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On quasi-equigenerated and Freiman cover ideals of graphs

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ABSTRACT

A quasi-equigenerated monomial ideal I in the polynomial ring $R = k[x_1, \ldots, x_n]$ is a Freiman ideal if $\mu(I^2) = l(I)\mu(I) - \binom{l(I)}{2}$ where $l(I)$ is the analytic spread of I and $\mu(I)$ is the number of minimal generators of I. Freiman ideals are special since there exists an exact formula computing the minimal number of generators of any of their powers. In this work, we address the question of characterizing which cover ideals of simple graphs are Freiman.

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1. Introduction

Given an homogeneous ideal I in a polynomial ring $R = k[x_1, \ldots, x_n]$, it is in general a difficult problem to exactly compute the number of generators of each power of I. In the case that I is generated by a regular sequence and has $\mu(I) = t$ generators, it is well-known that $\mu(I^m) = \binom{m+t-1}{t-1}$. This is the largest possible number of generators that I^m can achieve. At the other extreme, in some cases powers of an ideal can be generated by very few elements. For instance, the ideals with tiny square defined in [3] provide a family of monomial ideals whose members can be generated by arbitrarily many elements but satisfy $\mu(I^2) = 9$.

However, there are large classes of monomial ideals for which the number of minimal generators of I^n is well-behaved. If all generators of I have the same degree with respect to some standard (or non-standard) $\mathbb{N}$-grading of R, then I is called *equigenerated* (or *quasi-equigenerated*). Such an ideal cannot have a tiny square. In particular, when I is equigenerated or quasi-equigenerated a result of Herzog, Mohammadi Saem, and Zamani [7, Theorem 1.9] shows that $\mu(I^2) \geq l(I)\mu(I) - \binom{l(I)}{2}$ where $l(I)$ is the analytic spread of I. Furthermore in [8, Proposition 1.8], Herzog and Zhu show a similar inequality, describing a lower bound for the number of generators of any power of I. These inequalities are consequence of a famous theorem of Freiman [4], proved in the context of additive number theory and stating that, for a finite set $X \subseteq \mathbb{Z}^n$,

$$|2X| \geq (d+1)|X| - \binom{d+1}{2}$$

where d is the dimension of the smallest affine subspace of $\mathbb{Q}^n$ containing X and $2X = \{a + b :$

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$a, b \in X\}$. The lower bound for $\mu(I^2)$ is then obtained applying Freiman's theorem to the set of exponents of the generators of I.

In [8], Herzog and Zhu also consider the case in which this lower bound is met. They define a *Freiman ideal* to be an equigenerated monomial ideal I such that $\mu(I^2) = l(I)\mu(I) - \binom{l(I)}{2}$. In [6], Herzog, Hibi, and Zhu extend the same definition to quasi-equigenerated monomial ideals. What makes a Freiman ideal very interesting is the fact that the number of generators of any power can be computed by an exact formula in terms of the number of generators of the ideal and of the analytic spread (see [8, Corollary 1.9]). This formula generalizes the formula existing in the case of ideals generated by regular sequences, since for them $l(I) = \mu(I)$.

The two papers [6, 8] provide several characterizations of Freiman ideals in terms of their Hilbert polynomials and fiber cones. In particular [6, Theorem 1.3] shows that Freiman ideals are exactly those ideals with linear Hilbert polynomials (that is, $h_i = 0$ for all $i \geq 2$) and with fiber cones having minimal multiplicity. This is a very restrictive property which, for ideals arising from combinatorial structures, often guarantees strong combinatorial properties. Freiman ideals in the classes of principal Borel ideals, Hibi ideals, Veronese type ideals, matroid ideals, and edge ideals of graphs have been studied in [6, 8].

The aim of this article is to study Freiman ideals among cover ideal of graphs. Let G be a finite simple graph on n vertices with edge set E. Identifying each vertex with a variable $x_i \in R$, the cover ideal $J(G)$ is defined to be

$$J(G) := \bigcap_{\{x_i, x_j\} \in E} (x_i, x_j) \subseteq R.$$

Cover ideals are squarefree monomial ideals and have been studied by many authors in the last twenty years. For an overview we refer to [13]. For a more detailed study of the foundational results on cover ideals, we refer to [10].

In Section 2, we recall basic definitions and results about fiber cones, Freiman ideals and cover ideals. Furthermore, we introduce the notion of equivalent vertices in graphs and explore their relation to cover ideals.

In studying Freiman property for cover ideals, the first challenge is that cover ideals are often not quasi-equigenerated. In Section 3, we approach this challenge. It is difficult to find a purely combinatorial characterizations of quasi-equigeneratedness but we describe several criteria and give explicit characterization for cover ideals of some classes of graphs including trees, circulant graphs, and graphs with independence number one or two. Finally, we consider the behavior of cover ideals of graphs obtained through various constructions.

In Section 4, we consider the Freiman property for cover ideals. We describe when the join of two graphs is Freiman, then we relate squarefree Freiman ideals with ideals of minors of $2 \times n$ generic matrices and we prove that, in general, graphs that are close enough to be complete have Freiman cover ideal. Since the defining ideal of the fiber cone of Freiman ideal has a 2-linear free resolution, it is clear that, when is nonzero, it must be generated in degree 2. However, we also find graphs for which the defining ideal of the fiber cone of the cover ideal is generated in any possible degree $n \geq 3$, and hence their cover ideals are not Freiman.

Finally, in Section 5, we characterize Freiman cover ideals among different families of graphs. We consider the family of the pairs of complete graphs sharing a vertex, the family of circulant graphs, and the family of whiskered graphs. For all these families, we also explicitly compute the analytic spread of the cover ideals.

2. Preliminaries

In this section, we recall definitions and several preliminary results about fiber cones, Freiman ideals and cover ideals of graphs.

2.1. Fiber cones and Freiman ideals

Let I be an homogeneous ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$, where k is any field. The *fiber cone* of I is the standard-graded k-algebra

$$F(I) = \bigoplus_{i=0}^{\infty} \frac{I^i}{\mathfrak{m}I^i},$$

where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the homogeneous maximal ideal of R. The coefficients of the Hilbert series of the fiber cone are $\dim_k\left(\frac{I^i}{\mathfrak{m}I^i}\right) = \mu(I^i)$, where $\mu(I^i)$ indicates as usual the minimal number of generators of I^i. The Krull dimension of $F(I)$ is called *analytic spread* and it is usually denoted by $l(I)$; this invariant measures how fast the number of generators of the powers of I increase.

If the fiber cone of I is isomorphic to a polynomial ring in $\mu(I)$ variables, the ideal I is called of *linear type*. This happens if the generators of I form a regular sequence.

In order to study the Freiman property of ideals, we will consider (quasi-)equigenerated homogeneous ideals. We give the following definition.

Definition 2.1. Let $I = (f_1, \ldots, f_t) \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal. Write its monomial generators as $f_j = \prod_{i=0}^{n} x_i^{c_{i_j}}$. For $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$ we define

$$d_\alpha(f_j) = \sum_{i=0}^{n} a_i c_{i_j}.$$

We say that I is *quasi-equigenerated of degree d* if there exists $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n_{\geq 1}$, such that, for every j, k,

$$d_\alpha(f_j) = d_\alpha(f_k) = d.$$

We say that I is *equigenerated* if all its generators have the same degree (equivalently I is quasi-equigenerated taking $\alpha = (1, 1, \ldots, 1)$).

It is known that the fiber cone of an equigenerated homogeneous ideal is an integral domain because it is isomorphic to the k-algebra generated by its minimal generators. We recall this result because, up to changing the grading on the polynomial ring, it can be extended to quasi-equigenerated ideals.

Proposition 2.2 ([5, Proposition 4.8]). *Let $R = k[x_1, \ldots, x_n]$ be a (not necessarily standard) $\mathbb{N}$-graded polynomial ring in n variables over field k, and let $\mathfrak{m}$ denote its homogeneous maximal ideal. Suppose the ideal $I = (f_1, \ldots, f_n)$ is homogeneous and equigenerated of degree t with respect to the grading of R. Then $F(I_\mathfrak{m})$ (and thus $F(I)$) is a domain.*

Proof. Let $I_i = I^i \cap R_{it}$ be the ith graded component of I^i. Then $k \oplus I_1 \oplus I_2 \oplus \cdots = k[f_1, \ldots, f_n]$. Since I is equigenerated with respect to the grading of R, $I^i/\mathfrak{m}I^i \cong I_i$ for all i. Therefore

$$k[f_1, \ldots, f_n] \cong \bigoplus_{i=0}^{\infty} \left(\frac{I^i}{\mathfrak{m}I^i}\right) \cong \bigoplus_{i=0}^{\infty} \left(\frac{I^i_\mathfrak{m}}{\mathfrak{m}I^i_\mathfrak{m}}\right) \cong F(I_\mathfrak{m}).$$

Thus $F(I_\mathfrak{m})$ is a domain. □

We specialize now to the case of ideals that can be generated by monomials. Let $I = (f_1, \ldots, f_t) \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal. Using the homomorphism $T_i \to f_i$, we write the fiber cone as

$$F(I) \cong \frac{k[T_1, \ldots, T_t]}{\mathcal{I}}$$

where $k[T_1, \ldots, T_t]$ is a polynomial ring over the same field k. The ideal $\mathcal{I} \subseteq k[T_1, \ldots, T_t]$ is called the *defining ideal* of the fiber cone of I. We describe the structure of the defining ideal in the monomial case:

Proposition 2.3. *Let $I = (f_1, \ldots, f_t)$ be a quasi-equigenerated monomial ideal. The defining ideal $\mathcal{I} \subseteq k[T_1, \ldots, T_t]$ of the fiber cone F(I) is a homogeneous ideal, and when it is nonzero, it is generated by binomials of the form*

$$T_{i_1} \cdots T_{i_r} - T_{j_1} \cdots T_{j_r}$$

and such that $\{i_1, \ldots, i_r\} \cap \{j_1, \ldots, j_r\} = \emptyset$.

