Uncertainty in Boundary Conditions – An Interval Finite Element Approach

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Abstract In this work, we introduce an interval formulation that accounts for uncertainty in supporting conditions of structural systems. Uncertainty in structural systems has been the focus of a wide range of research. Different models of uncertain parameters have been used. Conventional treatment of uncertainty involves probability theory, in which uncertain parameters are modeled as random variables. Due to specific limitation of probabilistic approaches, such as the need of a prior knowledge on the distributions, lack of complete information, and in addition to their intensive computational cost, the rationale behind their results is under debate. Alternative approaches such as fuzzy sets, evidence theory, and intervals have been developed. In this work, it is assumed that only bounds on uncertain parameters are available and intervals are used to model uncertainty.

Here, we present a new approach to treat uncertainty in supporting conditions. Within the context of Interval Finite Element Method (IFEM), all uncertain parameters are modeled as intervals. However, supporting conditions are considered in idealized types and described by deterministic values without accounting for any form of uncertainty. In the current developed approach, uncertainty in supporting conditions is modeled as bounded range of values, i.e., interval value that capture any possible variation in supporting condition within a given interval. Extreme interval bounds can be obtained by analyzing the considered system under the conditions of the presence and absence of the specific supporting condition. A set of numerical examples is presented to illustrate and verify the accuracy of the proposed approach.

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1 Introduction

Uncertainties have been studied for improving the predictability of mathematical models. Among others, interval finite element method handles uncertainties in interval form due to the lack of knowledge about the system parameters [2, 4]. Within the context of IFEM, all uncertain parameters are defined as intervals. While uncertainty in load, materials, and geometry in structural system have been intensively studied [3, 2, 6, 10, 11], in spite of their importance, uncertainty in supporting conditions barely have had the needed attention. Modeling boundary conditions is one of the most sensitive steps in the formulation of the mathematical model of a given system. Usually, supporting conditions are idealized through simplifying the mechanical behavior of the actual structural supports. Such idealization, in addition to disregarding the actual supporting conditions and uncertainty involved, may lead to inaccurate evaluation of the actual behavior of the structure [1, 8].

The paper is structured as follows. First, to illustrate the proposed interval approach for handling uncertainty in supporting conditions, we briefly present key features of Element-By-Element formulation of interval finite element. Uncertainty bounds of supporting conditions is then obtained from the extreme conditions and incorporated in the formulation. Examples are finally presented and discussed.

2 Formulation

Usually, in structural systems, supporting conditions are described such as roller, pin, or fixed supports, and in some cases might be modeled as partial restraint. In all these cases, supporting conditions are described by deterministic values. For example, a pin support in plane problems considered to prevent two degrees of freedom (displacements), in other words those degrees of freedom are set equal to zero (Figure 1a). In real life, such conditions rarely can be met, and any variation in these conditions might happen. Thus, in order to capture uncertainty in a supporting condition, we are modeling the condition as interval value that capture any possible variation within a given interval. In the case of a pin support, for example, one degree of freedom can take the values between zero (lower bound, fully restraint) and a second value (upper bound, fully free) corresponding to the absence of the constraint in direction of the considered degree of freedom (Figure 1b).

The treatment of uncertain supporting conditions can be achieved by expanding our previous formulation of IFEM [10] through incorporating additional linear or rotational elements whose stiffness is interval, that depends on the specific type of the structure (approach 1). Experts can provide the interval value of the element stiffness. Another way to handle this uncertain situation is to solve the system in the absence of the specific constraint, under the condition that the system remains geometrically stable, and calculate the displacement in the direction of the removed constraint (approach 2). The obtained value of the displacement in the direction of the removed constraint can be used as an upper bound of prescribed displacement

