Journal of Algebra 560 (2020) 312–342

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Betti numbers of symmetric shifted ideals

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A R T I C L E   I N F O

Article history:
Received 7 September 2019
Available online 26 May 2020
Communicated by David Hernandez

MSC:
13A15
13A50
13D02

Keywords:
Betti numbers

A B S T R A C T

We introduce a new class of monomial ideals which we call symmetric shifted ideals. Symmetric shifted ideals are fixed by the natural action of the symmetric group and, within the class of monomial ideals fixed by this action, they can be considered as an analogue of stable monomial ideals within the class of monomial ideals. We show that a symmetric shifted ideal has linear quotients and compute its (equivariant) graded Betti numbers. As an application of this result, we obtain several consequences for graded

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https://doi.org/10.1016/j.jalgebra.2020.04.037

Equivariant resolution
Linear quotients
Shifted ideal
Star configuration
Symbolic power

Betti numbers of symbolic powers of defining ideals of star configurations.

1. Introduction

The study of graded Betti numbers of graded ideals in a polynomial ring is one of the central topics in commutative algebra. It has always been of great interest to find combinatorial formulas for these numbers for various families of monomial ideals. In this paper, we introduce a new class of monomial ideals, which we call symmetric shifted ideals, and compute their graded Betti numbers.

For the definition of these ideals, we first provide some necessary notation. Let $S = \Bbbk[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field $\Bbbk$. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 0}$, we write $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and $|\mathbf{a}| = a_1 + \cdots + a_n$. We consider an action of the symmetric group $\mathfrak{S}_n$ on S defined by permutations of the variables and focus on $\mathfrak{S}_n$-fixed monomial ideals $I \subset S$, that is to say, monomial ideals $I \subset S$ with $\sigma(I) = I$ for all $\sigma \in \mathfrak{S}_n$. Such ideals have recently attracted attention as elements of ascending chains of ideals that are invariant under actions of symmetric groups (see, e.g., [3,11,23–25,29]).

We say that a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative integers is a partition of d of length n, if $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ and $|\lambda| = d$. Let

$$\mathrm{P}_n = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n\}$$

be the set of partitions of length n. For a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ of degree d, we write $\operatorname{part}(u) \in \mathbb{Z}^n_{\geqslant 0}$ for the partition obtained from $(a_1, \dots, a_n)$ by permuting its entries in a suitable way. For example, $\operatorname{part}(x_1^2 x_2^0 x_3^1 x_4^2) = (0, 1, 2, 2)$. If a monomial ideal $I \subset S$ is $\mathfrak{S}_n$-fixed, then a monomial u is in I if and only if $x^{\operatorname{part}(u)}$ is in I. Thus, the set

$$\mathrm{P}(I) = \{\lambda \in \mathrm{P}_n : x^\lambda \in I\}$$

determines the monomials in I and the ideal itself. The central object of study of this note are:

Definition 1.1. Let $I \subset S$ be an $\mathfrak{S}_n$-fixed monomial ideal. We say that I is a **symmetric shifted ideal** if, for every $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathrm{P}(I)$ and $1 \leqslant k < n$ with $\lambda_k < \lambda_n$, one has $x^\lambda (x_k/x_n) \in I$. Also, we say that I is a **symmetric strongly shifted ideal** if, for every $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathrm{P}(I)$ and $1 \leqslant k < l \leqslant n$ with $\lambda_k < \lambda_l$, one has $x^\lambda (x_k/x_l) \in I$. We may also refer to these ideals simply as shifted and strongly shifted ideals.

In the following remarks, we note that the above properties can be defined purely in terms of partitions.

Remark 1.2. Let $I \subset S$ be an $\mathfrak{S}_n$-fixed monomial ideal. Denote by e_i the i-th standard basis vector of $\mathbb{Z}^n$. If for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{P}(I)$ with $j = \min\{k : \lambda_k = \lambda_n\}$ and for every $\lambda - e_j + e_i$ with $i < j$ that is also a partition (i.e., non-decreasing) we have $\lambda - e_j + e_i \in \mathrm{P}(I)$, then I is shifted.

Remark 1.3. For partitions $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n)$, we define

$$\mu \trianglelefteq \lambda \text{ if } \mu_k + \cdots + \mu_n \leqslant \lambda_k + \cdots + \lambda_n \text{ for all } k.$$

The partial order $\lhd$ is known in the literature as the **dominance order** (see, e.g., [34, §7.2]). An $\mathfrak{S}_n$-fixed monomial ideal I is strongly shifted if and only if, for every $\lambda, \mu \in \mathrm{P}_n$ with $|\lambda| = |\mu|$, $\lambda \in P(I)$ and $\mu \trianglelefteq \lambda$ imply $\mu \in \mathrm{P}(I)$.

The definition of shifted and strongly shifted ideals is inspired by the definition of stable and strongly stable ideals, which are important classes of monomial ideals since, e.g., in characteristic zero generic initial initials are strongly stable. Recall that Eliahou and Kervaire [12] constructed minimal graded free resolutions of stable ideals and gave a simple formula for their graded Betti numbers in terms of the data of their minimal systems of monomial generators. The main results of this paper are the following formulas for graded Betti numbers of symmetric shifted ideals.

(1) We prove that every symmetric shifted ideal I has linear quotients (Theorem 3.2). This allows us to give a formula for its graded Betti numbers in terms of its monomial generators $G(I)$ (Corollary 3.4).
(2) We also give a formula for the graded Betti numbers of a symmetric shifted ideal I in terms of its partition generators $\{\lambda \in P_n : x^\lambda \in G(I)\}$ (Corollary 5.7).
(3) We compute equivariant graded Betti numbers of a symmetric shifted ideal I. In other words, we determine the $\Bbbk[\mathfrak{S}_n]$-module structure of $\mathrm{Tor}_i(I, \Bbbk)_j$ (Theorem 6.2).

Our initial motivation for defining symmetric shifted ideals comes from the study of minimal graded free resolutions of symbolic powers of star configurations. A **codimension c star configuration** is a union of linear subspaces of a projective space $\mathbb{P}^N$ of the form

$$V_c = \bigcup_{1 \leqslant i_1 < \cdots < i_c \leqslant n} H_{i_1} \cap \cdots \cap H_{i_c},$$

where $H_1, \ldots, H_n$ are distinct hyperplanes in $\mathbb{P}^N$ such that the intersection of any j of them is either empty or has codimension j and where $1 \leqslant c \leqslant \min\{n, N\}$. The name is motivated by the special case of 10 points located at pairwise intersections of 5 lines in the projective plane, with the lines positioned in the shape of a star. Let $L_1, \ldots, L_n$ be defining linear forms of $H_1, \ldots, H_n$. Then the defining ideal of V_c is given by $I_{V_c} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} (L_{i_1}, \ldots, L_{i_c})$ and its m-th symbolic power can be written as

$$I_{V_c}^{(m)} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} (L_{i_1}, \ldots, L_{i_c})^m$$

because each ideal $(L_{i_1}, \ldots, L_{i_c})$ is a minimal prime of I_{V_c} and it is generated by a regular sequence; see, e.g., [37, Appendix 6, Lemma 5]. These ideals I_{V_c} and $I_{V_c}^{(m)}$ have been extensively studied from the point of view of algebra, geometry, and combinatorics in [1,2,4–7,9,13,14,19,28,30,32,35,36]. We recommend [17] as a great introduction to the subject. In particular, the Betti numbers of the defining ideal of a star configuration and its symbolic square have been determined in [17, Remark 2.11, Theorem 3.2]. Further motivation for studying these ideals can be found in [18], which considers generalizations where the linear forms are replaced by forms of arbitrary degree and also explores connections with Stanley-Reisner ideals of matroids.

Let $I_{n,c}$ be the monomial ideal of $S = \Bbbk[x_1, \ldots, x_n]$ defined by

$$I_{n,c} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} (x_{i_1}, \ldots, x_{i_c}).$$

Then the minimal graded free resolution of $I_{V_c}^{(m)}$ is completely determined by that of $I_{n,c}^{(m)}$; in particular, it was shown that these two ideals have the same graded Betti numbers (see [18, Example 3.4 and Theorem 3.6], where the more general case of hypersurface or matroid configurations is considered). The same reference also shows that both ideals are Cohen-Macaulay. Note that the ideal $I_{n,c}$ can also be described as the ideal generated by all squarefree monomials of degree $n - c + 1$ [18, Proposition 2.3]. Obviously the ideal $I_{n,c}^{(m)}$ is $\mathfrak{S}_n$-fixed. As one of our main results we prove (in Theorem 4.3) that:

Theorem. *The ideal $I_{n,c}^{(m)}$ is strongly shifted.*

Since we find a formula for the graded Betti numbers of symmetric shifted ideals, this result gives various information on graded Betti numbers of symbolic powers of star configurations, including their Castelnuovo-Mumford regularity, a simple formula for the Betti numbers in the top and bottom rows of the Betti table, an explicit formula for the Betti numbers of the symbolic cube, and more (see the results in Section 4). Our results for star configurations also apply to the computation of Betti numbers of fat point schemes (see Remark 4.9).

This paper is organized as follows: In Section 2, we study some combinatorial properties of symmetric shifted ideals. In Section 3, we prove that symmetric shifted ideals have linear quotients, and in Section 4 we apply the results in Section 3 to symbolic powers of star configurations. In Sections 5 and 6, we compute the (equivariant) graded Betti numbers of symmetric shifted ideals.

Acknowledgments

This work was started at the workshop "Ordinary and Symbolic Powers of Ideals" at Casa Matemática Oaxaca (CMO) in May 2017. We thank the organizers of the workshop and CMO for their kind invitation and warm hospitality.

We also thank the anonymous referee who provided several useful suggestions to clarify our exposition and improve the quality of our manuscript.

Shortly after this paper was posted on arXiv, another preprint appeared by Paolo Mantero [27] which also computes the graded Betti numbers of symbolic powers of star configurations. The results in Mantero's preprint were obtained independently from ours, and utilize new and interesting techniques. We are also grateful to Paolo for pointing out a mistake in an earlier version of Corollary 4.4. In December 2019, another preprint appeared on arXiv by Kuei-Nuan Lin and Yi-Huang Shen that uses and generalizes some of the results in our paper to a-fold product ideals [26].

The research of the fourth author is partially supported by KAKENHI 16K05102. The fifth author was partially supported by Simons Foundation grant #317096. The last author was supported by NSF grant DMS–1601024 and EPSCoR award OIA–1557417.

2. Symmetric shifted ideals

In this section, we discuss some basic properties of symmetric shifted ideals. For a monomial $u \in S$, we write $\mathfrak{S}_n \cdot u$ for the $\mathfrak{S}_n$-orbit of u in S, i.e., $\mathfrak{S}_n \cdot u = \{\sigma(u) : \sigma \in \mathfrak{S}_n\}$. The set P_n can be regarded as a poset with the order defined by $\lambda \geqslant \mu$ if x^μ divides x^λ. Then the set $\mathrm{P}(I)$ is a filter in the poset P_n, that is to say, for $\mu \in \mathrm{P}(I)$ and $\lambda \in \mathrm{P}_n$, one has $\lambda \in \mathrm{P}(I)$ if $\lambda \geqslant \mu$. The next lemma shows that the assignment $I \mapsto \mathrm{P}(I)$ defines a one-to-one correspondence between $\mathfrak{S}_n$-fixed monomial ideals in S and filters in P_n.

Lemma 2.1. *Let $\lambda, \mu \in \mathrm{P}_n$. There exist monomials $u \in \mathfrak{S}_n \cdot x^\mu$ and $w \in \mathfrak{S}_n \cdot x^\lambda$ such that u divides w if and only if x^μ divides x^λ.*

Proof. The "if" part is obvious. We prove the "only if" part. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$. The assumption says that there exists $\sigma \in \mathfrak{S}_n$ such that

$$\mu_{\sigma(1)} \leqslant \lambda_1, \dots, \mu_{\sigma(n)} \leqslant \lambda_n.$$

Since $\lambda_1 \leqslant \cdots \leqslant \lambda_n$, for each $k = 1, 2, \dots, n$ we have

$$\mu_{\sigma(1)} \leqslant \lambda_k, \mu_{\sigma(2)} \leqslant \lambda_k, \dots, \mu_{\sigma(k)} \leqslant \lambda_k.$$

This implies that the partition μ contains at least k entries smaller than or equal to λ_k. Therefore $\mu_k \leqslant \lambda_k$ for all k, and x^μ divides x^λ. $\square$

Throughout the rest of the paper, we will say that $\mu \in \mathrm{P}_n$ divides $\lambda \in \mathrm{P}_n$ if x^μ divides x^λ.

