

Standard Model parameters in the tadpole-free pure $\overline{\text{MS}}$ scheme

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We present an implementation and numerical study of the Standard Model couplings, masses, and vacuum expectation value (VEV), using the pure $\overline{\text{MS}}$ renormalization scheme based on dimensional regularization. Here, the $\overline{\text{MS}}$ Lagrangian parameters are treated as the fundamental inputs, and the VEV is defined as the minimum of the Landau gauge effective potential, so that tadpole diagrams vanish, resulting in improved convergence of perturbation theory. State-of-the-art calculations relating the $\overline{\text{MS}}$ inputs to on-shell observables are implemented in a consistent way within a public computer code library, **SMDR** (Standard Model in Dimensional Regularization), which can be run interactively or called by other programs. Included here for the first time are the full 2-loop contributions to the Fermi constant within this scheme and studies of the minimization condition for the VEV at 3-loop order with 4-loop QCD effects. We also implement, and study the scale dependence of, all known multi-loop contributions to the physical masses of the Higgs boson, the W and Z bosons, and the top quark, the fine structure constant and weak mixing angle, and the renormalization group equations and threshold matching relations for the gauge couplings, fermion masses, and Yukawa couplings.

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I. INTRODUCTION

With the discovery of the Higgs boson, the Standard Model is technically complete. This is despite indications that it will have to be extended to accommodate dark matter and to solve issues such as the hierarchy problem, the strong CP problem, and the cosmological constant problems. At this writing, the LHC continues to strengthen lower bounds on the masses of new particles in hypothetical ultraviolet completions such as supersymmetry. It is therefore plausible that we should view the Standard Model as a valid, complete effective field theory up to the TeV scale and perhaps well beyond, with non-renormalizable terms in the Lagrangian correspondingly highly suppressed. This paper is concerned with the ongoing program of determining, as accurately as possible, the relations between the renormalizable Lagrangian parameters that define the theory and the observables and on-shell quantities that are more directly connected to experimental results. This is part of a larger goal of improving our understanding of the Standard Model at the level of accuracy required to test it with future experiments.

A convenient method of handling the ultraviolet divergences of the Standard Model is provided by dimensional regularization [1–5] followed by renormalization by modified minimal subtraction, $\overline{\text{MS}}$ [6, 7]. To describe the effects of electroweak symmetry breaking induced by the Higgs VEV, there are at least two distinct ways to proceed. Consider the Higgs potential

$$V(\phi) = \Lambda + m^2 H^\dagger H + \lambda (H^\dagger H)^2, \quad (1.1)$$

where H is the canonically normalized complex Higgs doublet field. First, one may choose to organize perturbation theory by expanding the electrically neutral component of H around a tree-level VEV $v_{\text{tree}}/\sqrt{2}$, defined by:

$$v_{\text{tree}} \equiv \sqrt{-m^2/\lambda}. \quad (1.2)$$

This is used in many works, because it has the advantage that v_{tree} is manifestly independent of the choice of gauge-fixing. However, it has the disadvantage that Higgs tadpole loop diagrams do not vanish, and must be included order-by-order in perturbation theory. This comes with a parametrically slower convergence of perturbation theory, as the tadpole contributions to other calculated quantities will include powers of $1/\lambda$ due to their zero-momentum Higgs propagators.

We choose instead to expand the Higgs field around a loop-corrected VEV v , which is defined to be the minimum of the full effective potential [8–10] in Landau gauge. For the Standard Model (and indeed for a general renormalizable field theory), the effective potential has now been obtained at 2-loop [11, 12] and 3-loop [13, 14] orders, with the 4-loop contributions known [15] at leading order in QCD. The choice of Landau gauge is made because other gauge-fixing choices lead to unpleasant technical problems including kinetic mixing between the longitudinal components of the vector and the Goldstone scalar degrees of freedom.[†] The disadvantage of defining the VEV in this way is that calculations that make use of it are then restricted to Landau gauge. But the advantage of this

[†] The full 2-loop effective potential has been recently obtained in a large class of more general gauge-fixing schemes in ref. [16], but it is quite unwieldy, and extending it to 3-loop order is a daunting challenge.

choice is that the sum of all Higgs tadpole diagrams (including the tree-level tadpole) automatically vanishes, and there are no corresponding $1/\lambda^n$ contributions in perturbation theory.

Another issue to be dealt with is that the minimization condition for the effective potential requires resummation of Goldstone boson contributions, as explained in [17, 18], in order to avoid spurious imaginary parts and infrared divergences at higher loop orders. (For further perspectives and developments on this issue, see refs. [19–25].) The end result can be written as a relation between the tree-level and loop-corrected VEVs:

$$v_{\text{tree}}^2 = v^2 + \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \Delta_n, \quad (1.3)$$

with n -loop order contributions Δ_n that are free of spurious imaginary parts and infrared divergences and do not depend at all on the Goldstone boson squared mass. (The $1/\lambda$ in this equation is the source of the tadpole effects noted above if one chooses to expand in terms of v_{tree} rather than v .) The full 3-loop contributions were given in [14] in terms of 2-loop and 3-loop basis integrals that can be efficiently evaluated numerically using the computer code **3VIL** [26],[‡] and the 4-loop contribution was obtained at leading order in QCD in [15]. However, a numerical illustration of these effects was deferred. One of the purposes of the present paper is to remedy this by providing a numerical study of the 3-loop and 4-loop effects.

We also have a broader purpose here; to bring together in a coherent form, implemented as a public computer code, results obtained in recent years relating pole masses and other observables to the Lagrangian parameters in the tadpole-free pure $\overline{\text{MS}}$ scheme. The new code, called **SMDR** for Standard Model in Dimensional Regularization, is a software library written in C with functions callable from user C or C++ programs. It uses the $\overline{\text{MS}}$ input parameters that define[§] the Standard Model theory at a given renormalization scale Q :

$$v, \lambda, g_3, g, g', y_t, y_b, y_c, y_s, y_d, y_u, y_\tau, y_\mu, y_e, \Delta\alpha_{\text{had}}^{(5)}(M_Z). \quad (1.4)$$

All of these, except the last, are defined as running parameters in the non-decoupled (high-energy) Standard Model, with gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ with gauge couplings g_3, g , and g' respectively, and 6 active quarks. Note that the running $\overline{\text{MS}}$ Higgs squared mass parameter m^2 need not be included among these, because it is not independent, being determined in terms of λ, v , and the other parameters by the effective potential minimization condition eq. (1.3). Also, the hadronic light-quark contribution to the fine-structure constant is given by a parameter $\Delta\alpha_{\text{had}}^{(5)}(M_Z)$. In principle this is not independent of the others in eq. (1.4), but in practice it must (at least, at present) be treated as an independent input because it depends on non-perturbative

[‡] **3VIL** computes 3-loop vacuum basis integrals numerically using the differential equations method, except in special cases for which they can be computed analytically, including the cases found in refs. [27]–[47]. See ref. [48] for an alternative evaluation of 3-loop vacuum integrals based on dispersion relations.

[§] Cabibbo-Kobayashi-Maskawa mixing and neutrino mass and mixing effects are neglected in the present version. Including them would have a negligible effect on the quantities in eq. (1.5), compared to other sources of uncertainty.

physics. The code then provides computations of the following “on-shell” output quantities:

$$\begin{aligned}
\text{heavy particle pole masses:} & \quad M_t, M_h, M_Z, M_W, \\
\text{running light quark masses:} & \quad m_b(m_b), m_c(m_c), m_s(2 \text{ GeV}), m_d(2 \text{ GeV}), m_u(2 \text{ GeV}), \\
\text{lepton pole masses:} & \quad M_\tau, M_\mu, M_e, \\
\text{5-quark QCD coupling:} & \quad \alpha_S^{(5)}(M_Z), \\
\text{Fermi constant:} & \quad G_F = 1.1663787 \dots \times 10^{-5} \text{ GeV}^{-2}, \\
\text{fine structure constant:} & \quad \alpha_0 = 1/137.035999139 \dots \text{ and } \Delta\alpha_{\text{had}}^{(5)}(M_Z),
\end{aligned} \tag{1.5}$$

which can be viewed as dual to the $\overline{\text{MS}}$ inputs. (Even though G_F and α_0 are extremely accurately known from experiment, as indicated, they are considered as outputs from the point of view of the pure $\overline{\text{MS}}$ renormalization scheme.) However, note that M_W is actually extra, in the sense that the other parameters in eq. (1.5) are already sufficient to fix the $\overline{\text{MS}}$ quantities in eq. (1.4); therefore, the computation of M_W provides a consistency check on the Standard Model. The quantity $\Delta\alpha_{\text{had}}^{(5)}(M_Z)$ appears in both lists (1.4) and (1.5), due to its non-perturbative nature; it always is obtained from experiment rather than fits to other quantities. The **SMDR** code also computes the weak mixing angle as defined by the Particle Data Group’s Review of Particle Properties (RPP) [49] (which, unlike the present paper, uses a scheme with the top quark decoupled but the massive W boson active, corresponding to a non-renormalizable effective theory even when the Lagrangian couplings of negative mass dimension are neglected), but this is again extra, since it is not needed in order to fix the $\overline{\text{MS}}$ quantities.