Proof. By Proposition 2.2, $F(I) \cong k[f_1, \ldots, f_t]$, hence $F(I)$ is a toric ring and its defining ideal $\mathcal{I}$ is generated by binomials by [9, Proposition 10.1.1]. To see that $\mathcal{I}$ is homogeneous, assume by way of contradiction that it has a generator of the form $T_{i_1} \cdots T_{i_r} - T_{j_1} \cdots T_{j_s}$ with $r \neq s$. This it is equivalent to say that, if $I = (f_1, \ldots, f_t)$, then $f_{i_1} \cdots f_{i_r} = f_{j_1} \cdots f_{j_s}$. But, since I is quasi-equigenerated, we may find $\alpha \in \mathbb{N}^n_{\geq 1}$ such that $d_\alpha(f_i) = d_\alpha(f_j)$, for every i, j. This implies $rd_\alpha(f_{i_1}) = sd_\alpha(f_{i_1})$, and hence $d_\alpha(f_{i_1}) = 0$, that is a contradiction. □

The next proposition states a criterion that we are going to use trough this paper, to show when the ideals of $F(I)$ generated by monomials of degree one (the variables T_i) are primes.

Proposition 2.4. *Let $I = (f_1, \ldots, f_t) \subseteq k[x_1, \ldots, x_n]$ be a quasi-equigenerated monomial ideal. Let $S = k[T_1, \ldots, T_t]$, and $\mathcal{I} \subseteq S$ be the defining ideal of F(I). Let $\mathbf{T} = \{T_1, \ldots, T_t\}$ and let $\mathcal{J}$ be an ideal of S generated by some subset $V \subseteq \mathbf{T}$. Let J be the image of $\mathcal{J}$ in F(I). The following are equivalent:*

1. *J is prime.*
2. $\frac{(\mathcal{I}+\mathcal{J})}{\mathcal{J}}$ *is either* (0) *or is generated by binomials.*
3. $\frac{F(I)}{J} = \frac{S}{(\mathcal{I}+\mathcal{J})} \cong k\left[f_i \mid T_i \notin V\right]$.

Proof. $(1) \Rightarrow (2)$: Let $\psi : S \to S/\mathcal{J}$ and $\rho : S \to S/\mathcal{I}$ be the natural maps. By Proposition 2.3, $\mathcal{I}$ is generated by binomials. Let $\alpha - \beta$ be a binomial in $\mathcal{I}$ with α, β monomials. If $\psi(\alpha - \beta) \neq 0$, then either one or both of α, β are not in $\mathcal{J}$. If this happen for both, then $\psi(\alpha - \beta)$ is a binomial in $S/\mathcal{J}$. Suppose $\alpha \notin \mathcal{J}$, and $\beta \in \mathcal{J}$. Since $\rho(\mathcal{J}) = J$, we know that $\rho(\beta) \in J$ and, since $0 = \rho(\alpha - \beta) \in J$, also $\rho(\alpha) \in J$. But $\rho(\alpha)$ is a product of elements of the form $\rho(T_i)$ where $T_i \notin V$, thus if $\rho(\alpha) \in J$ then, the assumption of J a prime ideal implies also $\rho(T_i) \in J$ for some $T_i \notin V$. Since α is a monomial, so is $\rho(T_i)$. Since ρ is a degree preserving map, $\rho(T_i) = \rho(T)$ for some $T \in V$. Thus $T - T_i \in \mathcal{I}$. However, as the defining ideal of a fiber cone, $\mathcal{I}$ is generated in degree at least 2, a contradiction.

$(2) \Rightarrow (3)$ Consider the map $\phi : S/\mathcal{J} \to k\left[f_i \mid \mathrm{T_i} \notin \mathrm{V}\right]$ given by $\overline{T_i} \mapsto f_i$. Since $S/\mathcal{J} \cong k[\mathbf{T} \setminus V]$ we identify $S/\mathcal{J}$ with its isomorphic subring in S and view each nonzero $\overline{T_i} \in S/\mathcal{J}$ as $T_i \in S$. We know that $\ker\phi$ consists of all polynomials in T_i corresponding to relations in $\{f_i \mid \mathrm{T_i} \notin \mathrm{V}\}$. Such elements of S are contained in $(\mathcal{I} + \mathcal{J})/\mathcal{J}$, so $\ker(\phi) \subseteq \mathcal{I}$. Let g be a nonzero generator of $(\mathcal{I} + \mathcal{J})/\mathcal{J}$. Then g is a binomial and $g \in \mathcal{I} \setminus \mathcal{J}$. Since $\mathcal{I}$ is generated by binomials corresponding to the binomial relations on $\{f_1, \ldots, f_t\}$, g corresponds to a binomial relation on $\{f_i \mid \mathrm{T_i} \notin \mathrm{V}\}$. Therefore $g \in \ker(\phi)$, and thus $(\mathcal{I} + \mathcal{J})/\mathcal{J} = \ker(\phi)$.

$(3) \Rightarrow (1)$: This is clear since $k\left[f_i \mid \mathrm{T_i} \notin \mathrm{V}\right]$ is an integral domain. □

For a quasi-equigenerated monomial ideal, it has been shown in [6, Theorem 1.1] that

$$\mu(I^2) \geq l(I)\mu(I) - \binom{l(I)}{2}.$$

We will refer to this fact as the *Freiman inequality*. When the equality holds in the Freiman inequality, I is called a *Freiman ideal*. Quasi-equigenerated monomial ideals of linear type are always Freiman ideals for which $l(I) = \mu(I)$.

Freiman ideals are characterized by the following properties. A very interesting fact obtained as a consequence, is that, if an ideal is Freiman, the number of generators of any power can be exactly computed in term of $\mu(I)$ and $l(I)$.

Theorem 2.5 ([6, Theorem 1.3]). *Let I be a quasi-equigenerated monomial ideal. The following assertions are equivalent:*

1. *I is a Freiman ideal.*
2. $\mu(I^j) = \binom{l(I)+j-2}{j-1}\mu(I) - (j-1)\binom{l(I)+j-2}{j}$ *for every $j \geq 1$.*
3. $\mu(I^j) = \binom{l(I)+j-2}{j-1}\mu(I) - (j-1)\binom{l(I)+j-2}{j}$ *for some $j \geq 2$.*
4. *The Hilbert polynomial of F(I) is linear (that is, $h_i = 0$ for all $i \geq 2$).*
5. *F(I) has minimal multiplicity.*
6. *F(I) is Cohen-Macaulay and its defining ideal $\mathcal{I}$ has a 2-linear free resolution.*

2.2. Cover ideals of graphs

In the following we always consider simple finite graphs without multi-edges or loops.

Definition 2.6. Let G be a graph with vertex set $V = \{x_1, ..., x_n\}$ and edge set E. We say that a monomial $g \in R$ is a *(vertex) cover* of G if for each edge $\{x_i, x_j\} \in E$, either $x_i|g$ or $x_j|g$ (or both conditions happen). We say that a vertex cover is minimal if it is not divisible by any other vertex cover.

The generators of the cover ideal $J(G)$ are the minimal vertex covers of G. In other words

$$J(G) := \bigcap_{\{x_i, x_j\} \in E} (x_i, x_j) \subseteq R.$$

Definition 2.7. Given a graph G with $V = \{x_1, ..., x_n\}$, an *independent set* of G is a subset $U \subseteq V$ such that for every $x_i, x_j \in U, \{x_i, x_j\} \notin E$. We denote by $c(G)$ the *independence number* of G, that is the maximal cardinality of an independent set of G. An independent set of G is said *maximal* if it is not contained in any other independent set.

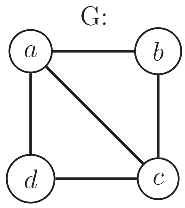
Remark 2.8. A set $U \subseteq V$ is an independent set of G if and only if the monomial $h_U = \prod_{x_i \notin U} x_i$ is a cover of G and is maximal if and only if h_U is a minimal cover.

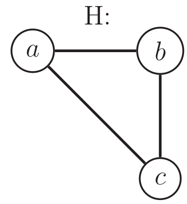
We recall the following well-known notation. Given a graph $G = (V, E)$ and a vertex x, the set of *neighbors* of x is the set $\mathcal{N}(x)$ containing all the vertices x_j adjacent to x (i.e. $\{x, x_j\}$ is an edge of G). We recall that the *degree* of a vertex is the number of adjacent vertices and a vertex of degree 1 is called a *leaf*. A graph on n vertices such that each vertex has degree n – 1 is called *complete*. Given a set of vertices $U \subseteq V$ in G, the *induced subgraph* on U is the graph with vertex set U, and edge set $\{\{x, y\} \in E | x, y \in U\}$.

Definition 2.9. We say that two vertices x and y of a graph G are *equivalent* if $\mathcal{N}(x) = \mathcal{N}(y)$.

Given two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ we say $G_1 \leq^\star G_2$ if $V_1 \subseteq V_2$, G_1 is the induced subgraph of G_2 on V_1, and every vertex $x \in V_2 \setminus V_1$ is equivalent to some vertex in V_1 as vertices of G_2. The relation $\leq^\star$ defines a partial order and we say that a minimal element with respect to it is a *reduced* graph. A reduced graph has no equivalent vertices.

Example 2.10. Vertice b and d are equivalent in graph G below. The graph $H \leq^\star G$ is reduced.





Now, we show that in order to study cover ideals we can reduce to consider reduced graphs.

Lemma 2.11. *Let G be a graph, I its cover ideal, and let x, y be two equivalent vertices of G. For any minimal cover f of G, x divides f if and only if y divides f.*

Proof. Let f be a minimal cover of G and assume that x divides f. Hence, by minimality, there exists $z \in \mathcal{N}(x) = \mathcal{N}(y)$ such that z does not divide f. If follows that y divides f. Clearly the converse follows by the same argument. □

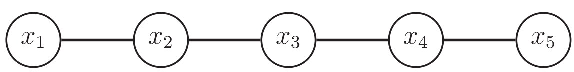
Theorem 2.12. *Let $G_1 \leq^\star G_2$ be two graphs and let I_1 and I_2 be their cover ideals. Then their fiber cones $F(I_1)$ and $F(I_2)$ are isomorphic.*

Proof. Let $V_1 = \{x_1, ..., x_n\}$ be the set of vertices of G_1. We may assume the set of vertices of G_2 to be $V_2 = \{x_1, ..., x_n, y\}$ (if there are more vertices in V_2 it is possible to iterate the same argument of this proof). Since $G_1 \leq^\star G_2$, there exists $x_i \in V_1$ equivalent to y. Say that $I_1 = (f_1, ..., f_t)$. Hence by Lemma 2.11, $I_2 = (g_1, ..., g_t)$ where $g_j = f_j$ if x_i does not divide f_j and $g_j = yf_j$ otherwise. Now observe that $f_{j_1} \cdots f_{j_l} = f_{h_1} \cdots f_{h_l}$ if and only if $g_{j_1} \cdots g_{j_l} = g_{h_1} \cdots g_{h_l}$. It clearly follows that $F(I_1)$ and $F(I_2)$ are isomorphic. □

3. Quasi-equigenerated cover ideals

In this section, we approach the question of understanding which cover ideals of graphs are quasi-equigenerated.