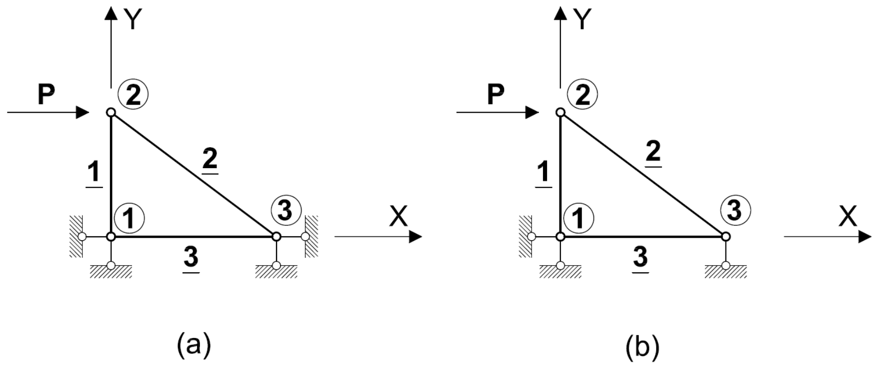

Fig. 1 Truss structure with different boundary conditions. (a) pin-pin at node 1 and 3, (b) pin at node 1 and roller at node 3.

for which the lower bound is zero.

The same approach can be used to account for uncertainty in the rigidity of the structures' joints. When joints connecting structural members are assumed to be rigid, the relative rotation of the adjacent sections of the intersecting members is considered equal to zero. By introducing the rigidity condition as interval, we can consider more realistic behavior of these joints. All interval quantities will be introduced in non-italic boldface font.

Our formulation of interval finite element method is based on the Element-By-Element formulation [6] where each element has its own set of nodes, but the set of elements is disassembled (Figure 2). A set of additional constraints is imposed to satisfy the conditions of compatibility, these constraints have the form of $CU = V$, where C, U, and V are a constraint matrix, displacement vector, and prescribed displacement values if any, respectively. The constraint matrix describes the connectivity conditions between local element degrees of freedom and corresponding global nodal degrees of freedom in addition to the imposed boundary conditions. For instance, for a joint demonstrated in Figure 2, the connectivity matrix has the following form,

$$C = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & 0 & 0 & \cdots & -\cos(\phi) & -\sin(\phi) & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots & \sin(\phi) & -\cos(\phi) & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad (1)$$

The matrix is obtained based on the following conditions

$$\begin{aligned} u_j - U_k \cos(\phi) - V_k \sin(\phi) &= 0 \\ v_j + U_k \sin(\phi) - V_k \cos(\phi) &= 0 \\ \theta_j - \Theta_k &= 0 \end{aligned} \qquad (2)$$

where ϕ is the angle of rotation between the local and global coordinate systems, u_j, v_j and θ_j are local horizontal, vertical and rotational degrees of freedom of node j that belongs to element e, respectively, and U_k, V_k and Θ_k are global horizontal, vertical and rotational degrees of freedom of the common node k. Then, the essential boundary conditions are imposed by setting the corresponding components in C equal to 1.

