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Discrete nonlocal nonlinear Schrödinger systems: Integrability, inverse scattering and solitons

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Abstract

A number of integrable nonlocal discrete nonlinear Schrödinger (NLS) type systems have been recently proposed. They arise from integrable symmetry reductions of the well-known Ablowitz–Ladik scattering problem. The equations include: the classical integrable discrete NLS equation, integrable nonlocal: *PT* symmetric, reverse space time (RST), and the reverse time (RT) discrete NLS equations. Their mathematical structure is particularly rich. The inverse scattering transforms (IST) for the nonlocal discrete *PT* symmetric NLS corresponding to decaying boundary conditions was outlined earlier. In this paper, a detailed study of the IST applied to the *PT* symmetric, RST and RT integrable discrete NLS equations is carried out for rapidly decaying boundary conditions. This includes the direct and inverse scattering problem, symmetries of the eigenfunctions and scattering data. The general linearization method is based on a discrete nonlocal Riemann–Hilbert approach. For each discrete nonlocal NLS equation, an explicit one soliton solution is provided. Interestingly, certain one soliton solutions of the discrete *PT* symmetric NLS equation satisfy nonlocal discrete analogs of discrete elliptic function/Painlevé-type equations.

Keywords: inverse scattering transform, integrable systems, discrete nonlinear Schrödinger, integrable nonlocal reductions
Mathematics Subject Classification numbers: 37K15, 37Q55, 35Q51

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1. Introduction

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}q_t = q_{xx} - 2\sigma q^2 q^*, \quad \sigma = \mp 1, \tag{1.1}$$

where q^* is the complex conjugate of q, is a prototypical dispersive nonlinear partial differential equation that has been derived in many areas of physics and analyzed in detail for over 50 years. The NLS equation arises in electromagnetics, fluid dynamics/water waves, magnetic spin systems, Bose–Einstein condensation amongst many others [1–6]. When $\sigma = -1$ the NLS equation exhibits modulational instability and contains localized solitary waves/solitons. On the other hand, for $\sigma = +1$, the NLS equation is modulationally stable; it has dark solitary waves/solitons which have nonzero boundary values at infinity [6]. Mathematically speaking, the NLS equation attains even broader significance since, in one plus one dimension, it is integrable via the inverse scattering transform (IST) [7–12].

While for the past few decades much attention have been directed at the mathematical structure and physical applications of the continuous NLS equation (1.1), in recent years, a relatively new research area devoted to discrete photonics has emerged [13–18]. Progress in the mathematics and physics of complex discrete systems was possible due to advances in technology along side the successful asymptotic derivation of reduced discrete models. Among such a discrete equation is the well known discrete nonlinear Schrödinger

$$\mathrm{i}\frac{\mathrm{d}q_n}{\mathrm{d}t} = q_{n+1} + q_{n-1} + |q_n|^2 q_n, \tag{1.2}$$

where $q_n(t)$ describes a time-dependent discrete envelope function, dot stands for time derivative and n is an integer. This model successfully predicts numerous important phenomena in the physical and biological sciences. Examples include wave propagation in coupled waveguide arrays [19–25], biophysical system [26], molecular crystals [27], atomic chains [27, 28] as well as many recent observations related to PT symmetric arrays of linearly and/or nonlinearly coupled optical waveguide.

Although the discrete NLS (1.2) is commonly used in modeling wave propagation in photonic systems and optical waveguide arrays, to date, it is widely believed to not be integrable. In a series of papers published in 1975/76, Ablowitz and Ladik (AL) [29, 30] discovered an integrable discretization of the continuous NLS equation (1.1) given by

$$\mathrm{i}\frac{\mathrm{d}Q_n(t)}{\mathrm{d}t} = Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t) - \sigma|Q_n(t)|^2[Q_{n+1}(t) + Q_{n-1}(t)], \tag{1.3}$$

also frequently called the integrable AL model. If one let $Q_n(t) \equiv hq_n(t)$ with constant h, then in the limit of $h \to 0$, the integrable discrete NLS equation (1.3) tends to the continuous NLS equation (1.1), recovering all the underlying integrable properties of the NLS plus new mathematical features.