Easy computations allow to observe that the cover ideal of any graph with 3 or 4 vertices is quasi-equigenerated while the unique graph with 5 vertices and non-quasi-equigenerated cover ideal is the path P_5 (see below).



Indeed the cover ideal of P_5 is

$$J(P_5) = (x_1x_3x_5, x_1x_3x_4, x_2x_4, x_2x_3x_5)$$

and, assuming the existence of $\alpha \in \mathbb{N}^n_{\geq 1}$ such that, $d_\alpha(f_j) = d_\alpha(f_k)$, we get the relations $a_4 = a_5$, $a_4 = a_3 + a_5$ deriving the contradiction $a_3 = 0$.

Example 3.1. Let G be a graph whose cover ideal I is generated in only two different degrees $m_1 < m_2$. Assume that there exists x_i dividing all the minimal covers of degree m_1 but not dividing any of the minimal covers of degree m_2. It follows that I is quasi-equigenerated setting $a_i = m_2 - m_1 + 1$ and $a_k = 1$ for $k \neq i$.

In general it seems not easy to find an exact classification of all the graphs having quasi-equigenerated cover ideal. We provide here several criteria and the complete characterization for some families of graphs. First we observe that for this study we can only consider graphs reduced in the sense of Definition 2.9. For a graph $G = (V, E)$ and a vertex $x \in V$, we denote by $G \setminus \{x\}$ the induced subgraph on $V \setminus \{x\}$.

Lemma 3.2. *Let G be a n-vertex graph with equivalent vertices x_n and x_{n-1}. Then $J(G)$ is quasi-equigenerated if and only if $J(G \setminus \{x_n\})$ is quasi-equigenerated.*

Proof. Suppose that $J(G \setminus \{x_n\}) = (f_1, \ldots, f_s)$ is quasi-equigenerated. Then there exists some $\alpha = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}_{\geq 1}$ such that $d_\alpha(f_i) = d_\alpha(f_j)$ for all $1 \leq i, j \leq s$. Let $\beta \in \mathbb{N}^n_{\geq 1}$ be defined by $(2a_1, \ldots, 2a_{n-2}, a_{n-1}, a_{n-1})$. We know by Lemma 2.11 that each generator of $J(G)$ is of the form $x_n^{t_i} f_i$ for some $1 \leq i \leq s$ where t_i is the highest power of x_{n-1} dividing f_i ($t_i \in \{0, 1\}$). Thus $d_\beta(x_n^{t_i} f_i) = a_{n-1}t_i + 2d_\alpha(f_i) - a_{n-1}t_i = 2d_\alpha(f_i)$ and therefore $J(G)$ is quasi-equigenerated.

Suppose $J(G) = (g_1, \ldots, g_s)$ is quasi-equigenerated. We know by Lemma 2.11 that $J(G \setminus \{x_n\}) = \left(\frac{g_1}{x_n^{t_1}}, \ldots, \frac{g_s}{x_n^{t_s}}\right)$ where for each i, t_i is the highest power of x_n dividing g_i. Since $J(G)$ is quasi-equigenerated, there exists some $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n_{\geq 1}$ such that $d_\alpha(g_i) = d_\alpha(g_j)$ for all $1 \leq i, j \leq s$. Let $\beta = (a_1, \ldots, a_{n-1} + a_n)$. Then $d_\beta\left(\frac{g_i}{x_n^{t_i}}\right) = d_\alpha(g_i)$ for all i, and hence $J(G \setminus \{x_n\})$ is quasi-equigenerated. □

The cover ideals of graphs of independence number two are always quasi-equigenerated.

Proposition 3.3. *Let G be a graph such that $c(G) = 2$ and let $I = J(G)$ be its cover ideal. Then I is quasi-equigenerated.*

Proof. After relabeling the vertices, let $x_1, \ldots, x_c$ be the vertices of G of maximal degree and let $F = x_1 x_2 \cdots x_n$ be the product of all the variables. The minimal covers of G are of the form Fx_i^{-1} for $i \leq c$ and $F(x_l x_j)^{-1}$ where $\{x_l, x_j\}$ is an independent set of G of cardinality two and $j, l > c$. In the case there are no vertices of G having maximal degree, then each vertex is contained in an independent set of cardinality two, and therefore I is equigenerated. Otherwise, set $a_i = 2$ for $i \leq c$ and $a_i = 1$ for $i > c$. Hence, setting $\alpha = (a_1, a_2, \ldots, a_n)$, we get for $i \leq c$,

$$d_\alpha(Fx_i^{-1}) = 2(c-1) + (n-c)$$

and for $j, l > c$,

$$d_\alpha\left(F(x_l x_j)^{-1}\right) = 2c + (n - c - 2).$$

This implies I quasi-equigenerated. □

Next result characterizes graphs with equigenerated cover ideals in term of independent sets.

Proposition 3.4. *Let $G = (V, E)$ be a graph of n vertices and let I be its cover ideal. The following conditions are equivalent:*

1. *I is equigenerated;*
2. *All the maximal independent sets of G have the same cardinality.*

Proof. It is a straightforward consequence of the definitions and of Remark 2.8. □

Next lemma describes a useful way to detect non-quasi-equigenerated cover ideals considering the cover ideals of particular induced subgraphs.

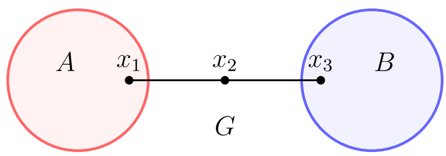
Definition 3.5. Let $G = (V, E)$ be a graph of n vertices and let $x_i \in V$. We call G_i the induced subgraph on the set $V \setminus (\mathcal{N}(x_i) \cup \{x_i\})$.

Lemma 3.6. *Let $G = (V, E)$ be a graph of n vertices and let $I = (f_1, \dots, f_t)$ be its cover ideal. Then if I is quasi-equigenerated (resp. equigenerated), the cover ideal $J(G_i)$ is quasi-equigenerated (resp. equigenerated) for every i.*

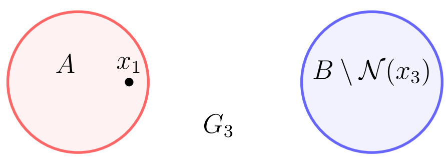
Proof. Let h_j be a minimal cover h_j of G_i and let h be the product of the neighbors of x_i in G. Hence $f_j = hh_j$ is a minimal cover of G and the minimal covers of G not divisible by x_i are all of this form. Let $V \setminus (\mathcal{N}(x_i) \cup \{x_i\}) = \{x_{i_1}, \dots, x_{i_c}\}$. It follows that, if I is quasi-equigenerated with $\alpha = (a_1, \dots, a_n)$, $J(G_i)$ is quasi-equigenerated with $(a_{i_1}, \dots, a_{i_c})$. □

The converse of this result is not true, since one can see that the cover ideal of the 6-cycle $J(C_6)$ is not quasi equigenerated, but $J((C_6)_i)$ is quasi-equigenerated for every of its vertices x_i. In Theorem 3.12, we explicitly characterize which circulant graphs have quasi-equigenerated cover ideals.

Remark 3.7. Lemma 3.6 gives rise to a criterion for non-quasi-equigeneratedness. Let G be a graph containing an induced subgraph A such that $J(A)$ is non-quasi-equigenerated.



If there exist vertices x_1, x_2, x_3 with $x_1 \in V(A)$, such that the induced subgraph on $\{x_1, x_2, x_3\}$ is P_3, then the graph G_3 (obtained following Definition 3.5) will have A as a connected component with non-quasi-equigenerated cover ideal.

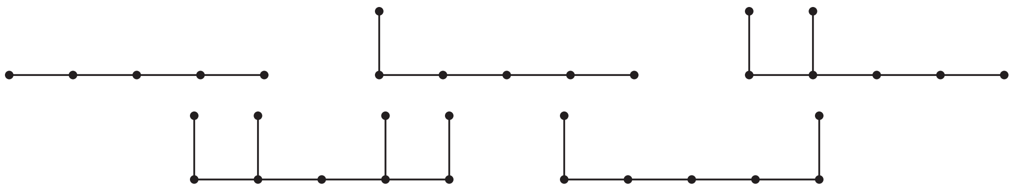


Thus by Lemma 3.6, $J(G)$ is not quasi-equigenerated.

We describe which trees have quasi-equigenerated cover ideal. We recall that a *tree* is a graph not containing any induced cyclic subgraph.

Theorem 3.8. *Let T be a tree. $J(T)$ is quasi-equigenerated if and only if every vertex of degree at least 2 is adjacent to a leaf.*

Proof. Suppose that T contains a vertex v of degree at least 2 which is not adjacent to any leaves. Then there exist vertices a, b, x, y in T such that (a, b), (b, v), (v, x), and (x, y) are edges in T. Let H be the set of all leaves adjacent to $\{a, b, x, y\}$, and consider the induced subgraph S of T on $\{a, b, v, x, y\} \cup H$. Let $x_1, \dots, x_s$ be the vertices of T having exactly distance two from at least one vertex among $\{a, b, v, x, y\}$. Following the notation of Definition 3.5, set $G^1 := T_1$ and for $j = 2, \dots s$, set $G^j := G_j^{j-1}$ (clearly $x_1, \dots, x_s$ are pairwise not adjacent since T is a tree and hence x_j is a vertex of G^{j-1}). The last graph obtained with this process is $G^s = S$. By iterated applications of Remark 3.7 to the graphs G^j we get that, if $J(S)$ is not quasi-equigenerated then also $J(T)$ is not quasi-equigenerated. By Lemma 3.2 we may assume that S has no two equivalent vertices. The assumption of v not adjacent to any leaf implies that S is one of the five following graphs.



By inspection, with the help of Lemma 3.6, we get that the cover ideals of these five graphs are not quasi-equigenerated and hence also $J(T)$ is not quasi-equigenerated.

