# How Do People Choose Among Rational Number Notations?

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Experiments 1, 2, and 4 of the current study were part of Jing Tian's doctoral dissertation, *Understanding Percentages*. The data and ideas of the study have never been presented at conferences or shared in other ways. All data and materials have been made publicly available via the Open Science Framework and can be accessed at https://osf.io/4zgwf/

**Abstract**

Three rational number notations -- fractions, decimals, and percentages -- have existed in their modern forms for over 300 years, suggesting that each notation serves a distinct function. However, it is unclear what these functions are and how people choose which notation to use in a given situation. In the present article, we propose quantification process theory to account for people's preferences among fractions, decimals, and percentages. According to this theory, the preferred notation for representing a ratio corresponding to a given situation depends on the processes used to quantify the ratio or its components. Quantification process theory predicts that if exact enumeration is used to generate a ratio, fractions will be preferred to decimals and percentages; in contrast, if estimation is used to generate the ratio, decimals and percentages will be preferred to fractions. Moreover, percentages will be preferred over decimals for representing ratios when approximation to the nearest percent is sufficiently precise, due to the lesser demands of using percentages. Experiments 1, 2, and 3 yielded empirical evidence regarding preferences that were consistent with quantification process theory. Experiment 4 indicated that the accuracy with which participants identified the numerical values of ratios when they used different notations generally paralleled their preferences. Educational implications of the findings are discussed.

**How Do People Choose Among Rational Number Notations?**

## 1. Introduction

Rational numbers—fractions, decimals, and percentages—are pervasively important. A trip to a supermarket might include paying $0.99 (a decimal) for 1/8 pound of prosciutto (a fraction) on sale at 10% off of its usual price (a percentage). Rational numbers are frequently used in the workplace as well. In a poll of a random sample of over 2,000 U.S. adults, including both high-skill and low-skill white-collar and blue-collar workers, 68% reported using rational numbers at their jobs (Handel, 2016), almost as many as said they used whole numbers and far more than said they used any more advanced mathematics. Rational numbers also play a crucial role in numerical development. They are essential to mathematics beyond elementary school, as reflected in their pervasive use in algebra, statistics, and trigonometry. Individual differences in rational number understanding are also important: Fractions knowledge in fifth grade predicts overall mathematics achievement in tenth grade, even after controlling for the effects of whole number arithmetic skill, verbal and non-verbal IQ, working memory, and socio-economic status (Siegler et al., 2012).

Unlike whole numbers, for which a single standard notation is almost always used, three different rational number notations are common: fractions, decimals, and percentages. Fractions have the most expressive power among the three: Every rational number can be represented as a fraction, because every rational number is, by definition, a ratio between two numbers, the numerator and the denominator. Decimals, on the other hand, can only represent rational numbers whose implicit denominators are powers of 10 (e.g., 0.7 signifies 7/10, 0.753 signifies 753/1000), and percentages can only represent rational numbers whose implicit denominator is 100 (e.g., 75% signifies 75/100). (For simplicity, we do not consider percentages with decimal components, such as 75.33%, which are percentage-decimal hybrids.)

The greater expressive power of fractions suggests the question of why decimals and percentages are needed at all. Fractions, decimals, and percentages have all existed in their modern forms for at least three centuries (Cajori, 1930). Thus, it appears likely that each serves a distinct function and could not easily be replaced by the others. But what are those distinct functions, and how do people choose which notation to use in a given situation?

The purpose of the present study is to propose and test *quantification process theory*, an approach to understanding people's preferences among fractions, decimals, and percentages. First, we review a previous proposal regarding preferences among rational number notations: semantic alignment theory (DeWolf, Bassok, & Holyoak, 2015). Then, we present quantification process theory and note similarities and differences between its predictions and those of semantic alignment theory. Finally, we describe four experiments designed to test predictions of the two theories.

### 1.1. Semantic Alignment Theory

DeWolf et al. (2015) applied the general theory of semantic alignment (Bassok, Chase, & Martin, 1998) to explain people's preferences, as well as their speed and accuracy, in using rational number notations to represent spatial displays. DeWolf et al. posited that when choosing among rational number notations to represent the ratio in a given display, people prefer to use the notation whose semantic structure matches that of the ratio displayed. Regarding the particular alignment, DeWolf et al. (2015) argued that fractions are bi-dimensional due to their “bipartite structure,” and that decimals are “inherently unidimensional because the implied denominator is fixed (base 10)” (p. 129). They further proposed that ratios involving discrete quantities are two-dimensional, but ratios involving continuous quantities are unidimensional. Therefore, fractions “naturally express a two-dimensional relationship between the cardinal values of sets” (p. 140), whereas decimals “provide a one-dimensional measure of a portion of a continuous unit” (p. 129).

To test this theory, DeWolf, Bassok, and Holyoak (2015) presented U.S. university students with several types of visual displays representing ratios between: a) two small sets of discrete objects, henceforth referred to as “small-number discrete displays” (Figure 1A); b) two small sets of units within continuous quantities, henceforth referred to as “small-number discretized displays” (Figure 1B); and c) two parts of a continuous quantity, henceforth referred to as “continuous displays” (Figure 1C). Consistent with semantic alignment theory, when asked whether the ratio depicted in each display should be represented with a fraction or a decimal, participants chose fractions more often than decimals to represent small-number discrete and discretized displays, but they chose decimals more often than fractions to represent continuous displays. Further, when asked to judge whether a given fraction or decimal represented the part-part ratio or the part-whole ratio in a given display (e.g., the ratio of red items to green items or the ratio of red items to all items in Figure 1A), participants were faster and more accurate with fractions than with decimals for small-number discrete and discretized displays, but they were faster with decimals than with fractions for continuous displays. Subsequent studies with participants from the U.S., Korea, and Russia yielded similar results (Lee, DeWolf, Bassok, & Holyoak, 2016; Plummer, DeWolf, Bassok, Gordon, & Holyoak, 2017; Rapp, Bassok, DeWolf, & Holyoak, 2015; Tyumeneva et al., 2018).

===================== Insert Figure 1 About Here =====================

Gray, DeWolf, Bassok, and Holyoak (2017) extended semantic alignment theory to include percentages. U.S. university students were shown small-number discrete, small-number discretized, and continuous displays, with each display followed by a fraction, a decimal, or a percentage. Their task was to indicate whether the number represented the part-part ratio (i.e., the ratio of one color to the other color) or the part-whole ratio in the display (i.e., the ratio of one color to the whole). Results for decimals and percentages did not differ on any of the three types of displays. Therefore, Gray et al. concluded that the "dominant interpretation" of percentages is similar to that of decimals because percentages "are one-dimensional like decimals" (pp14-15).

**1.2. Quantification Process Theory**

Quantification process theory, a specific application of strategy choice theory (Siegler, 1996), reflects an alternative analysis of preferences among rational number notations. The key assumption of quantification process theory is that the preferred notation for representing a ratio corresponding to a given situation depends on the processes used to quantify the ratio or its components, rather than the alignment of semantic structures. Quantification processes involve either exact enumeration of the numerator and denominator (via subitizing, counting, adding, or a combination of them) or estimation of the ratio as a single integrated magnitude. The theory predicts that if the exact numerator and denominator are to be enumerated, fractions will be preferred to decimals and percentages, because enumeration yields an exact numerator and an exact denominator that together constitute the relevant fraction. In contrast, the theory predicts that if a ratio is to be estimated, decimals and percentages will be preferred to fractions, because decimals and percentages permit directly representing an integrated magnitude as a single number. Thus, quantification process theory posits that the preferences documented in DeWolf et al. (2015) and follow-up studies reflect the quantification processes elicited by the spatial displays and instructions in those studies. It also predicts that eliciting different quantification processes through varying the spatial displays and instructions will produce choices of notations not predicted by semantic alignment theory.

According to quantification process theory, whether a ratio is quantified via estimation or exact enumeration is largely determined by properties of the spatial display being represented. Exact enumeration is likely to be used when small numbers of discrete or discretized items are involved, because they can be easily counted or subitized; estimation is likely to be used when the display involves continuous proportions, which cannot be subitized or counted. Thus, both theories imply that fractions will be preferred to decimals and percentages for representing ratios of small-number discrete and discretized displays and that decimals and percentages will be preferred to fractions for representing ratios of continuous quantities.