Importantly, the AL model arises from a compatibility condition applied to the following system:

$$\mathrm{i}\frac{\mathrm{d}Q_n(t)}{\mathrm{d}t} = Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t) - Q_n(t)R_n(t)[Q_{n+1}(t) + Q_{n-1}(t)], \tag{1.4}$$

$$-\mathrm{i}\frac{dR_n(t)}{\mathrm{d}t} = R_{n+1}(t) - 2R_n(t) + R_{n-1}(t) - Q_n(t)R_n(t)[R_{n+1}(t) + R_{n-1}(t)]. \tag{1.5}$$

Indeed, one recovers equation (1.3) from the symmetry reduction

$$R_n(t) = \sigma Q_n^*(t), \quad \sigma = \mp 1. \tag{1.6}$$

The integrable discrete AL model has been studied in various mathematical settings (see [31–34] and references therein). Among them is the quantization of the AL model and its solution method using the so-called quantum inverse scattering transform [35–37]; local Darboux-Bäcklund transformations for the AL model with general discretizations and solutions via dressing methods [38, 39]. In 2014, Ablowitz and Musslimani [40] discovered a new, *PT* symmetric reduction of the AL scattering problem (see equation (2.1) below). It is given by

$$R_n(t) = \sigma Q_{-n}^*(t), \tag{1.7}$$

giving rise to the so-called integrable discrete nonlocal *PT* symmetric NLS equation:

$$\mathrm{i}\frac{\mathrm{d}Q_n(t)}{\mathrm{d}t} = Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t) - \sigma Q_n(t)Q_{-n}^*(t)[Q_{n+1}(t) + Q_{n-1}(t)]. \tag{1.8}$$

Subsequently, two new integrable symmetry reductions were identified [41]. These are

$$R_n(t) = \sigma Q_{-n}(-t), \tag{1.9}$$

$$R_n(t) = \sigma Q_n(-t), \tag{1.10}$$

giving rise to the so-called integrable discrete reverse space time (RST) and reverse time (RT) NLS equations respectively

$$\mathrm{i}\frac{\mathrm{d}Q_n(t)}{\mathrm{d}t} = Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t) - \sigma Q_n(t)Q_{-n}(-t)[Q_{n+1}(t) + Q_{n-1}(t)], \tag{1.11}$$

$$\mathrm{i}\frac{\mathrm{d}Q_n(t)}{\mathrm{d}t} = Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t) - \sigma Q_n(t)Q_n(-t)[Q_{n+1}(t) + Q_{n-1}(t)]. \tag{1.12}$$

Notice that one can recover the *PT* symmetric, RST and RT continuous NLS limits

$$\mathrm{i}q_t = q_{xx} - 2\sigma q(x,t)^2 q^*(-x,t), \tag{1.13}$$

$$\mathrm{i}q_t = q_{xx} - 2\sigma q(x,t)^2 q(-x,-t), \tag{1.14}$$

$$\mathrm{i}q_t = q_{xx} - 2\sigma q(x,t)^2 q(x,-t), \tag{1.15}$$

by letting $Q_n(t) \equiv hq_n(t)$ and take the $h \to 0$ limit. Equations (1.13)–(1.15) were found in [41–43] as a nonlocal in space and/or in time integrable symmetry reductions of the well-known AKNS scattering problem [44]. Furthermore, equations (1.13)–(1.15) were recently shown to arise from an integrable nonlocal asymptotic reductions of physically significant nonlinear equations such as the cubic nonlinear Klein–Gordon, the Korteweg–de Vries and water wave equations [45].