The relationship between the Sommerfeld fine-structure constant α_0 appearing in eq. (1.5) and the couplings g and g' in eq. (1.4) can be expressed as (see, for example, refs. [50–53]):

$$\alpha_0 = \frac{g^2(M_Z)g'^2(M_Z)}{4\pi [g^2(M_Z) + g'^2(M_Z)]} \left[1 - \Delta\alpha_{\text{had}}^{(5)}(M_Z) - \Delta\alpha_{\text{pert}}^{\text{LO}} - \Delta\alpha_{\text{pert}}^{\text{HO}} \right], \tag{1.6}$$

where the sum of 1-loop contributions from t, W, τ, μ, e (but not b, c, s, d, u) are:

$$\begin{aligned}
\Delta\alpha_{\text{pert}}^{\text{LO}} = & \frac{\alpha_0}{4\pi} \left[\frac{202}{27} + 14 \ln(M_W/M_Z) - \frac{32}{9} \ln(M_t/M_Z) - \frac{8}{3} \ln(M_\tau/M_Z) \right. \\
& \left. - \frac{8}{3} \ln(M_\mu/M_Z) - \frac{8}{3} \ln(M_e/M_Z) \right],
\end{aligned} \tag{1.7}$$

and the higher-order perturbative contribution $\Delta\alpha_{\text{pert}}^{\text{HO}}$ has been given as an interpolating formula in eqs. (19)–(21) of ref. [53]. For the running $\alpha^{\overline{\text{MS}}}(Q)$ in the decoupled theories used for renormalization group (RG) running below M_Z [with the numbers of active (quarks, charged leptons) equal to (5, 3) or (4, 3) or (4, 2) or (3, 2)], we use the results obtained in [54], as discussed in the next section.

The pole masses $M_t, M_h, M_Z, M_W, M_\tau, M_\mu$, and M_e are each defined in terms of the complex pole in the renormalized propagator,

$$s_{\text{pole}} = M^2 - i\Gamma M. \tag{1.8}$$

For the top-quark pole mass, the pure QCD contributions were obtained at 1-loop, 2-loop, 3-loop, and 4-loop orders in refs. [55], [56], [57], and [58, 59], respectively. The non-QCD contributions to M_t at 1-loop and 2-loop orders had also been obtained in other schemes and approximations. At 1-loop order they were found in refs. [60–62], and mixed electroweak-QCD 2-loop contributions were obtained in [63–65]. Further 2-loop contributions in the gauge-less limit (in which the electroweak boson masses are taken to be small compared to the top-quark mass) were found in refs. [66–69]. Finally, the full 2-loop results for M_t were provided in the tree-level VEV scheme in ref. [70], and in the tadpole-free scheme used in the present paper in [71].

For the Higgs boson mass, we use our calculation in ref. [72], which contains all 2-loop contributions and the leading (in the limit $g^2, g'^2, \lambda \ll g_3^2, y_t^2$) 3-loop contributions in the tadpole-free pure $\overline{\text{MS}}$ scheme. Earlier works on M_h at the 2-loop level in other schemes and approximations include ref. [73] which included the mixed QCD/electroweak contributions to M_h , ref. [74] which used the gauge-less limit approximation at 2-loop order, and the full 2-loop approximation given as an interpolating formula in a hybrid $\overline{\text{MS}}$ /on-shell scheme in ref. [75].

For the W and Z boson pole masses, we use the full 2-loop calculations using the tadpole-free pure $\overline{\text{MS}}$ scheme given in refs. [76] and [77], respectively. Previous 2-loop calculations of the vector boson pole masses in other schemes (expanding around v_{tree} rather than v) appeared in refs. [78], [62], [53], and [70]. It is important to note that for the vector bosons $V = W$ and Z , the values usually quoted, including by the RPP, are not the pole masses but the variable-width Breit-Wigner masses. These can be related to the pole masses by [79–82]:

$$M_{V,\text{Breit-Wigner}}^2 = M_V^2 + \Gamma_V^2. \quad (1.9)$$

Thus, the Z - and W -boson pole masses defined by eq. (1.8) are, respectively, approximately 34.1 MeV and 27.1 MeV smaller than the Breit-Wigner masses that are usually quoted.

The charged lepton pole masses are computed at 2-loop order in QED, by converting the corresponding QCD formulas given in ref. [56] and including small effects from non-zero lighter fermion masses from ref. [83].

The running light-quark masses in eq. (1.5) are defined in appropriate $SU(3)_c \times U(1)_{\text{EM}}$ effective field theories in which the heavier particles have been decoupled. Although it is possible to evaluate the QCD contributions to the bottom-quark and charm-quark pole masses, this is deprecated, because there is no semblance of convergence of the perturbative series relating the pole masses to the running masses for bottom and charm (and obviously for the lighter quarks as well); see ref. [59]. Therefore we use running $\overline{\text{MS}}$ masses for all lighter quarks. Thus $m_b(m_b)$ is defined as an $\overline{\text{MS}}$ running mass in the 5-quark, 3-lepton QCD+QED effective theory, while $m_c(m_c)$ is similarly defined in the 4-quark, 2-lepton theory, and $m_s(2 \text{ GeV})$, $m_d(2 \text{ GeV})$, $m_u(2 \text{ GeV})$ are defined in the 3-quark, 2-lepton theory. We follow the RPP ref. [49] in choosing to evaluate the last three at, somewhat arbitrarily, $Q = 2 \text{ GeV}$, in order to avoid larger QCD effects at smaller Q .

To obtain the 5-quark, 3-lepton QCD+QED effective field theory, we simultaneously decouple the heavier Standard Model particles t, h, Z, W at a common matching scale, which can be chosen at will, but should presumably be in the range from about M_W to M_t . Because W and Z are decoupled from it, this low-energy effective theory is a renormalizable gauge theory supplemented by interactions with couplings of negative mass dimension (including the Fermi four-fermion inter-

actions). The decouplings of the bottom quark, tau lepton, and charm quark are then performed individually.

In one mode of operation, the **SMDR** code takes the $\overline{\text{MS}}$ input parameters of eq. (1.4) provided by the user, and outputs the on-shell quantities in eq. (1.5). Alternatively, in a dual mode of operation, the **SMDR** code instead takes user input for the on-shell quantities in eq. (1.5) (except for M_W), and determines as outputs the $\overline{\text{MS}}$ quantities in eq. (1.4) and then M_W , by doing a fit. The **SMDR** code also implements all known contributions to the running and decoupling of the gauge and Yukawa couplings.