However, the theories also make several differing predictions. One involves their predictions regarding ratios of large numbers of discrete objects, henceforth referred to as “large-number discrete displays” (Figure 1D), a type of ratio that was not examined in previous studies. Because ratios between large discrete sets have the same two-dimensional semantic structure as ratios between small discrete sets, semantic alignment theory implies that fractions should be preferred for representing ratios between discrete sets, regardless of the numbers of items in the sets. However, from the perspective of quantification process theory, large-number discrete displays are likely to elicit estimation rather than counting or subitizing, because exact enumeration becomes increasingly time-consuming and effortful as the number of items increases. Quantification process theory therefore predicts that for representing ratios between discrete sets, preferences for fractions over decimals and percentages will decrease as the number of items in the sets increases. For large sets of discrete items, decimals and percentages will be preferred to fractions.

Especially important for distinguishing the two theories, quantification process theory implies that directly manipulating quantification processes should break the link between display properties and notation preferences. For example, according to quantification process theory, an individual who complies with a request to use counting to quantify a ratio between large-number discrete displays should prefer fraction notation, even if the display elicits a preference for percentages or decimals in the absence of such instructions. Similarly, instructions to estimate, together with time limits sufficiently stringent to preclude accurate counting, should lead to estimation being used to quantify the ratio and therefore to a preference for percentages or decimals to represent displays that otherwise would elicit a preference for fractions. Semantic alignment theory does not make these predictions, because it does not consider the quantification process used to generate numerical representations of ratios as a determinant of notation preferences.

Quantification process theory also posits that the amount of effort required to represent a ratio with each notation influences preferences among notations. The predictions described above were based in part on this assumption. The prediction that fractions will be preferred to decimals or percentages when the exact numerator and denominator of a ratio are enumerated was based in part on the fact that in this situation, using decimals or percentages would require the extra effort of dividing the numerator by the denominator after each had been enumerated. The prediction that decimals or percentages will be preferred to fractions when the ratio is estimated was based in part on the fact that in this situation, using fractions requires the extra effort of explicitly choosing a denominator from among all other numbers. Finally, the prediction that increasing set size should give rise to decreasing preference for fractions over percentages and decimals was based on the fact that larger sets require more effort to count than do smaller sets, whereas the effort needed to estimate ratios remains roughly constant over set sizes. This analysis closely resembles that of the more general strategy choice model in numerous other domains (Siegler, 1996).

The assumption that effort affects notation preferences also led to another prediction: Percentages will be preferred to decimals for representing ratios if the ratios are estimated and the number of decimal digits is not constrained. Choosing a decimal to represent a set requires choosing an implicit denominator (tenths, hundredths, thousandths, etc.). For example, with a display having 16 red and 28 blue dots, should the proportion of reds be represented as .4, .36, .364, or some other decimal? Such choices require effort. In contrast, choosing a percentage to represent the set constrains the choices to ones with no greater precision than the nearest percent. Therefore, percentages should be preferred to both decimals and fractions for representing ratios that are quantified by estimation, unless there is a need to represent a ratio with more precision than percentages afford (i.e., more precision than the closest 1%). This prediction contrasts with the implication of semantic alignment theory that there should be no preference between percentages and decimals, because both are one-dimensional (Gray et al., 2017).

Table 1 summarizes the predictions of quantification process and semantic alignment theories that were tested in the four experiments in this article. A more complete set of predictions of both theories is presented in the Supplementary Information, Table A.

===================== Insert Table 1 About Here =====================

## 2. Experiment 1

In this experiment, participants were presented small-number discrete, small-number discretized, and continuous displays (Figures 1A-1C) and were asked on each trial which rational number notation they preferred to represent the ratio shown in the display. In the *two-choice condition,* the choice of notations was between fractions and decimals, as in DeWolf et al. (2015). In the *three-choice* condition, the choice was among fractions, decimals, and percentages.

A key prediction of quantification process theory (Prediction 1) was that percentages should be the notation of choice when ratios are quantified by estimation. Therefore, we expected that participants in the three-choice condition would choose percentages more often than decimals and fractions to represent continuous displays. They would prefer percentages to fractions because continuous displays only allow estimation, and they would prefer percentages to decimals because using percentages avoids the cognitive effort needed to decide how many decimal digits to use. We also expected that in the two-choice condition, participants would choose decimals more often than fractions to represent continuous displays, again because continuous displays only allow estimation of ratios (and also following the results of DeWolf et al., 2015). For small-number discrete and discretized displays in both the two-choice and the three-choice conditions, both theories predicted that participants would choose fractions more often than decimals and percentages.

## 2.1. Method

**2.1.1. Participants.** The participants were 47 Carnegie Mellon University students. Twenty-four participants were randomly assigned to the two-choice condition (fractions or decimals), and 23 were assigned to the three-choice condition (fractions, decimals, or percentages). Students received course credit or monetary compensation. This and the subsequent experiments were approved by the Institutional Review Board of the universities from which participants were recruited. Informed consent was obtained from participants in this and the subsequent experiments.

**2.1.2. Design.** Participants were individually tested in a quiet room on a laptop during a single session. Stimuli were presented in MATLAB (The MathWorks, 2015) with Psychophysics Toolbox extensions (Brainard, 1997; Kleiner, Brainard, & Pelli, 2007) here and in Experiments 2 and 4. All of the visual displays in all experiments were presented within a black rectangle at the center of the screen with an approximate visual angle of 3.5°×3.5°.

***2.1.2.1. Notation preference task.*** The procedure followed that of DeWolf et al. (2015). On each trial, participants saw one of three types of display: small-number discrete, small-number discretized, or continuous (Figures 1A-1C). Participants were instructed to choose the notation that they preferred to represent the part-whole ratio in each display.

Twenty ratios were used to create the displays, with each ratio being presented for each type of display (Supplementary Information, Table B1). The number of entities in the discrete and discretized displays was equal to the denominator of the ratio used to create the display and ranged from 7 to 13. The part of the display corresponding to the numerator was red, and the remainder was green. For example, the discretized display that corresponded to 7/8 included eight rectangles, seven of which were red and one of which was green (Figure 1B). Following DeWolf et al. (2015), rectangles, triangles, and several other shapes were used to create small-number discrete displays.

As in DeWolf et al. (2015), the task involved choices among notations rather than numerical values. Participants in the two-choice condition were asked to "choose which notation is a better representation of the depicted relation – a fraction or a decimal"; participants in the three-choice condition were asked to "choose which notation is the best representation of the depicted relation – a fraction, a percentage, or a decimal."

The two-choice condition replicated that in DeWolf et al. (2015); the three-choice condition, new to this study, assessed the effect of adding percentages as an option. Participants were asked to press the key associated with their preferred notation for each display (i.e., for participants in both conditions, "z" for decimal and "m" for fraction; for participants in the three-choice condition, "v" for percentage).

Participants were notified that they did not have to use both/all of the notations and that we were simply interested in their preferences. One example of each type of display was presented before starting the task. Displays were presented at the center of the screen until a response was detected; there was no pressure to respond quickly. A fixation cross was presented for 600ms after each trial was completed.

Each participant completed 60 trials (3 display types (small-number discrete, small-number discretized, or continuous) × 20 ratios). Stimuli were presented in a pseudo-random order for each participant, with the restriction that no more than two successive trials involve displays of the same type.

**2.2. Analyses**

A bootstrap procedure was used to estimate the 95% confidence interval (CI) of preferences between each pair of notations for each type of display in each condition. To illustrate this bootstrap procedure, we describe how we generated the 95% CI for the preference between fractions and decimals for small-discrete displays in the two-choice condition. (1) For each of the 24 participants in the two-choice condition, we calculated a preference score between decimals and fractions for trials involving small-discrete displays by subtracting the proportion of those trials on which the participant chose decimals from the proportion of those trials on which the participant chose fractions. This yielded 24 preference scores. (2) In one simulated experiment, we randomly sampled, with replacement, 24 times from the 24 preference scores, then calculated the mean preference score across those 24 samples. (3) We ran 10,000 simulated experiments as described in (2), yielding 10,000 mean preference scores. The 95% CI was the smallest range of scores that included at least 95% of these 10,000 scores.

Preferences between a given pair of notations, for a given type of display in a given condition, were considered significant if the 95% CI of the preference score excluded zero. All significant differences are reported. The bootstrap analyses were conducted in R (Canty & Ripley, 2019; Team R Core, 2018).