Next, we show that to check the conditions of symmetric (strongly) shifted ideals, it is enough to check them on generators. Let I be a monomial ideal. We write $G(I)$ for the unique set of minimal monomial generators of I. When I is $\mathfrak{S}_n$-fixed, we define

$$\Lambda(I) = \{\lambda \in P(I) : x^\lambda \in G(I)\}.$$

Note that $G(I) = \biguplus_{\lambda \in \Lambda(I)} \mathfrak{S}_n \cdot x^\lambda$, where $\biguplus$ denotes a disjoint union of sets. As the next statement shows, to check I is shifted it is enough to check the condition of Definition 1.1 for partitions in $\Lambda(I)$.

Lemma 2.2. *Let $I \subset S$ be an $\mathfrak{S}_n$-fixed monomial ideal. Then I is shifted if and only if, for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(I)$ and $1 \leqslant k < n$ with $\lambda_k < \lambda_n$, one has $x^\lambda(x_k/x_n) \in I$.*

Proof. The “only if” part is obvious. We prove the “if” part. Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathrm{P}(I)$ and $1 \leqslant k < n$ with $\mu_k < \mu_n$. We claim $x^\mu(x_k/x_n) \in I$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(I)$ be a partition that divides μ. If $\lambda_n = \mu_n$, then $w = x^\lambda(x_k/x_n) \in I$ by assumption and w divides $x^\mu(x_k/x_n)$. If $\lambda_n < \mu_n$, then $x^\lambda x_k$ divides $x^\mu(x_k/x_n)$. In both cases, $x^\mu(x_k/x_n) \in I$ as desired. $\square$

An analogous statement holds for symmetric strongly shifted ideals. We omit the proof since it is essentially the same as the one for symmetric shifted ideals.

Lemma 2.3. *Let $I \subset S$ be an $\mathfrak{S}_n$-fixed monomial ideal. Then I is strongly shifted if and only if, for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(I)$ and $1 \leqslant k < l \leqslant n$ with $\lambda_k < \lambda_l$, one has $x^\lambda(x_k/x_l) \in I$.*

Example 2.4. Let $I \subset \Bbbk[x_1, x_2, x_3]$ be the $\mathfrak{S}_3$-fixed monomial ideal with

$$\Lambda(I) = \{(1, 1, 1), (0, 1, 2), (0, 0, 4)\}.$$

The ideal I is strongly shifted and is minimally generated by the following ten monomials:

$$x_1x_2x_3, \quad x_2x_3^2, \quad x_2^2x_3, \quad x_1x_3^2, \quad x_1^2x_3, \quad x_1x_2^2, \quad x_1^2x_2, \quad x_1^4, \quad x_2^4, \quad x_3^4.$$

Example 2.5. Let $I \subset \Bbbk[x_1, x_2, x_3, x_4]$ be the $\mathfrak{S}_4$-fixed monomial ideal with

$$\Lambda(I) = \{(1, 1, 2, 2), (0, 2, 2, 2), (0, 1, 2, 3)\}.$$

Then I is shifted but not strongly shifted since $(0, 1, 2, 3) \in \mathrm{P}(I)$ but $(1, 1, 1, 3) \notin \mathrm{P}(I)$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we define the quantities

$$p(\lambda) = \#\{k : \lambda_k < \lambda_n - 1\},$$
$$r(\lambda) = \#\{k : \lambda_k = \lambda_n\}.$$

We also introduce the truncation of the partition λ by setting

$$\lambda_{\leqslant k} = (\lambda_1, \dots, \lambda_k)$$

for $k = 1, 2, \dots, n$. Let $<_{\text{lex}}$ be the total order on $\mathbb{Z}^n_{\geqslant 0}$ defined by

$$\mathbf{a} = (a_1, \dots, a_n) <_{\text{lex}} (b_1, \dots, b_n) = \mathbf{b}$$

if

(i) $|\mathbf{a}| < |\mathbf{b}|$, or
(ii) $|\mathbf{a}| = |\mathbf{b}|$ and the leftmost non-zero entry of $(a_1 - b_1, \dots, a_n - b_n)$ is positive.

Remark 2.6. Our definition of the order $<_{\text{lex}}$ is the opposite of the more familiar lexicographic order for monomials (cf. [10, Ch. 2 §2]). This is necessary to ensure compatibility with our definition of partitions as non-decreasing sequences.

We establish another result that will be used in later sections.

Lemma 2.7. *Let I be a symmetric shifted ideal. For every $\mu \in P(I)$, there is a unique $\lambda \in \Lambda(I)$ such that*

(a) λ divides μ, and
(b) $\lambda_{\leqslant p(\lambda)} = \mu_{\leqslant p(\lambda)}$.

Proof. Let $\mu = (\mu_1, \dots, \mu_n) \in \mathrm{P}(I)$. Let

$$\lambda = (\lambda_1, \dots, \lambda_n) = \min_{<_{\text{lex}}}\{\rho \in \Lambda(I) : \rho \text{ divides } \mu\}$$

and $p = p(\lambda)$. Clearly λ satisfies condition (a). We claim that λ fulfills also (b), that is to say, $\lambda_k = \mu_k$ for all $k \leqslant p$.

Suppose to the contrary that there is $k \leqslant p$ such that $\lambda_k < \mu_k$. Then $w = x^\lambda(x_k/x_n)$ divides x^μ and, by definition of symmetric shifted ideals, we have $w \in I$. Let $\lambda' = \mathrm{part}(w)$. Observe that λ' is constructed from λ by replacing the part λ_n with $\lambda_n - 1$, the part λ_k with $\lambda_k + 1$, and rearranging in non-decreasing order. By definition of $r = r(\lambda)$, the partition λ has r parts equal to λ_n. Now suppose that $\lambda' = \lambda$. Then λ' also has r parts equal to λ_n, so we must have $\lambda_k + 1 = \lambda_n$ or $\lambda_k = \lambda_n - 1$. However, the definition of $p = p(\lambda)$ implies that $\lambda_k < \lambda_n - 1$. We deduce that $\lambda' \neq \lambda$. Then λ' divides μ by Lemma 2.1 and there is a $\rho \in \Lambda(I)$ that divides $\lambda' \in \mathrm{P}(I)$. However, such a ρ satisfies $\rho \leqslant_{\text{lex}} \lambda' <_{\text{lex}} \lambda$, contradicting the minimality of λ.

Next, we prove uniqueness. Suppose that $\lambda, \lambda' \in \Lambda(I)$ satisfy conditions (a) and (b). We prove $\lambda = \lambda'$. Let $p = p(\lambda)$ and $p' = p(\lambda')$. We may assume $p \leqslant p'$. By condition (b), λ and λ' are of the form

$$\lambda = (\mu_1, \dots, \mu_p, \lambda_n - 1, \dots, \lambda_n - 1, \lambda_n, \dots, \lambda_n)$$

and

$$\lambda' = (\mu_1, \dots, \mu_p, \mu_{p+1} \dots, \mu_{p'}, \lambda'_n - 1, \dots, \lambda'_n - 1, \lambda'_n, \dots, \lambda'_n).$$

Suppose $p < p'$. Since λ divides μ, we have

$$\mu_k \geqslant \lambda_k \geqslant \lambda_n - 1 \quad \text{for } p < k \leqslant p',$$

and

$$\lambda'_n - 1 > \mu_{p'} \geqslant \lambda_n - 1.$$

But these inequalities say that λ properly divides λ', contradicting $\lambda, \lambda' \in \Lambda(I)$. Hence $p = p'$. However, given the shape of λ and λ', $p = p'$ implies that either λ divides λ' or λ' divides λ. Since $\lambda, \lambda' \in \Lambda(I)$, λ must be equal to λ' as desired. □

3. Symmetric shifted ideals have linear quotients

A monomial ideal $I \subset S$ is said to have **linear quotients** if there is an order $u_1, \dots, u_s$ of monomials in $G(I)$ such that the colon ideal

$$(u_1, \dots, u_{k-1}) : u_k$$

is generated by variables for all $k = 2, 3, \dots, s$. A nice consequence of having linear quotients is that we can easily compute the graded Betti numbers from the above colon ideals. Recall that, for a graded ideal $I \subset S$, graded Betti numbers of I are the numbers $\beta_{i,j}(I) = \dim_{\Bbbk} \operatorname{Tor}_i(I, \Bbbk)_j$. Herzog and Takayama produced a formula for the bigraded Poincaré series of a monomial ideal with linear quotients [22, Corollary 1.6]:

Theorem 3.1. *With the same notation as above,*

$$\beta_{i,i+j}(I) = \sum_{\deg(u_k)=j} \binom{|G((u_1, \dots, u_{k-1}) : u_k)|}{i}.$$

Next, we present our first main result about symmetric shifted ideals.

Theorem 3.2. *Symmetric shifted ideals have linear quotients.*

Using the same notation as in Section 2, we define a total order on the set of monomials in S. Let $\lambda, \mu \in \mathrm{P}_n$. For distinct monomials $v = \tau(x^\mu)$ and $u = \sigma(x^\lambda)$ in S, we define $v \prec u$ if

(i) $\mu <_{\mathrm{lex}} \lambda$, or
(ii) $\mu = \lambda$ and $v <_{\mathrm{lex}} u$.

Note in particular that if v strictly divides u, then $v \prec u$.

Proof of Theorem 3.2. Let $I \subset S$ be a symmetric shifted ideal and fix a monomial $u = \sigma(x^\lambda) \in G(I)$ with $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(I)$. Let $p = p(\lambda)$ and $r = r(\lambda)$. Thus, we have

$$u = \sigma(x^\lambda) = x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(p)}^{\lambda_p} x_{\sigma(p+1)}^{\lambda_n - 1} \cdots x_{\sigma(n-r)}^{\lambda_n - 1} x_{\sigma(n-r+1)}^{\lambda_n} \cdots x_{\sigma(n)}^{\lambda_n}.$$

We also define the quantity

$$\max(u) = \max\{\sigma(k) : \lambda_k = \lambda_n\} = \max\{\sigma(n-r+1), \dots, \sigma(n)\}.$$

Let

$$J = (v \in G(I) : v \prec u).$$

We claim that

$$J : u = (x_{\sigma(1)}, \dots, x_{\sigma(p)}) + (x_{\sigma(k)} : p+1 \leqslant k \leqslant n-r,\ \sigma(k) < \max(u)). \tag{3.1}$$

This proves that I has linear quotients.

We prove the containment "$\supseteq$" holds in (3.1). We first prove $x_{\sigma(k)} u \in J$ for $1 \leqslant k \leqslant p$. In this case we have $\lambda_k < \lambda_n - 1 < \lambda_n$. Together with the fact that $u \in I$, this implies $x^\lambda (x_k / x_n) \in I$ because I is shifted. It follows that the monomial

$$w = u(x_{\sigma(k)}/x_{\sigma(n)}) = \sigma(x^\lambda (x_k/x_n))$$

is also in I because I is $\mathfrak{S}_n$-fixed. Reasoning as in the existence part of Lemma 2.7, we have $\operatorname{part}(w) <_{\mathrm{lex}} \lambda$, so $w \prec u$. This implies $w \in J$. Therefore we have $x_{\sigma(k)} u = x_{\sigma(n)} w \in J$.

Next we prove $x_{\sigma(k)} u \in J$ whenever $p+1 \leqslant k \leqslant n-r$ and $\sigma(k) < \max(u)$. In this case, the monomial $w = u(x_{\sigma(k)}/x_{\max(u)}) \in I$ is obtained from u by permuting variables so $\operatorname{part}(w) = \lambda$. However, $\sigma(k) < \max(u)$ implies $w <_{\mathrm{lex}} u$. Again we have $w \prec u$ and $w \in J$, so $x_{\sigma(k)} u = x_{\max(u)} w \in J$ as desired.