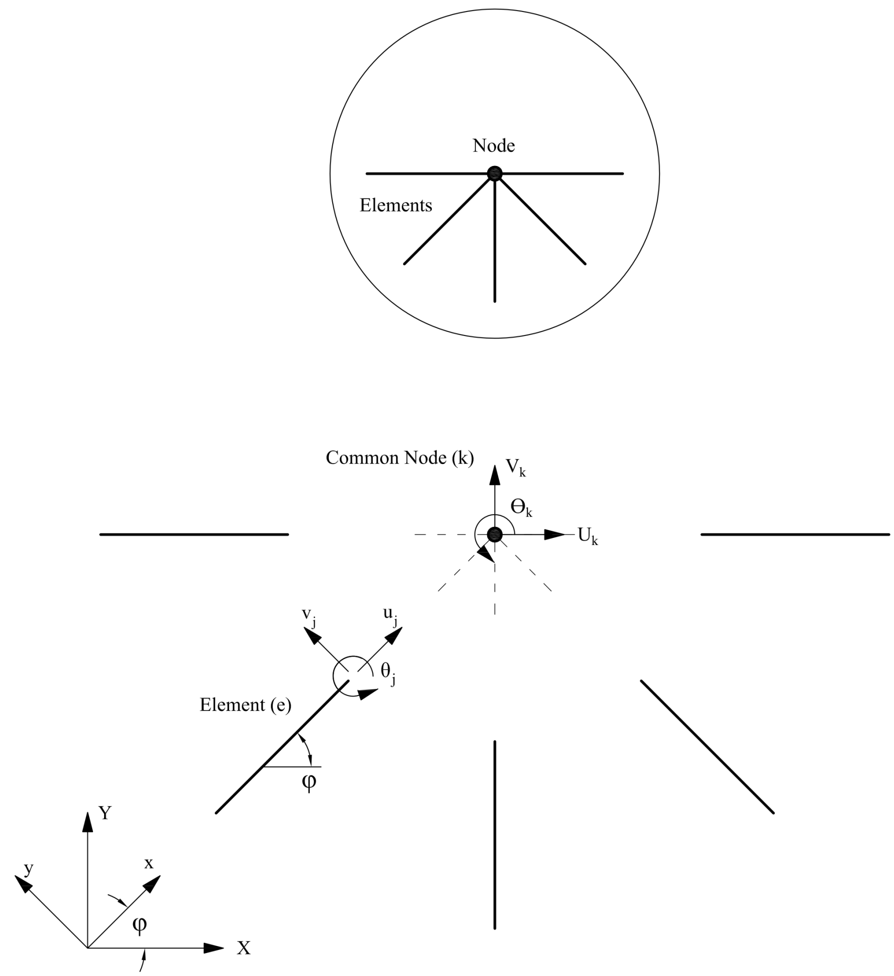

Fig. 2 Element- By-Element. Connection between element nodes and common nodes.

Under these constraints, the amended total potential energy using the Lagrange multiplier approach takes the following form:

$$\Pi^* = \frac{1}{2}U^T KU - U^T P + \lambda^T (CU - V) \tag{3}$$

where Π^*, K, U, P and λ are total potential energy, stiffness matrix, displacement vector, load vector, and Lagrange multipliers' vector, respectively. Invoking the stationarity of Π^*, that is $\delta\Pi^* = 0$, we obtain the following interval equilibrium equations, where in the present study, uncertainty is assumed only in the right-hand side,

$$\begin{bmatrix} \mathbf{K} & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{V} \end{bmatrix}. \tag{4}$$

The advantage of this formulation is that it allows handling any prescribed interval displacement as a part of the right-hand side, i.e., the load vector. Eliminating overestimation due to interval dependency in the load vector has been achieved in our previous work [3].

In approach 1, simply we can add a linear or rotation element with interval property in direction of the considered support accounting for the level of uncertainty. In approach 2, the system is solved after removing the constraint with uncertainty, the obtained displacement value in the direction of the constraint represents the upper bound of the uncertain supporting condition and the lower bound will be equal to zero. This interval range of displacement in direction of the selected degree of freedom is used as prescribed interval displacement in the corresponding entry of V in Equation 4.

In the case of uncertainty in joint rigidity (partially restrained connections) [5, 7] , the system is solved for two extreme deterministic cases: (i) The system with all joints considered rigid, and (ii) the system with hinges introduced at the joints with uncertain rigidity. By deterministically solving (i) and (ii), the rotation of the rigid joint and the rotations at each side of the hinge are calculated, respectively. These values are used to determine the bounds of intervals that model the joints' uncertainties. The obtained intervals are used as prescribed rotations in the right-hand side vector of the equilibrium system of Equation 4. In the next section, a number of examples are introduced to illustrate the numerical results of the developed formulation.

This formulation for handling uncertainty in supporting conditions and joints' rigidity is an integral part of the previously developed interval finite element that handles uncertainty in materials, geometry, and different loading conditions.

3 Example Problems

The proposed method is implemented in a MATLAB program using the interval toolbox INTLAB to handle interval computations [9]. Three example problems are chosen to illustrate and verify the present approach; a truss and two frames. The first example is an eleven-bar truss as shown in Figure 3. The numerical results of this example are obtained to illustrate the effect of uncertainty in boundary conditions.