**2.3. Results**

Separate analyses were conducted for the two- and three-choice conditions. Consistent with predictions of both theories and with previous findings, participants in the two-choice condition (Figure 2A) preferred fractions to decimals for representing both small-number discrete displays (mean preference score = 45%, 95% *CI* = [19%, 69%]) and small-number discretized displays (mean preference score = 54%, 95% *CI* = [29%, 76%]) but preferred decimals to fractions for continuous displays (mean preference score = 44%, 95% *CI* = [17%, 69%]).

===================== Insert Figure 2 About Here =====================

Also as predicted by both theories and by previous findings in the three-choice condition, fractions were the preferred notation for the small-number discrete and discretized displays (Figure 2B). Participants preferred fractions to decimals for both small-number discrete displays (mean preference score = 65%, 95% *CI* = [42%, 84%]) and small-number discretized displays (mean preference score = 41%, 95% *CI* = [12%, 67%]). Participants also preferred fractions to percentages for both small-number discrete (mean preference score = 65%, 95% *CI* = [46%, 81%]) and small-number discretized displays (mean preference score = 34%, 95% *CI* = [5%, 61%]).

However, consistent with Prediction 1 of quantification process theory, the option to choose percentages greatly changed preferences among notations for representing continuous displays. As shown in the rightmost column of Figure 2B, participants in the three-choice condition chose percentages to represent continuous displays on 77% of trials with continuous displays. This was far more often than they chose not only fractions (mean preference score = 70%, 95% *CI* = [50%, 85%]) but also decimals (mean preference score = 63%, 95% *CI* = [39%, 83%]).

**2.4. Discussion**

Adding the option of choosing percentages revealed notational preferences for representing continuous displays that were not evident in prior experiments: Decimals were the preferred notation when the only alternatives were decimals and fractions, but percentages were strongly preferred when they were an additional option in the three-choice condition. The differences were very large: Decimals were preferred on almost 75% of trials when fractions were the only alternative, but on only 15% of trials when percentages were also an option. Percentages were preferred on more than 75% of trials when they were an alternative. The strong preference for percentages over decimals for representing continuous displays in the three-choice condition was predicted by quantification process theory but not by semantic alignment theory.

An alternative interpretation of these findings was that the preference for percentages to represent continuous displays in the three-choice condition reflected the task providing three rather than two choices. To test this interpretation, we presented a new group of university students (*N* = 36) the same notation preference task but with only binary choices: fractions or percentages, decimals or percentages, and fractions or decimals. As in the three-choice condition of Experiment 1, when asked to represent continuous displays on the binary choice tasks, participants consistently preferred percentages to fractions (82% vs. 18%) and to decimals (75% vs. 25%). Moreover, the preference for percentages over fractions was greater than the preference for decimal over fractions (60% vs. 40%) for continuous displays. Thus, the option of choosing percentages yielded the preference for percentages for representing continuous displays regardless of whether the task involved two or three choices. Other findings from this new group of students were consistent with those reported above; a complete report of these findings is presented in the Supplementary Information, Part C.

## 3. Experiment 2

Consistent with Prediction 1 of quantification process theory, participants in Experiment 1 consistently preferred percentages over fractions and decimals for representing ratios in continuous displays. One purpose of Experiment 2 was to test whether this finding, which was obtained from U.S. university students, could be generalized to students from China, a nation with quite different educational and cultural practices than the U.S. The notation preference task presented to U.S. students in Experiment 1 was used, with participants randomly assigned to either the two-choice or the three-choice condition. We predicted that despite the educational and cultural differences between China and the US, Chinese students would show the same pattern of preferences as their U.S. peers, because notation preferences in both countries would be driven by the same quantification processes in the same way.

The other purpose of Experiment 2 was to test Prediction 2, that large-number discrete displays would produce preferences like those with continuous displays and unlike those with small-number discrete displays. This prediction was based on the logic that large-number discrete displays would elicit estimation rather than counting, due to the effort required to count the many objects in the sets. This reasoning, combined with Prediction 1, implied that participants would prefer percentages to represent these displays. The large-number discrete displays were presented under three-choice conditions to all participants, regardless of whether they earlier had been in the two-choice or three-choice condition. The reason was to maximize power to test whether participants preferred percentages to represent these displays.

## 3.1. Method

**3.1.1. Participants**. Forty-eight students at Beijing Normal University and Beijing University of Posts and Telecommunications in China were randomly assigned to the two-choice or the three-choice condition, 24 students in each condition.

**3.1.2. Design**. Participants first completed the notation preference task with small-number discrete displays, small-number discretized displays, and continuous displays, either under two-choice or three-choice conditions. Then, all participants were presented large-number discrete displays under three-choice conditions. The first author, a native

Chinese speaker, translated the instructions and the text into Chinese and administered the experiment.

***3.1.2.1. Notation preference tasks.*** The notation preference task with small-number discrete, small-number discretized, and continuous displays had the same design, stimuli, and procedure as the notation preference task in Experiment 1. The notation preference task with large-number discrete displays was identical to the three-choice condition of the notation preference task in Experiment 1 except that displays were more numerous and only included dots, rather than rectangles and other shapes. The large-number discrete displays were created using the same ratios as the small-number displays, so that the total number of dots and numbers of dots in the subsets were multiples of the numbers of shapes in the small-number discrete displays. Each large-number discrete display included between 70 and 91 dots, compared to between 6 and 13 in the small-number discrete displays. For example, where the small-number discrete display representing 7/8 contained 7 red squares and 1 green circle for a total of 8 shapes, the corresponding large-number discrete display contained 63 red dots and 9 green dots for a total of 72 dots (Figure 1D).

One large-number discrete display with dots of the same size and one large-number discrete display with dots of varied sizes were created for each of 20 ratios, resulting in 40 trials. Using dots of constant or variable sizes enabled us to investigate whether differences between area and numerical cues interacted with numerical notation (DeWind, Adams, Platt, & Brannon, 2015). Including variability of dot sizes in our analyses did not have any detectable effect; therefore, it is not included as a factor in the analyses here or in Experiment 4, which also employed variable-size dots displays.

**3.2. Results**

Chinese students' performance on the notation preference task showed highly similar results to those with US students. For small-number discrete, small-number discretized, and continuous displays in the two-choice condition (Figure 3A), Chinese participants, like those in the U.S., chose fractions more often than decimals for both small-number discrete displays (mean preference score = 30%, 95% *CI* = [2%, 59%]) and small-number discretized displays (mean preference score = 66%, 95% *CI* = [53%, 79%]). The proportion of trials on which Chinese participants chose decimals and fractions did not differ for continuous displays.

===================== Insert Figure 3 About Here =====================

Similarly, in the three-choice condition (Figure 3B), to represent small-number discrete displays, Chinese participants, like U.S. peers, preferred fractions over decimals (mean preference score = 49%, 95% *CI* = [24%, 71%]) and decimals over percentages (mean preference score = 32%, 95% *CI* = [3%, 61%]). Also like U.S. peers, to represent small-number discretized displays, Chinese participants chose fractions more often than decimals (mean preference score = 71%, 95% *CI* = [54%, 86%]) or percentages (mean preference score = 57%, 95% *CI* = [29%, 81%]). They also chose percentages somewhat more often than decimals for those displays (mean preference score = 14%, 95% *CI* = [2%, 28%]). Again like U.S. peers, to represent continuous displays, Chinese participants chose percentages more often than either fractions (mean preference score = 39%, 95% *CI* = [10%, 64%]) or decimals (mean preference score = 48%, 95% *CI* = [25%, 69%]).

Of particular interest, when presented large-number discrete displays, the Chinese students chose percentages more often than fractions (mean preference score = 30%, 95% *CI* = [15%, 44%]) or decimals (mean preference score = 40%, 95% *CI* = [29%, 52%]). The preference for percentages over decimals was consistent with Prediction 1 of quantification process theory; the fact that students also preferred percentages to fractions was consistent with Prediction 2 of the theory. As expected, preferences for large-number discrete displays (Figure 3C) closely resembled preferences for continuous displays.

### 3.3. Discussion

In addition to replicating the Experiment 1 findings with students from a different country with a different educational system and culture, results of Experiment 2 demonstrated a preference for percentages over decimals and fractions for representing ratios in large-number discrete displays. The preferences closely resembled those with continuous displays. Taken together, these results are consistent with the prediction of quantification process theory that percentages are the preferred notation for representing ratios when the ratios are quantified by estimation rather than enumeration and when there is no need to represent ratios with more precision than percentages allow.