We prove the containment "$\subseteq$" holds in (3.1). Let $z \in S$ be a monomial not divisible by any variable in the set

$$\{x_{\sigma(1)}, \dots, x_{\sigma(p)}\} \cup \{x_{\sigma(k)} : p+1 \leqslant k \leqslant n-r,\ \sigma(k) < \max(u)\}.$$

We can write zu as

$$zu = x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(p)}^{\lambda_p} x_{\sigma(p+1)}^{b_{p+1}} \cdots x_{\sigma(n-r)}^{b_{n-r}} x_{\sigma(n-r+1)}^{b_{n-r+1}} \cdots x_{\sigma(n)}^{b_n}, \tag{3.2}$$

where $b_i \geqslant \lambda_i \geqslant \lambda_n - 1$ for all $i \geqslant p+1$ and $b_i = \lambda_n - 1$ for all $p+1 \leqslant i \leqslant n-r$ with $\sigma(i) < \max(u)$. We must prove that $zu \notin J$.

Assume, by contradiction, that $zu \in J$. By Equation (3.2), we have that:

(a) λ divides $\mathrm{part}(zu)$, and
(b) $\lambda_{\leqslant p} = \mathrm{part}(zu)_{\leqslant p}$.

Since $\lambda = \mathrm{part}(u) \in \Lambda(I)$, Lemma 2.7 guarantees that λ is the unique partition in $\Lambda(I)$ satisfying properties (a) and (b). Now define the ideal

$$J' = (v \in G(I) : \mathrm{part}(v) <_{\mathrm{lex}} \lambda).$$

Note that J' is $\mathfrak{S}_n$-fixed and shifted because I is. Moreover $\Lambda(J') \subset \Lambda(I)$ and $\lambda \in \Lambda(I) \setminus \Lambda(J')$. Hence, $zu \notin J'$ by Lemma 2.7.

Since $zu \in J$, there is a monomial $w \in G(J)$ that divides zu. Because $zu \notin J'$, we deduce that $w \notin G(J')$ so $\mathrm{part}(w) \geqslant_{\mathrm{lex}} \lambda$. At the same time, $w \in G(J)$ gives $w \prec u$, so $\mathrm{part}(w) \leqslant_{\mathrm{lex}} \lambda$. This forces $\mathrm{part}(w) = \lambda$, therefore $w = \tau(x^\lambda)$ for some $\tau \in \mathfrak{S}_n$. More explicitly, w is of the form

$$w = x_{\tau(1)}^{\lambda_1} \cdots x_{\tau(p)}^{\lambda_p} x_{\tau(p+1)}^{\lambda_n - 1} \cdots x_{\tau(n-r)}^{\lambda_n - 1} x_{\tau(n-r+1)}^{\lambda_n} \cdots x_{\tau(n)}^{\lambda_n}.$$

Comparing with Equation (3.2), we get

$$\{\sigma(1), \dots, \sigma(p)\} = \{\tau(1), \dots, \tau(p)\}. \tag{3.3}$$

Observe that $\lambda_{\sigma^{-1}(k)}$ and $\lambda_{\tau^{-1}(k)}$ are the exponents of x_k in $u = \sigma(x^\lambda)$ and $w = \tau(x^\lambda)$, respectively. If $\lambda_{\sigma^{-1}(k)} = \lambda_{\tau^{-1}(k)}$ for all $1 \leqslant k \leqslant n$, then $u = w$, which contradicts $w \prec u$. Therefore it makes sense to define

$$\ell = \min\{k : \lambda_{\tau^{-1}(k)} \neq \lambda_{\sigma^{-1}(k)}\},$$

and to write $\ell = \sigma(q)$ for some $1 \leqslant q \leqslant n$. By Equation (3.3), we have

$$\{\lambda_{\sigma^{-1}(\ell)}, \lambda_{\tau^{-1}(\ell)}\} = \{\lambda_n - 1, \lambda_n\}.$$

Table 1
Linear quotients of a symmetric shifted ideal.

i	u_i	$I_{i-1} : (u_i)$	$\max(u_i)$	i	u_i	$I_{i-1} : (u_i)$	$\max(u_i)$
1	$x_1^2x_2^2x_3x_4$	-	2	18	$x_1^2x_2^3x_4$	(x_1, x_3, x_4)	2
2	$x_1^2x_2x_3^2x_4$	(x_2)	3	19	$x_1^2x_2x_3^3$	(x_1, x_2, x_4)	3
3	$x_1^2x_2x_3x_4^2$	(x_2, x_3)	4	20	$x_1^2x_2x_4^3$	(x_1, x_2, x_3)	4
4	$x_1x_2^2x_3^2x_4$	(x_1)	3	21	$x_1^2x_3^3x_4$	(x_1, x_2, x_4)	3
5	$x_1x_2^2x_3x_4^2$	(x_1, x_3)	4	22	$x_1^2x_3x_4^3$	(x_1, x_2, x_3)	4
6	$x_1x_2x_3^2x_4^2$	(x_1, x_2)	4	23	$x_1x_2^3x_3^2$	(x_1, x_4)	2
7	$x_1^2x_2^2x_3^2$	(x_4)	3	24	$x_1x_2^3x_4^2$	(x_1, x_3)	2
8	$x_1^2x_2^2x_4^2$	(x_3)	4	25	$x_1x_2^2x_3^3$	(x_1, x_2, x_4)	3
9	$x_1^2x_3^2x_4^2$	(x_2)	4	26	$x_1x_2^2x_4^3$	(x_1, x_2, x_3)	4
10	$x_2^2x_3^2x_4^2$	(x_1)	4	27	$x_1x_3^3x_4^2$	(x_1, x_2)	3
11	$x_1^3x_2^2x_3$	(x_3, x_4)	1	28	$x_1x_3^2x_4^3$	(x_1, x_2, x_3)	4
12	$x_1^3x_2^2x_4$	(x_3, x_4)	1	29	$x_2^3x_3^2x_4$	(x_1, x_4)	2
13	$x_1^3x_2x_3^2$	(x_2, x_4)	1	30	$x_2^3x_3x_4^2$	(x_1, x_3)	2
14	$x_1^3x_2x_4^2$	(x_2, x_3)	1	31	$x_2^2x_3^3x_4$	(x_1, x_2, x_4)	3
15	$x_1^3x_3^2x_4$	(x_2, x_4)	1	32	$x_2^2x_3x_4^3$	(x_1, x_2, x_3)	4
16	$x_1^3x_3x_4^2$	(x_2, x_3)	1	33	$x_2x_3^3x_4^2$	(x_1, x_2)	3
17	$x_1^2x_2^3x_3$	(x_1, x_3, x_4)	2	34	$x_2x_3^2x_4^3$	(x_1, x_2, x_3)	4

Since $w = \tau(x^\lambda) <_{\text{lex}} \sigma(x^\lambda) = u$ by the definition of $\prec$, we actually have $\lambda_{\tau^{-1}(\ell)} = \lambda_n$ and $\lambda_q = \lambda_{\sigma^{-1}(\ell)} = \lambda_n - 1$. Also, since $w \neq u$, there is $m > \ell$ such that $\lambda_{\tau^{-1}(m)} = \lambda_n - 1$ and $\lambda_{\sigma^{-1}(m)} = \lambda_n$. This shows that

$$\sigma(q) = \ell < m = \sigma(\sigma^{-1}(m)) \leqslant \max(u),$$

and therefore $b_q = \lambda_n - 1$. However, this contradicts the fact that w divides zu since the exponent of x_ℓ in w is λ_n but the exponent of $x_\ell = x_{\sigma(q)}$ in zu is $b_q = \lambda_n - 1$. $\square$

Example 3.3. Let $I \subset \Bbbk[x_1, x_2, x_3, x_4]$ be the symmetric shifted ideal with

$$\Lambda(I) = \{(1, 1, 2, 2), (0, 2, 2, 2), (0, 1, 2, 3)\}.$$

The ideal I has 34 generators. We arrange them in an increasing sequence using the order $\prec$ described at the beginning of this section, and we denote them $u_1, \dots, u_{34}$. We also set $I_{i-1} = (u_1, \dots, u_{i-1})$ for $2 \leqslant i \leqslant 34$. Table 1 shows all the linear quotients $I_{i-1} : (u_i)$ of the ideal I in the given order of the generators. All computations were performed using Macaulay2 [20].

Using these results together with Theorem 3.2, we can give a formula for graded Betti numbers of symmetric shifted ideals. For a monomial $u = \sigma(x^\lambda)$, recall our earlier notations:

$$p = p(\lambda) = \#\{k : \lambda_k < \lambda_n - 1\},$$
$$r = r(\lambda) = \#\{k : \lambda_k = \lambda_n\},$$
$$\max(u) = \max\{\sigma(k) : \lambda_k = \lambda_n\},$$

and let

$$C(u) = \{x_{\sigma(1)}, \ldots, x_{\sigma(p)}\} \cup \{x_{\sigma(k)} : p+1 \leqslant k \leqslant n-r,\ \sigma(k) < \max(u)\}.$$

The next result follows from Theorem 3.1 and Equation (3.1).

Corollary 3.4. *Let I be a symmetric shifted ideal. Then for all i, j one has*

$$\beta_{i,i+j}(I) = \sum_{u \in G(I),\ \deg u = j} \binom{|C(u)|}{i}.$$

Example 3.5. Consider the ideal I of Example 3.3. Using Corollary 3.4 and the information in Table 1, we obtain the following Betti table for I.

	0	1	2	3
total:	34	72	51	12
6:	34	72	51	12

4. Star configurations

In this section, we apply the results in the previous section to symbolic powers of star configurations. Recall that $I_{n,c}$ is the monomial ideal of $S = \Bbbk[x_1, \ldots, x_n]$ defined by

$$I_{n,c} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} (x_{i_1}, \ldots, x_{i_c})$$

and the m-th symbolic power of $I_{n,c}$ is given by

$$I_{n,c}^{(m)} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} (x_{i_1}, \ldots, x_{i_c})^m. \tag{4.1}$$

We will show that $I_{n,c}^{(m)}$ is actually a symmetric strongly shifted ideal.

Proposition 4.1. *For every integer $m \geqslant 1$, the ideal $I_{n,c}^{(m)}$ is $\mathfrak{S}_n$-fixed. Moreover*

$$P(I_{n,c}^{(m)}) = \{\lambda \in P_n : |\lambda_{\leqslant c}| \geqslant m\},$$
$$\Lambda(I_{n,c}^{(m)}) = \{\lambda \in P_n : |\lambda_{\leqslant c}| = m, \forall i > c\ \lambda_i = \lambda_c\}.$$

Proof. Equation (4.1) immediately implies that $I_{n,c}^{(m)}$ is $\mathfrak{S}_n$-fixed because each element of $\mathfrak{S}_n$ acts by permuting the primary components of the ideal.

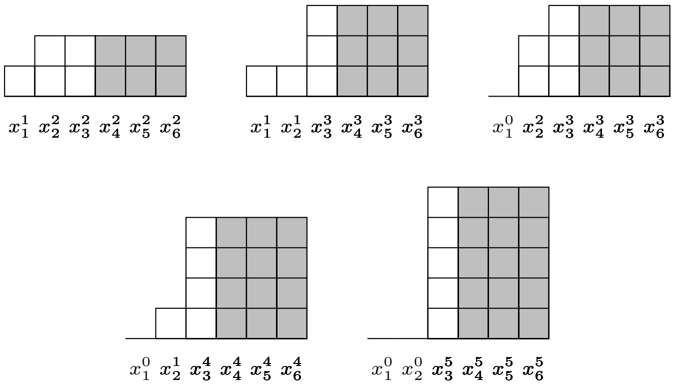


Fig. 1. Partitions and monomials generating $I_{6,3}^{(5)}$.

If $x^\lambda \in I_{n,c}^{(m)}$, then $x^\lambda \in (x_1, \dots, x_c)^m$, which gives $|\lambda_{\leqslant c}| \geqslant m$. Conversely, if $|\lambda_{\leqslant c}| \geqslant m$, then for all $1 \leqslant i_1 < \cdots < i_c \leqslant n$ and all $1 \leqslant j \leqslant c$, we have $\lambda_j \leqslant \lambda_{i_j}$ because λ is a partition. This implies

$$\sum_{j=1}^{c} \lambda_{i_j} \geqslant \sum_{j=1}^{c} \lambda_j = |\lambda_{\leqslant c}| \geqslant m.$$

Hence, $x^\lambda \in (x_{i_1}, \dots, x_{i_c})^m$. Thus, the statement about $P(I_{n,c}^{(m)})$ is proved.