The second example is one story single-bay frame with uncertainty in a horizontal boundary condition. This simple example is demonstrating the general applicability of the proposed approach in different types of structures. The third numerical example is the same frame structure analyzed in the second example but we illustrate the effect of the uncertainty in joint rigidity.

These examples do not include any material or load uncertainty, which can be handled straight forward by the previously developed IFEM. However, the numerical results illustrate how the proposed extension to IFEM approach provides exact values of system response under uncertainty in structures' supporting conditions and joints' rigidity.

3.1 Example 1: Planar truss with uncertainty in boundary condition

The eleven-bar truss shown in Figure 3 is subjected to a horizontal concentrated load $P_1 = 50$ kN at node 2, and three downward concentrated loads of $P_2 = 100$ kN and $P_3 = P_4 = 50$ kN applied at nodes 4, 2 and 6, respectively. The cross-sectional areas and the nominal value of Young's modulus of all elements are $A_i = 0.01$ m^2 and $E_i = 2 \times 10^{11}$ N/m^2, respectively. The dimensions are also illustrated in the figure.

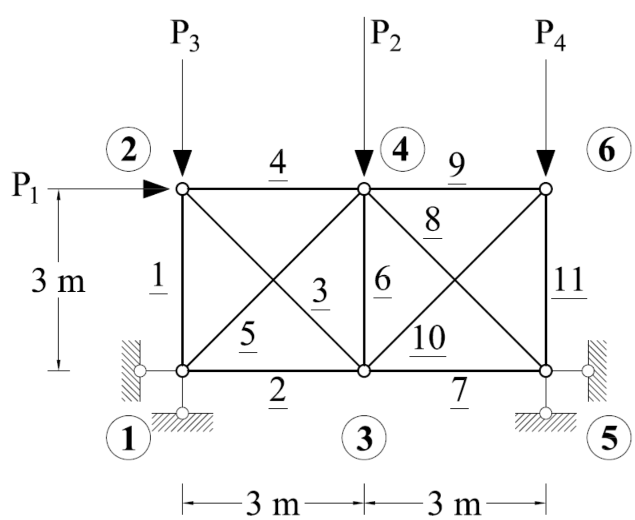

Fig. 3 Eleven-bar truss, pin supported at nodes 1 and 5.

To investigate uncertainty in the horizontal constraint of node 5, the problem is solved in the absence of that specific constraint (Figure 4) to obtain the upper bound of prescribed displacement of that constraint.

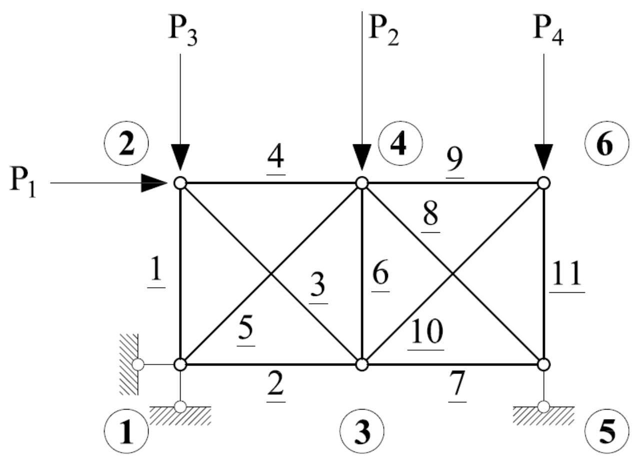

Fig. 4 Eleven-bar truss, roller support at node 5.

Table 1 presents displacement values obtained by solving these two deterministic cases; namely, pin-pin and pin-roller supporting conditions.

Table 1 Truss nodal displacements, pin-pin and pin-roller supporting conditions.