In the numerical studies below, we employ a benchmark model point, chosen to yield the central values of the quantities in eq. (1.5) (other than M_W , as noted above), as given in the 2019 update of the 2018 edition of the Review of Particle Properties ref. [49]:

$$\begin{aligned}
M_t &= 173.1 \text{ GeV}, & M_h &= 125.1 \text{ GeV}, & M_{Z, \text{Breit-Wigner}} &= 91.1876 \text{ GeV}, \\
G_F &= 1.1663787 \times 10^{-5} \text{ GeV}^2, & \alpha_0 &= 1/137.035999139, & \alpha_S^{(5)}(M_Z) &= 0.1181, \\
m_b(m_b) &= 4.18 \text{ GeV}, & m_c(m_c) &= 1.27 \text{ GeV}, & m_s(2 \text{ GeV}) &= 0.093 \text{ GeV} \\
m_d(2 \text{ GeV}) &= 0.00467 \text{ GeV}, & m_u(2 \text{ GeV}) &= 0.00216 \text{ GeV}, & M_\tau &= 1.77686 \text{ GeV}, \\
M_\mu &= 0.1056583745 \text{ GeV}, & M_e &= 0.000510998946 \text{ GeV}, \\
\Delta\alpha_{\text{had}}^{(5)}(M_Z) &= 0.02764,
\end{aligned} \tag{1.10}$$

The $\overline{\text{MS}}$ input quantities that do this are found (with default scale choices for evaluations in **SMDR**) to be:

$$\begin{aligned}
Q_0 &= 173.1 \text{ GeV}, \\
v(Q_0) &= 246.60109 \text{ GeV}, & \lambda(Q_0) &= 0.12603842, \\
g_3(Q_0) &= 1.1636241, & g_2(Q_0) &= 0.64765961, & g'(Q_0) &= 0.35853877, \\
y_t(Q_0) &= 0.93480082, & y_b(Q_0) &= 0.015480097, & y_\tau(Q_0) &= 0.0099944422, \\
y_c(Q_0) &= 0.0033820038, & y_s(Q_0) &= 0.00029094484, & y_\mu(Q_0) &= 0.00058837986, \\
y_d(Q_0) &= 1.4609792 \times 10^{-5}, & y_u(Q_0) &= 6.7227779 \times 10^{-6}, \\
y_e(Q_0) &= 2.7929820 \times 10^{-6}.
\end{aligned} \tag{1.11}$$

This set of values obviously includes more significant digits than justified by the experimental and theoretical uncertainties; this is for the sake of reproducibility and checking when changes are made to the code, or to the default choices of matching or evaluation scales. Equation (1.11) will be referred to below as the reference model point, and a sample input file included with the **SMDR** distribution provides for automatic loading of these parameters. As future versions of the RPP with new experimental results become available, corresponding new versions of the reference model file will be included in new **SMDR** distributions; they can also be constructed easily by using functions provided. All of the figures appearing below are made using short programs (included with the **SMDR** distribution) that employ the **SMDR** library functions, in order to illustrate how the latter should be used.

II. RENORMALIZATION GROUP RUNNING AND DECOUPLING

The $\overline{\text{MS}}$ renormalization group equations for the Standard Model used in this paper, and by default in the **SMDR** code, are the state-of-the-art ones. These include the 2-loop [84–88] and 3-loop [89–97] order contributions for all parameters, including the gauge couplings, the fermion Yukawa couplings, the Higgs self-coupling λ , VEV v , and negative squared mass m^2 . In addition, for the strong coupling, the contributions to the beta function at 4-loop order in the limit $g^2, g'^2 \ll g_3^2, y_t^2, \lambda$ [98–102] and pure QCD 5-loop order [103, 104] are included. Similarly, the higher-order QCD contributions to the beta functions of the quark Yukawa couplings are included, using results found at 4-loop order in refs. [105, 106] and at 5-loop order in ref. [107]. Finally, the leading QCD 4-loop contribution to the beta function of the Higgs self-coupling λ is included from refs. [15, 108].

Using the reference model of eq. (1.11) as inputs, the renormalization group running of the couplings are illustrated in Figure 2.1 for the range $10^2 \text{ GeV} < Q < 10^{19} \text{ GeV}$. The left panel shows the inverse gauge couplings $1/\alpha_3 = 4\pi/g_3^2$ and $1/\alpha_2 = 4\pi/g^2$ and (in a Grand Unified Theory [GUT] normalization) $1/\alpha_1 = (3/5)4\pi/g'^2$, while the right panel shows the Yukawa couplings for all of the Standard Model charged fermions.

For lower scales, we use the results given in ref. [54] to simultaneously decouple the top quark, Higgs boson, Z boson, and W boson at a common matching scale, so that the low-energy effective field theory is renormalizable and has gauge group $SU(3)_c \times U(1)_{\text{EM}}$. The common matching scale is, in principle, arbitrary; by default the **SMDR** code uses $Q = M_Z$ for the matching but this can be modified at run time by the user. The matching results include the 2-loop matching found in [54] for the electromagnetic $\overline{\text{MS}}$ coupling $\alpha(Q)$ in the theory with 5 quarks and 3 leptons, as well as the matching relation for the 5-quark QCD coupling $\alpha_S(Q)$ at 1-loop [109, 110], 2-loop [111, 112], 3-loop [113, 114], and 4-loop [115, 116] orders together with the complete Yukawa and electroweak 2-loop contributions obtained first in ref. [117] (and verified and written in a different way compatible with the present paper in ref. [54]). The pure QCD corrections to the quark mass matching relations were given at 3-loop order in ref. [113, 114] and 4-loop order in ref. [118].

For the QCD parts of the matching relations and beta functions, complete results had been calculated and incorporated long ago into the **RunDec** and **CRunDec** [119–121] codes. In addition, the 2-loop mixed QCD/electroweak and pure electroweak contributions to matching of the running b, c, s, d, u and τ, μ, e fermion masses were obtained in refs. [69, 70, 122–124] and [54]. They are implemented in **SMDR** using the formulas provided in ref. [54] consistent with the conventions of the present paper.

The running and decoupling of the QCD and QED gauge couplings and running fermion masses are shown in Figure 2.2 for the sequence of effective theories with 5 quarks and 3 charged leptons (for $m_b(m_b) \leq Q \leq M_Z$), with 4 quarks and 3 charged leptons (for $M_\tau \leq Q \leq m_b(m_b)$), with 4 quarks and 2 charged leptons (for $m_c(m_c) \leq Q \leq M_\tau$), and with 3 quarks and 2 charged leptons (for $Q \leq m_c(m_c)$). The boundaries between these effective theories are somewhat arbitrary, and correspond to the default points within the **SMDR** code, which can be adjusted by the user. At each of the matching points $Q = m_b(m_b)$ and M_τ and $m_c(m_c)$, the parameters are actually discontinuous due to the matching mentioned above due to changing effective theories, but this cannot be discerned with the resolution of the plots.

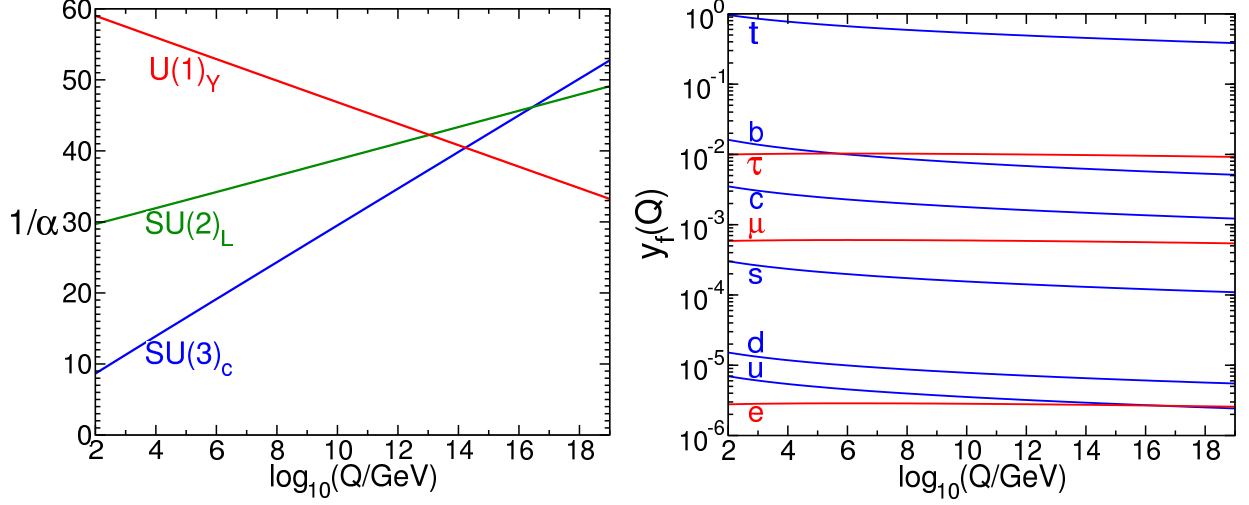


FIG. 2.1: Renormalization group running of the $\overline{\text{MS}}$ inverse gauge couplings $1/\alpha_3$, $1/\alpha_2$, and $1/\alpha_1$ in a grand unified theory normalization (left panel) and charged fermion Yukawa couplings (right panel), as functions of the renormalization scale Q . The input parameters are given by the reference model point defined in eq. (1.11) at $Q_0 = 173.1$ GeV.

