jq}\right)e^P\right)\Big|\Omega\right]\\ &\quad + O(N^{-\frac{1}{2}+\epsilon}),\end{aligned} \tag{5.52}$$

where $\kappa_2^{V_{ij}^G}$ denotes the second cumulant of V_{ij}^G. We now put (5.48) and (5.52) into (5.32) conditioning on Ω. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[e^{P(t)}\Big|\,\Omega\right] = (\sin t)\sum_{l=1}^{3}(\cos^l t)I_l - (\cos t\sin t)I_1^G + O(N^{-\frac{1}{2}+\epsilon}), \tag{5.53}$$

where we define I_l by

$$\sum_{i,j}\frac{\kappa_{l+1}^{V_{ij}}}{l!}\mathbb{E}\left[\left(\frac{\partial}{\partial M_{ij}}\right)^l\left(\left(t_1\Re(G^2)_{ij} + t_2\Im(G^2)_{ij} + \frac{t_3}{\sqrt{N}}\sum_{p,q}\hat{G}_{pi}\hat{G}_{jq}\right)e^P\right)\Big|\Omega\right] \tag{5.54}$$

and I_1^G by

$$\sum_{i,j}\kappa_2^{V_{ij}^G}\mathbb{E}\left[\frac{\partial}{\partial M_{ij}}\left(\left(t_1\Re(G^2)_{ij} + t_2\Im(G^2)_{ij} + \frac{t_3}{\sqrt{N}}\sum_{p,q}\hat{G}_{pi}\hat{G}_{jq}\right)e^P\right)\Big|\Omega\right]. \tag{5.55}$$

In the following, we will evaluate I_l for $l = 1, 2, 3$ separately. We may omit the conditioning on Ω for the ease of notation.

5.2.1 Estimate for $I_1 - I_1^G$

Since $\kappa_2^{V_{ij}^G} = \kappa_2^{V_{ij}} = \frac{1}{N}$ for $i \ne j$, we only need to consider the contribution from the diagonal entries to $I_1 - I_1^G$. By (5.54) and (5.55), $I_1 - I_1^G$ equals

$$\sum_i(\kappa_2^{V_{ii}} - \kappa_2^{V_{ii}^G})\,\mathbb{E}\left[\frac{\partial}{\partial M_{ii}}\left(\left(t_1\Re(G^2)_{ii} + t_2\Im(G^2)_{ii} + \frac{t_3}{\sqrt{N}}\sum_{p,q}\hat{G}_{pi}\hat{G}_{iq}\right)e^P\right)\right]. \tag{5.56}$$

From (5.50), we find that

$$\frac{t_3}{\sqrt{N}} \frac{\partial}{\partial M_{ii}} \sum_{p,q} \hat{G}_{pi} \hat{G}_{iq} = O\left(N^{-\frac{1}{2}+\epsilon}\right).$$

Similarly, it can be checked that all terms in the right-hand side of (5.56) involving $\hat{G}$ are $O(N^{-\frac{1}{2}+\epsilon})$. Collecting the terms of order 1 only, (5.56) becomes

$$\begin{aligned} \frac{1}{N} \sum_i (w_2 - 2)\, \mathbb{E}\Big[\Big(& 2t_1 \Re\left((G^2)_{ii} G_{ii}\right) + 2t_2 \Im\left((G^2)_{ii} G_{ii}\right) \\ & + (t_1 \Re(G^2)_{ii} + t_2 \Im(G^2)_{ii})^2 \Big) e^P \Big] + O\left(N^{-\frac{1}{2}+\epsilon}\right). \end{aligned} \tag{5.57}$$

Using the estimate $|G_{ij} - \delta_{ij} s_1|, |(G^2)_{ij} - \delta_{ij} s_1'| \le N^{-\frac{1}{2}+\epsilon}$ on Ω_4, we conclude that $I_1 - I_1^G$ equals

$$(w_2 - 2)\left(2t_1 \Re(s_1' s_1) + 2t_2 \Im(s_1' s_1) + (t_1 \Re(s_1') + t_2 \Im(s_1'))^2\right) \mathbb{E}\left[e^P\right] + O(N^{-\frac{1}{2}+\epsilon}). \tag{5.58}$$