The new results and organization of the paper are summarized as follows:

- The inverse scattering transforms are developed for the integrable discrete *PT*, RST and RT nonlinear Schrödinger equations with rapidly decaying boundary conditions:

$$\lim_{n\to\pm\infty} Q_n = 0, \quad \lim_{n\to\pm\infty} R_n = 0. \tag{1.16}$$

More general integrable boundary conditions, such as $\lim_{n\to\pm\infty} Q_n = Q_{\pm\infty} \neq 0$ and $\lim_{n\to\pm\infty} R_n = R_{\pm\infty} \neq 0$ is still an open problem and will be studied in the future. Some recent relevant results for the AL model and the behavior of its solitons (with non-zero boundary conditions) are presented in [46–49].

- This includes a detailed study of the discrete direct and inverse scattering theory for general potentials Q_n, R_n. In particular we derive all symmetries satisfied by the eigenfunctions, scattering data and norming constants for all three reduction cases. We note that all relevant symmetry conditions are very different than the classical integrable AL model in the sense that they are nonlocal and their derivation requires a forward and a backward scattering problems. This paper also formulates the IST in order to also solve the discrete RST and RT symmetric nonlocal systems.
- Soliton solutions for all three nonlocal integrable discrete NLS equations are obtained. Their properties are discussed particularly the issue of singularity formation in finite time. We note that physical systems can exhibit finite-time blow up singularities as evidenced by the physically significant two dimensional nonlinear Schrödinger equation [50]. As shown in this paper, interesting blow up solutions can occur in these nonlocal discrete systems.
- Novel reconstruction formulae for the potentials are found. This in turn enables one to observe the integrable symmetry on the inverse side simply by looking at the functional form of both potentials R_n and Q_n.
- Trace formulae are developed for the RST and *PT* symmetric cases and used to find an explicit expressions for the norming constants in terms of scattering data.
- Discrete and continuous RT and RST nonlocal Painlevé equations are introduced.
- Sections 2–6 discuss the direct scattering associated with the above integrable discrete *PT*, RST and RT nonlinear Schrödinger equations: (1.8), (1.11) and (1.12), and their associated scattering space symmetry relations. Sections 8–10 details the inverse scattering, time dependence and reconstruction formulae. Sections 11 and 12 discuss norming constants, trace formulae and symmetries needed to compute soliton solutions. Section 13 provides one soliton solutions for all cases. In section 14 remarks about nonlocal Painlevé equations are provided and section 15 is the conclusion.

2. Linear pairs and integrability

We start by considering the Ablowiz–Ladik scattering problem

$$v_{n+1}(z,t) = \begin{pmatrix} z & Q_n(t) \\ R_n(t) & z^{-1} \end{pmatrix} v_n(z,t), \tag{2.1}$$

$$\frac{\mathrm{d}v_n(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{i}Q_n(t)R_{n-1}(t) - \frac{\mathrm{i}}{2}(z-z^{-1})^2 & -\mathrm{i}(zQ_n(t) - z^{-1}Q_{n-1}(t)) \\ \mathrm{i}(z^{-1}R_n(t) - zR_{n-1}(t)) & -\mathrm{i}R_n(t)Q_{n-1}(t) + \frac{\mathrm{i}}{2}(z-z^{-1})^2 \end{pmatrix} v_n(t), \tag{2.2}$$

where $v_n(t) \equiv (v_n^{(1)}(t), v_n^{(2)}(t))^T$ is a complex valued function of $t \geqslant 0$ and $n \in \mathbb{Z}$; $Q_n(t)$ and $R_n(t)$ are complex valued potentials that rapidly decay to zero as $n \to \pm\infty$. Here, z is a spectral parameter taken to be (in general) complex and independent of t, n. The discrete compatibility condition $\frac{\mathrm{d}}{\mathrm{d}t}v_{n+1} = (\frac{\mathrm{d}}{\mathrm{d}t}v_m)_{m=n+1}$ yields the system of equations (1.4) and (1.5). As mentioned earlier, all of the above integrable discrete equations, i.e. (1.3), (1.8), (1.11) and (1.12) are obtained from the symmetry reductions between $R_n(t)$ and $Q_n(t)$ given by equations (1.6), (1.7), (1.9) and (1.10). As such, they all form integrable infinite-dimensional Hamiltonian dynamical systems; their conserved quantities are given in [34, 40].