Suppose every degree 2 vertex of T is adjacent to a leaf. By Lemma 3.2, we may assume T to be reduced in the sense of Definition 2.9. Thus every vertex of T is either a leaf or adjacent to exactly one leaf. Therefore T is a whiskered graph. In [14] it is proved that cover ideals of whiskered graphs are equigenerated (we prove again this fact here in Proposition 5.7). □

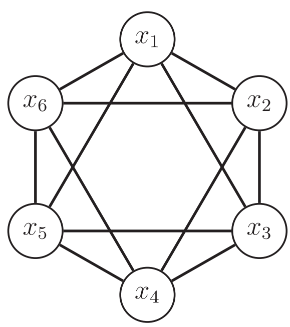
3.1. Quasi-equigenerated circulant graphs

It is easy to observe that complete graphs have equigenerated cover ideal and that the cyclic graph C_n has equigenerated cover ideal only for $n = 3, 4, 5, 7$. A natural generalization of complete graphs and cyclic graphs is given by the class of circulant graphs. Results about ideals related to circulant graphs are given for instance in [1, 11, 12]. Here we are interested in characterizing which circulant graph has quasi-equigenerated cover ideal.

Definition 3.9. Let n be a positive integer and let $1 \leq s \leq \lfloor n/2 \rfloor$. Denote by $\mathbb{Z}_n$ the cyclic group with n elements let $S = \{1, 2, ..., s\} \subseteq \mathbb{Z}_n$.

The circulant graph $C_n(1, 2, ..., s)$ is defined as the graph with vertex set $\{x_1, ..., x_n\}$ and with edge set formed by the edges $\{x_i, x_{i+j}\}$ such that $j \in \{\pm 1, \pm 2, ..., \pm s\}$ with the sums taken modulo n. Note that the cycle C_n is equal to $C_n(1)$ and the complete graph K_n is $C_n(1, 2, ..., \lfloor n/2 \rfloor)$.

The graph in the next picture is the circulant graph $C_6(1, 2)$.



Our strategy is to apply Lemma 3.6 and consider the induced subgraphs of circulant graphs of the form G_i. For this purpose, we need to study the cover ideals of the family of graphs we now introduce.

Definition 3.10. Given two positive integers $n, s \geq 1$, we define the graph $P_{(n,s)} = (V, E)$ where $V = \{x_1, ..., x_n\}$ and

$$E := \left\{ (x_i, x_j) \mid |i - j| \leq s \right\}.$$

Notice that for $s = 1$, $P_{(n,1)} = P_n$ is the path on n vertices.

Proposition 3.11. *The cover ideal of the graph $P_{(n,s)}$ is equigenerated if and only if either $n \leq s+1$ or $n = 2s+2$.*

Proof. When $n \leq s+1$, the graph is complete and hence its cover ideal is equigenerated of degree $n-1$. If $n = 2s+2$ we argue in the following way: observing that $\mathcal{N}(x_{s+1}) = V \setminus \{x_{2s+2}\}$, there exists a unique minimal cover of $P_{(2s+2,s)}$ not divisible by x_{s+1} an it has degree $2s$. For the same reason, since $\mathcal{N}(x_{s+2}) = V \setminus \{x_1\}$, also the unique minimal cover not divisible by x_{s+2} has degree $2s$. All the remaining minimal covers are of the form $x_{s+1}x_{s+2}h_1h_2$ where h_1 is a minimal cover of the induced subgraph on $\{x_1, ..., x_s\}$ and h_2 is a minimal cover of the induced subgraph on $\{x_{s+3}, ..., x_{2s+2}\}$. But both these induced subgraphs are complete graphs with s vertices and hence

$$\deg(x_{s+1}x_{s+2}h_1h_2) = 2 + \deg(h_1) + \deg\ (h_2) = 2 + 2(s-1) = 2s.$$

To show that all the other graphs of the family are not equigenerated first consider the case where $s+1 \leq n \leq 2s+1$. If n is even we have for $1 \leq j \leq n$,

$$\left|\frac{n}{2} - j\right| \leq \left|\frac{2s+1}{2} - j\right| = \left|s + \frac{1}{2} - j\right| \leq s$$

and thus $(x_{\frac{n}{2}}, x_j) \in E$ for every j. Using the symmetry of the graph, we get also $(x_{\frac{n}{2}+1}, x_j) \in E$ for every j. Now, the unique minimal cover not divisible by $x_{\frac{n}{2}}$ has degree $n-1$ but, since the induced subgraphs on $\{x_1, ..., x_{\frac{n}{2}-1}\}$ and on $\{x_{\frac{n}{2}+2}, ..., x_n\}$ are complete (and have the same number of vertices), any minimal cover divisible by $x_{\frac{n}{2}}x_{\frac{n}{2}+1}$ has degree $2 + 2\left(\frac{n}{2} - 2\right) = n-2$ and therefore the cover ideal is not equigenerated. Similarly, if n is odd we get $(x_{\frac{n+1}{2}}, x_j) \in E$ for every j and we show that the cover ideal is not equigenerated in an analogous way.

Finally, we observe that if $G = P_{(n,s)}$, then, for $0 \leq j \leq s$,

$$G_{n-j} = P_{(n-s-1-j,s)}.$$

It follows that by Lemma 3.6, since $J\left(P_{(s+2,s)}\right)$ is not equigenerated, then $J\left(P_{(s+2+(s+1+j),s)}\right)$ is not equigenerated and therefore $J\left(P_{(2s+3,s)}\right), J\left(P_{(2s+4,s)}\right), ..., J\left(P_{(3s+2,s)}\right)$ are not equigenerated. Since $2s+3+(s+1+j) \geq 3s+4$, we can iterate this last process and conclude that for $n \geq 2s+3, J\left(P_{(n,s)}\right)$ is not equigenerated. □

By convention we assume the cover ideals of a graph without edges to be equigenerated.

Theorem 3.12. *Let I be the cover ideal of a circulant graph $G = C_n(1, 2, ..., s)$. The following conditions are equivalent:*

1. *I is equigenerated;*
2. *I is quasi-equigenerated;*
3. *$J(G_1)$ is equigenerated;*
4. *Either $s \geq \frac{n-1}{3}$ or $s = \frac{n-3}{4}$.*

Proof. $(1) \Rightarrow (2)$ is trivial.
$(1) \Rightarrow (3)$ follows by Lemma 3.6.
$(2) \Rightarrow (1)$ By Lemma 3.6, since I is quasi-equigenerated with $\alpha = (a_1, ..., a_n), J(G_1)$ is quasi-equigenerated with $(a_{i_1}, ..., a_{i_c})$ correspondent to the vertices $\{x_{i_1}, ..., x_{i_c}\}$ of G_1. But, by the symmetry of the circulant graphs, after relabeling the vertices, $G_1 = G_i$ for every i. Thus, we can set the same values a_{i_l} on the vertices of G_2 preserving the order given to those of G_1. Now observe that the vertices of G_2 are

$$\{x_{i_1+1}, ..., x_{i_c+1}\}$$

where the sums $i_l + k$ are taken modulo n. Hence $a_{i_l} = a_{i_l+1}$ for every $1 \le l \le c$. Using inductively this argument, we find that $a_i = a_l$ for every i, l and therefore I is equigenerated.
(3) $\Rightarrow$ (1) Any minimal cover of G not divisible by the variable x_1 is of the form hh_j where h is the product of the neighbors of x_1 and h_j is a minimal cover of G_1. Since $J(G_1)$ is equigenerated, we get these last covers are all equigenerated. Since all the vertices x_i, have the same number of neighbors and $G_1 = G_i$ for every i, we get that I is equigenerated.
(3) $\Longleftrightarrow$ (4) Observe that $G_1 = P_{(n-2s-1,s)}$ and conclude applying Proposition 3.11. □

3.2. Quasi-equigenerated join of graphs

Our next aim is to characterize how quasi-equigeneratedness of the cover ideal behaves with respect to graph operations. We consider the operation of adding edges between vertices of two graphs.

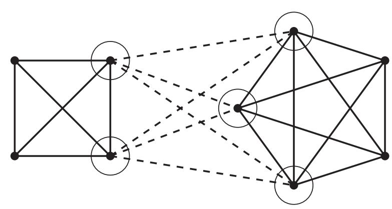
Definition 3.13. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs where $V_1 = \{x_1, ..., x_n\}$ and $V_2 = \{y_1, ..., y_m\}$ are two disjoint sets of vertices. Given two non-empty subsets $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$, we define the graph

$$G_1 \oplus_{U_1, U_2} G_2 := (V_1 \cup V_2, E_1 \cup E_2 \cup D)$$

where

$$D := \{(x_i, y_j) \mid x_i \in U_1, y_j \in U_2\}.$$

When $U_1 = V_1$ and $U_2 = V_2$ we simply denote $G_1 \oplus G_2 := G_1 \oplus_{V_1, V_2} G_2$. This last graph is sometimes called the *join* of G_1 and G_2.



Definition 3.14. We say that $G_1 \oplus_{U_1, U_2} G_2$ is *linked with covers* if $p_1 = \prod_{x_i \in U_1} x_i$ and $p_2 = \prod_{y_j \in U_2} y_j$ are respectively covers of G_1 and G_2 (not necessarily minimal).

Proposition 3.15. *Let $G_1 \oplus_{U_1, U_2} G_2$ be linked with covers and let $I_1 = (f_1, ..., f_s) \subseteq k[x_1, ..., x_n]$ and $I_2 = (g_1, ..., g_t) \subseteq k[y_1, ..., y_m]$ be respectively the cover ideals of G_1 and G_2. The cover ideal I of G is contained in $k[x_1, ..., x_n, y_1, ..., y_m]$ and it is generated by:*

$$I = (f_1p_2, ..., f_sp_2, g_1p_1, ..., g_tp_1).$$

The monomial p_1p_2 is a minimal cover if and only if p_1 and p_2 are both minimal covers of G_1 and G_2. The others are all minimal.