## 4. Experiment 3

Consistent with Prediction 2 of quantification process theory, participants in Experiment 2 preferred percentages for representing ratios between large discrete sets, as they earlier had been found to prefer them for representing ratios of continuous quantities. Experiment 3 tested a related hypothesis, Prediction 3: For ratios of discrete objects, preferences for fractions should decrease, and preferences for percentages, decimals, or both should increase, as the number of items in the sets increases. To test this prediction, we administered the notation preference task using discrete displays of 31 to 76 dots, a range within which pilot testing indicated that notation preferences vary.

Experiment 3 also tested Prediction 4: Instructions to enumerate should produce a preference for fractions, and instruction to estimate should produce a preference for percentages or decimals to represent ratios in discrete displays. To test this prediction, we used a different task, the *number generation task*, in which participants were asked to generate a fraction, a decimal, or a percentage to represent the part-whole ratio in each display, either using counting (*counting instructions condition*) or estimation (*estimation instructions* condition). When participants were instructed to count, we expected fractions to be preferred. When participants were instructed to estimate, we expected percentages to be preferred, based on Predictions 1 and 4 and the results of Experiments 1 and 2.

The reason that we employed the number generation task to test Prediction 4 was to increase the likelihood that participants would use the instructed quantification process (i.e., counting or estimating). Display characteristics strongly influence choices among quantification processes (Boyer, Levine, & Huttenlocher, 2008; Plummer et al., 2017), and if asked to choose a notation without generating a number (as in Experiments 1 and 2), participants might ignore the instructions and choose a notation consistent with the quantification process they would typically use for that type of display. However, the requirement of the number generation task to state a specific number to represent the ratio was expected to increase the likelihood of participants using the requested quantification process.

In this experiment, the same displays were used in the notation preference task and in the number generation task. In contrast to our Prediction 3, that number of items in the displays would affect preferences on the *notation preference task*, we predicted that to the extent that participants adhered to the instructions, number of items would have little or no impact on notation choices on the *number generation task* (Prediction 4). The reason was that by manipulating the quantification strategy directly, the instructions would sever the usual link between set size and notation preferences on the number generation task. According to quantification process theory, effects of display properties—such as number of items—on notation preferences are mediated by quantification processes. Therefore, on the number generation task, regardless of set size, instructions to count were expected to produce predominant choices of fractions, whereas instructions to estimate were expected to produce predominant choices of percentages.

## 4.1. Method

**4.1.1. Participants.** The participants were 40 Florida State University students who received course credit for participating.

**4.1.2. Design.** Participants were individually tested in a quiet room on a laptop during a single session. Stimuli were presented in PsychoPy3 (Peirce et al., 2019). All participants first performed the notation preference task and then performed the number generation task under both counting and estimation instructions conditions. The notation preference task was presented before the number generation task so that instructions to use a particular quantification strategy in the number generation task would not affect responses in the notation preference task.

Both the notation preference task and the number generation task involved discrete displays of red and blue dots. Two sets of ratios, with 12 ratios in each set, were used to create the displays, with each ratio used once (Supplementary Information, Table B2). The number of red dots in each display was equivalent to the numerator of the ratio used to create the display, and the total number of dots in each display was equivalent to the denominator. Denominators of the ratios ranged from 32 to 75 in one set and 31 to 76 in the other. In each set of ratios, three ratios fell into each quartile of the range 0 – 1.

***4.1.2.1. Notation preference task.*** The procedure of this task was identical to that in the three-choice condition of Experiment 1. Each participant was presented each of the 24 displays once in a random order (not blocked by stimulus set).

***4.1.2.2. Number generation task.*** Each participant performed this task in both the counting instructions condition and the estimation instructions condition. On each trial, participants were shown a display on a screen, asked to determine the part-whole ratio in the display, and asked to write down their answer on printed answer sheets in whichever rational number notation they preferred.

In the counting instructions condition, participants were asked to determine the ratio in each display by counting the dots; they were given as much time as needed for each trial. The display remained on the screen until the participant pressed the *Space* key to proceed to a screen that asked them to write their answer for that display on the answer sheet. Pressing the *Space* key again started the next trial.

In the estimation instructions condition, participants were asked to estimate the ratio in each display. A display appeared on the screen for 2 seconds. This time limit was imposed to prevent participants from counting the dots. A fixation cross appeared at the center of the screen after the display disappeared. As in the counting instructions condition, participants were asked to press the *Space* key after they wrote each answer to start the next trial.

In each condition, participants finished two practice trials without feedback before the test trials. Four ratios that were not in the two sets of ratios used to create stimuli for the main task were used to create practice trial displays (Supplementary Information Table B2). Each participant saw displays created from one set of ratios in the counting instructions condition and displays created from the other set of ratios in the estimation instructions condition. Each display appeared once per participant. The order of the two conditions and the sets of displays used in them were counterbalanced across participants.

Symbols for all three rational number notations were printed on the answer sheets for each trial so that participants only needed to fill in the numerals (see Supplementary Information Part D, for an example answer sheet). This manipulation was intended to eliminate differences in the effort needed to write rational numbers in different notations. Locations of symbols for the three notations were randomized for each participant.

### 4.2. Analyses

**4.2.1. Notation preference task.** To examine the relation between the number of dots in each display and participants' notation preferences on that display, we fitted a mixed-effects logistic regression model for each notation to predict whether that notation was chosen on each trial, with number of dots as a fixed effect and participant as a random effect. Models were fitted with R (Team R Core, 2018) and the *lme4* package

(Bates, Mächler, Bolker, & Walker, 2014). *P*-values were obtained by likelihood ratio tests comparing the model including the effect of interest to the model without it.

**4.2.2. Number generation task.** The same bootstrap procedure used to analyze data on the notation preference task in Experiments 1 and 2 was used to estimate pairwise differences in the proportion of trials on which the participant generated answers in each notation in each condition. Then, to test our prediction that number of items would have little or no effect once quantification method was specified, we fitted mixed-effects logistic regression models to predict whether or not each particular notation was chosen, with number of dots as a fixed effect and participant as a random effect. Separate analyses were conducted for each combination of task condition and notation (six in total).

## 4.3. Results

**4.3.1. Notation preference task.** Consistent with Prediction 3 of quantification process theory, on the notation preference task, increased number of dots predicted higher likelihood of choosing percentages, $\boldsymbol{B} = 0.03$, $\chi^2(1) = 26.64$, $p < .001$ (Figure 4A), and lower likelihood of choosing fractions, $\boldsymbol{B} = -0.03$, $\chi^2(1) = 22.21$, $p < .001$ (Figure 4B). The number of dots did not influence likelihood of choosing decimals, $\beta = -.006$, $\chi^2(1) = 0.66$, $p = 0.42$.

===================== Insert Figure 4 About Here =====================

**4.3.2. Number generation task.** Consistent with Prediction 4 of quantification process theory, in the counting instructions condition of the number generation task, participants generated fractions far more often than percentages (mean preference score =

86%, 95% CI = [72%, 96%]) or decimals (mean preference score = 91%, 95% CI = [83%, 97%]). Also consistent with Prediction 4, in the estimation instructions condition, participants generated percentages far more often than fractions (mean preference score = 53%, 95% CI = [30%, 73%]). Consistent with Prediction 1, participants also chose percentages far more often than decimals (mean preference score = 68%, 95% CI = [53%, 81%]). Moreover, mixed-effect logistic regressions for each combination of condition and notation, with use of a given notation as the dependent variable, number of dots as a fixed effect, and participant as a random effect, did not indicate an effect of number of dots for any combination of condition and notation, *p*s > 0.09.

**4.4. Discussion**

Consistent with Prediction 3, on the notation preference task, preferences among rational number notations were dependent on the number of items in the display to be quantified. However, directly manipulating quantification strategy in the number generation task eliminated this effect of number of items. When instructed to count, participants more often represented the ratio with fractions than percentages, but when instructed to estimate, they more often represented the ratio with percentages than fractions.

These findings provide strong evidence for quantification process theory. They show that when the quantification process is directly manipulated, that process controls the choice of rational number notation to represent ratios in sets of discrete objects. Number of items in the set, which influences choice of notations when quantification process *is not* manipulated, loses its influence when quantification process *is* manipulated. These findings were not predicted by semantic alignment theory, which posits that choice of notations to represent ratios in spatial displays depends on the semantic structure of the displays. All displays in Experiment 3 had the same two-dimensional semantic structure, but choice of notation varied greatly with instructions to count or to estimate.