Node	Pin-Pin		Pin-Roller	
	u (mm)	v (mm)	u (mm)	v (mm)
1	0.00000	0.00000	0.00000	0.00000
2	0.12789	-0.06873	0.23008	-0.07832
3	0.01658	-0.16232	0.10918	-0.20861
4	0.05916	-0.18295	0.15175	-0.24842
5	0.00000	0.00000	0.18519	0.00000
6	0.03226	-0.10190	0.11526	-0.11149

Therefore, the interval prescribed horizontal displacement of the support at node 5 is $[0, 0.18519] \times 10^{-3}$ m. By imposing this condition in the corresponding entry of vector V as prescribed displacement, the interval solution is obtained. Table 2 presents the numerical values of displacements due to 100% uncertainty in the horizontal constraint of node 5.

The results indicate that the bounds of nodal displacement match exactly the displacement results obtained for extreme values given in Table 1. This observation

Table 2 Interval displacements due to 100% uncertainty in the support at node 5.

Node	u (mm)		v (mm)	
	Lower	Upper	Lower	Upper
1	0.00000	0.00000	0.00000	0.00000
2	0.12789	0.23008	-0.07832	-0.06873
3	0.01658	0.10918	-0.20861	-0.16232
4	0.05916	0.15175	-0.24842	-0.18295
5	0.00000	0.18519	0.00000	0.00000
6	0.03226	0.11526	-0.11149	-0.10190

holds true for internal and reaction forces. For instance, the axial force in elements 6 and 7 denoted by N_6 and N_7, respectively, and the horizontal reaction forces at node 1 and 5 denoted by R_1 and R_5, respectively, are presented in Table 3.

Table 3 Selected internal and reaction forces of the truss in Figure 4 due to 100% uncertainty.

Forces (kN)	Deterministic		Interval	
	Pin	Roller	Lower	Upper
N_6	-13.75631	-26.54092	-26.54092	-13.75631
N_7	-11.05606	50.67348	-11.05606	50.67348
R_1	18.12184	-50.00000	-50.00000	18.12184
R_5	-68.12184	0.00000	-68.12184	0.00000

3.2 Example 2: Planar frame with uncertainty in boundary condition

The next example is a single-story frame subjected to a horizontal concentrated load $P = 5$ kN applied at node 2. The modulus of elasticity, cross sectional area and moment of inertia of all elements are equal to $E = 2\times10^{11}$ N/m^2, $I = 3\times10^{-4}$ m^4 and $A = 8.37\times10^{-3}$ m^2, respectively. In this frame, the effect of uncertainty in the horizontal support at node 4 is studied. To calculate the prescribed displacement interval, the horizontal support at this node is removed by replacing the fixed support with a horizontal slider support (Figure 5) and the deterministic solution is obtained. Table 4 presents the results of two cases: fixed support at node 4 and horizontal slider support at node 4.

Therefore, the associated interval prescribed horizontal displacement of support at node 4 is $[0, 1.73611]\times10^{-3}$ m. The obtained interval solution is an exact enclosure of the deterministic solutions of the two extreme cases: Fixed support and slider support at node 4. Numerical results are introduced in Table 5, where u_2 and θ_3 denote the horizontal displacement and rotation of node 2 and 3, respectively.

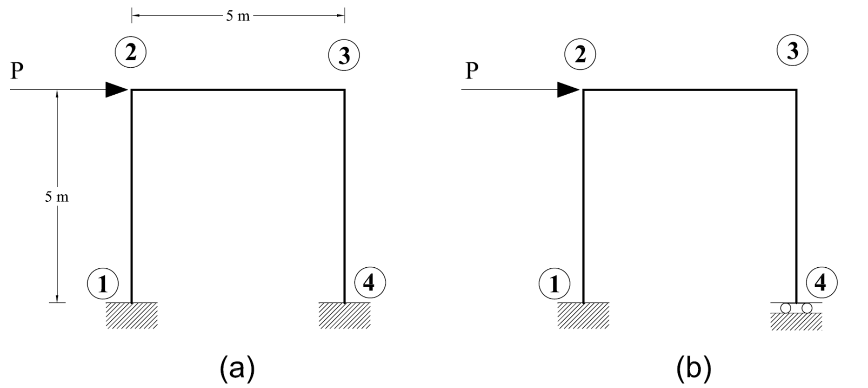

Fig. 5 Planar frame with uncertainty in support condition. (a) Fixed supports at nodes 1 and 4 (b) Fixed at node 1 and slider support at node 4.