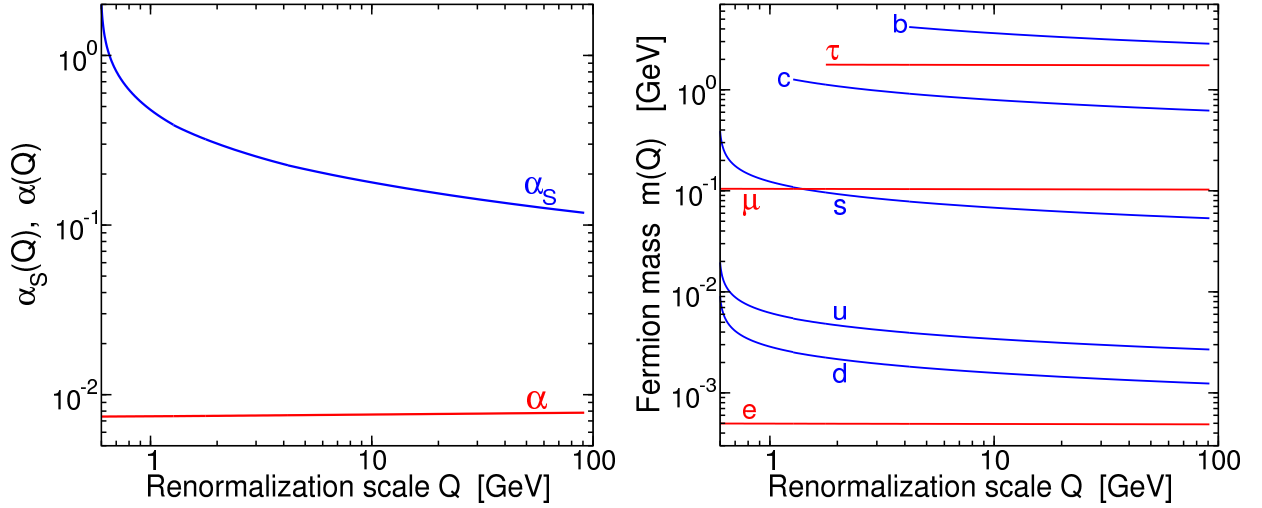


FIG. 2.2: Renormalization group running of the $\overline{\text{MS}}$ QCD and QED gauge couplings α_S and α (left panel) and fermion masses (right panel), as functions of the renormalization scale Q . The beta functions used are 5-loop order in QCD and 3-loop order in QED, with active fermion contents as follows: 5-quark, 3-lepton for $m_b(m_b) \leq Q \leq 91.1876$ GeV; 4-quark, 3-lepton for $M_\tau \leq Q \leq m_b(m_b)$; 4-quark, 2-lepton for $m_c(m_c) \leq Q \leq M_\tau$; and 3-quark, 2-lepton for $Q \leq m_c(m_c)$. The matchings at $Q = m_b(m_b)$ and M_τ and $m_c(m_c)$ are done at 4-loop order for the QCD coupling, 2-loop order for the QED coupling, and the fermion mass matchings include effects at 3-loop order in QCD and 2-loop order in QED. The input parameters are defined by the reference model point given in eq. (1.11), with t, h, Z, W simultaneously decoupled at $Q = 91.1876$ GeV.

III. MINIMIZATION OF THE EFFECTIVE POTENTIAL AND THE VACUUM EXPECTATION VALUE

We first consider a numerical illustration of the minimization condition for the effective potential, eq. (1.3), which can be used to trade m^2 for v , when all of the other $\overline{\text{MS}}$ parameters are taken to be known inputs. The quantities Δ_n have been given up to 3-loop order in ref. [14] and the 4-loop order contribution at leading order in QCD is found in ref. [15].

In Figure 3.1, we start with the $\overline{\text{MS}}$ quantities taken to be their benchmark reference point values defined at $Q = Q_0 = 173.1$ in eq. (1.11). From eq. (1.3), the value of m^2 at Q_0 for the reference model is then found to be (again including more significant digits than justified by the uncertainties):

$$m^2(Q_0) = -(92.878850 \text{ GeV})^2. \quad (3.1)$$

At other renormalization group scales Q , we determine $m^2(Q)$ in two different ways. For the first way, we renormalization-group run all of the other parameters to Q , where $m^2(Q)_{\text{min}}$ is then determined by again applying eq. (1.3). The results are shown in the left panel of Figure 3.1, in various approximations (as labeled) for the minimization condition. The second way is to directly RG run $m^2(Q)_{\text{run}}$ starting with eq. (3.1) as its boundary condition. In the right panel, we show the ratio of $m^2(Q)_{\text{min}}/m^2(Q)_{\text{run}}$ as a function of Q . This provides a scale-invariance check yielding a lower bound on the error, because in the idealized case of calculations to all orders in perturbation theory, the ratio should be exactly 1. We find that in the case of the full 3-loop plus QCD 4-loop approximation, the deviation of the ratio from unity is less than 10^{-4} for the entire range shown from 70 GeV to 220 GeV, and over most of this range the deviation is actually much smaller. Without including the 4-loop QCD contribution, the scale dependence is still quite good, but is a few times 10^{-4} . In both cases, the parametric uncertainties from experimentally measured quantities would seem to be probably larger than the theoretical uncertainties, although we emphasize that the scale-dependence check can only give a lower bound on the theoretical error.

In Figure 3.2, we perform the inverse of the preceding analysis. This time, we take $m^2(Q_0)$ as an input given by eq. (3.1) and determine $v(Q)$ as an output. Of course, at $Q = Q_0$, the result is exactly as given in eq. (1.11). At other Q , we obtain $v(Q)_{\text{min}}$ by first running all of the other $\overline{\text{MS}}$ quantities from Q_0 to Q and then apply eq. (1.3) again. The results are shown in the left panel of Figure 3.2. We also obtain $v(Q)_{\text{run}}$ by directly running it using its RG equations from Q_0 . The ratio $v(Q)_{\text{min}}/v(Q)_{\text{run}}$ is shown in the right panel of Figure 3.2. Again, in the best available approximation, the scale dependence of the ratio is much smaller than 10^{-4} over the entire range.

IV. THE FERMI DECAY CONSTANT

The Fermi weak decay constant is closely related to the vacuum expectation value, with $G_F = 1/\sqrt{2}v^2$ at tree-level. Including radiative corrections, one can write:

$$G_F = \frac{1 + \Delta\bar{r}}{\sqrt{2}v_{\text{tree}}^2} = \frac{1 + \Delta\tilde{r}}{\sqrt{2}v^2}. \quad (4.1)$$

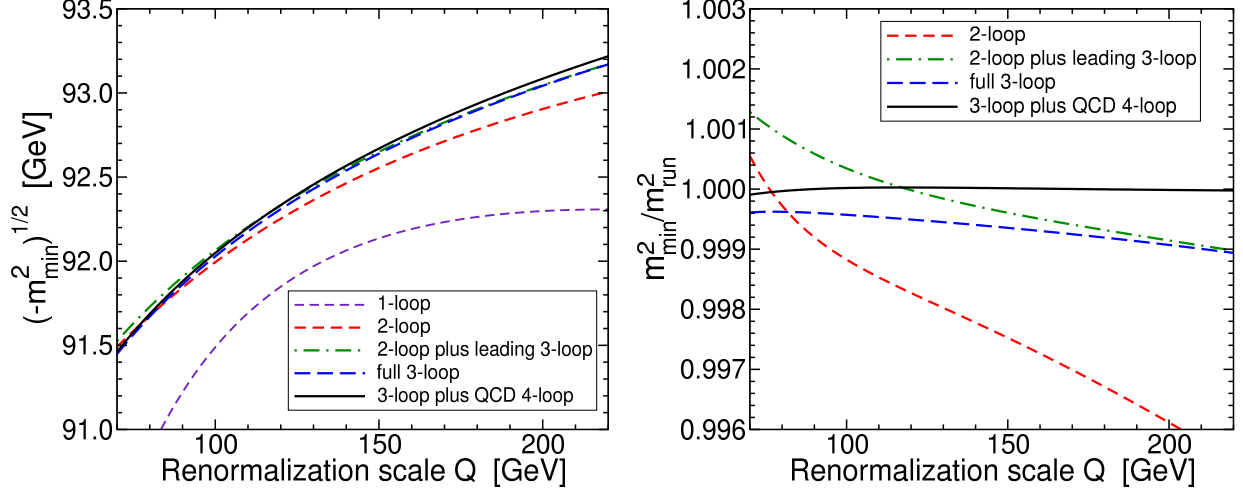


FIG. 3.1: The $\overline{\text{MS}}$ Higgs squared mass parameter, as a function of the renormalization scale Q , for the reference model point defined at $Q_0 = 173.1$ GeV in eq. (1.11). The other input parameters, including the VEV $v(Q)$, are obtained from the reference model by evolving them using their RG equations to the scale Q , where the Landau gauge effective potential is then required to be minimized to determine $m^2(Q)_{\min}$. In the left panel, results are shown for the 1-loop, 2-loop, 2-loop plus leading 3-loop, full 3-loop, and 3-loop plus QCD 4-loop approximations to the effective potential minimization condition. The right panel shows the results for $m^2(Q)_{\min}/m^2(Q)_{\text{run}}$, where $m^2(Q)_{\min}$ is determined as in the left panel, and $m^2(Q)_{\text{run}}$ is obtained directly by renormalization running its input value from the reference scale $Q_0 = 173.1$ GeV.