Proof. It is clear by the above definitions that $p_1p_2 \in I$. If p_1 and p_2 are both minimal covers of the respective graphs, p_1p_2 is a minimal cover (hence $p_1p_2 = f_ip_2 = p_1g_j$ for some i, j). Assume instead that one of them, say p_1, is not minimal and it is divisible by a cover h of G_1. It follows that hp_2 is a cover of G dividing p_1p_2. For $f_i \neq p_1$, the monomial f_ip_2 is clearly a minimal cover since there must exist x_l dividing p_1 but not f_i, and therefore we cannot remove any variable y_j

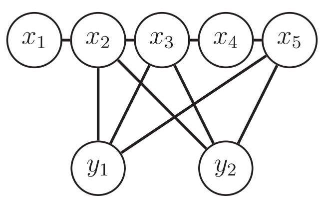
from p_2, since otherwise we would uncover the edge (x_l, y_j). Similarly we can see that all the monomials of the form g_jp_1 are minimal covers if $g_j \neq p_2$. It is easy to observe that any cover not of this form is not minimal. □

Theorem 3.16. *Let $G_1 \oplus_{U_1,U_2} G_2$ be linked with covers and take p_1, p_2 as in Definition 3.14. Call I the cover ideal of G and I_i the cover ideal of G_i for $i = 1, 2$. Assume p_1, p_2 are either both minimal covers of the respective graphs or both non-minimal. The following conditions are equivalent:*

1. *I is quasi-equigenerated;*
2. *Both I_1 and I_2 are quasi-equigenerated*

Proof. (1) $\Rightarrow$ (2) We argue by way of contradiction and assume $I_1 = (f_1, \dots, f_s)$ not quasi-equigenerated. Assuming p_1, p_2 both minimal or both non-minimal, we get by Proposition 3.15 that for every i, f_ip_2 is a minimal cover of G. Since I_1 is not quasi-equigenerated the linear system defined by the equations $d_\alpha(f_i) = d_\alpha(f_j)$ has no solutions among the nonzero positive integers. It follows that also the linear system defined by equations $d_\alpha(f_ip_2) = d_\alpha(f_jp_2)$ has no solutions, and hence I is not quasi-equigenerated.
(2) $\Rightarrow$ (1) Let $I_1 = (f_1, \dots, f_s)$ be quasi-equigenerated with $\alpha = (a_1, \dots, a_n)$ and $I_2 = (g_1, \dots, g_t)$ with $\beta = (b_1, \dots, b_m)$. If p_1, p_2 are both minimal covers, by Proposition 3.15, it is easy to observe that I is quasi-equigenerated with $(a_1, \dots, a_n, b_1, \dots, b_m)$, since $d_\alpha(f_i) = d_\alpha(p_1)$ and $d_\beta(g_j) = d_\beta(p_2)$ for every i, j. Instead, if p_1, p_2 are both non-minimal, the numbers $A = d_\alpha(p_1) - d_\alpha(f_i)$ and $B = d_\beta(p_2) - d_\beta(g_j)$ are both greater than zero. Hence, the ideal I is quasi-equigenerated with $(ca_1, ca_2, \dots, ca_n, db_1, db_2, \dots, db_m)$ where $c = \frac{B}{\gcd(A,B)}$ and $d = \frac{A}{\gcd(A,B)}$. □

Remark 3.17. Observe that the argument used to prove Theorem 3.16 actually shows that if $J(G_1)$ and $J(G_2)$ are not quasi-equigenerated then $J(G)$ cannot be quasi-equigenerated. Anyway, if one of them has quasi-equigenerated cover ideal, the assumptions on minimality of p_1 and p_2 are needed, since it is possible to produce a counterexample in the case one is minimal as a cover and the other is not. The counterexample is the graph $G = P_5 \oplus_{U_1,V_2} P_2$ where $U_1 = \{x_2, x_3, x_5\} \subseteq \{x_1, x_2, x_3, x_4, x_5\}$ and $V_2 = \{y_1, y_2\}$.



Indeed $J(G)$ is quasi-equigenerated setting $d_\alpha(x_2) = 2$ and $d_\alpha(x_i) = d_\alpha(y_j) = 1$ for all j and $i \neq 2$.

4. Freiman cover ideals

In this section, we study Freiman property for quasi-equigenerated cover ideals by giving a general structure theorem and characterizing Freiman cover ideals among some classes of graphs. Our settings and notations, where not differently specified, will be the following: I will be the cover ideal (quasi-equigenerated) of a graph G on n vertices or more generally, a quasi-equigenerated squarefree monomial ideal. We express the fiber cone of I as the ring

$$F(I) = \frac{k[T_1, ..., T_t]}{\mathcal{I}},$$

where the variables T_i correspond to minimal generators of I via the usual ring homomorphism. We consider the minimal covers of G using the equivalent notation induced by the maximal independent sets described in Remark 2.8. For U a maximal independent of G, we call T_U the correspondent variable in $k[T_1, ..., T_t]$. We set $t = \mu(I)$ as the number of minimal generators of I and $l(I)$ as its analytic spread. Often, we will use the notation T_i, T_U also for their corresponding images in $F(I)$ instead of writing $\overline{T_i}, \overline{T_U}$.

Set $a := t - l(I)$ and call b the number of minimal generators of $\mathcal{I}$ having degree 2 with respect to the grading of the polynomial ring $k[T_1, ..., T_t]$. The next lemma translates the Freiman condition in term of the invariants a and b.

Lemma 4.1. *Let I be a squarefree quasi-equigenerated monomial ideal and let $\mathcal{I}$ be the defining ideal of the fiber cone $F(I)$. For a, b defined as above, $b \le \binom{a+1}{2}$ and I is Freiman if and only if $b = \binom{a+1}{2}$.*

Proof. Since $\mathcal{I}$ is generated by binomials, $\mu(I^2) = \binom{t+1}{2} - b$. Thus we have

$$\binom{t+1}{2} - b \ge (t-a)t - \binom{t-a}{2}$$

and the equality holds if and only if I is Freiman. A straightforward computation leads to our thesis. □

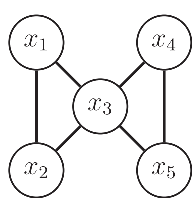
Remark 4.2. In Theorem 2.5, it is stated that the fiber cone of a Freiman ideal has a 2-linear free resolution. An immediate consequence of this fact is that, when I is Freiman, the ideal $\mathcal{I}$ is generated in degree 2.

Example 4.3. We observe the following facts.

1. The cover ideals of complete graphs are Freiman of linear type. One can check this fact easily, but we shall prove a more general result in Theorem 4.10.
2. The graph in the following picture has cover ideal generated by $f_1 = x_1x_2x_4x_5, f_2 = x_1x_3x_4, f_3 = x_1x_3x_5, f_4 = x_2x_3x_4, f_5 = x_2x_3x_5$. Its fiber cone is isomorphic to the ring

   $$\frac{k[T_1, T_2, T_3, T_4, T_5]}{(T_2T_5 - T_4T_3)}$$

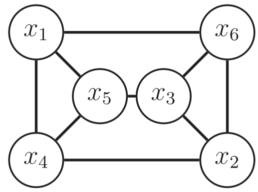
   and hence the cover ideal is Freiman by Lemma 4.1.



3. Call H_3 the graph in next picture, that is the graph on 6 vertices $x_1, ..., x_6$ whose independent sets are $\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_1, x_3\}, \{x_2, x_5\}, \{x_4, x_6\}$. The fiber cone of this graph is isomorphic to the ring

$$\frac{k[T_1, T_2, T_3, T_4, T_5, T_6]}{(T_1T_2T_3 - T_4T_5T_6)}$$

and hence the cover ideal is not Freiman by Remark 4.2.



4. Starting with H_3, one may inductively construct a family of graphs whose fiber cones have principal defining ideal generated by a binomial of degree n for each $n \geq 3$ (clearly the cover ideals of the graphs of this family are not Freiman). Indeed, for $n \geq 3$, define H_{n+1} as the graph on the vertices $x_1, \ldots, x_{2n+2}$, having the same independent sets as H_n except $\{x_{2n-2}, x_{2n}\}$ and having also the independent sets $\{x_{2n+1}, x_{2n+2}\}, \{x_{2n-2}, x_{2n+1}\}, \{x_{2n}, x_{2n+2}\}$.

As application of Lemma 4.1, we can get a complete characterization of when the join of two graphs is Freiman.

Theorem 4.4. *Let G_1 and G_2 be two simple graphs and let $G := G_1 \oplus G_2$. Let I_1, I_2, I be respectively the cover ideals of G_1, G_2, G. Suppose I_1 and I_2 to be quasi-equigenerated. The following conditions are equivalent:*

1. *I is a Freiman ideal;*
2. *I_1 and I_2 are both Freiman ideals and at least one of them is of linear type.*

Proof. Since I_1 and I_2 are quasi-equigenerated, so it is I by Theorem 3.16. Assume as in Definition 3.13, $I_1 = (f_1, \ldots, f_t) \subseteq k[x_1, \ldots, x_n]$ and $I_2 = (g_1, \ldots, g_s) \subseteq k[y_1, \ldots, y_m]$. Write their fiber cones as

$$F(I_1) \cong \frac{k[T_1, \ldots, T_t]}{\mathcal{I}_1}$$

and

$$F(I_2) \cong \frac{k[U_1, \ldots, U_s]}{\mathcal{I}_2}.$$

We claim that

$$F(I) \cong \frac{k[T_1, \ldots, T_t, U_1, \ldots, U_s]}{\mathcal{I}_1 + \mathcal{I}_2}.$$

Set $F_1 = x_1x_2\cdots x_n, F_2 = y_1y_2\cdots y_m$ and observe that by Proposition 3.15, the generators of I are of the form f_iF_2 and g_jF_1. Hence, using the usual homomorphism defined by $T_i \to f_iF_2$ and $U_j \to g_jF_1$, we identify $F(I)$ with the quotient of $k[T_1, \ldots, T_t, U_1, \ldots, U_s]$ by a defining ideal $\mathcal{I}$. Moreover, notice that if $q \in \mathcal{I}_1$, then $F_2q \in \mathcal{I}$ and analogously if $q \in \mathcal{I}_2$, then $F_1q \in \mathcal{I}$. We only need to show that these are all the generators of $\mathcal{I}$. To do this, let

$$p = T_{i_1}\cdots T_{i_{r_1}}U_{j_1}\cdots U_{j_{r_2}} - T_{k_1}\cdots T_{k_{r_3}}U_{l_1}\cdots U_{l_{r_4}} \in \mathcal{I}$$

(observe that by Proposition 2.3, $r_1 + r_2 = r_3 + r_4$), hence we must have

$$f_{i_1}\cdots f_{i_{r_1}}F_2^{r_1}g_{j_1}\cdots g_{j_{r_2}}F_1^{r_2} = f_{k_1}\cdots f_{k_{r_3}}F_2^{r_3}g_{l_1}\cdots g_{l_{r_4}}F_1^{r_4}.$$

Separating the variables, this implies $f_{i_1}\cdots f_{i_{r_1}}F_1^{r_2} = f_{k_1}\cdots f_{k_{r_3}}F_1^{r_4}$. Assuming by way of contradiction $r_2 < r_4$ (or analogously $r_4 < r_2$), we get $f_{i_1}\cdots f_{i_{r_1}} = f_{k_1}\cdots f_{k_{r_3}}F_1^{r_4-r_2}$ and, since I_1 is quasi-equigenerated, we can find $\alpha \in \mathbb{N}^n_{\geq 1}$ such that

$$r_1 d_\alpha(f_{i_1}) = r_3 d_\alpha(f_{i_1}) + (r_4 - r_2)r_1 d_\alpha(F_1).$$

But this is a contradiction since $r_4 - r_2 = r_1 - r_3$ and since f_{i_1} properly divides F_1. Therefore, we must have $r_2 = r_4$, $r_1 = r_3$, and hence p is in the ideal generated by other generators of $\mathcal{I}$ of the form F_2q and F_1q for $q \in \mathcal{I}_1$ or $q \in \mathcal{I}_2$ and this proves our claim.