Now the partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is in $\Lambda(I_{n,c}^{(m)})$ if and only if $\lambda \in P(I_{n,c}^{(m)})$ and the partition obtained from λ by decreasing any λ_i is not in $P(I_{n,c}^{(m)})$. This forces $|\lambda_{\leqslant c}| = m$ and $\forall i > c\ \lambda_i = \lambda_c$. $\square$

Example 4.2. Fig. 1 illustrates generators of $I_{n,c}^{(m)}$ when $n = 6$, $c = 3$ and $m = 5$.

We now discuss the main result of this section.

Theorem 4.3. *For every integer $m \geqslant 1$, the ideal $I_{n,c}^{(m)}$ is strongly shifted.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in P(I_{n,c}^{(m)})$ and $1 \leqslant k < l \leqslant n$ with $\lambda_k < \lambda_l$. Define $v = x^\lambda(x_k/x_l)$ and $\mu = \mathrm{part}(v)$. We have $|\mu_{\leqslant c}| \geqslant |\lambda_{\leqslant c}|$. Thus, by Proposition 4.1, we get $|\mu_{\leqslant c}| \geqslant m$ and $v \in I_{n,c}^{(m)}$. $\square$

Using Corollary 3.4, we can derive several consequences for the Betti tables of the ideals $I_{n,c}^{(m)}$.

Corollary 4.4.

(1) The degree j strand in the Betti table of $I_{n,c}^{(m)}$ is nonzero only when j belongs to $\{m + k(n - c) : k = 1, 2, \dots, m\}$, that is,

$$\beta_{i,i+j}(I_{n,c}^{(m)}) = 0 \quad \textit{for all } i \geqslant 0 \textit{ and } j \notin \{m + k(n-c) : k = 1, 2, \ldots, m\}.$$

(2) For all $i \geqslant 0$ and $k \geqslant \frac{m}{2} + 1$, one has

$$\beta_{i,i+m+k(n-c)}(I_{n,c}^{(m)}) = \binom{c-1}{i} \beta_{0,m+k(n-c)}(I_{n,c}^{(m)}).$$

In particular, $\beta_{i,i+m+k(n-c)}(I_{n,c}^{(m)})$ only depends on the number of generators of $I_{n,c}^{(m)}$ of degree $m + k(n-c)$ when $k \geqslant \frac{m}{2} + 1$.

(3) The Castelnuovo-Mumford regularity of $I_{n,c}^{(m)}$ is $m(1 + n - c)$. Moreover, if $m \geqslant 2$, then the bottom row in the Betti table of $I_{n,c}^{(m)}$ is given by the following formula:

$$\beta_{i,i+m(1+n-c)}(I_{n,c}^{(m)}) = \binom{n}{c-1}\binom{c-1}{i} \quad \textit{for all } i \geqslant 0.$$

(4) If $m \leqslant c$, then

$$\beta_{i,i+m+n-c}(I_{n,c}^{(m)}) = \binom{n}{c-m-i}\binom{m+n-c+i-1}{i} \quad \textit{for all } i \geqslant 0.$$

(5) All nonzero rows in the Betti table of $I_{n,c}^{(m)}$ have length $c - 1$, with the exception of the top one.

Proof. It follows from Proposition 4.1, that any element in $G(I_{n,c}^{(m)})$ has degree $m+k(n-c)$ with $1 \leqslant k \leqslant m$. Then Corollary 3.4 proves (1).

Next we prove (2) and (3). Let $u \in G(I_{n,c}^{(m)})$ be a monomial of degree greater than or equal to $m+(m/2+1)(n-c)$. Corollary 3.4 says that, to prove (2), it is enough to show that $|C(u)| = c - 1$. By Proposition 4.1, u must be a monomial of the form

$$u = \sigma(x_1^{\lambda_1} \cdots x_{c-1}^{\lambda_{c-1}} x_c^{\lambda_c} x_{c+1}^{\lambda_c} \cdots x_n^{\lambda_c})$$

with $\lambda_1 + \cdots + \lambda_c = m$, $\lambda_c \geqslant (m/2 + 1)$ and $\sigma \in \mathfrak{S}_n$. This says $\lambda_{c-1} \leqslant m/2 - 1 \leqslant \lambda_c - 2$ and therefore $C(u) = \{x_{\sigma(1)}, \ldots, x_{\sigma(c-1)}\}$, as desired. Then statement (3) follows from (2) together with the fact that the monomials of degree $m(1 + m - c)$ in $G(I_{n,c}^{(m)})$ are precisely those in the set $\{\sigma(x_c^m x_{c+1}^m \cdots x_n^m) : \sigma \in \mathfrak{S}_n\}$.

To prove (4), let $m \leqslant c$. It is easy to see that the minimum degree of monomials in $G(I_{n,c}^{(m)})$ is $m + n - c$. Also, the monomials of degree $m + n - c$ in $G(I_{n,c}^{(m)})$ are precisely those in the set

$$X = \{\sigma(x_{c-m+1}x_{c-m+2} \ldots x_n) : \sigma \in \mathfrak{S}_n\}.$$

The monomials in X are solely responsible for the top row in the Betti table of $I_{n,c}^{(m)}$. Note that X is the set of all squarefree monomials of degree $m+n-c$. The Betti numbers of the ideal generated by X can be found in [15, Thm. 2.1], leading to the formula in (4).

Finally, to see (5), let q be the quotient of m divided by c. Observe that for every $q+1 < k \leqslant m$, there is a monomial $u_k \in G(I_{n,c}^{(m)})$ with $\deg(u_k) = m + k(n-c)$ of the form

$$u_k = x_1^{\lambda_1} \cdots x_{c-1}^{\lambda_{c-1}} x_c^k \cdots x_n^k$$

with $\lambda_1 \leqslant \cdots \leqslant \lambda_{c-1} \leqslant k-1$. Then $C(u_k) = \{x_1, \ldots, x_{c-1}\}$ and, by Corollary 3.4, row $m + k(n-c)$ in the Betti table of $I_{n,c}^{(m)}$ has length $c-1$. $\square$

The previous corollary immediately gives a closed formula for the Betti numbers of symbolic squares of star configurations. Note that this result was first established in [17, Theorem 3.2]. Our assumption about codimension eliminates degenerate cases where some minimal generators disappear along with corresponding rows of the Betti table.

Corollary 4.5. *If $c \geqslant 2$, then*

$$\beta_{i,i+j}(I_{n,c}^{(2)}) = \begin{cases} \binom{n}{c-2-i}\binom{n-c+1+i}{i}, & j = n-c+2, \\ \binom{n}{c-1}\binom{c-1}{i}, & j = 2(n-c+1). \end{cases}$$

Using Corollary 3.4, we can even give a closed formula for the Betti numbers of the symbolic cube of star configurations.

Corollary 4.6. *If $c \geqslant 3$, then*

$$\beta_{i,i+j}(I_{n,c}^{(3)}) = \begin{cases} \binom{n}{c-3-i}\binom{n-c+2+i}{i}, & j = n-c+3, \\ \binom{n}{c-2}\left(\binom{c-2}{i} + (n-c+1)\binom{c-1}{i}\right), & j = 2(n-c+1)+1, \\ \binom{n}{c-1}\binom{c-1}{i}, & j = 3(n-c+1). \end{cases}$$

Proof. The top and bottom row of the Betti table are computed as in Corollary 4.4.

By Proposition 4.1, the minimal generators of $I_{n,c}^{(3)}$ with degree $2(n-c+1)+1$ are the ones in the set

$$\{\sigma(x_{c-1}x_c^2x_{c+1}^2 \cdots x_n^2) : \sigma \in \mathfrak{S}_n\}.$$

In particular, these are monomials $u = \sigma(x^\lambda)$ with

$$\lambda = (\underbrace{0, \ldots, 0}_{c-2}, 1, \underbrace{2, \ldots, 2}_{n-c+1}),$$

and $p(\lambda) = c-2$, $r(\lambda) = n-c+1$. We also have $n-c+1 \leqslant \max(u) \leqslant n$. It follows that $|C(u)| = c-2$ if $\sigma(c-1) \geqslant \max(u)$, and $|C(u)| = c-1$ if $\sigma(c-1) < \max(u)$. We count how many monomials we have in each case.

To produce a monomial u with $|C(u)| = c - 2$, we can first choose which variables have degree zero. Among the remaining variables, the single one having degree one must appear last. This can be accomplished in $\binom{n}{c-2}$ ways.

To produce a monomial u with $|C(u)| = c - 1$, we can first choose which variables have degree zero. Next we can choose any one the remaining variables except the last one to appear with degree one. This can be accomplished in $\binom{n}{c-2}(n - c + 1)$ ways.

The statement now follows from Corollary 3.4. $\square$

Example 4.7. The Betti table of $I_{9,4}^{(3)}$ is

	0	1	2	3
total:	345	980	936	300
8:	9	8	.	.
9:	.	.	.	.
10:	.	.	.	.
11:	.	.	.	.
12:	.	.	.	.
13:	252	720	684	216
14:	.	.	.	.
15:	.	.	.	.
16:	.	.	.	.
17:	.	.	.	.
18:	84	252	252	84

Remark 4.8. It follows immediately from Corollary 4.6 that the third symbolic defect of $I_{n,c}$ is $\binom{n}{c-2}(n - c + 2)$. We refer the reader to [16] for the definition of symbolic defect and [16, Corollary 3.17] for a previously known bound.

Remark 4.9. The ideal $I_{n,n-1}$ can be thought of as the defining ideal of the set of the n points

$$e_1 = [1 : 0 : 0 : \ldots : 0], e_2 = [0 : 1 : 0 : \ldots : 0], \ldots, e_n = [0 : 0 : 0 : \ldots : 1] \in \mathbb{P}_{\Bbbk}^{n-1}$$

where $\mathbb{P}_{\Bbbk}^{n-1}$ denotes the $(n - 1)$-dimensional projective space over $\Bbbk$. Similarly, $I_{n,n-1}^{(m)}$ can be thought of as the ideal defining the fat point scheme $me_1 + me_2 + \cdots + me_n$. For an introduction to fat points, we invite the reader to consult [8]. If $\{p_1, \ldots, p_n\} \subset \mathbb{P}_{\Bbbk}^{n-1}$ is a set of n points in general linear position, then there is a linear automorphism of $\mathbb{P}_{\Bbbk}^{n-1}$ taking e_i to p_i. Algebraically, this corresponds to an invertible linear change of coordinates that preserves Betti numbers. In particular, it follows that the results of Corollary 4.4 provide information about the Betti numbers of the fat point scheme $mp_1 + mp_2 + \cdots + mp_n$ in $\mathbb{P}_{\Bbbk}^{n-1}$. For more complete information, the Betti numbers of this fat point scheme can be computed by combining Proposition 4.1 and our later Corollary 5.7. We are grateful to Brian Harbourne for clarifying this connection.

5. Decompositions of symmetric shifted ideals

For $r \leqslant n$, let $J_{[n],r}$ be the ideal of $S = \Bbbk[x_1, \dots, x_n]$ generated by all squarefree monomials of degree r. The ideal $J_{[n],r}$ is actually the same as the ideal $I_{n,n+1-r}$ in the previous section, but we introduce a new notation to simplify the proofs in Sections 5 and 6. Note that $J_{[n],r}$ is $\mathfrak{S}_n$-fixed and shifted. Its equivariant resolution has been described in [15, Theorem 4.11]. In the following two sections we extend this result to an arbitrary symmetric shifted ideal I (see Proposition 6.1). This will be done in two steps. In this section, we establish a coarse decomposition of $\mathrm{Tor}_i(I, \Bbbk)_{i+d}$ (see Theorem 5.5). This will be refined in the next section.