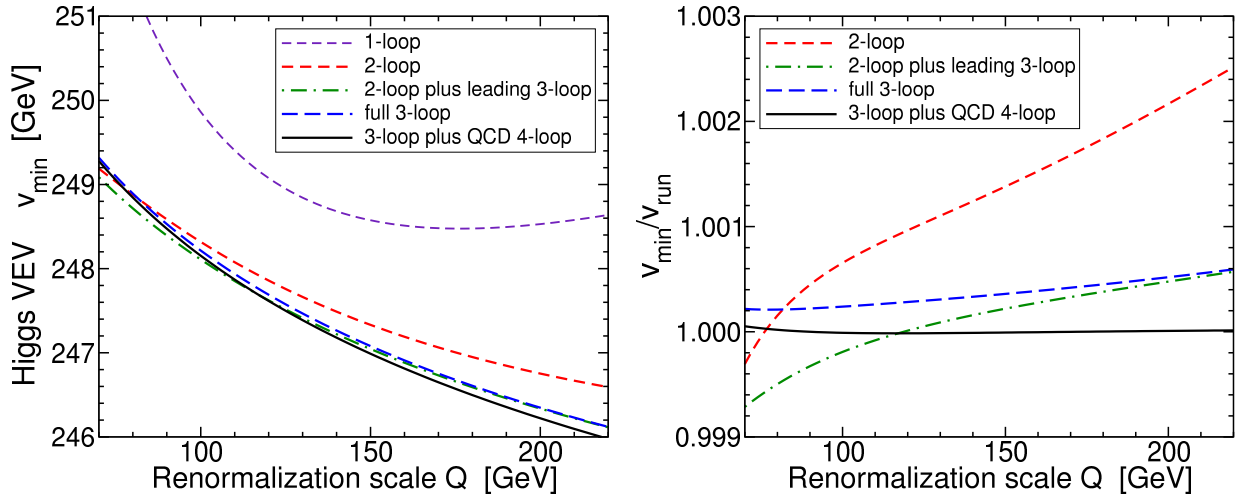


FIG. 3.2: The $\overline{\text{MS}}$ Higgs VEV, as a function of the renormalization scale Q , for the reference model point defined at $Q_0 = 173.1$ GeV in eq. (1.11). The other input parameters, including $m^2(Q)$, are obtained from the reference model by evolving them using their RG equations to the scale Q , where the Landau gauge effective potential is minimized to obtain $v(Q)_{\min}$. In the left panel, results are shown for the 1-loop, 2-loop, 2-loop plus leading 3-loop, full 3-loop, and 3-loop plus QCD 4-loop approximations to the effective potential minimization condition. The right panel shows the results for $v(Q)_{\min}/v(Q)_{\text{run}}$, where $v(Q)_{\text{run}}$ is obtained directly by renormalization running from the reference scale $Q_0 = 173.1$ GeV.

Expressions for $\Delta\bar{r}$ have been given at 2-loop order in the so-called gauge-less limit ($g^2, g'^2 \ll g_3^2, y_t^2, \lambda$) in ref. [69] and ref. [70], using expansions in terms of $\overline{\text{MS}}$ and on-shell quantities respectively, but in both cases determined in terms of the tree-level VEV. The full 2-loop version of $\Delta\bar{r}$ is quite lengthy, and to our knowledge has not appeared in print, but was obtained and presented within the public computer code `mr` [124]. We have obtained the corresponding complete 2-loop result for $\Delta\tilde{r}$ in terms of v ,

$$\Delta\tilde{r} = \frac{1}{16\pi^2}\Delta\tilde{r}^{(1)} + \frac{1}{(16\pi^2)^2}\Delta\tilde{r}^{(2)} + \dots \quad (4.2)$$

The 1-loop order part is

$$\begin{aligned} \Delta\tilde{r}^{(1)} = & \frac{3}{4}(g^2 + g'^2)[A(Z) - A(W)]/(Z - W) + \frac{3}{4}[(4g^2 - 24\lambda)A(W) - g^2A(h)]/(h - W) \\ & + 3[y_t^2A(t) - y_b^2A(b)]/(t - b) + 2A(\tau)/v^2 - (3g^2 + g'^2)/8 + (3y_t^2 + 3y_b^2 + y_\tau^2)/2, \end{aligned} \quad (4.3)$$

where

$$Z = (g^2 + g'^2)v^2/4, \quad W = g^2v^2/4, \quad h = 2\lambda v^2, \quad (4.4)$$

$$t = y_t^2v^2/2, \quad b = y_b^2v^2/2, \quad \tau = y_\tau^2v^2/2, \quad (4.5)$$

are the running $\overline{\text{MS}}$ squared masses, and

$$A(x) = x\overline{\ln}(x) - x \quad (4.6)$$

with

$$\overline{\ln}(x) = \ln(x/Q^2). \quad (4.7)$$

The 2-loop part is

$$\Delta\tilde{r}^{(2)} = g_3^2 y_t^2 [8\zeta_2 - 17 - 16A(t)/t - 12A(t)^2/t^2] + \Delta\tilde{r}_{\text{non-QCD}}^{(2)}, \quad (4.8)$$

where $\Delta\tilde{r}_{\text{non-QCD}}^{(2)}$ is again rather lengthy, and so is provided in its complete form as an ancillary file `Deltatilde.txt` distributed with this paper, rather than in text form here. It has the form:

$$\Delta\tilde{r}_{\text{non-QCD}}^{(2)} = \sum_j C_j^{(2)} I_j^{(2)} + \sum_{j \leq k} C_{j,k}^{(1,1)} I_j^{(1)} I_k^{(1)} + \sum_j C_j^{(1)} I_j^{(1)} + C^{(0)} \quad (4.9)$$

where the lists of 2-loop and 1-loop basis integrals required are:

$$\begin{aligned} I^{(2)} = & \{\zeta_2, I(h, h, h), I(h, t, t), I(0, h, t), I(0, h, W), I(0, h, Z), I(0, t, W), \\ & I(0, t, Z), I(0, W, Z), I(h, h, W), I(h, W, W), I(h, W, Z), I(h, Z, Z), \end{aligned}$$

$$I(t, t, W), I(t, t, Z), I(W, W, W), I(W, W, Z), I(W, Z, Z)\}, \quad (4.10)$$

$$I^{(1)} = \{A(t), A(h), A(Z), A(W)\}, \quad (4.11)$$

with the 2-loop vacuum integral function $I(x, y, z)$ as defined as in previous papers e.g. [26, 125, 126], and the coefficients $C_j^{(2)}$, $C_{j,k}^{(1,1)}$, $C_j^{(1)}$, and $C^{(0)}$ are rational functions of t , h , Z , W , and v . (The v dependence is $1/v^4$ in each case.) The Goldstone boson contributions in $\Delta\tilde{r}$ have been resummed, so that, as explained in refs. [14, 17], the Higgs squared mass appearing here is $h \equiv 2\lambda v^2$, and not $m^2 + 3\lambda v^2$. Also, note that $\Delta\tilde{r}^{(1)}$ is well-defined in the formal limits $W \rightarrow Z$, $W \rightarrow h$, and $b \rightarrow t$, despite denominators that vanish in those limits. Furthermore, although $\Delta\tilde{r}^{(2)}$ has several individual terms with λ in the denominator, once can check that the whole expression for $\Delta\tilde{r}$ is finite in the limit $\lambda \rightarrow 0$, unlike $\Delta\tilde{r}$. This illustrates the absence of $1/\lambda$ effects in the tadpole-free scheme based on v ; more generally, the absence of $1/\lambda$ effects provides useful checks on calculations. We have also checked that $\Delta\tilde{r}^{(2)}$ is well-defined in the formal limits where $Z - 4t$ and $h - W$ and $W - Z$ and $h - 4Z$ and $h - 4W$ vanish, despite many of the individual coefficients having denominators containing factors of these quantities. Furthermore, we have checked that $G_F = (1 + \Delta\tilde{r})/\sqrt{2}v^2$ is RG scale invariant through 2-loop order, as required by its status as a physical observable.