Now, it is easy to observe that the analytic spread $l(I) = l(I_1) + l(I_2)$. Following the notation of Lemma 4.1, denote by b_1, b_2, b the number of minimal generators of degree 2, respectively of $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}$. Also write $a_1 := n - l(I_1), a_2 := m - l(I_2), a := n + m - l(I)$. Hence, clearly $b = b_1 + b_2$ and $a = a_1 + a_2$. Thus

$$b \leq \binom{a_1+1}{2} + \binom{a_2+1}{2} \leq \binom{a+1}{2}.$$

The first inequality is an equality if and only if both I_1 and I_2 are Freiman ideals, while the second one is an equality if and only if one among a_1 and a_2 is zero, meaning that $F(I_1)$ or $F(I_2)$ is a polynomial ring. By Lemma 4.1, I is a Freiman ideal if and only if both these conditions hold. □

In [6, Theorem 1.3], it is shown that a quasi-equigenerated ideal I is Freiman if and only if $F(I)$ has minimal multiplicity. We want to use this fact, together with the classification of homogeneous domain (among quotients of polynomial rings) of minimal multiplicity given in [2, Section 4], in order to describe Freiman squarefree quasi-equigenerated monomial ideals as ideals generated by minors of certain matrices.

Lemma 4.5. *Let $\mathbf{T} = \{T_1, ..., T_s\}$ for some $s > 4$ be a set of indeterminants, and let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \subset \mathbf{T}$ such that $T_1 \notin \mathbf{A}, T_2 \notin \mathbf{B}, T_3 \notin \mathbf{C}$, and $T_4 \notin \mathbf{D}$. Let $\ell_1 \in k[\mathbf{A}], \ell_2 \in k[\mathbf{B}], \ell_3 \in k[\mathbf{C}]$, and $\ell_4 \in k[\mathbf{D}]$ be linear forms. Suppose that the matrix*

$$M = \begin{pmatrix} aT_1 + \ell_1 & bT_2 + \ell_2 \\ cT_3 + \ell_3 & dT_4 + \ell_4 \end{pmatrix}$$

satisfies $\det M = \alpha(T_1T_4 - T_2T_3)$ for some $\alpha \in k$. Then $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 0$.

Proof. Since $\ell_1 \in k[\mathbf{A}], \ell_1 = \sum_{A\in\mathbf{A}} a_A A$ where each $a_A \in k$. Similarly $\ell_2 = \sum_{B\in\mathbf{B}} b_B B, \ell_3 = \sum_{C\in\mathbf{C}} c_C C$, and $\ell_4 = \sum_{D\in\mathbf{D}} d_D D$ where each $b_B, c_C, d_D \in k$.

We may assume that $\mathbf{A} \cap \mathbf{D} = \emptyset = \mathbf{B} \cap \mathbf{C}$. Indeed, suppose that $\mathbf{A} \cap \mathbf{D}$ is nonempty. Then there is some term $A \in \mathbf{A}$ also contained in $\mathbf{D}$. Since the terms of det$M$ are squarefree, we conclude that $A \in \mathbf{B} \cap \mathbf{C}$ as well. Thus, by applying a row operation to M, we can obtain an new matrix,

$$M' = \begin{pmatrix} aT_1 + \ell_1 & bT_2 + \ell_2 \\ cT_3 + \ell'_3 & dT_4 + \ell'_4 \end{pmatrix}$$

with the same determinant as M but such that ℓ_1 and ℓ_4 share one fewer variables than in M. Repeating this process we can eliminate all elements of $\mathbf{A} \cap \mathbf{D}$. Working symmetrically, the same conclusion holds for $\mathbf{B} \cap \mathbf{C}$.

We also may assume that $a = d = 1$, since scaling rows of matrices scales determinants by constants.

Suppose $\ell_4 \neq 0$. Since ℓ_4 is nonzero then $(T_1 + \ell_1)(T_3 + \ell_4)$ contains terms T_1D_i for each $1 \leq i \leq v$. As these terms do not appear in the determinant, they must be canceled by terms of $(bT_2 + \ell_2)(cT_3 + \ell_3)$. Thus $T_1 \in \mathbf{B} \cup \mathbf{C}$, and so $\mathbf{D} \subseteq \mathbf{B}$. Since $T_1 \in \mathbf{C}, (bT_2 + \ell_2)(cT_3 + \ell_3)$ contains a term of the form T_1B for each $B \in \mathbf{B}$. Since these terms do not appear in the determinant, they must be canceled with terms from the main diagonal. Thus $\mathbf{D} \subseteq \mathbf{B}$ and so $\mathbf{B} = \mathbf{D}$. Since terms of the form $a_A T_4 A$ occur on the product of the main diagonal, but not in the determinant, they must cancel with terms of $(bT_2 + \ell_2)(cT_3 + \ell_3)$. Thus $T_4 \in \mathbf{B} \cup \mathbf{C}$. Since $T_4 \notin \mathbf{D} = \mathbf{B}, T_4 \in \mathbf{C}$. However, this means that $\alpha T_4 B$ is a term of the product of the antidiagonal which doesn't appear in the determinant and thus must be canceled with a term from $(T_1 + \ell_1)(T_3 + \ell_4)$. This cannot happen because $B \in \mathbf{D}$ and thus $B \notin \mathbf{A}$.

Thus $\ell_4 = 0$ and by symmetric arguments, $\ell_1 = \ell_2 = \ell_3 = 0$. □

A *generic* $n \times m$ matrix over a field k (or over a ring) is a matrix whose entries are distinct independent variables over k. We are now ready to prove the following theorem:

Theorem 4.6. *Let $I \subseteq k[x_1, ..., x_n]$ be a squarefree quasi-equigenerated monomial ideal. Then I is a Freiman ideal if and only if the defining ideal of its fiber cone $\mathcal{I}$ is either (0) or it is generated by the minors of a generic $2 \times m$ matrix over k.*

Proof. When $\mathcal{I} = (0)$ or when it is generated by the minors of a $2 \times m$ matrix, by the results in [2, Section 4], $F(I)$ has minimal multiplicity and hence I is Freiman.

Conversely assume $F(I)$ to have minimal multiplicity and $\mathcal{I} \neq (0)$. By [2, Theorem 4.3], and since $\mathcal{I}$ is generated by binomials, either $\mathcal{I}$ is the ideal generated by the minors of a $2 \times m$ matrix of linear forms or is the ideal generated by the 2×2 minors of a generic 3×3 symmetric matrix of linear forms (notice that the case in which $F(I)$ is an hypersurface generated by a quadratic binomial is included in the first case). But, since I is squarefree, for any choice of three monomials f, g, h among the minimal generators of I, $f^2 \neq gh$, and thus the case of minors of a symmetric 3×3 matrix has to be excluded.

Suppose that $I = I_2(M)$ for some $2 \times m$ matrix M, of linear forms. Since I is generated by squarefree binomials of degree 2, we may assume by Lemma 4.5 that the entries of M are monomials, and hence they are generic. □

Next result will be useful in the following to study the Freiman property for the cover ideals of some classes of graphs.

Proposition 4.7. *Let G be a graph and I its cover ideal. Consider a vertex x belonging to only one maximal independent set U of G. Then, the variable T_U, associated to the minimal cover h_U, is a prime element in the fiber cone $F(I)$.*

Proof. Consider the defining ideal of the fiber cone $\mathcal{I} = (q_1, ..., q_s)$. By Proposition 2.4, the variable T_U is a prime element in $F(I)$ if and only if it does not divide any monomial of the binomials $q_1, ..., q_s$. Let $q = \alpha - \beta$ be a minimal generator of $\mathcal{I}$ with α, β monomials, and assume by way of contradiction T_U divides α. By Proposition 2.3, $\mathcal{I}$ is an homogeneous ideal and it follows that there exist some minimal covers h_i, h_j of G and an integer $u \geq 1$ such that

$$h_U^u h_{i_1} ... h_{i_r} = h_{j_1} ... h_{j_{r+u}}.$$

But, since x is contained in only one independent set, then it divides every minimal cover of G different from h_U and this induces a contradiction since the degree of x in $h_U^u h_{i_1} ... h_{i_r}$ is now strictly less than the degree of x in $h_{j_1} ... h_{j_{r+u}}$. □

An easy application of last proposition shows that the elements of a family of graphs including complete graphs have cover ideal of linear type.

Definition 4.8. A graph of $n+1$ vertices is said to be *almost complete* if it has an induced complete subgraph of n vertices. An almost complete graph having vertices $x_1, x_2, ..., x_n, y$ is of the form

$$G = K_n \oplus_{U,\{y\}} \{y\}$$

where K_n is the complete graph on the vertices $x_1, x_2, ..., x_n$ and U is a subset of this same set of vertices.

Remark 4.9. A graph G is complete if and only if $c(G) = 1$ and it is almost complete if and only if $c(G) = 2$ and the intersection of all the independent sets containing two elements is non-empty.

Let G be an almost complete graph with $n \geq 4$ vertices and let I be its cover ideal. Let $d = n - |\mathcal{N}(y)|$, where $\mathcal{N}(y)$ is the set of neighbors of y.

1. If $d = 0$, $G = K_{n+1}$.
2. If $d = 1$, there exists a unique vertex x_i not adjacent to y, and $\mathcal{N}(x_i) = \mathcal{N}(y)$. Thus, by Theorem 2.12, I has fiber cone isomorphic to the fiber cone of $J(K_n)$.