Due to the nonlocal (in space) nature of the symmetry reductions (1.7) and (1.9), it proves crucial in the analysis of the direct scattering problem (particularly for obtaining the symmetries in scattering space) to supplement the AL system (2.1) with a 'backward' scattering problem (obtained by inverting equation (2.1)) defined by

$$w_{n-1}(z,t) = \begin{pmatrix} z^{-1} & -Q_n(t) \\ -R_n(t) & z \end{pmatrix} w_n(z,t). \tag{2.3}$$

Importantly, any solution $v_n(z,t)$ of (2.1) can be related to a solution of (2.3) via the transformation

$$w_n(z,t) = f_n(t)v_{n+1}(z,t), \quad f_n(t) \equiv \prod_{k=-\infty}^{n} \frac{1}{1-Q_k(t)R_k(t)}. \tag{2.4}$$

Since both scattering problems (2.1) and (2.3) are linear, the presence of the factor $f_n(t)$ in equation (2.4) suggests that the solution set $\{v_n(t), w_n(t)\}$ need to be 'chosen' in such a way that the correct boundary conditions are satisfied (see section 3). For the rest of the paper, and to avoid any confusion, we shall explicitly highlight the time-dependence of the eigenfunctions, scattering data, all symmetry relations, and potentials for the RST and RT nonlocal NLS cases. Furthermore, we shall suppress the time dependence of any equation that depend locally on time.

3. Direct scattering problem: general consideration

In this section, we provide the main ingredients necessary to solve the AL scattering problem for generic potentials. Since the discrete potentials Q_n, R_n vanish rapidly as $n \to \pm\infty$, the scattering problem (2.1) is defined by the following boundary conditions [34]:

$$\lim_{n\to-\infty} \phi_n(z) = z^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lim_{n\to-\infty} \overline{\phi}_n(z) = z^{-n} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.1}$$

$$\lim_{n\to+\infty} \psi_n(z) = z^{-n} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lim_{n\to+\infty} \overline{\psi}_n(z) = z^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.2}$$

where each $\phi_n(z), \overline{\phi}_n(z)$ and $\psi_n(z), \overline{\psi}_n(z)$ individually satisfy equation (2.1). Furthermore, the scattering problem (2.3) is subject to the same boundary conditions. In essence, transformation (2.4) implies that each eigenfunction need to be properly 'normalized' such that the corresponding boundary conditions between v_n and w_n match. Throughout the rest of the paper, star is used to indicate complex conjugation (and not bar). Clearly, the eigenfunction pair $\{\phi_n(z), \overline{\phi}_n(z)\}$ are linearly independent (similarly for $\{\psi_n(z), \overline{\psi}_n(z)\}$). Thus, since the scattering problem (2.1) is second order, the two eigenfunctions sets are related. Mathematically speaking, this fact is expressed as

$$\phi_n(z) = b(z)\psi_n(z) + a(z)\overline{\psi}_n(z), \tag{3.3}$$

$$\overline{\phi}_n(z) = \overline{a}(z)\psi_n(z) + \overline{b}(z)\overline{\psi}_n(z), \tag{3.4}$$

where $a(z), \overline{a}(z), b(z), \overline{b}(z)$ are the scattering data given by the relations

$$a(z) = c_n W\left(\phi_n(z),\ \psi_n(z)\right), \tag{3.5}$$

$$\overline{a}(z) = c_n W\left(\overline{\psi}_n(z),\ \overline{\phi}_n(z)\right), \tag{3.6}$$

$$b(z) = c_n W\left(\overline{\psi}_n(z),\ \phi_n(z)\right), \tag{3.7}$$

$$\overline{b}(z) = c_n W\left(\overline{\phi}_n(z),\ \psi_n(z)\right). \tag{3.8}$$