## 5. Experiment 4

Results of Experiments 1-3 were consistent with Prediction 1, that percentages are preferred to decimals for representing estimated ratios if there is no need for precision greater than 1%. The basis for this prediction was that although both decimals and percentages permit representing an estimated ratio as a single number, decimals—but not percentages—require determination of an implicit denominator, which can be any power of 10. This assumption holds implications not only for preferences, but also for performance. Specifically, if representing estimated ratios is more cognitively demanding when decimals rather than percentages are used, then accuracy and speed should be greater with percentages than decimals.

Experiment 4 tested this prediction. Participants completed a *number matching task*, in which they were asked to choose which of two specific fractions, two specific decimals, or two specific percentages more accurately represented ratios in small-number discrete, small-number discretized, large-number discrete, and continuous displays. The key prediction was that responses would be slower and/or less accurate with decimals than with percentages for the types of displays on which we expected estimation to be used, namely, large-number discrete and continuous displays.

This prediction contrasts with findings from a previous study (Gray et al., 2017) in which no performance differences between decimals and percentages emerged. However, in that study, all decimals had the same number of decimal digits (two). From the perspective of quantification process theory, the presence or potential for varying numbers of decimal digits, and therefore varying implicit denominators, is what makes decimals more difficult to use than percentages. Therefore, in Experiment 4, we examined whether performance was better with percentages than with decimals when number of decimal digits varied.

## 5.1. Method

**5.1.1. Participants.** Thirty-eight Carnegie Mellon University students participated in the experiment and received course credit or monetary compensation for participating.

**5.1.2. Design.** Participants were individually tested in a quiet room during a single session. They first completed the number matching task with small-number discrete displays, small-number discretized displays, and continuous displays, and then with large-number discrete displays. The task was performed on a laptop.

***5.1.2.1 Number matching task.*** On the number matching task, sixteen pairs of ratios were used to create the stimuli (Supplementary Information, Table B3). Each ratio pair consisted of a *target ratio* and a *foil ratio*; each foil ratio differed from the corresponding target ratio by between 0.12 and 0.18. Only the target ratios were used to create the displays shown to participants; the target and foil ratios together were used to generate the response options, as described below.

For each target ratio, we created one small-number discrete display, one small-number discretized display, one continuous display, and two large-number discrete displays with the same number of dots as each other (one with dots of constant size and one with dots of varied sizes). The displays were created as described in Experiments 1-3. The number of entities in small-number discrete and discretized displays was equal to the denominator of the ratio used to create the display. The number of dots in each large-number display was a multiple of the denominator of the ratio that was used to create the display and ranged from 70 to 91.

For each target-foil pair, three pairs of response options were created—one pair of fractions, one pair of decimals, and one pair of percentages (as shown in Supplementary Information, Table B3). The fractions within each pair were exactly equal to the corresponding target and foil ratios. The decimals within each pair were within 0.03 of the corresponding target and foil ratios; also, the decimals within each pair had different numbers of decimal digits, ranging from one to three. The percentages within each pair were equal to the corresponding target and foil ratios rounded to the nearest hundredth. The distance between the target and foil ratio was very close (within .01) to being constant over the fraction, decimal, and percentage response option pairs. For example, on the trial with response options 7/8 and 5/7, the corresponding options with percentages was 88% and 71%, and with decimals 0.9 and 0.73.

On each trial, participants were shown a display and a pair of numbers—either two fractions, two decimals, or two percentages. One of the numbers was either equal (in the case of fractions) or approximately equal (in the case of decimals and percentages) to the target ratio used to create the display; the other number was equal or approximately equal to the corresponding foil ratio. The task was to choose the number that better represented the part-whole ratio in the display; the correct answer was the number that was equal or approximately equal to the target ratio.

The number matching task with small-number discrete, small-number discretized, and continuous displays employed a 3 (notation: fraction, decimal, or percentage) by 3 (display type: small-number discrete, small-number discretized, or continuous) design. Both factors were within-subjects. Each combination of notation, display type, and ratio pair was presented twice: once with the target rational number on the left and once with it on the right. Thus, each participant completed 288 test trials (3 rational number notations × 3 display types × 16 ratio pairs × 2 locations for the target number). Problems were completed in two blocks of 144 trials each. The order of stimuli was pseudo-randomized for each participant, with the constraints that each combination of notation, display type, and ratio pair appeared only once in each block, and a ratio pair would not appear again until all 16 pairs were presented.

The number matching task with large-number discrete displays employed a 3 (notation: fraction, decimal, or percentage) by 2 (dot size: constant or variable) design. Both factors were within-subjects. Each combination of notation, ratio pair, and dot size was presented twice—once with the target fraction, decimal, or percentage shown on the left and once with it on the right—resulting in 192 test trials (3 notations × 16 ratio pairs × 2 dot sizes × 2 locations for the target fraction, decimal, or percentage). The trials were completed in two blocks of 96 trials each. The order of stimuli was pseudo-randomized for each participant, with the same constraints as for the number matching task.

Participants were told to choose as quickly as possible without sacrificing accuracy, and that the next trial would start if they did not respond within 5s. Six practice trials were presented before the main task. Feedback was given on the correctness of participants' answers on practice trials but not on test trials.

**5.2. Analyses**

Data were analyzed using ANOVAs. Significant effects of three-level factors (notation and display type) were investigated using pairwise post-hoc comparisons; *p*-values for these comparisons were adjusted using the Bonferroni correction. Trials on which participants did not respond within 5s were excluded from analysis (< 1% of all trials). A small number of participants (three in the number matching task with continuous, small-number discrete, and small-number discretized displays, and four in the number matching task with large-number discrete displays) were excluded due to large numbers of no-response trials, extremely low accuracy, or technical failures (details are provided in the Supplementary Information, Part E).

Analyses focused on accuracy. Response time patterns on correct trials generally converged with accuracy patterns or showed no effects of notation; there was no evidence of speed-accuracy tradeoffs. Analyses of RTs are reported in the Supplementary Information, Part F.

**5.3. Results**

A 3 (notation) by 3 (display type) ANOVA on accuracy yielded main effects of notation, $F$ (2, 68) = 17.73, $p < .001$, $\eta_p^2 = .34$, and display type, $F$ (2, 68) = 20.37, $p < .001$, $\eta_p^2 = .37$, and an interaction between the two , $F$ (4, 136) = 26.89, $p < .001$, $\eta_p^2 = .44$.

Post-hoc comparisons indicated that, as predicted, accuracy was lower when choosing between decimals (*M* = 81%, *SD* = 7%) than when choosing between percentages (*M* = 85%, *SD* = 8%), *F* (1, 34) = 11.25, $p$ = .005, $\eta_p^2$ = .25. Accuracy was also lower when choosing between decimals than between fractions (*M* = 87%, *SD* = 8%), *F* (1, 34) = 35.26, $p$ < .001, $\eta_p^2$ = .51. Participants were more accurate in identifying the correct ratio for small-number discretized displays (*M* = 87%, *SD* = 8%) than for continuous displays (*M* = 85%, *SD* = 8%), *F* (1, 34) = 6.57, $p$ = .045, $\eta_p^2$ = .16, and more accurate on continuous than on small-number discrete displays (*M* = 82%, *SD* = 8%), *F* (1, 34) = 12.97, $p$ = .003 .045, $\eta_p^2$ = .28.

For continuous displays, accuracy with percentages (*M* = 88%, *SD* = 9%) was higher than with fractions (*M* = 81%, *SD* = 10%), *F* (1, 34) = 18.76, $p$ < .001, $\eta_p^2$ = .36. Accuracy with decimals (*M* = 85%, *SD* = 9%) was also higher than with fractions, *F* (1, 34) = 8.61, $p$ = .02, $\eta_p^2$ = .20. These findings were consistent with quantification process theory and with previous findings (DeWolf et al., 2015; Gray et al., 2017; Lee et al., 2016). Participants also were more accurate with percentages than decimals, as predicted by quantification process theory, but the difference was not significant, *F* (1, 34) = 3.11, $p$ = 0.26, $\eta_p^2$ = 0.08.