This numerical result for G_F in terms of the $\overline{\text{MS}}$ quantities is shown in Figure 4.1 for the benchmark reference model as a function of the scale Q at which it is computed. The scale variation is less than 1 part in 10^{-4} for Q between 100 and 220 GeV. By default, the **SMDR** code evaluates G_F at $Q = M_t$, and so the benchmark point there agrees exactly with the experimental value. The results can also be compared to those of formulas relating G_F to M_W given by Degrandi, Gambino, and Giardino in ref. [53], which is larger by a fraction of about 0.0002 (or 0.0001), provided that Q in our calculation is taken to be close to M_t (or M_Z). This corresponds to a difference in the physical W -boson mass of about 8 MeV (or 4 MeV), less than the current experimental uncertainty in M_W . A further reduction in the purely theoretical sources of uncertainty in our approach could come about from including the leading (in g_3 and y_t) 3-loop contributions to G_F , M_Z , and M_W . There appear to be no technical obstacles to performing these calculations; when they become available, they will be included in the **SMDR** code.

V. PHYSICAL MASSES OF HEAVY PARTICLES

For the case of the benchmark reference model defined in eq. (1.11), we show the pole masses of t and h and the Breit-Wigner masses of W and Z in various approximations, as a function of the renormalization scale Q used for the computation, in Figure 5.1. The results shown are obtained using **SMDR**, which implements the formulas found in refs. [71, 72, 76, 77] for the tadpole-free pure $\overline{\text{MS}}$ scheme. These papers make use of the **TSIL** software library in order to numerically evaluate the required two-loop self-energy basis integrals, using the differential equations method as described in [126], and analytical special cases found in refs. [56, 63, 127–135] and [126].

In the case of the Higgs boson pole mass, the Q dependence is seen to be of order several tens of MeV in Figure 5.1, for the best available approximation, which includes the full 2-loop and leading (in g_3 and y_t) 3-loop contributions. However, as we argued in ref. [72], in the specific case of M_h ,

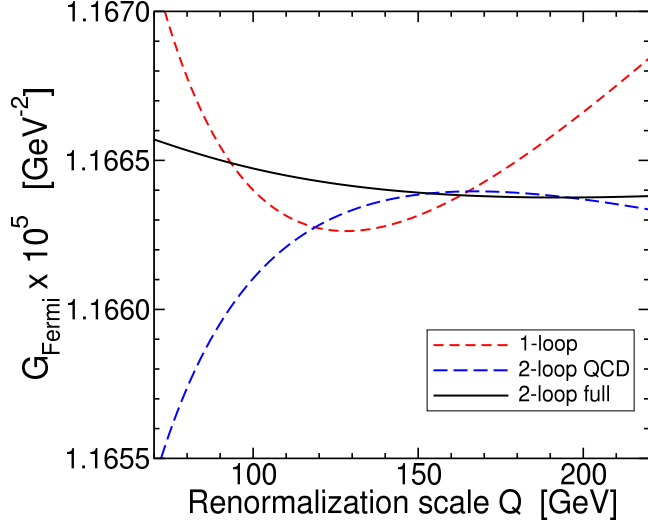


FIG. 4.1: The Fermi constant G_F , as a function of the renormalization scale Q at which it is computed from the $\overline{\text{MS}}$ input parameters, for the reference model point defined at $Q_0 = 173.1$ GeV in eq. (1.11). The short-dashed, long-dashed, and solid lines show the results of including the 1-loop, 1-loop plus 2-loop QCD, and full 2-loop contributions, respectively.

a renormalization scale close to $Q = 160$ GeV should be made in order to minimize the error from other 3-loop contributions, and this choice is used by default in **SMDR**.

In the case of the top-quark pole mass, in Figure 5.1 we start with the known 4-loop pure QCD approximation. Although other works often treat the top-quark pole mass using only QCD effects, the neglect of electroweak corrections is certainly not justified. Indeed, the 4-loop pure QCD approximation is seen to have a very large scale dependence of about 1.7 GeV as Q is varied from 70 GeV to 200 GeV. This shows that failing to include the electroweak contributions at 1-loop order contributes a very large and scale-dependent error, although this is obscured if one also neglects the corresponding non-QCD contributions in the renormalization group running of the parameters. Even the 2-loop mixed QCD/electroweak and non-QCD effects are roughly of order 200 MeV and 100 MeV, and scale dependent. By default, the **SMDR** code uses a scale choice $Q = M_t$ when computing M_t , but this can be changed by the user, as for example when making Figure 5.1.

The lower two panels of Figure 5.1 show the dependences of the Breit-Wigner M_Z and M_W on the scale Q at which they are computed, based on the full 2-loop calculations in refs. [76, 77]. The Q dependences are seen to be greatly reduced by the inclusion of the 2-loop contributions, as expected. The reference model shown was chosen to reproduce the experimental value of M_Z , for $Q = 160$ GeV. The result for M_W is then a prediction, since it was not used at all in the determination of the model parameters in eq. (1.11). Note that the range of values obtained in Figure 5.1 is lower than the current world average from the Review of Particle Properties in ref. [49], which is $M_W = 80.379 \pm 0.012$ GeV. This reflects the well-known observation that the predicted central value of M_W in the Standard Model is somewhat lower than the observed range, but not by enough to draw any firm conclusions about the validity of the minimal Standard Model. (There is a long history of calculation of higher-loop contributions [32, 66, 128, 136–155] to the ρ parameter, which gives the W boson mass in terms of the Z boson mass and other on-shell parameters.) By default, **SMDR** uses a choice $Q = 160$ GeV when computing both the Z and W physical masses, but these choices can again be modified independently by the user at run time, as of course was done when making Figure 5.1.

The information from the Higgs boson mass M_h can be inverted to obtain the self-coupling λ , assuming the minimal Standard Model. This is illustrated in the left panel of Figure 5.2 where