5.2.2 Estimate for I_2

We decompose I_2 into

$$\begin{aligned} I_2 &= \sum_{i,j} \frac{W_3}{2N^{\frac{3}{2}}} \mathbb{E}\left[\left(\frac{\partial}{\partial M_{ij}} \right)^2 \left(\left(t_1 \Re(G^2)_{ij} + t_2 \Im(G^2)_{ij} + \frac{t_3}{\sqrt{N}} \sum_{p,q} \hat{G}_{pi} \hat{G}_{jq} \right) e^P \right) \right] \\ &:= I_{2,0} + 2I_{1,1} + I_{0,2}, \end{aligned} \tag{5.59}$$

where $I_{r,2-r}$ is defined by

$$\begin{aligned} \sum_{i,j} \frac{W_3}{2N^{\frac{3}{2}}} \mathbb{E}\Big[& \left(\frac{\partial}{\partial M_{ij}} \right)^r \Big(t_1 \Re(G^2)_{ij} + t_2 \Im(G^2)_{ij} \\ & + \frac{t_3}{\sqrt{N}} \sum_{p,q} \hat{G}_{pi} \hat{G}_{jq} \Big) \cdot \left(\frac{\partial}{\partial M_{ij}} \right)^{2-r} e^P \Big]. \end{aligned} \tag{5.60}$$

We first consider the case $i \neq j$ in the summand of $I_{r,2-r}$ for $r = 0, 1, 2$. Recall that all terms of $O(N^{-\frac{1}{2}+\epsilon})$ are negligible in the sense that they can be absorbed into the error term in the right-hand side of (5.53).

(i) For $I_{2,0}$, we note that the terms arising from the derivatives of the G^2 are negligible, which can be checked by following the argument in the proof of Theorem 3.3 in [22], especially the estimate of T_3 in (3.53) of [22]. For example, one of such terms is bounded by

$$\left| N^{-\frac{3}{2}} \sum_{i,j} \frac{W_3}{2} \mathbb{E}\left[t_1 \Re(G_{ii} G_{jj} (G^2)_{ij}) e^P \right] \right| \le \frac{C}{\eta^4 \sqrt{N}}. \tag{5.61}$$

To prove it, we consider a vector $\boldsymbol{u} = (G_{11}, G_{22}, \ldots, G_{NN})$ and proceed as

$$\left|\sum_{i,j} G_{ii}G_{jj}(G^2)_{ij}\right| = \left|\langle \overline{\boldsymbol{u}}, G^2\boldsymbol{u}\rangle\right| \le \|G^2\|\|\boldsymbol{u}\|^2 \le N\|G^2\|\|G\|^2 \le \frac{N}{\eta^4}.$$

On the other hand,

$$\begin{aligned}\left(\frac{\partial}{\partial M_{ij}}\right)^2 \hat{G}_{pi}\hat{G}_{jq} = 6(\hat{G}_{pi}\hat{G}_{ji}^2\hat{G}_{jq} + \hat{G}_{pj}\hat{G}_{ii}\hat{G}_{ji}\hat{G}_{jq} + \hat{G}_{pi}\hat{G}_{ji}\hat{G}_{jj}\hat{G}_{iq}) \\ + \hat{G}_{ii}\hat{G}_{jj}(4\hat{G}_{pi}\hat{G}_{jq} + 2\hat{G}_{pj}\hat{G}_{iq}).\end{aligned} \tag{5.62}$$

From the estimate $|\hat{G}_{ij} - \delta_{ij}s_2| \le N^{-\frac{1}{2}+\epsilon}$ on Ω_3 the concentration of $\hat{G}_{\boldsymbol{ee}}$ on Ω_2, we then claim that

$$\begin{aligned}I_{2,0} =& \frac{W_3t_3}{2N^2}\sum_{i,j}\mathbb{E}\left[6\hat{G}_{ii}\hat{G}_{jj}\left(\sum_p \hat{G}_{pi}\right)\left(\sum_q \hat{G}_{qj}\right)e^P\right] + O\left(N^{-\frac{1}{2}+\epsilon}\right) \\ =& 3W_3t_3s_2^2\,\mathbb{E}\left[\hat{G}_{\boldsymbol{ee}}^2 e^P\right] + O\left(N^{-\frac{1}{2}+\epsilon}\right) = 3W_3t_3s_2^4\,\mathbb{E}[e^P] + O\left(N^{-\frac{1}{2}+\epsilon}\right).\end{aligned} \tag{5.63}$$