Theorem 4.10. *Let $G = (V, E)$ be an almost complete graph with $n \geq 4$ vertices and let I be its cover ideal. Then I is a quasi-equigenerated Freiman ideal of linear type.*

Proof. Let $U \subseteq V$ be the set of neighbors of y. Set $F := x_1x_2 \cdots x_ny$. It is easy to check that the cover ideal I of an almost complete graph $G = (V, E)$ is generated by the monomials $h_j = Fx_j^{-1}$ for $x_j \in U$, correspondent to the maximal independent set $\{x_j\}$ and $g_l = Fx_l^{-1}y^{-1}$ for $x_l \in V \setminus U$, correspondent to the maximal independent set $\{x_l, y\}$. In particular $c(G) = 2$ and therefore I is quasi-equigenerated by Proposition 3.3.

By Proposition 4.7, all the minimal covers of G are prime elements in the fiber cone of I. It follows that I is of linear type. □

5. Families of graphs and Freiman cover ideals

In this section, we consider some different families of graphs having quasi-equigenerated cover ideal and we characterize when their cover ideal are Freiman. We also provide explicit computations of their relations and analytic spreads.

5.1. Pairs of complete graphs sharing a vertex

Definition 5.1. Let $m \geq n \geq 2$. We define the graph $A_{n,m} = K_n \oplus_{\{x_n\}, V(K_{m-1})} K_{m-1}$ as the graph having vertex set

$$V = \{x_1, x_2, ..., x_n, y_2, y_3, ..., y_m\}$$

and edge set given by the union

$$E = \bigcup_{1 \leq i < j}^{n} (x_i, x_j) \cup \bigcup_{2 \leq i < j}^{m} (y_i, y_j) \cup \bigcup_{j=2}^{m} (x_n, y_j).$$

Theorem 5.2. *Let $m \geq n \geq 2$ and let I be the cover ideal of the graph $A_{n,m}$. Then, I is quasi-equigenerated and:*

1. *The analytic spread of I is $n + m - 2$;*
2. *I is a Freiman ideal if and only if $n \leq 3$.*

Proof. Observe that $c(G) = 2$ and hence I is quasi-equigenerated by Proposition 3.3. If $n = 2$, the graph is almost complete and therefore I is Freiman of linear type by Theorem 4.10. Let $F = x_1x_2\cdots x_ny_2y_3\cdots y_m$. The minimal generators of I are Fx_n^{-1} and the others are of the form $Fx_i^{-1}y_j^{-1}$ for every possible choice of $1 \le i \le n-1$ and $2 \le j \le m$. Here we use a slightly different notation for the fiber cone (with respect to Section 4), associating T to Fx_n^{-1} and $T_{i,j}$ to $Fx_i^{-1}y_j^{-1}$, and writing $F(I)$ as the quotient of the polynomial ring $k[T, T_{1,2}, \dots, T_{1,m}, T_{2,2}, \dots, T_{n-1,m}]$ by a homogeneous binomial ideal $\mathcal{I}$.

Let $q = \alpha - \beta \in \mathcal{I}$ where α, β are monomials. Observe that by Proposition 4.7, the image of T is a prime element of $F(I)$ and T divides α if and only if divides also β. Therefore, assuming q part of a set of minimal generators, we get that α, β are of the form $T_{i_1,j_1}T_{i_2,j_2}\cdots T_{i_r,j_r}$. Moreover, notice the following important facts:

$(*)$ $T_{i,j}$ divides α if and only if there exists $k \neq j$ such that $T_{i,k}$ divides β.
$(**)$ For every $i_1 \neq i_2, j_1 \neq j_2$, the binomial $T_{i_1,j_1}T_{i_2,j_2} - T_{i_1,j_2}T_{i_2,j_1}$ is a minimal generator of $\mathcal{I}$.

If $\deg(q) = 2$ and $\alpha = T_{i,j}T_{i,k}$ for the same i, then β is clearly forced to be equal to α, making $q = 0$. Hence all the minimal generators of $\mathcal{I}$ of degree 2 are of the form described in $(**)$. The number of these generators is

$$b = \binom{n-1}{2}\binom{m-1}{2}.$$

Now, we compute the analytic spread $l(I)$ of I. With an abuse of notation we write also the elements of $F(I)$ in the form $T, T_{i,j}$ instead of writing their classes modulo $\mathcal{I}$. Define for $i = 1, \dots, n-2$ the ideals:

$$\mathcal{I}_i := (T_{i,2}, T_{i,3}, \dots, T_{i,m}) \subseteq F(I)$$

and

$$\mathcal{P}_i = \sum_{k=1}^{i} \mathcal{I}_i \subseteq F(I).$$

The fact $(*)$ together with Proposition 2.4, shows that each ideal $\mathcal{P}_i$ is prime. Moreover we want to prove by induction that the height of $\mathcal{P}_i$ is i. For $i = 1$, consider the localization $F(I)_{\mathcal{P}_1}$ and observe that T and all the elements $T_{i,j}$ with $i \neq 1$ are units in this ring. By the fact $(**)$, it is easy to see that, for every j, k, $T_{1,j}$ and $T_{1,k}$ are associated in $F(I)_{\mathcal{P}_1}$, and hence $F(I)_{\mathcal{P}_1} \cong k[T_{1,2}]_{(T_{1,2})}$ has dimension one. Assume now for some $i \le n-2$ that $\mathcal{P}_{i-1}$ has height i – 1 and show that $\mathcal{P}_i$ has height i. The ideal $J := \frac{\mathcal{P}_i}{\mathcal{P}_{i-1}}$ is prime in the ring $R := \frac{F(I)}{\mathcal{P}_{i-1}}$ and the binomials of the form $T_{i,j}T_{l,k} - T_{i,k}T_{l,j} \in \mathcal{I} \setminus \mathcal{P}_{i-1}$ for every j, k, and $i < l \le n-1$. Hence, again by the fact $(**)$, $T_{i,j}$ and $T_{i,k}$ are associated in R_J, and hence, as in the previous case, R_J has dimension one, implying $F(I)_{\mathcal{P}_i}$ to have dimension i. Finally, observe that

$$\frac{F(I)}{\mathcal{P}_{n-2}} \cong k[T, T_{n-1,2}, \dots, T_{n-1,m}]$$

is a polynomial ring in m variables and therefore $l(I) = n - 2 + m$. Set

$$a := \mu(I) - l(I) = ((n-1)(m-1) + 1) - (n + m - 2) = (n-2)(m-2).$$

By Lemma 4.1, comparing b with $\binom{a+1}{2}$, we get

$$\binom{n-1}{2}\binom{m-1}{2} \leq \binom{(n-2)(m-2)+1}{2}$$

and I is a Freiman ideal if and only if the equality holds. If $n = 2$, 3, the equality holds, otherwise for $m, n \geq 4$, a straightforward computation shows that the inequality is strict and therefore I is not Freiman in such cases. $\square$

Remark 5.3. Following the notation of the preceding theorem, the defining ideal of the fiber cone of the cover ideal of $A_{n,m}$ can be seen to be generated by the minors of the $n \times m$ matrix

$$\begin{pmatrix} T_{i_1,j_1} & T_{i_1,j_2} & \dots & T_{i_1,j_m} \\ T_{i_2,j_1} & T_{i_2,j_2} & \dots & T_{i_2,j_m} \\ \vdots & \vdots & \vdots & \vdots \\ T_{i_n,j_1} & T_{i_n,j_2} & \dots & T_{i_n,j_m} \end{pmatrix}.$$

This gives another argument to characterize when these ideals are Freiman using Theorem 4.6.

5.2. Circulant graphs

In Theorem 3.12, we classified which circulant graphs have (quasi-)equigenerated cover ideals. Here we describe which of their cover ideals are Freiman. We recall that given a graph G and a vertex $x_i \in V(G)$, we call G_i the induced subgraph of G on the set $V(G) \setminus (\mathcal{N}(x_i) \cup \{x_i\})$.

Theorem 5.4. *Let $G = C_n(1, \dots, s)$. The cover ideal $J(G)$ is Freiman if and only if $s > \frac{n-4}{2}$ or $G \in \{C_5, C_7\}$.*

Proof. Both $J(C_5)$ and $J(C_7)$ are of linear type and thus are Freiman. Since we only want to consider graphs with equigenerated cover ideal, we suppose $s \geq 2$. First assume $s \leq \frac{n-4}{2}$. Then G_1 has vertex set $\{x_{s+1}, x_{s+2}, \dots, x_{n-s}\}$ of size at least 3 and $x_{n-s}, x_{n-s-1}, x_{n-s-2}$ are contained in a clique of G_1. There exists a maximal independent set H in G_1 with $x_{n-s-2} \in H$.

Let $\phi : V \to V$ be defined by $\phi(x_i) = x_{i+1}$ (modulo n). Note that ϕ preserves adjacency of vertices. Since $x_{n-s-2} \in H$ and $s \geq 2, x_{n-s-1}, x_{n-s} \notin H$, hence $H_1 = \{x_1\} \cup H, H_2 = \{x_1\} \cup \phi(H)$, and $H_3 = \{x_1\} \cup \phi^2(H)$ are all independent sets of G. Similarly, we see that $H_4 = \{x_2\} \cup \phi(H), H_5 = \{x_2\} \cup \phi^2(H), H_6 = \{x_2\} \cup \phi^3(H), H_7 = \{x_3\} \cup \phi^2(H)$, and $H_8 = \{x_3\} \cup \phi^3(H)$ are all maximal independent sets of G. However we see then that

$$h_{H_2}h_{H_6}h_{H_7} - h_{H_3}h_{H_4}h_{H_8}$$

is a minimal generator of $\mathcal{I}$ of degree 3. Thus $J(G)$ is not Freiman by Remark 4.2.

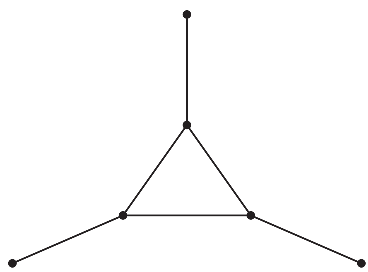
Suppose that $s > \frac{n-4}{2}$. Then either $|G_i| = 2, |G_i| = 1$, or $|G_i| = 0$ for all i. If each $|G_i| = 0$, then G is complete and $J(G)$ Freiman. If each $|G_i| = 1$ then $J(G)$ is generated by elements of the form $F(x_a x_{a+s+1})^{-1}$. For each x_a, there is exactly one element of this generating set not divisible by x_a, and thus $J(G)$ is Freiman by Proposition 4.7.