Results for small-number discrete and discretized displays also were consistent with previous findings (DeWolf et al., 2015; Gray et al., 2017; Lee et al., 2016), as well as with the predictions of the present theory. For small-number discrete displays, accuracy with fractions (*M* = 90%, *SD* = 9%) was higher than with percentages (*M* = 79%, *SD* = 10%), *F* (1, 34) = 64.26, $p$ < .001, $\eta_p^2$ = .65 or decimals (*M* = 76%, *SD* = 9%), *F* (1, 34) = 69.03, $p < .001$, $\eta_p^2 = .67$. For small-number discretized displays, accuracy with fractions (*M* = 90%, *SD* = 10%) also was higher than with decimals (*M* = 82%, *SD* = 8%), *F* (1, 34) = 28.48, $p < .05$, $\eta_p^2 = .46$. Accuracy with fractions on these displays was slightly higher than accuracy with percentages (*M* = 87%, *SD* = 9%), but the difference was not significant, *F* (1, 34) = 3.10, $p = 0.26$, $\eta_p^2 = 0.08$.

On the number matching task with large-number discrete displays, trials for 3 of the 32 displays were excluded from analyses because the displays were not exactly equal to the target ratios. Analyses that included these flawed stimuli yielded the same effects as reported in here (Supplementary Information, Part G). An ANOVA on accuracy, with notation as a within-subjects factor, yielded an effect of notation, *F* (2, 66) = 13.99, $p < .001$, $\eta_p^2 = .30$. Consistent with predictions of quantification process theory, post-hoc comparisons indicated that participants were more accurate on large-number discrete displays when the choice was between two percentages (*M* = 78%, *SD* = 8%) than when it was between two decimals (*M* = 74%, *SD* = 9%), *F* (1, 33) = 9.29, $p = 0.013$, $\eta_p^2 = 0.22$, or two fractions (*M* = 71%, *SD* = 7%), *F* (1, 33) = 27.80, $p < .001$, $\eta_p^2 = 0.48$. Participants also tended to be more accurate when the choice involved two decimals than two fractions, *F* (1, 33) = 5.15, $p = .090$, $\eta_p^2 = 0.14$.

**5.4. Discussion**

Consistent with quantification process theory, accuracy using rational numbers to represent ratios in spatial displays paralleled notation preferences. Critically, with large-number discrete displays, participants performed the number matching task more accurately with percentages than with decimals. This result is analogous to Experiment

2's finding that percentages were preferred to decimals for representing ratios in large-number discrete displays. Both findings can be explained by quantification process theory's assumption that using decimals incurs a greater cognitive demand than using percentages to represent estimated ratios.

## 6. General Discussion

In this study, we raised the question of how people choose among the three rational number notations in a given situation, proposed quantification process theory to explain people's preferences, and reported empirical data from four experiments that tested predictions of the theory. In this concluding section, we discuss findings about each notation's distinct functions, discuss how quantification process theory accounts for the findings, and then consider implications of the findings and theory for how rational numbers should be taught to students.

### 6.1. Percentages

The most striking findings to emerge from the present study relate to percentages. On the notation preference task, both U.S. and Chinese participants strongly preferred percentages over decimals and fractions for representing ratios in continuous displays and large-number discrete displays. As the number of items in discrete displays increased, participants' preference for percentages also increased. On the number generation task, when asked to estimate ratios, participants generated more answers in percentage notation than in fraction or decimal notation for all types of spatial displays. Finally, on the number matching task, accuracy was higher with percentages than with fractions or decimals on large-number discrete displays, and higher with percentages than with fractions for continuous displays. These findings were consistent with our premise that percentage notation endures because people prefer it to other notations when using estimation to approximate ratios.

Previous research on children's and adults' understanding of rational numbers has focused almost entirely on fractions and decimals; percentages have rarely been studied (Tian & Siegler, 2017). One reason why percentages have received so little attention is that they are mathematically interchangeable with decimals if hybrid representations such as 12.34% are allowed. This mathematical equivalence might be assumed to yield psychological equivalence. However, neither U.S. nor Chinese participants in the present study viewed decimals and percentages as interchangeable. If they had, they would not have shown strong, systematic preference for percentages over decimals for representing ratios in continuous and large-number discrete displays.

Analyses based on quantification process theory implied that percentages are preferred over decimals for representing estimated ratios when precision beyond the nearest percent is not required, which was the case in the present study. The rationale for this prediction was that percentages require less cognitive effort than decimals when decimals can have varying numbers of decimal digits, as is usually true in real-world situations. With a fixed implicit denominator of 100, percentages only require choice of a whole number numerator to complete the ratio, whereas using a decimal also requires a choice of how many decimal digits to include. Our assumption that choosing the number of decimal digits imposes a cognitive cost led to the predictions that participants would prefer percentages to decimals in situations that did not require highly precise answers, and that they would perform more accurately with percentages than with decimals if the decimal alternatives had varying numbers of digits. Both of these predictions proved accurate.

Looking beyond the types of tasks examined to date, percentages are used in a vast range of contexts, from the everyday to the arcane. They are used to convey information about price discounts, sales and income tax rates, gratuities in restaurants, demographic descriptions, survey and poll results, growth rates, probabilities, and variance explained in statistical analyses. Percentages also appear in common phrases such as “give 100 percent,” “120% effort,” and “the 1 percent,” and they are implicit in phrases such as “the odds are 50-50.” Future research should further explore the range of contexts in which percentages are the rational number notation of choice and the variables that determine those contexts.

**6.2. Fractions**

Many findings of the present study pertain to differences between fractions on the one hand, and decimals and percentages on the other. Previous investigators concluded that for representing ratios between discrete sets, fractions are preferred to, and yield better performance than, decimals and percentages (DeWolf et al., 2015; Gray et al., 2017; Lee et al., 2016; Plummer et al., 2017; Rapp et al., 2015). The present study replicated these results, but only when the discrete sets contained small numbers of items or when participants were instructed to use counting to determine the ratios, conditions hypothesized to elicit counting[1]. When discrete sets contained large numbers of items that made counting implausible within the time provided, or when participants were instructed to quantify the ratios by estimating, participants strongly preferred percentages to fractions. Also, when participants performed the number matching task with large-number discrete sets (Experiment 4), accuracy was lower with fractions than percentages.

The preference for, and greater accuracy with, fractions over decimals and percentages for representing ratios between small discrete sets have previously been interpreted as evidence for an inherent semantic alignment between the bipartite structure of fractions and ratio relations between sets. If this interpretation were correct, these preferences should be present regardless of set sizes, instructions, and time limits. In contrast, quantification process theory posits that preferences for, and advantages of, fractions for representing ratios between discrete sets should disappear with sets that do not allow precise enumeration in the allocated time or when participants are instructed to use estimation. The predictions of quantification process theory were the ones consistent with the data in the present experiments.

**6.3. Decimals**

On the notation preference and number generation tasks presented in Experiments 1-3, fractions or percentages were preferred to decimals for every type of display. On the number matching task in Experiment 4, fractions or percentages afforded higher accuracy

[1] Plummer et al.'s (2017) analyses of eye movements showed that small-number discrete displays tend to elicit counting and that continuous displays tend to elicit estimation.

than decimals in most cases in which differences between notations were found. These results invite the question: "Decimals: What are they good for?"

One answer is that decimals are a straightforward extension of the base-10 system for expressing whole numbers. With whole numbers, the digit immediately to the left invariably expresses a base that is 10 times as great as the one to its right. Decimals permit representation of rational numbers that follow the same principle.

Another rationale for the use of decimals is that they are particularly useful for representing measurements, especially with metric units, because their implicit denominators can be any power of 10. Although decimals cannot represent all ratios exactly, they can represent any ratio to any desired degree of precision. This property may make decimals particularly advantageous for measurement, because different measurement situations require different degrees of precision. When weighing checked luggage in an airport, measurements may be rounded to the nearest 0.1 or 0.01 pound; when weighing items to be carried into orbit by a rocket, decimals permit much greater precision. Perhaps for this reason, decimals are frequently used to express measures of mass, length, area, volume, speed, force, energy, pressure, and other dimensions.

Where decimals seem especially likely to be preferred for representing metric measures, fractions seem most likely to be preferred for representing imperial measures. The imperial system employs a variety of conversion ratios: 1ft = 12in, 1 yd = 3 ft, 1lb = 16oz, 1qt = 4cups, etc. Fractions provide useful flexibility for representing imperial system measures (an inch is 1/12 of a foot, an ounce is 1/16 of a pound, a cup is 1/4 of a quart, etc.), because their denominators can be any number.