All the other terms in $I_{2,0}$ arising from $\left(\frac{\partial}{\partial M_{ij}}\right)^2 \sum_{p,q}\hat{G}_{pi}\hat{G}_{jq}$ are negligible. For example, one of such terms is bounded by

$$\begin{aligned}&\left|\frac{W_3t_3}{2N^2}\sum_{i,j}\mathbb{E}\left[\left(\sum_p \hat{G}_{pj}\right)\hat{G}_{ii}\hat{G}_{ji}\left(\sum_q \hat{G}_{jq}\right)e^P\right]\right| \\ &\le \frac{2|W_3||t_3|}{(\hat{J}-2)N^{\frac{5}{2}-3\epsilon}}\sum_{i,j}\mathbb{E}\left[|e^P|\right] = O\left(N^{-\frac{1}{2}+3\epsilon}\right)\end{aligned} \tag{5.64}$$

where we use the definitions of Ω_1, Ω_2 and Ω_3.

(ii) For $I_{1,1}$, the estimates for the negligible terms can be done by using the argument similar to (5.64) and (5.61). The remaining $O(1)$-terms are

$$\begin{aligned}&\frac{W_3t_3}{N^2}\sum_{i,j}\mathbb{E}\Bigg[\sum_{p,q}\hat{G}_{pi}\hat{G}_{jq}\Big(t_1\Re\left(G_{ii}(G^2)_{jj} + G_{jj}(G^2)_{ii}\right) \\ &\quad + t_2\Im\left(G_{ii}(G^2)_{jj} + G_{jj}(G^2)_{ii}\right)\Big)e^P\Bigg].\end{aligned}$$

Using the definitions of Ω_2 and Ω_4, we write

$$\begin{aligned}I_{1,1} &= 2W_3t_3\left(t_1\Re(s_1s_1') + t_2\Im(s_1s_1')\right)\mathbb{E}\left[\hat{G}_{\boldsymbol{ee}}^2 e^P\right] + O\left(N^{-\frac{1}{2}+\epsilon}\right) \\ &= 2W_3t_3\left(t_1\Re(s_1s_1') + t_2\Im(s_1s_1')\right)s_2^2\,\mathbb{E}\left[e^P\right] + O\left(N^{-\frac{1}{2}+\epsilon}\right).\end{aligned} \tag{5.65}$$

(iii) For $I_{0,2}$, from the same analysis as for $I_{1,1}$,

$$I_{0,2} = 2W_3t_3\left(t_1\Re(s_1s_1') + t_2\Im(s_1s_1')\right)s_2^2\,\mathbb{E}\left[e^P\right] + O\left(N^{-\frac{1}{2}+\epsilon}\right). \tag{5.66}$$

Again, the estimate can be done in a similar manner.

For the case $i = j$, since there are only N terms in the summation in I_2, all terms are negligible due to the priori bounds on $\|G\|$ and $\sum_p \hat{G}_{pi}$.

Collecting the terms in (5.63), (5.65), and (5.66), we obtain that I_2 equals

$$\begin{aligned}\sum_{i,j} \frac{\kappa_3^{V_{ij}}}{2!}\mathbb{E}\left[\left(\frac{\partial}{\partial M_{ij}}\right)^2\left(\left(t_1\Re(G^2)_{ij}+t_2\Im(G^2)_{ij}+\frac{t_3}{\sqrt{N}}\sum_{p,q}\hat{G}_{pi}\hat{G}_{jq}\right)e^P\right)\right] \\ = W_3\left[3t_3s_2^4+6t_1t_3\Re(s_1s_1')(s_2)^2+6t_2t_3\Im(s_1s_1')(s_2)^2\right]\mathbb{E}\left[e^P\right]+O\left(N^{-\frac{1}{2}+\epsilon}\right).\end{aligned} \tag{5.67}$$

5.2.3 Estimate for I_3

Note that any term in I_3 involving $\hat{G}$ is negligible due to the extra $N^{-\frac{1}{2}}$ factor. Estimating as in the previous subsection, we obtain that I_3 equals

$$\begin{aligned}\sum \frac{\kappa_4^{V_{ij}}}{3!}\mathbb{E}\left[\left(\frac{\partial}{\partial M_{ij}}\right)^3\left(\left(t_1\Re(G^2)_{ij}+t_2\Im(G^2)_{ij}+\frac{t_3}{\sqrt{N}}\sum_{p,q}\hat{G}_{pi}\hat{G}_{jq}\right)e^P\right)\right] \\ = -4(W_4-3)\left[t_1\Re(s_1^3s_1')+t_2\Im(s_1^3s_1')+\left(t_1\Re(s_1s_1')+t_2\Im(s_1s_1')\right)^2\right]\mathbb{E}\left[e^P\right] \\ +O(N^{-\frac{1}{2}+\epsilon}).\end{aligned} \tag{5.68}$$

We remark that $O(1)$-terms in I_3 contribute only to the corrections of linear statistics.