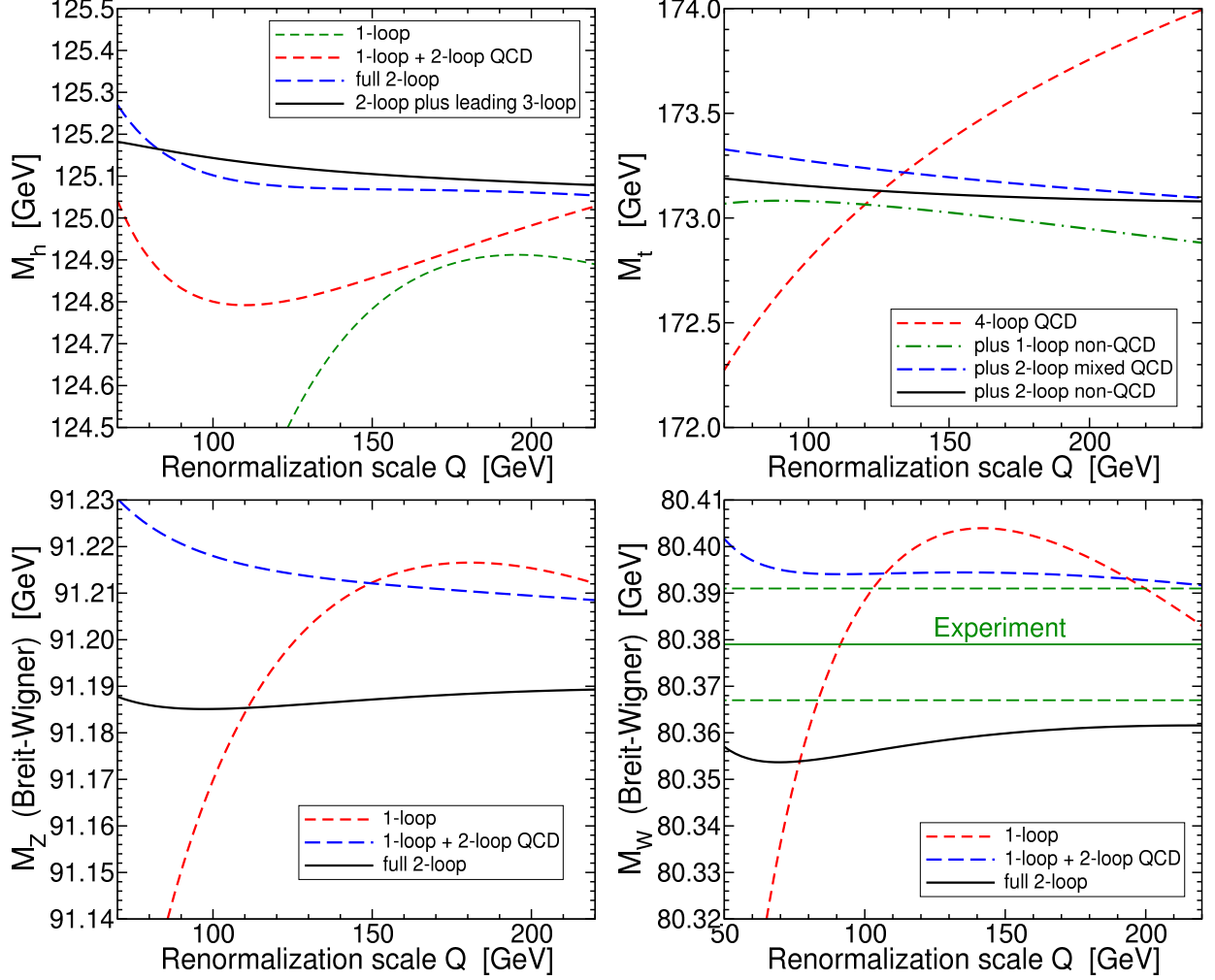


FIG. 5.1: Physical masses of the Higgs boson, top quark, Z boson, and W boson, as functions of the renormalization scale Q at which they are computed, in various approximations as labeled. The $\overline{\text{MS}}$ input parameters at Q are determined by RG evolution from the reference model point defined at $Q_0 = 173.1$ GeV in eq. (1.11). In the case of M_W , we also show the present experimental central (horizontal solid line) and $\pm 1\sigma$ (horizontal dashed lines) values.

we compute $\lambda(Q)$ at the renormalization scale Q by requiring it to give $M_h = 125.10$ GeV, using various approximations for the calculation of the latter. In the right panel, we then show the ratio of the value λ_{M_h} obtained in this way to the value λ_{run} obtained by RG running it from the value in the reference model at $Q_0 = 173.1$ GeV. This ratio is exactly 1 by construction at $Q = Q_0$ in the approximation used to define the reference model. In this approximation, the ratio remains less than 1 part in 10^4 over the entire range shown for Q . The parameters $\lambda(Q)$ and $m^2(Q)$ can also be run up to very high scales using the RG equations. These results are shown in Figure 5.3, including the central value fit as well as the envelopes resulting from varying each of M_h , M_t , and α_S independently within their 1-sigma and 2-sigma experimentally allowed ranges. As is now well-known (see for example refs. [156] and [73–75] and references therein), in the best-fit case with M_h near 125 GeV, $\lambda(Q)$ runs negative at a scale intermediate between the weak scale

and the Planck mass, indicating that our vacuum state may be quasi-stable if one makes the bold assumption that there is really no new physics all the way up to mass scales comparable to the scale Q where $\lambda(Q) < 0$.

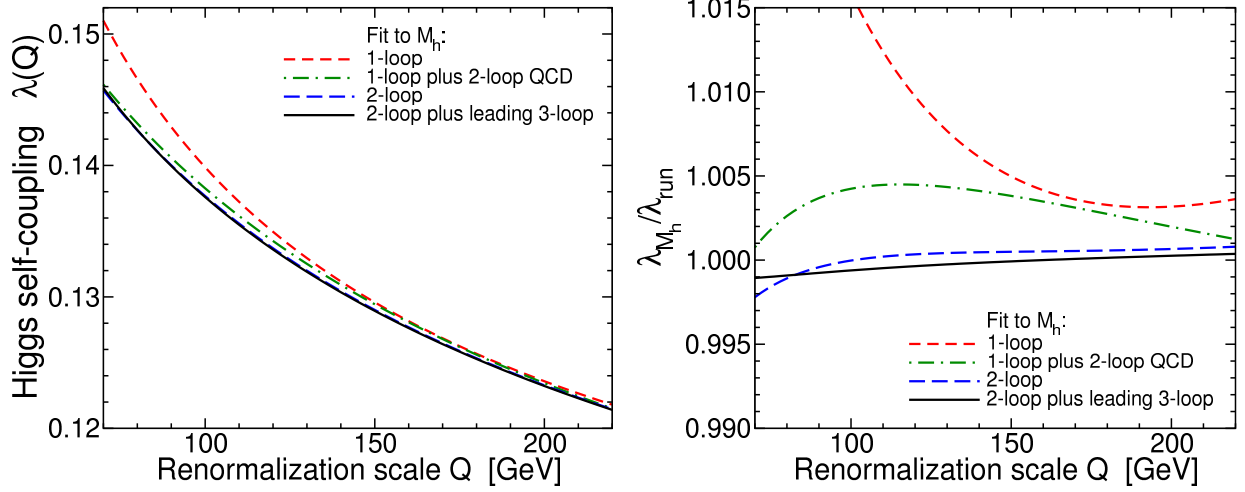


FIG. 5.2: The $\overline{\text{MS}}$ Higgs self-coupling λ , as a function of the renormalization scale Q , for the reference model point defined at $Q_0 = 173.1$ GeV in eq. (1.11). The other input parameters are obtained from the reference model by evolving them using their RG equations to the scale Q , where $\lambda(Q)$ is then obtained by requiring the Higgs pole mass to be 125.10 GeV. In the left panel, results are shown when the calculation of M_h is done in the 1-loop, 1-loop plus 2-loop QCD, full 2-loop, and 2-loop plus leading 3-loop approximations. The right panel shows the results for $\lambda(Q)_{M_h}/\lambda(Q)_{\text{run}}$, where $\lambda(Q)_{M_h}$ is determined as in the left panel, and $\lambda(Q)_{\text{run}}$ is obtained directly by renormalization running from the reference scale $Q_0 = 173.1$ GeV.

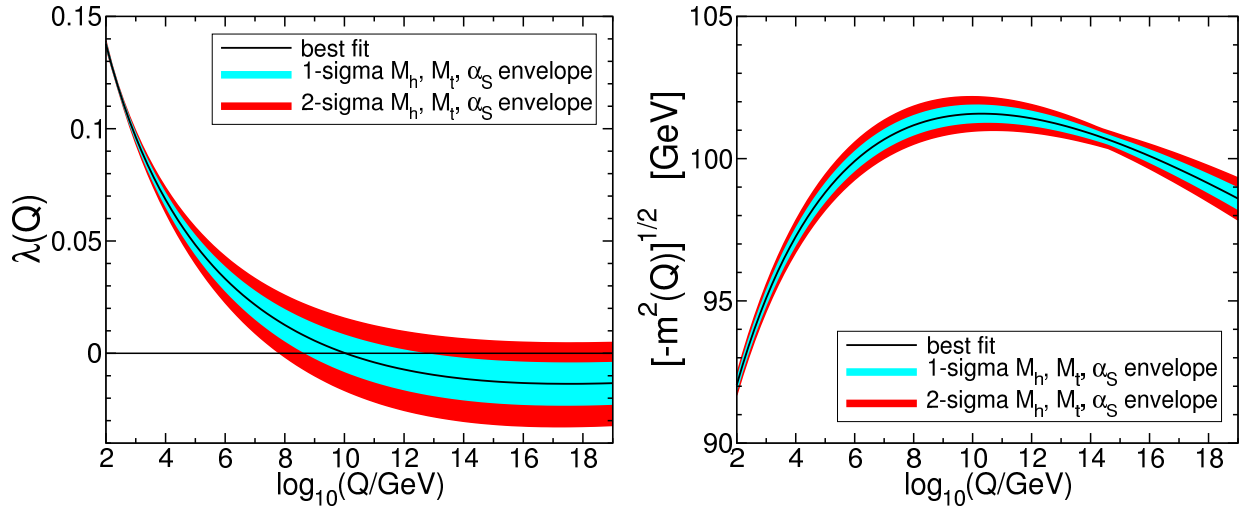


FIG. 5.3: Renormalization group running of the $\overline{\text{MS}}$ Higgs potential parameters λ and $\sqrt{-m^2}$, as a function of the renormalization scale Q . The black lines are the central values obtained from present experimental inputs. Also shown are the envelopes obtained by varying M_t , M_h , and $\alpha_S^{(5)}(M_Z)$ within 1-sigma (blue shaded region) and 2-sigma (red shaded region) of their central values. The slight “pinch” in the envelopes in the right panel near $Q = 10^{14}$ GeV is due to a focusing behavior of the α_S dependence of the $m^2(Q)$ renormalization group equation.