An analysis of fraction and decimal word problems in mathematics textbooks—specifically, kindergarten to eighth grade textbooks of *Addison-Wesley Mathematics* --yielded results consistent with this analysis. Rapp et al. (2015) found that when textbook problems involved metric units (e.g., centimeters, grams, liters), decimals were used more often than fractions. In contrast, when textbook problems involved imperial units (e.g., inches, pounds, gallons), fractions were used more often than decimals. It seems likely that preferences between decimals and fractions for different types of measurement units are influenced by the variability of denominators in the notations. Further research is needed to test this hypothesis and to identify other influential factors involving notation preferences.

**6.4. Quantification Process Theory and Semantic Alignment Theory**

Semantic alignment, the process of aligning multiple semantic structures, has much broader application than explaining preferences among rational number notations. For example, it has been applied to understanding reasoning by analogy (Gentner, 1983; Hummel & Holyoak, 2003), solving and constructing problems mathematically (Bassok, Chase, & Martin, 1998; Novick & Holyoak, 1991), modeling business processes (Brockmans, Ehrig, Koschmider, Oberweis, & Studer, 2006), and analyzing biological pathways (Gamalielsson & Olsson, 2008). The present findings are not intended to challenge semantic alignment theory in general, but rather to challenge its specific application for explaining preferences among rational number notations.

It might seem that semantic alignment theory could accommodate some of the present findings if supplemented by assumptions about how semantic structure varies with stimulus features and context. For example, the finding that increasing the number of items involved in a discrete ratio leads to decreased preference for fractions to represent the ratio could be explained by assuming that large numbers of items are likely to be perceived as continuous quantities, so that ratios between large numbers are likely to be perceived as one-dimensional. Similarly, the finding that preference for fractions decreases when participants are instructed to quantify ratios by estimating rather than by counting could be explained by assuming that estimation leads to the ratios being perceived as one-dimensional.

However, these explanations implicitly acknowledge that quantification processes shape preferences among rational number notations. In contrast, both the present and previous findings, especially ones involving notation preferences, can be accounted for by considering the quantification processes used in the situation, without reference to semantic alignment. Thus, quantification process theory appears to offer a more parsimonious account of findings regarding choices among rational number notations than does semantic alignment theory.

### 6.5. Quantification Process Theory and Strategy Choice Theory

Quantification process theory assumes that the preferred notation for representing a ratio depends on the strategy employed to quantify it. Choices among quantification strategies, in turn, depend on contextual variables, such as the number of items in the display and instructions to count or estimate. These and other determinants of strategy choices follow from Siegler's (1996) more general strategy choice theory, which asserts that choices among strategies depend on task characteristics, situational variables, frequency of input of various types of problems, and past experience with the strategies being chosen among.

This perspective implies that other variables that affect strategy choices in general, including variables that were not manipulated in the present study, would also affect rational number notation preferences. For example, a need for high precision (a task characteristic) would be expected to increase preference for decimals over percentages; a short time limit (a situational variable) would be expected to increase preference for estimation over counting, and therefore influence choice of rational number notations (as it, together with the instructions, did in Experiment 3 of the present study); exposure to percentage discounts in advertisements (an experiential variable) should increase preference for percentages. Future research should test these and other implications of quantification process theory and link the specific theory of choices among rational number notations more closely to the general strategy choice theory.

**6.6. Implications for Education**

The present findings also have implications for how rational numbers should be taught. Addressing this issue is particularly urgent because many children struggle with rational numbers (Lortie-Forgues, Tian, & Siegler, 2015; Siegler & Braithwaite, 2017); because rational number knowledge in elementary school is predictive of mathematics achievement in high school, even after controlling for numerous relevant variables (Siegler et al., 2012); and because rational numbers have been found to be pervasively used in the workplace (Handel, 2016).

Existing mathematics textbooks focus primarily on using fractions, decimals, and percentages to represent quantities exactly; mathematics instruction rarely address the use of rational numbers for approximation (Siegler & Booth, 2005). However, representing quantities that are difficult or impossible to determine exactly, and that therefore must be estimated, is also an important function of rational numbers. An implication is that rational number instruction should devote greater attention to the use of rational numbers for approximating quantitative values; doing so could better prepare children to apply rational numbers to the full range of tasks for which they are suitable.

Another educational implication relates to the sequence in which different notations should be taught. Children in kindergarten to fourth grade can reason about ratios between continuous quantities earlier than they can reason about ratios between small sets of discrete objects (Boyer et al., 2008). For example, when asked to choose which of two pictures showed a ratio matching that in a target picture, kindergartners were more accurate when the pictures involved continuous quantities than second graders were when the pictures involved countable discretized quantities. This finding, together with the present ones, suggests that it might be advantageous to introduce ratios between continuous quantities, using decimals and/or percentages, earlier in the curriculum than ratios between small discrete sets, using fractions.

This conclusion runs counter to most U.S. mathematics curricula, which typically introduce fractions years before decimals and percentages (National Governors Association Center for Best Practices, 2010). However, the conclusion dovetails with findings from studies of an experimental rational number curriculum (Kalchman, Moss, & Case, 2001; Moss & Case, 1999). In a five-month intervention with fourth graders, percentages were introduced first, decimals second, and fractions last. When compared to peers in a control condition, who received business-as-usual instruction that introduced the three notations in the opposite order, children who received the experimental curriculum showed greater gains in rational number knowledge. The current findings provide additional reasons to explore whether introducing percentages and decimals earlier in the curriculum than is now typical leads to improved learning of rational numbers among younger children.

### 6.7. Conclusion

The present study suggests that there are good reasons why multiple notations for representing rational numbers have endured for more than 300 years. Each notation serves a valuable function, and people prefer different notations in different situations. Fractions are the preferred notation for representing ratios whose exact numerators and denominators are needed, known, or easily established. Decimals and percentages are preferred to fractions for representing estimates of ratios. Percentages are preferred to decimals when precision beyond the nearest percent is not required. Decimals seem to be preferred for expressing metric measures. Considering quantification processes involved in determining ratios, as well as properties of each notation, can help provide a unified account of how, when, and why people use percentages, decimals, and fractions, as well as linking this specific strategy choice to more general theories of strategy choice.

**Open Practice**

All data and materials have been made publicly available via the Open Science Framework and can be accessed at https://osf.io/4zgwf/

**Declaration of Interests**

The authors declare that there is no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Author Contributions**

**Jing Tian**: Conceptualization, Methodology, Software, Formal analysis, Investigation, Data curation, Writing – Original draft, Visualization, Project administration. **David W. Braithwaite**: Conceptualization, Methodology, Validation, Investigation, Resources, Writing – Original draft, Visualization, Project administration. **Robert S. Siegler**: Conceptualization, Methodology, Resources, Writing – Review & Editing, Supervision, Funding acquisition.

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**Table 1**

*Divergent Predictions of Quantification Process Theory and Semantic Alignment Theory*

| Predictions of Quantification Process Theory | Corresponding Predictions of Semantic Alignment Theory |
|---|---|
| *Prediction 1.* Both percentages and decimals should be preferred to fractions for representing ratios in continuous displays, because such displays elicit estimation, but percentages should be preferred to decimals because estimating percentages entails less effort. | Both percentages and decimals should be preferred to fractions for representing ratios in continuous displays, and they should be equally preferred, because percentages, decimals, and ratios in continuous displays all have a one-dimensional structure. |
| *Prediction 2.* Notation preferences for large-number discrete displays should be similar to preferences for continuous displays, rather than preferences for small-number discrete displays, because large-number displays (like continuous displays) elicit estimation rather than exact enumeration. | Notation preferences for large-number discrete displays should be similar to preferences for small-number discrete displays rather than preferences for continuous displays, because large-number displays (like small-number discrete displays) are two-dimensional rather than one-dimensional. |
| *Prediction 3.* For representing discrete displays, the degree to which fractions are preferred over decimals and percentages should decrease, and the degree to which percentages or decimals are preferred over fractions should increase, as the number of items in the displays increases. | For representing discrete displays, the degree to which fractions are preferred over decimals and percentages should not change as the number of items in the displays increases, because the discrete displays are always two-dimensional regardless of the number of items included. |
| *Prediction 4.* Instruction to enumerate should produce a preference for fractions to represent large-number discrete displays that otherwise would elicit a preference for decimals or percentages. Instruction to estimate, and time limits short enough to preclude counting, should produce a preference for percentages or decimals to represent discrete displays that otherwise would elicit a preference for fractions. | Instructions to enumerate or estimate and time limits should not affect notational preferences, because they do not affect the one-dimensional or two-dimensional quality of the displays or the notations. |