5.2.4 Proof of Theorem 2.2 for general case

Define $\tilde{P}(t)$ by

$$\begin{aligned}P(t)-(W_2-2)(\cos t)^2\left(t_1\Re(s_1's_1)+t_2\Im(s_1's_1)+\frac{1}{2}\left(t_1\Re(s_1')+t_2\Im(s_1')\right)^2\right) \\ +W_3(\cos t)^3\left(t_3s_2^4+2t_1t_3\Re(s_1s_1')s_2^2+2t_2t_3\Im(s_1s_1')s_2^2\right) \\ -(W_4-3)(\cos t)^4\left(t_1\Re(s_1^3s_1')+t_2\Im(s_1^3s_1')+(t_1\Re(s_1s_1')+t_2\Im(s_1s_1'))^2\right).\end{aligned} \tag{5.69}$$

Then, plugging (5.58), (5.67), and (5.68) into (5.48), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[e^{\tilde{P}}|\Omega] = O(N^{-\frac{1}{2}+\epsilon}), \tag{5.70}$$

which implies that

$$\mathbb{E}[e^{\tilde{P}(0)}|\Omega] = \mathbb{E}[e^{\tilde{P}(\frac{\pi}{2})}|\Omega] + O(N^{-\frac{1}{2}+\epsilon}). \tag{5.71}$$

Thus,

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}\left[e^P(0)\right] &= \lim_{N\to\infty}\left(\mathbb{E}\left[e^{P(0)}|\Omega\right]\mathbb{P}(\Omega)+\mathbb{E}\left[e^{P(0)}|\Omega^c\right]\mathbb{P}(\Omega^c)\right) \\ &= e^{P(0)-\tilde{P}(0)}\lim_{N\to\infty}\mathbb{E}[e^{\tilde{P}(0)}|\Omega] = e^{P(0)-\tilde{P}(0)}\lim_{N\to\infty}\mathbb{E}[e^{P(\frac{\pi}{2})}].\end{aligned} \tag{5.72}$$

Here we use the fact that Ω holds with high probability and $\tilde{P}(\frac{\pi}{2}) = P(\frac{\pi}{2})$. We can now conclude that $(\Re\xi_N(z), \Im\xi_N(z), n_N)$ converges to a multivariate Gaussian vector in distribution

as $N \to \infty$. By direct calculation, we also find that

$$\begin{pmatrix} \xi_N(z) \\ n_N \end{pmatrix} \Rightarrow \mathcal{N}\left(\begin{pmatrix} b(z) \\ -W_3 s_2^4 \end{pmatrix}, \begin{pmatrix} V(z_1) & -2W_3 s_1 s_1' s_2^2 \\ -2W_3 s_1 s_1' s_2^2 & \frac{2}{J^2(J^2-1)} \end{pmatrix} \right) \tag{5.73}$$

with $b(z)$ and $V(z)$ are defined in Lemma 4.2. Now, using (5.26), we arrive at

$$\begin{pmatrix} \xi_N(z) \\ \chi_N \end{pmatrix} \Rightarrow \mathcal{N}\left(\begin{pmatrix} b(z) \\ \frac{W_3}{J^2}\left(1-\frac{1}{J^2}\right) \end{pmatrix}, \begin{pmatrix} V(z_1) & 2W_3 s_1 s_1' \left(1-\frac{1}{J^2}\right) \\ 2W_3 s_1 s_1' \left(1-\frac{1}{J^2}\right) & 2\left(1-\frac{1}{J^2}\right) \end{pmatrix} \right). \tag{5.74}$$

Hence, the asymptotic Gaussianity of $(\mathcal{N}_N^{(2)}(\varphi), \chi_N)$ follows. For (2.10) and (2.11), the mean and the variance of $\mathcal{N}^{(2)}[\varphi]$ is given in Theorem 2.1. The limiting covariance is given by

$$-2W_3\left(1-\frac{1}{J^2}\right)\oint_\Gamma \varphi(z)s(z)s'(z)\frac{\mathrm{d}z}{2\pi \mathrm{i}} = 2W_3\left(1-\frac{1}{J^2}\right)\tau_1(\varphi). \tag{5.75}$$

where we use the change of variables $z \mapsto s$ mapping $\mathbb{C} \setminus [-2, 2]$ to the disk $|s| < 1$ with $s + \frac{1}{s} = -z$ and (4.16) in [5]. This completes the proof of Theorem 2.2 for general case.