If each $|G_i| = 2$ then $J(G)$ is generated by elements of the form $h_{a,b} = F(x_a x_b)^{-1}$ where $b \in \{a+s+1, a+s+2\}$. As usual we consider $T_{a,b}$ to be the image of $h_{a,b}$ in the fiber cone of $J(G)$. Let $\alpha - \beta$ be a binomial minimal generator of $\mathcal{I}$, where α, β are monomials. If $T_{a,a+s+1}$ divides α, then $T_{a,a+s+2}$ divides β, since $T_{a,a+s}$ and $T_{a,a+s+2}$ are the only generators of $J(G)$ which x_a does not divide. Since the highest power of x_{a+s+1} dividing $T_{a,a+s+1}$ is one less than the highest power dividing $T_{a,a+s+2}$, there must exist another $T_{c,d}$ dividing β and not divisible by x_{a+s+1}. As $|G_{a+s+1}| = 2$, we know that the only such element is $T_{a+s+1,a-1}$. But then the highest power of x_{a-1} dividing β is one lower than the highest power dividing α unless another generator of $J(G)$

which is not divisible by x_{a-1} divides α. Continuing this process, we see that all the generators of $J(G)$ must divide both α and β, and therefore the fiber cone is a polynomial ring. Thus $J(G)$ is Freiman. □

5.3. Whiskered graphs

Definition 5.5. Given a graph G with $V(G) = \{x_1, \ldots, x_n\}$, the whiskering of G is the graph $\tilde{G}$ with $V(\tilde{G}) = V(G) \cup \{y_1, \ldots, y_n\}$ and $E(\tilde{G}) = E(G) \cup \{\{x_i, y_i\} | i = 1, \ldots, n\}$. The graph in next picture is the whiskering of C_3.



Definition 5.6. Given a graph G, let $I = J(G)$ be its cover ideal. For any squarefree cover $f \in I$ (non-necessarily minimal), we define $A_f = \{i \mid x_i \text{ does not divide } f\}$ and

$$f_w := f \prod_{i \in A_f} y_i.$$

Proposition 5.7. *Let G be a graph on n vertices and let $\tilde{G}$ be its whiskering. Then $J(\tilde{G})$ is minimally generated by the set*

$$\{f_w \mid f \in J(G) \text{ squarefree cover}\}$$

and it is equigenerated of degree n.

Proof. Clearly, by definition, f_w is a cover of $\tilde{G}$. To show that they are all minimal covers of $\tilde{G}$, take $f, h \in J(G)$ squarefree covers. Hence, we may find x_i dividing f and not h (or we may find the opposite case). It follows that x_i divides f_w and not h_w, while y_i divides h_w and not f_w, and therefore f_w and h_w are both minimal. Obviously each f_w has degree n, making $J(\tilde{G})$ equigenerated in degree n. □

Let G be a graph on n vertices and let $J(\tilde{G})$ be the cover ideal of the whiskering of G. We make use of the following notation, using Proposition 5.7, to describe the minimal generators of $J(\tilde{G})$. Since $F = x_1 x_2 \cdots x_n$ is a cover of G, then $F = F_w$ is also a minimal cover of $\tilde{G}$. The others generators of $J(\tilde{G})$ are of the form

$$h_U := \left(F \prod_{x_i \in U} x_i^{-1} \right)_w$$

where U is an independent set of G. In the case $U = \{x_i\}$, we denote

$$h_i := \left(F x_i^{-1} \right)_w.$$

By these observations, it follows that $\mu(J(\tilde{G})) = n + 1 + d$ where d denotes the number of independent sets of G of cardinality at least two.

Theorem 5.8. *Let G be a graph on n vertices and let $I = J(\tilde{G})$ be the cover ideal of the whiskering of G. The analytic spread of I is $l(I) = n+1$.*

Proof. We proceed along the line of the computation of the analytic spread done in the proof of Theorem 5.2. Let us associate the variable T to F and, for any independent set U of G, the variable T_U to the cover h_U (for $U = \{x_i\}$ we call the variables $T_U := T_i$). Thus, we present $F(I)$ as the quotient of the polynomial ring $k[T, T_1, ..., T_n, T_{U_1}, ..., T_{U_d}]$ by a homogeneous binomial ideal $\mathcal{I}$. The generators of $\mathcal{I}$ are of the form

$$T_{U_{l_1}} T_{U_{l_2}} \cdots T_{U_{l_r}} T^c - T_{U_{j_1}} T_{U_{j_2}} \cdots T_{U_{j_s}} \tag{5.1}$$

where $c + r = s$, U_{l_v}, U_{j_k} are independent sets of G, and $U_{l_v} \neq U_{j_k}$ for every l_v, j_k. With an abuse of notation we write also the elements of $F(I)$ in the form T_U instead of writing their classes modulo $\mathcal{I}$. Define now for $i = 1, ..., n$ the ideals

$$\mathcal{I}_i := (T_U \mid U \text{ is an independent set of } G \text{ and } x_i \in U) \subseteq F(I).$$

Notice that $\mathcal{I}_i$ always contains T_i and it is principal if and only if x_i is connected to any other vertex of G. We define, for $i = 1, ..., n$,

$$\mathcal{P}_i = \sum_{k=1}^{i} \mathcal{I}_k$$

and

$$\mathcal{P}_{n+1} = \mathcal{P}_n + (T),$$

and we observe that since any minimal cover of $\tilde{G}$ belongs to some of the ideals $\mathcal{I}_i$, the homogeneous maximal ideal of $F(I)$ is equal to $\mathcal{P}_{n+1}$. We need to show that $\mathcal{P}_i$ is a prime ideal of $F(I)$ of height i and this will imply our thesis.

Consider a binomial q minimal generator of $\mathcal{I}$. Clearly, since q is of the form given in (5.1), $x_i \in U_{l_v}$ for some v if and only if $x_i \in U_{j_k}$ for some k, hence either $q \in \mathcal{P}_i$ or all the independent sets U_{l_v}, U_{j_k} do not contain the vertices $x_1, ..., x_i$. By Proposition 2.4, it follows that for every i, $\mathcal{P}_i$ is a prime ideal.

We prove by induction that the height of $\mathcal{P}_i$ is i. Consider the localization $F(I)_{\mathcal{P}_1}$ and observe that T and all the elements T_U such that $x_1 \notin U$ are units in this ring. Since, if $\{x_1\} \subsetneq U$, the binomial $T_U T - T_1 T_{U \setminus \{x_1\}} \in \mathcal{I}$, then T_1 is associated to T_U in $F(I)_{\mathcal{P}_1}$, and hence $F(I)_{\mathcal{P}_1} \cong k[T_1]_{(T_1)}$ has dimension one. Applying the same argument inductively, as done in the proof of Theorem 5.2, we get that $F(I)_{\mathcal{P}_i} \cong k[T_1, ..., T_i]_{(T_1, ..., T_i)}$ has dimension i and therefore $\mathcal{P}_i$ has height i. □

Theorem 5.9. *Let G be a graph on n vertices. Let $I = J(\tilde{G})$ be the cover ideal of the whiskering of G. Then I is Freiman if and only if G is almost complete.*

Proof. We keep the same notation used the proof of Theorem 5.8 for the fiber cone $F(I)$ and its defining ideal $\mathcal{I}$. Let b be the number of minimal generators of $\mathcal{I}$ of degree 2, and let d be the number of independent sets (not necessarily maximal) of G of cardinality at least 2. By Theorem 5.8, we have $\mu(I) - l(I) = (n+1+d) - (n+1) = d$, and therefore by Lemma 4.1,

$$b \leq \binom{d+1}{2} = d + \binom{d}{2}$$

and I is Freiman if and only if the equality holds.

First assume $c(G) = w \geq 3$. Let $U = \{x_{i_1}, x_{i_2}, ..., x_{i_w}\}$ be an independent set of G of maximal cardinality. Observe that

$$T_U T^{w-1} - T_{i_1} T_{i_2} \cdots T_{i_w} \tag{5.2}$$

is a minimal generator of $\mathcal{I}$ of degree greater than 2. It follows by Remark 4.2, that I is not Freiman.

Assume now $c(G) = 2$. Hence d denotes the number of independent sets of G containing exactly two elements. If $U = \{x_i, x_j\}$ is one of these sets, we have that

$$T_U T - T_i T_j \tag{5.3}$$

is a minimal generator of $\mathcal{I}$, and there are exactly d minimal generators of this form. Given two independent sets $U_1 = \{x_i, x_j\}$ and $U_2 = \{x_i, x_k\}$ sharing the vertex x_i, we get that also

$$T_{U_1} T_k - T_{U_2} T_j \tag{5.4}$$

is a minimal generator of $\mathcal{I}$. Observe that G is almost complete if and only if all the independent sets containing two elements share the same fixed vertex. In this case, there are exactly $\binom{d}{2}$ minimal generators of $\mathcal{I}$ of this second form, implying $b = d + \binom{d}{2}$ and completing part of the proof. Naturally in this case, calling x_i the common vertex of all the independent sets, the binomials in (5.3) and (5.4) are the minors of the $2 \times (d+1)$ matrix:

$$\begin{pmatrix} T & T_1 & \dots & T_d \\ T_i & T_{U_1} & \dots & T_{U_d} \end{pmatrix}.$$

In the other case, assuming G not almost complete, there must be two independent sets U_1, U_2 of cardinality 2, such that $U_1 \cap U_2 = \emptyset$, indeed, if by way of contradiction this does not happen, there must exist three independent sets of the form $\{x_i, x_j\}, \{x_j, x_k\}, \{x_k, x_i\}$. But this would imply that $\{x_i, x_j, x_k\}$ is an independent set, contradicting the fact that $c(G) = 2$.

Now, set $U_1 = \{x_{i_1}, x_{j_1}\}$ and $U_2 = \{x_{i_2}, x_{j_2}\}$ and assume $U_1 \cap U_2 = \emptyset$. Clearly, the two binomials $T_{U_1} T - T_{i_1} T_{j_1}$ and $T_{U_2} T - T_{i_2} T_{j_2}$ cannot be minors of the same $2 \times m$ matrix, and therefore I is not Freiman by Theorem 4.6. □

Corollary 5.10. *Let T be a reduced tree. The cover ideal of T is Freiman if and only if $T = P_2, P_4, \tilde{P}_3$.*

Proof. Apply Theorems 3.8 and 5.9. □

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