We need some further notation. Let $\mathrm{Mon}(S)$ be the set of all monomials in S. For monomial ideals $I \supset J$ of S, we write

$$\mathrm{Mon}(I/J) = \{u \in \mathrm{Mon}(S) : u \in I,\ u \notin J\}.$$

When both I and J are $\mathfrak{S}_n$-fixed, we write

$$\mathrm{P}(I/J) = \{\lambda \in \mathrm{P}_n : \lambda \in \mathrm{P}(I),\ \lambda \notin \mathrm{P}(J)\}.$$

We note that $\mathrm{Mon}(I/J)$ is a $\Bbbk$-basis of I/J.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition and $p = p(\lambda)$. The following S-module N^λ plays an important role in our results:

$$N^\lambda = (\sigma(x^\lambda) : \sigma \in \mathfrak{S}_n)/(\sigma(x_i x^\lambda) : 1 \leqslant i \leqslant p, \sigma \in \mathfrak{S}_n).$$

We start by discussing some basic properties of the module N^λ. For $A = \{i_1, \dots, i_k\} \subset [n]$ with $i_1 < \dots < i_k$, we write $x_A = x_{i_1} \cdots x_{i_k}$, $\overline{A} = [n] \setminus A$, $\mathfrak{S}_A$ for the set of permutations on A, $S_A = \Bbbk[x_{i_1}, \dots, x_{i_k}]$, $\mathfrak{m}_A = (x_{i_1}, \dots, x_{i_k}) \subset S_A$ the maximal ideal of S_A, and $J_{A,r} \subset S_A$ the ideal of S_A generated by all squarefree monomials of degree r in S_A. For $\mathbf{a} = (a_1, \dots, a_k)$, we write as above $x_A^{\mathbf{a}} = x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$.

We set out to describe $\mathrm{Mon}(N^\lambda)$ starting with a preliminary example.

Example 5.1. Let $\lambda = (0, 1, 1, 2, 2, 3, 3)$, so $p = p(\lambda) = 3$ and $r = r(\lambda) = 2$. In this case, $N^\lambda = I/J$ where

$$\begin{aligned} I &= (\sigma(x_1^0 x_2^1 x_3^1 x_4^2 x_5^2 x_6^3 x_7^3) : \sigma \in \mathfrak{S}_n), \\ J &= (\sigma(x_1^1 x_2^1 x_3^1 x_4^2 x_5^2 x_6^3 x_7^3) : \sigma \in \mathfrak{S}_n) + (\sigma(x_1^0 x_2^1 x_3^2 x_4^2 x_5^2 x_6^3 x_7^3) : \sigma \in \mathfrak{S}_n). \end{aligned}$$

The monomial $x_2 x_3 x_4^2 x_5^2 x_6^3 x_7^3$ is an example of a monomial in I but not in J. We can represent it as $m(x_4 x_5 x_6 x_7)^2 u$ where $m = x_2 x_3$ and $u = x_6 x_7$. This splits the indices of the variables into two sets: $A = \{1, 2, 3\}$ and its complement $\overline{A} = \{4, 5, 6, 7\}$. Note that $m = x_A^{\lambda_{\leqslant p}}$ is in the polynomial ring $S_A = \Bbbk[x_1, x_2, x_3]$, while u is one of the minimal generators of $J_{\overline{A},r}$ in $S_{\overline{A}} = \Bbbk[x_4, x_5, x_6, x_7]$. Also, the middle term $(x_{\overline{A}})^2 = (x_4x_5x_6x_7)^2$ has exponent $\lambda_7 - 1$. Now notice that we can replace m with any permutation $\sigma(x_A^{\lambda_{\leqslant p}})$ where $\sigma \in \mathfrak{S}_A$ and still obtain a monomial in I and not in J; for example,

$$x_1x_2x_4^2x_5^2x_6^3x_7^3 = (x_1x_2)(x_4x_5x_6x_7)^2u.$$

Similarly, we can replace u by another generator (in fact, any monomial) of $J_{\overline{A},r}$ and still obtain a monomial in I and not in J; for example,

$$x_2x_3x_4^2x_5^3x_6^2x_7^3 = m(x_4x_5x_6x_7)^2(x_5x_7),$$
$$x_2x_3x_4^2x_5^3x_6^4x_7^5 = m(x_4x_5x_6x_7)^2(x_5x_6x_7^2).$$

In addition, we could operate the same reasoning on any monomial obtained by permuting the variables in $x_2x_3x_4^2x_5^2x_6^3x_7^3$, leading to a similar split but with a different choice of index set A. As we illustrate next, all elements of $\mathrm{Mon}(N^\lambda)$ can be obtained by combining these observations.

Lemma 5.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, $p = p(\lambda)$ and $r = r(\lambda)$. Then*

$$\mathrm{Mon}(N^\lambda) = \biguplus_{A\subset[n],\ |A|=p} \left(\biguplus_{m\in\mathfrak{S}_A\cdot x_A^{\lambda_{\leqslant p}}} \{m(x_{\overline{A}})^{\lambda_n-1}u : u \in \mathrm{Mon}(J_{\overline{A},r})\} \right), \tag{5.1}$$

and

$$\mathrm{P}(N^\lambda) = \{(\lambda_1, \ldots, \lambda_p, \mu_{p+1}, \ldots, \mu_n) \in \mathrm{P}_n : \mu_{p+1} \geqslant \lambda_n - 1,\ \mu_{n-r+1} \geqslant \lambda_n\}. \tag{5.2}$$

Proof. Equation (5.2) easily follows from (5.1). Hence, we only need to show (5.1). We first prove the inclusion "⊂". Let $u \in \mathrm{Mon}(N^\lambda)$. Then there is $\sigma \in \mathfrak{S}_n$ such that $\sigma(x^\lambda) = x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n}$ divides u. We write $u = x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n}$. Since u is not contained in the ideal $(\sigma(x_ix^\lambda) : 1 \leqslant i \leqslant p, \sigma \in \mathfrak{S}_n)$, we have $a_1 = \lambda_1, \ldots, a_p = \lambda_p$. Also, since $\sigma(x^\lambda)$ divides u, $a_k \geqslant \lambda_n - 1$ for $p < k \leqslant n - r$, and $a_k \geqslant \lambda_n$ for $k \geqslant n - r + 1$. These inequalities imply that, by setting $A = \{\sigma(1), \ldots, \sigma(p)\}$,

$$u = \sigma(x_1^{\lambda_1} \cdots x_p^{\lambda_p})(x_{\overline{A}})^{\lambda_n-1}(x_{\sigma(n-r+1)} \cdots x_{\sigma(n)})w$$

for some $w \in \mathrm{Mon}(S_{\overline{A}})$, which shows that u is contained in the right hand side of (5.1).

Next, we prove the inclusion "⊃" in (5.1). Let $u = m(x_{\overline{A}})^{\lambda_n-1}w$ with $m \in \mathfrak{S}_A \cdot x_A^{\lambda_{\leqslant p}}$ and $w \in \mathrm{Mon}(J_{\overline{A},r})$. By taking a permutation $\tau \in \mathfrak{S}_n$ appropriately,

$$\tau(u) = x_1^{\lambda_1} \cdots x_p^{\lambda_p}x_{p+1}^{\lambda_n-1} \cdots x_n^{\lambda_n-1}\alpha \tag{5.3}$$

with $\alpha \in \mathrm{Mon}(J_{\overline{[p]},r})$. Moreover, we may choose τ so that α is divisible by $x_{n-r+1} \cdots x_n$. Then x^λ divides $\tau(u)$ and $u \in (\sigma(x^\lambda) : \sigma \in \mathfrak{S}_n)$. We claim that u is not contained in the ideal $J = (\sigma(x_i x^\lambda) : 1 \leqslant i \leqslant p, \sigma \in \mathfrak{S}_n)$. We already see in (5.3) that if $\mu = \mathrm{type}(u)$ then μ is of the form

$$\mu = (\lambda_1, \dots, \lambda_p, \mu_{p+1}, \dots, \mu_n).$$

Observe that

$$\Lambda(J) = \{\lambda + e_i : 1 \leqslant i \leqslant p, \lambda + e_i \in \mathrm{P}_n\},$$

where e_i is the i-th standard basis vector of $\mathbb{Z}^n$. Since no element in $\Lambda(J)$ divides μ, by Lemma 2.1 the monomial $u \in \mathfrak{S}_n \cdot x^\mu$ is not contained in J.

We finally show that the right-hand side of (5.1) is indeed a disjoint union. To show this, it is enough to prove that for each $u = x_1^{a_1} \cdots x_n^{a_n}$ that is contained in the right-hand side of (5.1) there is a unique subset $A \subset [n]$ with $|A| = p$ and $m \in \mathfrak{S}_A \cdot x_A^{\lambda_{\leqslant p}}$ such that $u = m\alpha$ with $\alpha \in S_{\overline{A}}$. Indeed, since $|\{k : a_k < \lambda_n - 1\}| = p$ by the shape of monomials in the right-hand side of (5.1), such a set A must be equal to the set $\{k : a_k < \lambda_n - 1\}$, and a monomial m must be $\prod_{i \in A} x_i^{a_i}$. $\square$

Next, we decompose N^λ into smaller modules which have a simpler structure but are not fixed by the action of $\mathfrak{S}_n$. Let $\lambda \in \mathrm{P}_n$ be a partition, $p = p(\lambda)$ and $r = r(\lambda)$. For $A \subset [n]$ with $|A| = p$ and $m \in \mathfrak{S}_A \cdot x_A^{\lambda_{\leqslant p}}$, we define

$$N^\lambda_{A,m} = \left(m\tau \left(x_{\overline{A}}^{(\lambda_{p+1},\dots,\lambda_n)} \right) : \tau \in \mathfrak{S}_{\overline{A}} \right) / \left(x_i m\tau \left(x_{\overline{A}}^{(\lambda_{p+1},\dots,\lambda_n)} \right) : \tau \in \mathfrak{S}_{\overline{A}},\ i \in A \right).$$

Recall that, for $X \subset [n]$, $J_{X,r}$ is the monomial ideal of S_X generated by all squarefree monomials of degree r in S_X and $\mathfrak{m}_X = (x_i : i \in X)$ is the maximal ideal of S_X. Since

$$(\lambda_{p+1}, \dots, \lambda_n) = (\lambda_n - 1, \dots, \lambda_n - 1, \lambda_n, \dots, \lambda_n)$$

where λ_n appears r times, $N^\lambda_{A,m}$ is generated by monomials $\{m(x_{\overline{A}})^{\lambda_n - 1} x_T : T \subset \overline{A}, |T| = r\}$ and every monomial in $N^\lambda_{A,m}$ is divisible by $m(x_{\overline{A}})^{\lambda_n - 1}$. Thus, by the map $f \to f/(m(x_{\overline{A}})^{\lambda_n - 1})$, we have an isomorphism

$$\begin{aligned} N^\lambda_{A,m} &\cong (x_T : T \subset \overline{A}, |T| = r)/(x_i x_T : T \subset \overline{A}, |T| = r, i \in A) \\ &\cong \big(S/(x_i : i \in A)\big) \otimes_S (J_{\overline{A},r} S) \\ &\cong S_A/\mathfrak{m}_A \otimes_{\Bbbk} J_{\overline{A},r} \end{aligned} \tag{5.4}$$

where we consider that the last module is a module over $S = S_A \otimes_{\Bbbk} S_{\overline{A}}$.

Lemma 5.3. *Let $\lambda \in \mathrm{P}_n$ be a partition, $p = p(\lambda)$, $r = r(\lambda)$, $A \subset [n]$ with $|A| = p$, and $m \in \mathfrak{S}_A \cdot x_A^{\lambda_{\leqslant p}}$. Then*

(i) $\mathrm{Mon}(N^\lambda_{A,m}) = \{m(x_{\overline{A}})^{\lambda_n-1}u : u \in \mathrm{Mon}(J_{\overline{A},r})\}$.
(ii) $N^\lambda_{A,m}$ *is an* S*-submodule of* N^λ.
(iii) $N^\lambda = \bigoplus_{A\subset[n],|A|=p} \bigoplus_{m\in\mathfrak{S}_A\cdot x_A^{\lambda_{\leqslant p}}} N^\lambda_{A,m}$ *(as* S*-modules).*

Proof. Statement (i) follows from (5.4). To prove (ii), it is enough to show that for any $\{j_1,\dots,j_r\} \subset \overline{A}$, one has

$$\mathrm{ann}_{N^\lambda}(m(x_{\overline{A}})^{\lambda_n-1}x_{j_1}\cdots x_{j_r}) = (x_i : i \in A),$$

where $\mathrm{ann}_M(h) = \{f \in S : hf = 0\}$ for an S-module M and $h \in M$. The inclusion "$\supset$" is clear from the definition of $N^\lambda_{A,m}$. To see the inclusion "$\subset$", we must prove that for any monomial u in $S_{\overline{A}}$, $m(x_{\overline{A}})^{\lambda_n-1}x_{j_1}\cdots x_{j_r}u$ is non-zero in N^λ, and this follows from Lemma 5.2.