6 Matching

In the transitional regime, we took $2\beta = \frac{1}{J} + \frac{B}{\sqrt{N}}$. The ferromagnetic regime and the paramagnetic regime correspond to the limiting cases $2\beta > J$ and $2\beta < J$, respectively. In this section, we will consider formal limits $B \to \pm\infty$ of the formula given in the main result, Theorem 1.2, and check the consistency with the results for ferromagnetic and paramagnetic regimes obtained in [5].

Theorem 1.2 states that the free energy F_N is close to the random variable

$$\mathcal{F}_N^{\mathrm{tran}} := \frac{1}{4J^2} + \frac{B}{2J\sqrt{N}} + \frac{\log N}{4N} + \frac{B^2J^2}{4N} + \frac{1}{N}\mathcal{G}_1 + \frac{1}{N}Q(\mathcal{G}_2) \tag{6.1}$$

in an appropriate sense. Here, $(\mathcal{G}_1, \mathcal{G}_2)$ is a Gaussian vector independent of B. The function $Q(x)$ is given by (1.19). In ferromagnetic and paramagnetic regimes, [5] shows that the free energy is close to

$$\mathcal{F}_N^{\mathrm{ferro}} := \beta\left(J+\frac{1}{J}\right) - \frac{1}{2}\log(2\beta J) - \frac{1}{4J^2} - \frac{1}{2} + \frac{\beta - \frac{1}{2J}}{\sqrt{N}}\mathcal{N}(f_2', \alpha_2') \tag{6.2}$$

and

$$\mathcal{F}_N^{\mathrm{para}} := \beta^2 + \frac{1}{N}\mathcal{N}(f_1, \alpha_1), \tag{6.3}$$

respectively, where $\mathcal{N}(f, \alpha)$ denotes a Gaussian distribution of mean f and variance α. The parameters for the Gaussians are (see (4) of [4] which corrected an error in [5])

$$\begin{aligned} f_2' &= W_3(J^{-2} - J^{-4}), \\ \alpha_2' &= 2(1 - J^{-2}) \end{aligned} \tag{6.4}$$

and (see (1.11) and (1.12) of [5]; we set $J' = J$)

$$\begin{aligned} f_1 &= \frac{1}{4}\log(1-4\beta^2) + \beta^2(w_2-2) + 2\beta^4(W_4-3) - \frac{1}{2}\log(1-2\beta J), \\ \alpha_1 &= -\frac{1}{2}\log(1-4\beta^2) + \beta^2(w_2-2) + 2\beta^4(W_4-3). \end{aligned} \tag{6.5}$$

The function $Q(x)$ in (6.1) is given by

$$Q(x) = \frac{s(x)}{2(s(x)-x)} - \frac{s(x)^2}{4(J^2-1)} + \frac{\log(s(x)-x)}{2} + \log \mathbf{I}\left(\frac{(s(x)-x)^2}{J^2-1}\right) \tag{6.6}$$

where (recall the formula (1.20))

$$s(x) = \frac{x - B(J^2-1) + \sqrt{(x+B(J^2-1))^2 + 4(J^2-1)}}{2}. \tag{6.7}$$

From the formula, for $x = O(1)$,

$$s(x) = \begin{cases} x + \frac{1}{B} + O(B^{-2}) & \text{as } B \to +\infty, \\ -B(J^2-1) - \frac{1}{B} + O(B^{-2}) & \text{as } B \to -\infty. \end{cases} \tag{6.8}$$

Note that since we set $2\beta = \frac{1}{J} + \frac{B}{\sqrt{N}}$ in the transitional regime, we regard $B = O(\sqrt{N})$ when we take $B \to \pm\infty$.