Table 4 Frame nodal displacements pin-pin and pin-roller supporting conditions.

Node	Fixed-Fixed			Fixed-Slider		
	u ($\times 10^{-3}$ m)	v ($\times 10^{-3}$ m)	θ ($\times 10^{-3}$ rad)	u ($\times 10^{-3}$ m)	v ($\times 10^{-3}$ m)	θ ($\times 10^{-3}$ rad)
1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2	0.62922	0.00637	-0.07733	1.49355	0.00637	-0.25020
3	0.62178	-0.00637	-0.07584	1.49355	-0.00637	0.09702
4	0.00000	0.00000	0.00000	1.73611	0.00000	0.00000

The forces N_2 and M_2 are the axial force and bending moment at the right-end of element 2, respectively, and RM_4 denotes the reaction moment at node 4.

Table 5 Effect of 100% uncertainty in horizontal support of frame of Figure 5a.

Forces (kN)	Deterministic		Interval	
	Fixed	Slider	Lower	Upper
$u_2(\times 10^{-3}$ m)	0.62922	1.49355	0.62922	1.49355
$\theta_3(\times 10^{-3}$ rad)	-0.07584	0.09702	-0.07584	0.09702
N_2(kN)	-2.48929	0.00000	-2.48929	0.00000
M_2(kN-m)	-5.31309	-1.16427	-5.31309	-1.16427
RM_4(kN-m)	7.13337	-1.16427	-1.16427	7.13337

Numerical results show the large variability in displacements, internal forces, and reactions due to uncertainty in supporting conditions.

3.3 Example 3: Planar frame with uncertainty in joint rigidity

This example represents a case study of what is known in structural design as partially restrained connections [5, 7]. The same frame of Figure 5 is used to illustrate this case. The prescribed interval values at node 2 are obtained by replacing the rigid joint at node 2 by a hinge (Figure 6b).

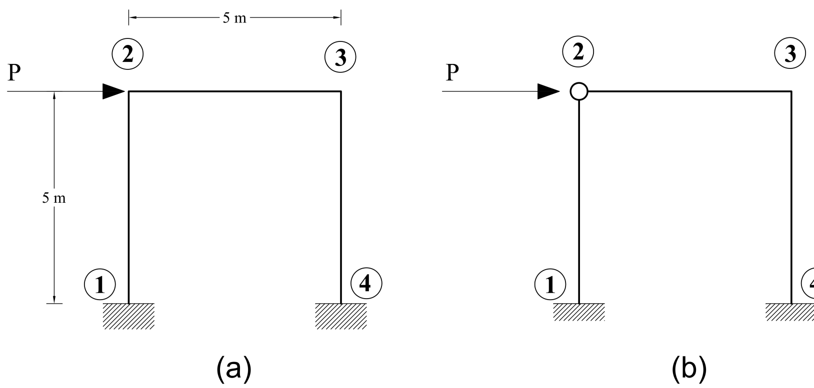

Fig. 6 Planar frame with uncertainty in joint rigidity. (a) Original (b) Hinge at node 2.

Solving the two extreme deterministic cases; the displacements and rotations for the structure with rigid joints and the structure with a hinge at node 2 are obtained. In the case of a hinge at node 2, two different rotation values are obtained for the sections adjacent to the hinge, see Table 6. Thus, the prescribed interval rotations are obtained as $[-0.31980, -0.07733] \times 10^{-3}$ rad and $[-0.0773, 0.08849] \times 10^{-3}$ rad, which are imposed in the corresponding entries of vector V as prescribed displacements.

Table 6 Deterministic rotations of node 2 for the rigid and hinge cases of frames in Figure 6.