Statement (iii) follows from (i) and Lemma 5.2. □

Corollary 5.4. *The* S*-modules* $N^\lambda_{A,m}$ *and* N^λ *have linear resolutions.*

Proof. By the isomorphism in (5.4), the tensor product of a minimal graded free resolution of $J_{\overline{A},r}$ and one of $S_A/\mathfrak{m}_A$ is isomorphic to a minimal graded free resolution of $N^\lambda_{A,m}$. Since $J_{\overline{A},r}$ and $S_A/\mathfrak{m}_A$ have linear resolutions, the module $N^\lambda_{A,m}$ has a linear resolution. Then N^λ also has a linear resolution by Lemma 5.3(iii). □

We now prove the main result of this section.

Theorem 5.5. *If* $I \subset S$ *is a symmetric shifted ideal, then as* $\Bbbk[\mathfrak{S}_n]$*-modules we have*

$$\mathrm{Tor}_i(I,\Bbbk)_{i+d} \cong \bigoplus_{\lambda\in\Lambda(I),\ |\lambda|=d} \mathrm{Tor}_i(N^\lambda,\Bbbk).$$

To prove the above theorem, we first show the following statement.

Lemma 5.6. *Let* I *be a symmetric shifted ideal and* $\Lambda(I) = \{\lambda^{(1)},\dots,\lambda^{(t)}\}$ *with* $\lambda^{(1)} <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} \lambda^{(t)}$. *Let* $I_{\leqslant k} \subseteq I$ *be the* $\mathfrak{S}_n$*-fixed monomial ideal with* $\Lambda(I_{\leqslant k}) = \{\lambda^{(1)},\dots,\lambda^{(k)}\}$ *for* $k = 1,2,\dots,t$. *Then, for* $k = 1,2,\dots,t$,

$$I_{\leqslant k}/I_{\leqslant k-1} \cong N^{\lambda^{(k)}}$$

as S*-modules, where* $I_{\leqslant 0} = (0)$.

Proof. Observe that $I_{\leqslant k}$ is shifted and

$$I_{\leqslant k}/I_{\leqslant k-1} = \left(\sigma(x^{\lambda^{(k)}}) : \sigma \in \mathfrak{S}_n\right) / \left((\sigma(x^{\lambda^{(k)}}) : \sigma \in \mathfrak{S}_n) \cap I_{\leqslant k-1}\right).$$

Since for monomial ideals I, J, J' with $I \supset J$ and $I \supset J'$, we have $I/J = I/J'$ if and only if $\mathrm{Mon}(I/J) = \mathrm{Mon}(I/J')$, it is enough to prove $\mathrm{P}(I_{\leqslant k}/I_{\leqslant k-1}) = \mathrm{P}(N^{\lambda^{(k)}})$. Let $\lambda^{(k)} = (\lambda_1, \dots, \lambda_n)$, $p = p(\lambda^{(k)})$ and $r = r(\lambda^{(k)})$. Then we have

$$\begin{aligned}\mathrm{P}(I_{\leqslant k}/I_{\leqslant k-1}) &= \mathrm{P}(I_{\leqslant k}) \setminus \mathrm{P}(I_{\leqslant k-1})\\ &= \{\mu = (\mu_1, \dots, \mu_n) \in \mathrm{P}_n : \lambda^{(k)} \text{ divides } \mu, \lambda^{(k)}_{\leqslant p} = \mu_{\leqslant p}\}\\ &= \{(\lambda_1, \dots, \lambda_p, \mu_{p+1}, \dots, \mu_n) \in \mathrm{P}_n : \mu_{p+1} \geqslant \lambda_n - 1,\ \mu_{n-r+1} \geqslant \lambda_n\}\\ &= \mathrm{P}(N^{\lambda^{(k)}}),\end{aligned}$$

where we use Lemma 2.7 for the third equality and Lemma 5.2 for the last one. □

Proof of Theorem 5.5. Let $\Lambda(I) = \{\lambda^{(1)}, \dots, \lambda^{(t)}\}$ with $\lambda^{(1)} <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} \lambda^{(t)}$ and let $I_{\leqslant k}$ be as in Lemma 5.6. Then we have the short exact sequence

$$0 \longrightarrow I_{\leqslant k-1} \longrightarrow I_{\leqslant k} \longrightarrow I_{\leqslant k}/I_{\leqslant k-1} \cong N^{\lambda^{(k)}} \longrightarrow 0. \tag{5.5}$$

We prove $\mathrm{Tor}_i(I_{\leqslant k}, \Bbbk) \cong \bigoplus_{l=1}^{k} \mathrm{Tor}_i(N^{\lambda^{(l)}}, \Bbbk)$ using induction on k. Note that by Corollary 5.4 this implies the desired statement.

By the definition of the shifted property, the partition $\lambda^{(1)}$ must be a partition of the form $\lambda^{(1)} = (a, a, \dots, a, a+1, \dots, a+1)$. Thus, $p(\lambda^{(1)}) = 0$ and

$$I_{\leqslant 1} = (\sigma(x^{\lambda^{(1)}}) : \sigma \in \mathfrak{S}_n) = N^{\lambda^{(1)}}.$$

Hence, the assertion holds when $k = 1$.

Suppose $k > 1$. Since $|\lambda^{(1)}| \leqslant \cdots \leqslant |\lambda^{(t)}|$, using the inductive hypothesis we get

$$\mathrm{reg}(I_{\leqslant k-1}) = \max\{|\lambda^{(1)}|, \dots, |\lambda^{(k-1)}|\} \leqslant |\lambda^{(k)}|,$$

because $N^{\lambda^{(l)}}$ is generated in degree $|\lambda^{(l)}|$ and has a linear resolution by Corollary 5.4. The short exact sequence in (5.5) induces the exact sequence

$$\begin{aligned}&\mathrm{Tor}_{i+1}(N^{\lambda^{(k)}}, \Bbbk)_{i+1+(j-1)} \longrightarrow \mathrm{Tor}_i(I_{\leqslant k-1}, \Bbbk)_{i+j} \longrightarrow \mathrm{Tor}_i(I_{\leqslant k}, \Bbbk)_{i+j}\\ &\longrightarrow \mathrm{Tor}_i(N^{\lambda^{(k)}}, \Bbbk)_{i+j} \longrightarrow \mathrm{Tor}_{i-1}(I_{\leqslant k-1}, \Bbbk)_{i-1+(j+1)}.\end{aligned} \tag{5.6}$$

Let $d = |\lambda^{(k)}|$. By Corollary 5.4,

$$\mathrm{Tor}_i(N^{\lambda^{(k)}}, \Bbbk)_{i+j} = 0 \quad \text{for } j \neq d. \tag{5.7}$$

Also, since $\mathrm{reg}(I_{\leqslant k-1}) \leqslant d$,

$$\mathrm{Tor}_{i-1}(I_{\leqslant k-1}, \Bbbk)_{i-1+(j+1)} = 0 \quad \text{for } j \geqslant d. \tag{5.8}$$

Then (5.6), (5.7) and (5.8) imply

$$\mathrm{Tor}_i(I_{\leqslant k}, \Bbbk)_{i+d} \cong \mathrm{Tor}_i(I_{\leqslant k-1}, \Bbbk)_{i+d} \bigoplus \mathrm{Tor}_i(N^{\lambda^{(k)}}, \Bbbk)_{i+d}$$

and

$$\mathrm{Tor}_i(I_{\leqslant k}, \Bbbk)_{i+j} \cong \mathrm{Tor}_i(I_{\leqslant k-1}, \Bbbk)_{i+j} \quad \text{for } j \neq d.$$

These isomorphisms prove the desired statement. $\square$

Using Theorem 5.5, it is possible to give a closed formula of graded Betti numbers of a symmetric shifted ideal I in terms of its partition generator $\Lambda(I)$. Let $\lambda \in \mathrm{P}_n$ with $|\lambda| = d$, $p = p(\lambda)$ and $r = r(\lambda)$. Then by (5.4) we have

$$\begin{aligned}\beta_{i,i+d}(N^{\lambda}_{A,m}) &= \beta_i(J_{\overline{A},r} \otimes_{\Bbbk} (S_A/\mathfrak{m}_A)) \\ &= \sum_{k+l=i} \beta_k(J_{\overline{A},r})\beta_l(S_A/\mathfrak{m}_A) \\ &= \sum_{k+l=i} \binom{n-p}{r+k}\binom{r+k-1}{k}\binom{p}{l},\end{aligned}$$

where we use the fact that

$$\beta_k(J_{\overline{A},r}) = \binom{n-p}{r+k}\binom{r+k-1}{k}$$

(see, e.g., [15, Theorem 2.1]). For $c = (c_1, c_2, \dots, c_n) \in \mathbb{Z}^n_{\geqslant 0}$, let $c! = c_1!c_2! \cdots c_n!$. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with $|\lambda| = d \geqslant 0$, its **type** $\mathrm{type}(c) = (t_0, t_1, \dots, t_d)$ is defined by $t_i = |\{k : \lambda_k = i\}|$. It is well-known that

$$|\mathfrak{S}_n \cdot x^{\lambda}| = \frac{n!}{\mathrm{type}(\lambda)!}.$$

Hence, by Lemma 5.3(iii)

$$\begin{aligned}\beta_i(N^{\lambda}) &= \binom{n}{p}\frac{p!}{\mathrm{type}(\lambda_{\leqslant p})!}\beta_i(N^{\lambda}_{A,m}) \\ &= \sum_{k+l=i} \frac{p!}{\mathrm{type}(\lambda_{\leqslant p})!}\binom{n}{p}\binom{n-p}{r+k}\binom{r+k-1}{k}\binom{p}{l}.\end{aligned}$$

Thus, using Theorem 5.5, we obtain the following formula, which may be viewed as a more explicit version of the formula given in Corollary 3.4.

Corollary 5.7. *If $I \subset S$ is a symmetric shifted ideal, then*

$$\beta_{i,i+d}(I) = \sum_{\lambda \in \Lambda(I), |\lambda|=d} \left(\sum_{k+l=i} \frac{p(\lambda)!}{\mathrm{type}(\lambda_{\leqslant p(\lambda)})!} \binom{n}{p(\lambda)} \binom{n-p(\lambda)}{r(\lambda)+k} \binom{r(\lambda)+k-1}{k} \binom{p(\lambda)}{l} \right).$$

Example 5.8. Let $I = J^{(2)}_{[n],r} (= I^{(2)}_{n,n+1-r})$. Then I is generated by two partitions $\lambda = (0^{n-r-1}, 1^{r+1})$ and $\mu = (0^{n-r}, 2^r)$, where a^i denotes $(a, a, \dots, a) \in \mathbb{Z}^i$. In this case, $p(\lambda) = 0$, $r(\lambda) = r+1$, $p(\mu) = n-r$ and $r(\mu) = r$. Corollary 5.7 says

$$\beta_{i,i+r+1}(I) = \sum_{k+l=i} \frac{0!}{0!} \binom{n}{0} \binom{n}{r+1+k} \binom{r+k-1}{k} \binom{0}{l} = \binom{n}{r+1+i} \binom{r+i-1}{i}$$

and

$$\beta_{i,i+2r}(I) = \sum_{k+l=i} \frac{(r+1)!}{(r+1)!} \binom{n}{n-r} \binom{n-n+r}{r+k} \binom{r+k-1}{k} \binom{n-r}{l} = \binom{n}{n-r} \binom{n-r}{i}.$$

This recovers Corollary 4.5.

6. Equivariant Betti numbers

While Corollary 5.7 gives a closed formula for the graded Betti numbers of symmetric shifted ideals, the formula is not simple. To understand these numbers better, we refine the decomposition of $\mathrm{Tor}_i(I, \Bbbk)$ given in Theorem 5.5. In this section, we give an explicit description of the $\Bbbk[\mathfrak{S}_n]$-module structure of $\mathrm{Tor}_i(I, \Bbbk)$ for a symmetric shifted ideal I by using Theorem 5.5, and explain how it helps to determine Betti numbers of these ideals by examples. We refer the reader to [33] for some basics on representation theory, such as induced representations and Specht modules.