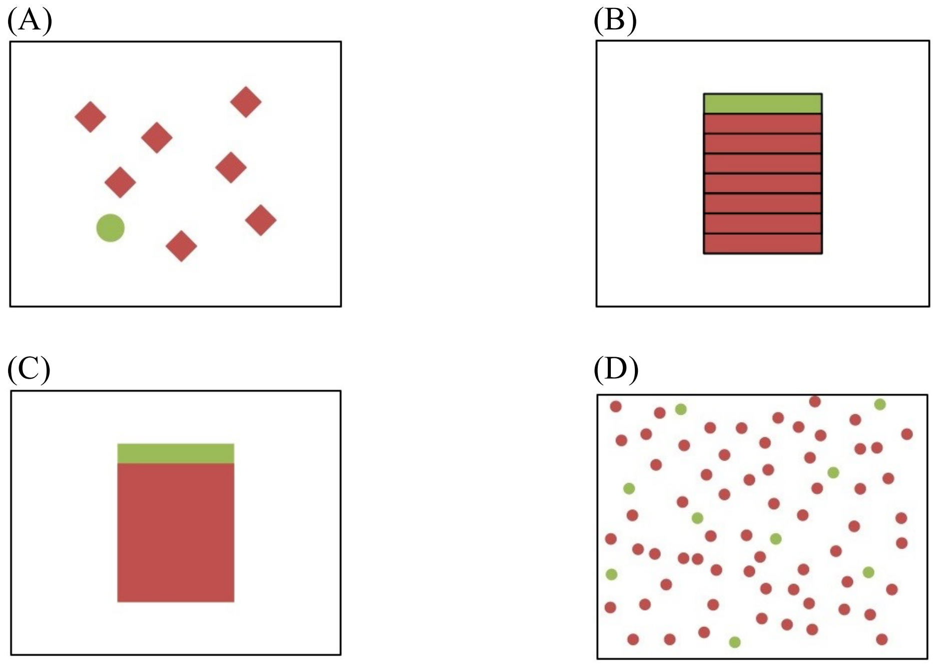


*Figure 1*. Visual displays of ratios used in the present study to express 7/8. (A) Small-number discrete display. (B) Small-number discretized display. (C) Continuous display. (D) Large-number discrete display. Displays (A), (B), and (C) were modeled after stimuli from DeWolf, Bassok, and Holyoak (2015). Display (D) was new to this study.

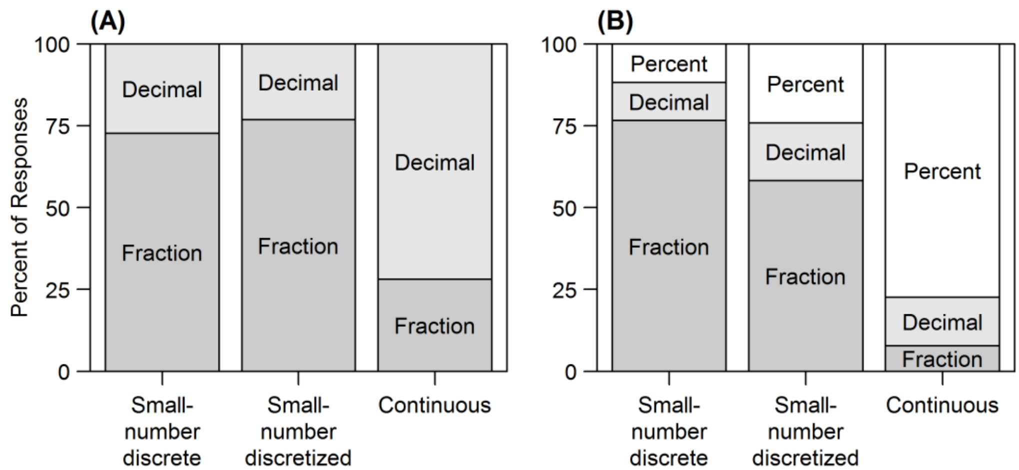


*Figure 2*. Notation preferences of U.S. students on small-number discrete, small-number discretized, and continuous displays in Experiment 1 in (A) the two-choice condition and (B) the three-choice condition.

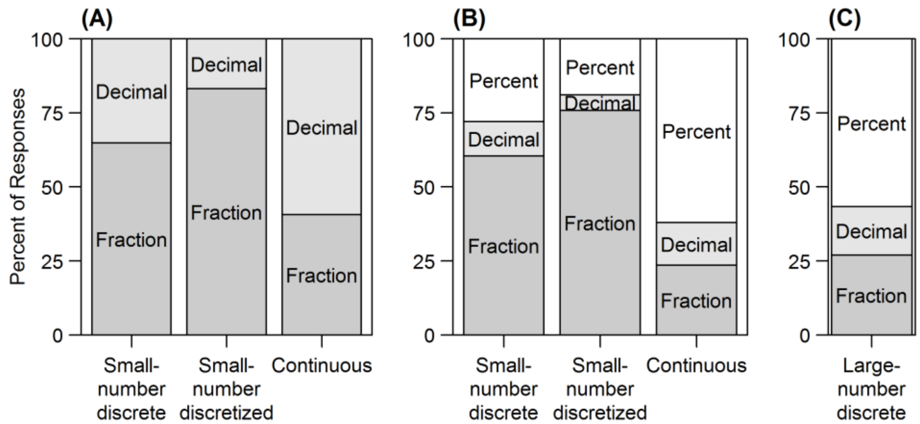


*Figure 3*. Notation preferences of Chinese students on small-number discrete, small-number discretized, and continuous displays in Experiment 2 in (A) the two-choice condition and (B) the three-choice condition. Results of the notation preference task with large-number discrete displays, in which all participants had three choices, are shown in (C).

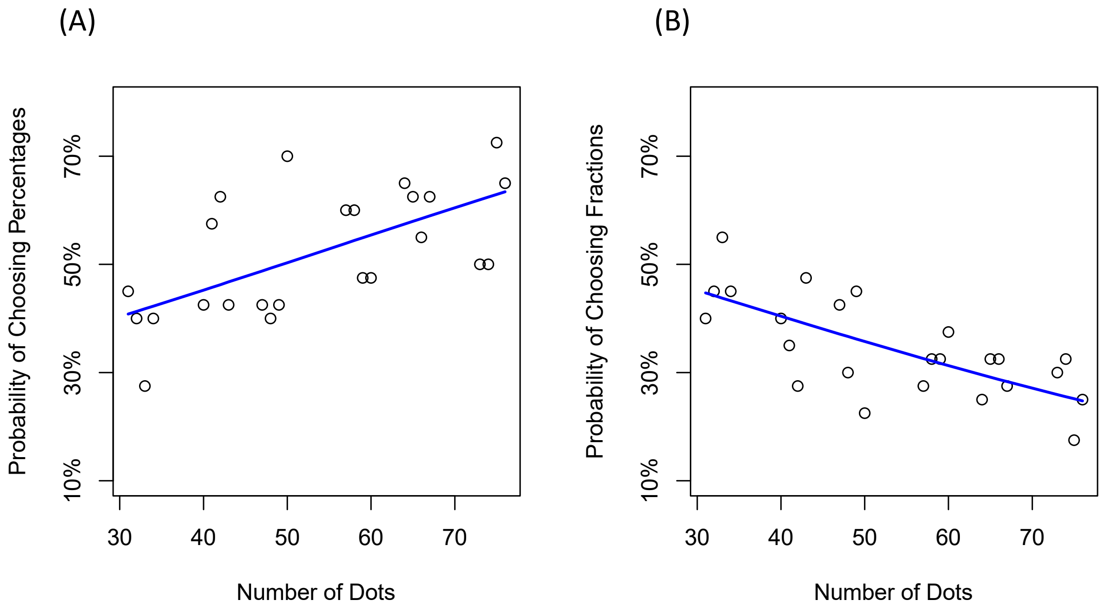


*Figure 4.* Probability of choosing percentages (A) or fractions (B) as a function of number of dots in the display on the notation preference task in Experiment 3. Circles indicate empirically observed probabilities; blue lines indicate probabilities predicted by the logistic regression models described in the text.