6.1 $B \to +\infty$

Using (6.8), we find that for $x = O(1)$,

$$Q(x) = \frac{Bx}{2} + O(\log B). \tag{6.9}$$

Hence, since $\mathcal{G}_1$ does not depend on B, we see that as $B = O(\sqrt{N})$ with $B > 0$,

$$\mathcal{F}_N^{\text{tran}} = \frac{1}{4J^2} + \frac{B}{2J\sqrt{N}} + \frac{B^2J^2}{4N} + \frac{B}{2N}\mathcal{G}_2 + O\left(\frac{\log B}{N}\right) + O\left(\frac{\log N}{N}\right). \tag{6.10}$$

where $O(f(B,N))$ represents a random variable X such that the moments of $\frac{X}{f(B,N)}$ are all bounded by constants independent of B and N.

We compare the above formula with the ferromagnetic case (6.2). If we set $2\beta = \frac{1}{J} + \frac{B}{\sqrt{N}}$, then

$$\mathcal{F}_N^{\text{ferro}} = \frac{1}{4J^2} + \frac{B}{2J\sqrt{N}} + \frac{B^2J^2}{4N} + \frac{B}{2N}\mathcal{N}(f_2', \alpha_2') + O(N^{-3/2}). \tag{6.11}$$

We note that (see (6.4) and (1.25)) the mean and variance are $f_2' = \mathbb{E}[\mathcal{G}_2]$ and $\alpha_2' = \text{Var}[\mathcal{G}_2]$. The above formula of $\mathcal{F}_N^{\text{tran}}$ is thus consistent with $\mathcal{F}_N^{\text{ferro}}$.

6.2 $B \to -\infty$

Consider (6.6). Recall that $\mathbf{I}(\alpha) = \sqrt{\frac{4\pi}{\alpha}}(1 + O(\alpha^{-1}))$ as $\alpha \to +\infty$ from (3.25). Hence, if $x = O(1)$ and $s(x) \to \infty$, then

$$Q(x) = -\frac{s(x)^2}{4(J^2-1)} + \log\sqrt{\frac{4\pi(J^2-1)}{s(x)}} + \frac{1}{2} + O\left(\frac{1}{s(x)}\right). \tag{6.12}$$

Using (6.8), we find that for $x = O(1)$,

$$Q(x) = -\frac{B^2(J^2-1)}{4} + \log\sqrt{\frac{4\pi}{|B|}} + O(B^{-1}). \tag{6.13}$$

Hence, the two leading terms of $Q(\mathcal{G}_2)$ do not depend on $\mathcal{G}_2$. Therefore, for $B = O(\sqrt{N})$ with $B < 0$,

$$\mathcal{F}_N^{\text{tran}} = \frac{1}{4}\left(\frac{1}{J} + \frac{B}{\sqrt{N}}\right)^2 + \frac{1}{2N}\log\left(\frac{4\pi\sqrt{N}}{|B|}\right) + \frac{1}{N}\mathcal{G}_1 + O\left(\frac{1}{NB}\right). \tag{6.14}$$

On the other hand, in the paramagnetic regime, if we set $2\beta = \frac{1}{J} + \frac{B}{\sqrt{N}}$ with $B < 0$, then the parameters in (6.5) satisfy (see (1.23))

$$\begin{aligned} f_1 &= \frac{1}{4}\log(1-J^{-2}) + \frac{1}{4J^2}(w_2-2) + \frac{1}{8J^4}(W_4-3) - \frac{1}{2}\log\left(\frac{|B|J}{\sqrt{N}}\right) + O(N^{-1/2}) \\ &= \mathbb{E}[\mathcal{G}_1] + \frac{1}{2}\log\left(\frac{4\pi\sqrt{N}}{|B|}\right) + O(N^{-1/2}) \end{aligned} \tag{6.15}$$

and

$$\alpha_1 = -\frac{1}{2}\log(1-J^{-2}) + \frac{w_2-2}{4J^2} + \frac{W_4-3}{8J^4} + O(N^{-1/2}) = \text{Var}[\mathcal{G}_1] + O(N^{-1/2}). \tag{6.16}$$

Thus, if we set $2\beta = \frac{1}{J} + \frac{B}{\sqrt{N}}$ with $B < 0$, then

$$\mathcal{F}_N^{\text{para}} = \frac{1}{4}\left(\frac{1}{J} + \frac{B}{\sqrt{N}}\right)^2 + \frac{1}{2N}\log\left(\frac{4\pi\sqrt{N}}{|B|}\right) + \frac{1}{N}\mathcal{N}(\mathbb{E}[\mathcal{G}_1], \text{Var}[\mathcal{G}_1]) + O(N^{-3/2}). \tag{6.17}$$

This is consistent with the formula of $\mathcal{F}_N^{\text{tran}}$.

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