For a monomial $m \in S_A$, let $\mathcal{M}_A(m) = \mathrm{span}_{\Bbbk}\{\sigma(m) : \sigma \in \mathfrak{S}_A\}$. We denote by $\mathrm{Ind}^{\mathfrak{S}_{k+l}}_{\mathfrak{S}_k \times \mathfrak{S}_l}(K \boxtimes K')$ the induced representation of the tensor product of a $\Bbbk[\mathfrak{S}_k]$-module K and a $\Bbbk[\mathfrak{S}_l]$-module K'. Let I be a symmetric shifted ideal. By Theorem 5.5, we know $\mathrm{Tor}_i(I, \Bbbk) \cong \bigoplus_{\lambda \in \Lambda(I)} \mathrm{Tor}_i(N^\lambda, \Bbbk)$. Thus, to understand the $\Bbbk[\mathfrak{S}_n]$-module structure of $\mathrm{Tor}_i(I, \Bbbk)$ it is enough to consider the $\Bbbk[\mathfrak{S}_n]$-module structure of $\mathrm{Tor}_i(N^\lambda, \Bbbk)$.

Let $\lambda \in \mathrm{P}_n$, $p = p(\lambda)$ and $r = r(\lambda)$. For each subset $A \subset [n]$ with $|A| = p$, fix a permutation $\rho_A \in \mathfrak{S}_n$ such that $\rho_A([p]) = A$. The set $\{\rho_A \in \mathfrak{S}_n : A \subset [n], |A| = p\}$ is a set of representatives of $\mathfrak{S}_n / (\mathfrak{S}_p \times \mathfrak{S}_{n-p})$.

By Lemma 5.3(iii) and (5.4), we have an isomorphism (up to shift of degrees)

$$\begin{aligned} N^\lambda &= \bigoplus_{A \subset [n],\ |A|=p} \ \bigoplus_{m \in \mathfrak{S}_A \cdot x_A^{\lambda_{\leqslant p}}} N^\lambda_{A,m} \\ &\cong \bigoplus_{A \subset [n],\ |A|=p} \left(\bigoplus_{m \in \mathfrak{S}_A \cdot x_A^{\lambda_{\leqslant p}}} \left(m (S_A/\mathfrak{m}_A) \otimes_{\Bbbk} (x_{\overline{A}})^{\lambda_n - 1} J_{\overline{A},r}\right) \right) \end{aligned}$$

$$
\begin{aligned}
&\cong \bigoplus_{A\subset[n],\ |A|=p} \left[\left(\mathcal{M}_A(x_A^{\lambda_{\leqslant p}}) \otimes_{\Bbbk} (S_A/\mathfrak{m}_A)\right) \otimes_{\Bbbk} (x_{\overline{A}})^{\lambda_n-1} J_{\overline{A},r}\right] \\
&\cong \bigoplus_{A\subset[n],\ |A|=p} \rho_A\left[\left(\mathcal{M}_{[p]}(x_{[p]}^{\lambda_{\leqslant p}}) \otimes_{\Bbbk} (S_{[p]}/\mathfrak{m}_{[p]})\right) \otimes_{\Bbbk} \left((x_{p+1}\cdots x_n)^{\lambda_n-1} J_{\overline{[p]},r}\right)\right] \\
&\cong \operatorname{Ind}_{\mathfrak{S}_p\times\mathfrak{S}_{n-p}}^{\mathfrak{S}_n}\left[\left(\mathcal{M}_{[p]}(x_{[p]}^{\lambda_{\leqslant p}}) \otimes_{\Bbbk} S_{[p]}/\mathfrak{m}_{[p]}\right) \boxtimes ((x_{p+1}\cdots x_n)^{\lambda_n-1} J_{\overline{[p]},r})\right] \\
&\cong \operatorname{Ind}_{\mathfrak{S}_p\times\mathfrak{S}_{n-p}}^{\mathfrak{S}_n}\left[\left(\mathcal{M}_{[p]}(x_{[p]}^{\lambda_{\leqslant p}}) \otimes_{\Bbbk} S_{[p]}/\mathfrak{m}_{[p]}\right) \boxtimes J_{\overline{[p]},r}\right], \qquad (6.1)
\end{aligned}
$$

where $\mathcal{M}_{[p]}(x_{[p]}^{\lambda_{\leqslant p}}) \otimes_{\Bbbk} S_{[p]}/\mathfrak{m}_{[p]} = \Bbbk$ if $p = 0$. Hence, we conclude that N^λ is isomorphic to the module (6.1) as $\Bbbk[\mathfrak{S}_n]$-modules. Note that as an $S_{[p]}$-module, $\mathcal{M}_{[p]}(x_{[p]}^{\lambda_{\leqslant p}}) \otimes_{\Bbbk} S_{[p]}/\mathfrak{m}_{[p]}$ is the direct sum of $|\mathfrak{S}_p \cdot x_{[p]}^{\lambda_{\leqslant p}}|$ copies of $S_{[p]}/\mathfrak{m}_{[p]}$. Recall that, for an S_A-module N and an $S_{\overline{A}}$-module M, there is an isomorphism

$$
\operatorname{Tor}_i^S(N \otimes_{\Bbbk} M, \Bbbk) \cong \bigoplus_{k+l=i} \operatorname{Tor}_k^{S_A}(N, \Bbbk) \otimes_{\Bbbk} \operatorname{Tor}_l^{S_{\overline{A}}}(M, \Bbbk).
$$

Then the decomposition in (6.1) shows that we have an isomorphism of $\Bbbk[\mathfrak{S}_n]$-modules

$$
\begin{aligned}
\operatorname{Tor}_i^S(N^\lambda, \Bbbk) \cong \bigoplus_{k+l=i} \Big[&\operatorname{Ind}_{\mathfrak{S}_p\times\mathfrak{S}_{n-p}}^{\mathfrak{S}_n}\left(\mathcal{M}_{[p]}(x_{[p]}^{\lambda_{\leqslant p}}) \otimes_{\Bbbk} \operatorname{Tor}_k^{S_{[p]}}(S_{[p]}/\mathfrak{m}_{[p]})\right) \\
&\boxtimes \left(\operatorname{Tor}_l^{S_{\overline{[p]}}}(I_{\overline{[p]},r}, \Bbbk)\right)\Big].
\end{aligned}
$$

Let S^λ be the Specht module associated to the partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ with $\lambda_1 > 0$ (see, e.g., [33, §2.3] or [15, §3]). For an integer $l \geqslant p$, set

$$
U_l^\lambda = \operatorname{Ind}_{\mathfrak{S}_p\times\mathfrak{S}_{l-p}}^{\mathfrak{S}_l} S^\lambda \boxtimes S^{(l-p)}.
$$

Galetto [15, Corollary 4.12] proved

$$
\operatorname{Tor}_i^{S_{[n]}}(J_{[n],r}, \Bbbk) \cong U_n^{(1^i,r)} \qquad (6.2)
$$

as $\Bbbk[\mathfrak{S}_n]$-modules. This says

$$
\operatorname{Tor}_i^{S_{[p]}}(S_{[p]}/\mathfrak{m}_{[p]}, \Bbbk) \boxtimes \operatorname{Tor}_l^{S_{\overline{[p]}}}(J_{\overline{[p]},r}, \Bbbk) \cong U_p^{(1^i)} \boxtimes U_{n-p}^{(1^i,r)}
$$

as $\Bbbk[\mathfrak{S}_p \times \mathfrak{S}_{n-p}]$-modules. Combining all these facts, we get the following.

Proposition 6.1. *Let $\lambda \in \mathrm{P}_n$, $p = p(\lambda)$ and $r = r(\lambda)$. As $\Bbbk[\mathfrak{S}_n]$-modules,*

$$
\operatorname{Tor}_i(N^\lambda, \Bbbk) \cong \bigoplus_{k+l=i}\left(\operatorname{Ind}_{\mathfrak{S}_p\times\mathfrak{S}_{n-p}}^{\mathfrak{S}_n}\left(\mathcal{M}_{[p]}(x_{[p]}^{\lambda_{\leqslant p}}) \otimes_{\Bbbk} U_p^{(1^k)}\right) \boxtimes U_{n-p}^{(1^l,r)}\right).
$$

We note that $\mathcal{M}(x^\lambda)$ is isomorphic to a $\Bbbk[\mathfrak{S}_n]$-module known as a permutation module [33, §2.1].

Theorem 6.2. *Let I be a symmetric shifted ideal. Then as $\Bbbk[\mathfrak{S}_n]$-modules*

$$\operatorname{Tor}_i(I,\Bbbk)_{i+d} \cong \bigoplus_{\substack{\lambda\in\Lambda(I)\\ |\lambda|=d}} \bigoplus_{k+l=i} \left(\operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{p(\lambda)}\times\mathfrak{S}_{n-p(\lambda)}} \left(\mathcal{M}_{[p(\lambda)]}(x^{\lambda_{\leqslant p(\lambda)}}_{[p(\lambda)]}) \otimes_{\Bbbk} U^{(1^k)}_{p(\lambda)}\right) \boxtimes U^{(1^l,r(\lambda))}_{n-p(\lambda)}\right).$$

In the rest of this section, we explain how Theorem 6.2 is useful to write down Betti numbers of symmetric shifted ideals. To do this, we identify S^λ with the Ferrers diagram corresponding to partition λ. Also, for simplicity, we write

$$\operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_p\times\mathfrak{S}_{n-p}} N\boxtimes M = N\boxtimes M \text{ and } \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_p\times\mathfrak{S}_{n-p}} N\boxtimes S^{(n-p)} = N_{\uparrow n}.$$

By Theorem 6.2, the $\Bbbk[\mathfrak{S}_n]$-module structure of $\operatorname{Tor}(N^\lambda,\Bbbk)$ only depends on $p(\lambda), r(\lambda)$ and $\lambda_{\leqslant p(\lambda)}$. We write

$$\operatorname{info}(\lambda) = (p(\lambda), r(\lambda), \lambda_{\leqslant p(\lambda)}).$$

Example 6.3. Let $I = J_{[n],r} \subset \Bbbk[x_1,\dots,x_n]$ be the monomial ideal generated by all squarefree monomials of degree r. As we already mentioned in (6.2), we have

$$\operatorname{Tor}_i(I,\Bbbk) \cong U^{(1^i,r)}_n$$

for all i. Here we check that our formula in Theorem 6.2 coincides with this. In this case, $\Lambda(I) = \{(0^{n-r},1^r)\}$. Let $\lambda = (0^{n-r},1^r)$. Then since $\operatorname{info}(\lambda) = (0,r,\emptyset)$, we have

$$\bigoplus_{k+l=i} \left(\mathcal{M}_{[p(\lambda)]}(x^{\lambda_{\leqslant p(\lambda)}}) \otimes_{\Bbbk} U^{(1^k)}_{p(\lambda)}\right) \boxtimes U^{(1^l,r(\lambda))}_{n-p(\lambda)} = U^{(1^i,r)}_n$$

and Theorem 6.2 yields

$$\operatorname{Tor}_i(I,\Bbbk) \cong \operatorname{Tor}_i(N^\lambda,\Bbbk) \cong U^{(1^i,r)}_n.$$

Graded Betti numbers of an S-module N are often presented by a Betti table, i.e., the table whose (i,j)-th entry is $\beta_{i,i+j}(N)$.

For a module N^λ and a symmetric shifted ideal, we present their graded Betti numbers by the table whose (i,j)-th entry is the $\Bbbk[\mathfrak{S}_n]$-module given in Theorem 6.2. We call such table an **equivariant Betti table**.