Node	Rigid	Hinge
	θ ($\times 10^{-3}$ rad)	θ ($\times 10^{-3}$ rad)
1	0.00000	0.00000
2	-0.07733	(-0.31980, 0.08849)
3	-0.07584	-0.18163
4	0.00000	0.00000

The selected results are presented in Table 7. The comparison between interval and deterministic solutions reveals that the interval solution is an exact enclosure of the deterministic analysis of extreme cases.

Table 7 Effect of 100% uncertainty in joint 2 rigidity on displacements and forces of the frame in Figure 6.

Forces (kN)	Deterministic		Interval	
	Fixed	Slider	Lower	Upper
$u_2(\times 10^{-3}$ m)	0.62922	1.06598	0.62922	1.06598
$\theta_3(\times 10^{-3}$ rad)	-0.07584	-0.18163	-0.18163	-0.07584
N_2(kN)	-2.48929	-3.46498	-3.46498	-2.48929
M_2(kN-m)	-5.34878	0.00000	0.00000	5.34878
RM_4(kN-m)	7.13337	10.84202	7.13337	10.84202

4 Conclusion

This paper presents a new approach in interval finite element method to study the effect of uncertainty in boundary conditions and joint rigidity. This approach treats the uncertainty in supporting conditions as prescribed interval displacements defined in the right-hand side of the system of equations. Exact enclosures of the correct solution are obtained and any form of overestimation is eliminated. The numerical examples illustrate that the proposed approach can successfully capture the effect of uncertainty in supporting conditions and joints rigidity. Numerical results show the large variability in displacements, internal forces, and reactions due to uncertainty in supporting conditions. In addition, this formulation represent an integral part of the IFEM developed previously by the author, and allows to consider uncertainties in materials, geometry, loads, and boundary conditions.

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References

1. Bursi, O.S., Abbiati, G., Caracoglia, L., Reza, M.S.: Effects of Uncertainties in Boundary Conditions on Dynamic Characteristics of Industrial Plant Components. ASME 2014 Pressure Vessels and Piping Conference. V008T08A028–V008T08A028 (2014)
2. Muhanna, R.L., Mullen, R.L.: Uncertainty in mechanics problems - Interval–based approach. Journal of Engineering Mechanics. **6**, 557–566 (2001)
3. Mullen, R.L., Muhanna, R.L.: Bounds of structural response for all possible loading combinations. Journal of Structural Engineering. **1**, 98–106 (1999)
4. Neumaier, A., Pownuk, A.: Linear systems with large uncertainties, with applications to truss structures. Reliable Computing. **2**, 149–172 (2007)
5. Nielson, B.G., McCormac, J.C.: Structural Analysis: Understanding Behavior. Wiley. (2017)
6. Rao, M.R., Mullen, R.L., Muhanna, R.L.: A new interval finite element formulation with the same accuracy in primary and derived variables. International Journal of Reliability and Safety. **5**, 336–357 (2011)
7. Ricles J., Bjorhovde R., Iwankiw N. : Frames with Partially Restrained Connections. Structural Stability Research Council, Workshop Proceedings. (1998-2000)

8. Ritto, T.G., Sampaio, R., Cataldo, E.: Timoshenko Beam with Uncertainty on the Boundary Conditions. Journal of the Brazilian Society of Mechanical Sciences and Engineering. **4**, 295–303 (2008)
9. Rump S.M.: INTLAB - INTerval LABoratory. In Tibor Csendes, editor, Developments in Reliable Computing, Kluwer Academic Publishers, Dordrecht. 77–104 (1999)
10. Xiao, N., Muhanna, R.L., Fedele, F., Mullen, R.L.: Interval finite elements for uncertainty in frame structures. 11th International Conference on Structural Safety & Reliability, New York. 16–20 (2013)
11. Xiao, N., Muhanna, R.L., Fedele, F., Mullen, R.L.: Uncertainty analysis of static plane problems by intervals. SAE International Journal of Materials and Manufacturing. **8**, 374–381 (2015)