VI. THE SMDR CODE

As noted above, we have collected our results and methods in the form of a public software library written in C, which can be used interactively or incorporated into other software, and which is modular enough to be easily modified and updated.[†] A full description of how to use **SMDR**, and some example programs, are included with the distribution, which is available for download at [157]. For comprehensive information, we refer the reader to the file **README.txt**. In this section we give only a brief listing of some of the more common user interface variables and functions available. Note that these always begin with **SMDR_** to avoid naming conflicts with user code.

- The input values of Q and the $\overline{\text{MS}}$ parameters in eq. (1.4) are specified by global variables **SMDR_Q_in**, **SMDR_v_in**, **SMDR_lambda_in**, **SMDR_g3_in** etc. These can be set or adjusted by the user at any time, but typically remain fixed as multiple different tasks are performed, with corresponding temporary global variables **SMDR_Q**, **SMDR_v**, **SMDR_lambda**, **SMDR_g3** etc. used for renormalization group running to various other scales Q and subsequent individual calculations.
- Renormalization group running in the full, non-decoupled theory is done with the function **SMDR_RGeval_SM()**. In the decoupled QCD+QED theory with 5 quarks and 3 charged leptons, the evaluation of running parameters (with simultaneous decoupling of t, h, Z, W at a scale of choice) is done by **SMDR_RGeval_QCDQED_53()**. Similarly, evaluation of running parameters at lower scales including the sequential decoupling of the bottom quark, the tau lepton, and the charm quark, is done by **SMDR_RGeval_QCDQED_43()**, **SMDR_RGeval_QCDQED_42()**, and **SMDR_RGeval_QCDQED_32()**, respectively, where (5, 3) and (4, 3) and (4, 2) and (3, 2) refer to the numbers of active quarks and leptons.
- Minimization of the effective potential to find $m^2(Q)$ from $v(Q)$, or vice versa, are accomplished with functions **SMDR_Eval_m2()** or **SMDR_Eval_vev()**, respectively. These make use of the quantity $\Delta = \sum_n \Delta_n / (16\pi^2)^n$ appearing in eq. (1.3), which can also be computed separately with **SMDR_Eval_vevDelta()**.
- Evaluation of the complex pole masses of the four heavy particles is done with functions **SMDR_Eval_Mt()**, **SMDR_Eval_Mh()**, **SMDR_Eval_MZ()**, and **SMDR_Eval_MW()**. The last two functions also evaluate the variable-width Breit-Wigner masses of Z and W , which are the traditional ways of reporting those masses. In each case, one can specify the scale Q at which the computation is performed.
- Evaluation of the Fermi decay constant is done with the function **SMDR_Eval_GFermi()**, again with the computation performed at any specified choice of Q .
- The single function **SMDR_Eval_Gauge()** simultaneously evaluates the Sommerfeld fine structure constant α_0 and the RPP “ $\overline{\text{MS}}$ ” scheme (with only the top-quark decoupled) values $\hat{\alpha}(Q)$ and $\hat{s}_W^2(Q)$.

[†] The code **SMDR** subsumes and replaces our earlier program **SMH**, which evaluated only the Higgs pole mass and was described in ref. [72].

- The light quark $\overline{\text{MS}}$ masses $m_b(m_b)$, $m_c(m_c)$, $m_s(2 \text{ GeV})$, $m_d(2 \text{ GeV})$, and $m_u(2 \text{ GeV})$ are evaluated using `SMDR_Eval_mbmb()`, `SMDR_Eval_mcmc()`, and `SMDR_Eval_mquarks_2GeV()`.
- The charged lepton physical masses can be evaluated using `SMDR_Eval_Mtau_pole()`, `SMDR_Eval_Mmuon_pole()`, and `SMDR_Eval_Melectron_pole()`.
- A function `SMDR_Fit_Inputs()` performs a simultaneous fit to all of the $\overline{\text{MS}}$ quantities in eq. (1.4), for specified values of the on-shell observable quantities (except for M_W) in eq. (1.5), providing the results at a specified choice of Q .
- Various utility functions exist for reading parameters from and writing to electronic files.
- Our programs TSIL [125] for 2-loop self-energy integrals and 3VIL [26] for 3-loop vacuum integrals are included within the SMDR distribution, and so need not be downloaded separately.
- Interfaces for calling SMDR from external C or C++ code are included.
- A command-line program `calc_all` takes the $\overline{\text{MS}}$ inputs of eq. (1.4) and outputs all of the on-shell observables of eq. (1.5).
- Another command-line program `calc_fit` takes the on-shell observables of eq. (1.5) as inputs, and outputs the results of a fit to the $\overline{\text{MS}}$ inputs of eq. (1.4), by using the function `SMDR_Fit_Inputs()` mentioned above. This was used to obtain eq. (1.11).

As examples, the short C programs that produced all of the data used in the figures in this paper are included within the SMDR distribution. We also include several other command line programs. These should serve to illustrate how to incorporate SMDR into new programs.

VII. OUTLOOK

In this paper, we have studied the map between the $\overline{\text{MS}}$ Lagrangian parameters of the Standard Model and the observables to which they most closely correspond. In doing so, we have assumed that the minimal Standard Model is really the correct theory up to some high mass scale, so that new physics contributions effectively decouple. With the present absence of evidence at the LHC for new physics, this is at least a tenable hypothesis, and plausibly will remain so for quite some time. We therefore suggest that in the future the Review of Particle Properties should provide the best-fit values of the $\overline{\text{MS}}$ Lagrangian parameters of the Standard Model in the non-decoupled theory, since these fundamentally define the best model that we have to describe particle physics.

Another useful software package with rather similar aims to SMDR but a different implementation (including expansion around what we call v_{tree} rather than v) is `mr` [124]. There is also a very large number of works that test the whole space of electroweak precision observables in different ways; for an incomplete set of recent references and reviews on this approach, see refs. [158–167]. We emphasize that our primary goal here, of obtaining the best fit to the $\overline{\text{MS}}$ Lagrangian parameters, is different and complementary to that of testing the whole space of electroweak precision observables, as we are not considering possible non-negligible contributions from physics beyond the Standard Model. However, one application is to the matching to new physics models

(for example, supersymmetry) characterized by some mass scale much larger than the electroweak scale. This will necessitate a matching between the high energy theory and the Standard Model as an effective field theory, including with non-renormalizable operators. For a very incomplete sample of recent works on this subject, see refs. [168–183].

New theoretical refinements as well as more accurate experimental measurements will certainly come. We have therefore chosen a modular framework in which it should be straightforward to incorporate such new developments into the **SMDR** code. For example, we have avoided using numerical interpolating formulas from approximate fits to analytic formulas, instead opting to provide and use analytical calculations directly, up to the level of loop integrals that must then be evaluated numerically. This of course results in longer computation times, but is more transparent and easier to update. Most of the results presented in this paper are based on calculations that have appeared before, but we have provided for the first time a study of the impact of the 3-loop contributions to the effective potential on the relation between the loop-corrected VEV and the other Lagrangian parameters. We have also provided (in section IV and an ancillary file, as well as in the **SMDR** code) the full 2-loop relation between the loop-corrected VEV and the Fermi constant, as an alternative to the relation between G_F and the tree-level VEV that was found in refs. [69, 70, 124]. It is clear that significant advances will be needed in order to match the accuracy that can be obtained at proposed future e^+e^- colliders; for a recent review, see ref. [167]. Future work in the tadpole-free pure $\overline{\text{MS}}$ scheme will likely include the leading 3-loop corrections to M_W , M_Z , and G_F . These and $\Delta\alpha_{\text{had}}^{(5)}(M_Z)$ and M_t are the present bottlenecks to accuracy.

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