For example, the equivariant Betti table of $I_{6,3}$ is

	0	1	2	3
3	$\ydiagram{3}_{\uparrow 6}$	$\ydiagram{1,3}_{\uparrow 6}$	$\ydiagram{1,1,3}_{\uparrow 6}$	$\ydiagram{1,1,1,3}$

Example 6.4. Let $I = J^{(2)}_{[n],r}$ be the second symbolic power of the squarefree Veronese ideal with $n \geqslant r+1$. Then $\Lambda(I) = \{\lambda, \mu\}$, where $\lambda = (0^{n-r-1}, 1^{r+1})$ and $\mu = (0^{n-r}, 2^r)$. Using $\operatorname{info}(\lambda) = (0, r+1, \emptyset)$ and $\operatorname{info}(\mu) = (n-r, r, (0^{n-r}))$, we obtain

$$\operatorname{Tor}_i(N^\lambda, \Bbbk) \cong U_n^{(1^i, r+1)}$$

and

$$\operatorname{Tor}_i(N^\mu, \Bbbk) \cong \bigoplus_{k+l=i} \left(\left(\mathcal{M}_{[n-r]}(x^{(0^{n-r})}) \otimes U_{n-r}^{(1^k)} \right) \boxtimes U_r^{(1^l, r)} \right) = U_{n-r}^{(1^i)} \boxtimes U_r^{(r)} = (U_{n-r}^{(1^i)})_{\uparrow n}.$$

The equivariant Betti table of N^λ and N^μ when $n = 6$ and $r = 3$ are:

N^λ	0	1	2
4	$\ydiagram{4}_{\uparrow 6}$	$\ydiagram{1,4}_{\uparrow 6}$	$\ydiagram{1,1,4}$

N^μ	0	1	2	3
6	$(\emptyset_{\uparrow 3})_{\uparrow 6}$	$(\ydiagram{1}_{\uparrow 3})_{\uparrow 6}$	$\left(\ydiagram{1,1}_{\uparrow 3}\right)_{\uparrow 6}$	$\ydiagram{1,1,1}_{\uparrow 6}$

The equivariant Betti table of I is given by the sum of the two tables above as follows:

I	0	1	2	3
4	$\ydiagram{4}_{\uparrow 6}$	$\ydiagram{1,4}_{\uparrow 6}$	$\ydiagram{1,1,4}$	
6	$(\emptyset_{\uparrow 3})_{\uparrow 6}$	$(\ydiagram{1}_{\uparrow 3})_{\uparrow 6}$	$\left(\ydiagram{1,1}_{\uparrow 3}\right)_{\uparrow 6}$	$\ydiagram{1,1,1}_{\uparrow 6}$

Example 6.5. Let $I = I_{n,r}^{(3)}$ be the third symbolic power of the squarefree Veronese ideal with $n \geqslant r+2$. Then $\Lambda(I) = \{\lambda, \mu, \rho\}$ with $\lambda = (0^{n-r-2}, 1^{r+2}), \mu = (0^{n-r-1}, 1, 2^r), \rho = (0^{n-r}, 3^r)$. Using that $\operatorname{info}(\lambda) = (0, r+2, \emptyset), \operatorname{info}(\mu) = (n-r-1, r, (0^{n-r-1}))$ and $\operatorname{info}(\rho) = (n-r, r, (0^{n-r}))$, we have

$$\begin{aligned}
\operatorname{Tor}_i(N^\lambda, \Bbbk) &\cong U_n^{(1^i, r+2)} \\
\operatorname{Tor}_i(N^\mu, \Bbbk) &\cong \bigoplus_{k+l=i} (\mathcal{M}_{[n-r-1]}(x^{(0^{n-r-1})}) \otimes U_{n-r-1}^{(1^k)}) \boxtimes U_{r+1}^{(1^l, r)} \\
&= \left(U_{n-r-1}^{(1^i)} \boxtimes U_{r+1}^{(r)}\right) \bigoplus \left(U_{n-r-1}^{(1^{i-1})} \boxtimes U_{r+1}^{(1,r)}\right), \\
\operatorname{Tor}_i(N^\rho, \Bbbk) &\cong \bigoplus_{k+l=i} (\mathcal{M}_{[n-r]}(x^{(0^{n-r})}) \otimes U_{n-r}^{(1^k)}) \boxtimes U_r^{(1^l, r)} = U_{n-r}^{(1^i)} \boxtimes U_r^{(r)} = (U_{n-r}^{(1^i)})_{\uparrow n}.
\end{aligned}$$

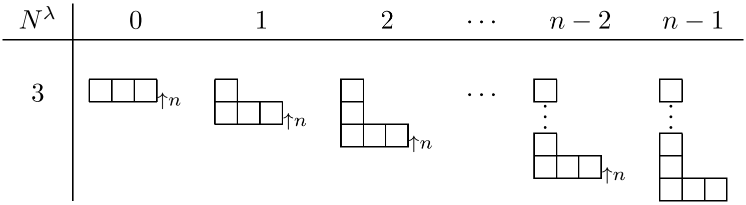
The equivariant Betti table of I is the sum of the equivariant Betti table of N^λ, N^μ and N^ρ. The following tables are the equivariant Betti tables of these three modules when $n = 6$ and $r = 3$.

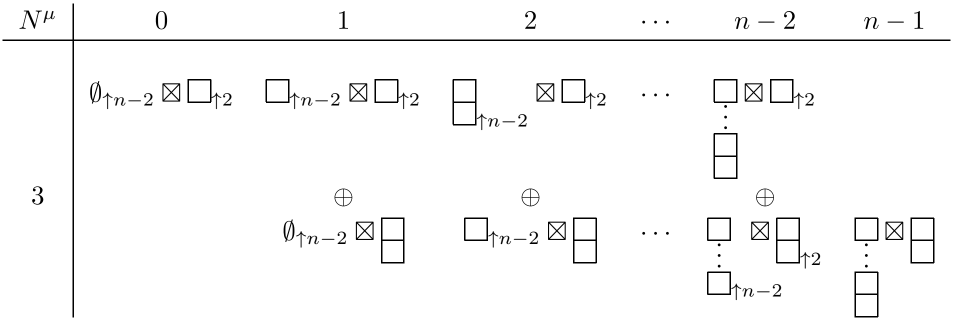
N^λ	0	1
5	$\ydiagram{5}_{\uparrow 6}$	$\ydiagram{1,5}$

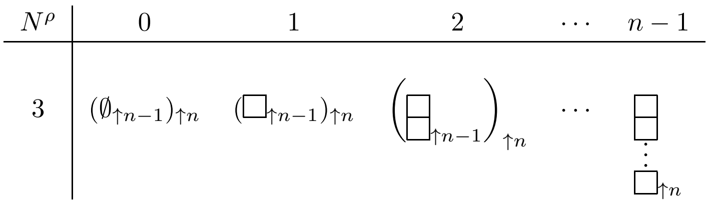
N^μ	0	1	2	3
7	$\emptyset_{\uparrow 2} \boxtimes \ydiagram{3}_{\uparrow 4}$	$\ydiagram{1}_{\uparrow 2} \boxtimes \ydiagram{3}_{\uparrow 4}$ $\oplus$ $\emptyset_{\uparrow 2} \boxtimes \ydiagram{1,3}$	$\ydiagram{1,1} \boxtimes \ydiagram{3}_{\uparrow 4}$ $\oplus$ $\ydiagram{1}_{\uparrow 2} \boxtimes \ydiagram{1,3}$	$\ydiagram{1,1} \boxtimes \ydiagram{1,3}$

N^ρ	0	1	2	3
9	$(\emptyset_{\uparrow 3})_{\uparrow 6}$	$(\ydiagram{1}_{\uparrow 3})_{\uparrow 6}$	$\left(\ydiagram{1,1}_{\uparrow 3}\right)_{\uparrow 6}$	$\ydiagram{1,1,1}_{\uparrow 6}$

Example 6.6. Let $I = (x_1, \ldots, x_n)^3 \subset \Bbbk[x_1, \ldots, x_n]$ with $n \geqslant 3$. Then $\Lambda(I) = \{\lambda, \mu, \rho\}$ with $\lambda = (0^{n-3}, 1^3), \mu = (0^{n-2}, 1, 2), \rho = (0^{n-1}, 3)$. A computation similar to Example 6.6 shows that the equivariant Betti tables of N^λ, N^μ and N^ρ are







Example 6.7. Let $I = (I_{n,2})^2$ with $n \geqslant 4$. Then $\Lambda(I) = \{\lambda, \mu, \rho\}$ with

$$\lambda = (0^{n-4}, 1^4), \mu = (0^{n-3}, 1^2, 2), \rho = (0^{n-2}, 2^2)$$

and

$$\begin{aligned}
\mathrm{Tor}_i(N^\lambda, \Bbbk) &\cong U_n^{(1^i,4)}, \\
\mathrm{Tor}_i(N^\mu, \Bbbk) &\cong \bigoplus_{k+l=i} \left(\mathcal{M}_{[n-3]}(x^{(0^{n-3})}) \otimes U_{n-3}^{(1^k)}\right) \boxtimes U_3^{(1^{l+1})} \\
&\cong \left(U_{n-3}^{(1^i)} \boxtimes U_3^{(1)}\right) \bigoplus \left(U_{n-3}^{(1^{i-1})} \boxtimes U_3^{(1^2)}\right) \bigoplus \left(U_{n-3}^{(1^{i-2})} \boxtimes U_3^{(1^3)}\right), \\
\mathrm{Tor}_i(N^\rho, \Bbbk) &\cong U_{n-2}^{(1^i)} \boxtimes U_2^{(2)} = (U_{n-2}^{(1^i)})_{\uparrow n}.
\end{aligned}$$

7. Other considerations

7.1. Weakly polymatroidal ideals

Our definition of symmetric shifted ideals is inspired by stable monomial ideals, which also have linear quotients (see [21, §7]), but almost all stable monomial ideals are not fixed by an action of the symmetric group. Besides stable monomial ideals, another famous class of monomial ideals which have linear quotients are (weakly) polymatroidal ideals (see [21, §12] for more details). A monomial ideal $I \subset S$ is said to be **weakly polymatroidal** if for any two monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n} \in G(I)$ such that $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $a_t > b_t$ for some t, there is $j > t$ such that $v(x_t/x_j) \in I$.

One may wonder whether $I_{n,c}^{(m)}$ is a weakly polymatroidal ideal and the fact that it has linear quotients follows from the weakly polymatroidal property. The next example shows this is not the case.

Example 7.1. Consider the ideal $I = I_{6,3}^{(5)}$ which we also studied in Example 4.2. Recall that this ideal is generated by the $\mathfrak{S}_6$-orbits of the following five monomials

$$x_1x_2^2x_3^2x_4^2x_5^2x_6^2, \quad x_1x_2x_3^3x_4^3x_5^3x_6^3, \quad x_2^2x_3^3x_4^3x_5^3x_6^3, \quad x_2x_3^4x_4^4x_5^4x_6^4, \quad x_3^5x_4^5x_5^5x_6^5. \tag{7.1}$$

Then the two monomials

$$u = x_1^{a_1} \cdots x_6^{a_6} = x_1^7x_2^4x_3^4x_4^4x_5^1x_6^0 \text{ and } v = x_1^{b_1} \cdots x_6^{b_6} = x_1^5x_2^5x_3^5x_4^5x_5^0x_6^0$$

are contained in I. Clearly $a_1 > b_1$, but for any $j > 1$ the monomial $v(x_1/x_j)$ must belong to the $\mathfrak{S}_6$-orbit of $x_3^4x_4^5x_5^5x_6^6$. However, the monomial $x_3^4x_4^5x_5^5x_6^6$ is not divisible by any monomial listed in (7.1), so I is not weakly polymatroidal.

7.2. Open questions

Finally, we give a few open problems relating to symmetric shifted ideals. We give a formula for (equivariant) Betti numbers of symmetric shifted ideals, but we could not construct their minimal graded free resolutions. On the other hand, an explicit $\mathfrak{S}_n$-equivariant minimal graded free resolutions of $I_{n,c}$ is constructed in [15].

Problem 7.2. Construct explicit $\mathfrak{S}_n$-equivariant minimal graded free resolutions of symmetric shifted ideals.

Symmetric shifted ideals give a class of $\mathfrak{S}_n$-fixed monomial ideals having linear resolutions. However, we do not know if there is an $\mathfrak{S}_n$-fixed monomial ideal which is not shifted but has a linear resolution. This prompts the following:

Problem 7.3. Find a combinatorial characterization of $\mathfrak{S}_n$-fixed monomial ideals having linear resolutions.

Remark 7.4. After this paper was posted on arXiv, Claudiu Raicu [31] gave an answer to Problem 7.3. He proves that if an $\mathfrak{S}_n$-fixed monomial ideal has a linear resolution then it must be a symmetric shifted ideal. In particular, Theorem 3.2 and his result imply that an $\mathfrak{S}_n$-fixed monomial ideal has linear quotients if and only if it is a symmetric shifted ideal.

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