Here, W is the Wronskian defined by

$$W(v_n, w_n) = v_n^{(1)} w_n^{(2)} - v_n^{(2)} w_n^{(1)}, \tag{3.9}$$

where $v_n = (v_n^{(1)},\ v_n^{(2)})^T$ and $w_n = (w_n^{(1)},\ w_n^{(2)})^T$ and

$$c_n = \prod_{k=n}^{+\infty} (1 - Q_k R_k). \tag{3.10}$$

The above scattering data also satisfies the unitarity condition (see [34])

$$a(z)\overline{a}(z) - b(z)\overline{b}(z) = c_{-\infty}. \tag{3.11}$$

As we shall see later, the scattering data $a(z), \overline{a}(z)$ and the product $b(z)\overline{b}(z)$ turn out to be time-independent (see [34] for further details.) This fact implies $c_{-\infty}$ be time-independent as well, thus making it a constant of motion. In the following analysis, it is convenient to consider functions with constant boundary conditions. We define the bounded eigenfunctions as follows:

$$M_n(z) = z^{-n}\phi_n(z), \quad \overline{M}_n(z) = z^n \overline{\phi}_n(z), \tag{3.12}$$

$$N_n(z) = z^n \psi_n(z), \quad \overline{N}_n(z) = z^{-n}\overline{\psi}_n(z). \tag{3.13}$$

For the convenience of the reader, we write down the boundary conditions associated with these eigenfunctions:

$$\lim_{n\to-\infty} M_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lim_{n\to-\infty} \overline{M}_n(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.14}$$

$$\lim_{n\to+\infty} N_n(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lim_{n\to+\infty} \overline{N}_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.15}$$

In [34] it was shown that if $\|Q\|_1 = \sum_{-\infty}^{\infty} |Q_n| < \infty$ and $\|R\|_1 = \sum_{-\infty}^{\infty} |R_n| < \infty$, then $M_n(z), N_n(z), a(z)$ are analytic for $|z| > 1$ and continuous for $|z| \geqslant 1$. Furthermore, $\overline{M}_n(z), \overline{N}_n(z), a(z)$ are analytic for $|z| < 1$ and continuous for $|z| \leqslant 1$. The scattering data $a(z), \overline{a}(z)$ are even functions of z while $b(z), \overline{b}(z)$ are odd functions of z. The eigenfunctions admit the following asymptotic expansions as $z \to \infty, z \to 0$ (which will be used when solving the inverse problem)

$$M_n(z) = \begin{pmatrix} 1 + O(z^{-2}), \text{even powers of } z \text{ only} \\ z^{-1}R_{n-1} + O(z^{-3}), \text{odd powers of } z \text{ only} \end{pmatrix} \quad \text{as} \quad |z| \to \infty, \tag{3.16}$$

$$N_n(z) = \begin{pmatrix} -z^{-1}c_n^{-1}Q_n + O(z^{-3}), \text{odd powers of } z \text{ only} \\ c_n^{-1} + O(z^{-2}), \text{even powers of } z \text{ only} \end{pmatrix} \quad \text{as} \quad |z| \to \infty, \tag{3.17}$$

$$\overline{M}_n(z) = \begin{pmatrix} zQ_{n-1} + O(z^3), \text{odd powers of } z \text{ only} \\ 1 + O(z^2), \text{even powers of } z \text{ only} \end{pmatrix} \quad \text{as} \quad z \to 0, \tag{3.18}$$

$$\overline{N}_n(z) = \begin{pmatrix} c_n^{-1} + O(z^2), \text{even powers of } z \text{ only} \\ zc_n^{-1}R_n + O(z^3), \text{odd powers of } z \text{ only} \end{pmatrix} \quad \text{as} \quad z \to 0. \tag{3.19}$$

The scattering data have the following asymptotic behavior for large and small z:

$$a(z) = 1 + O(z^{-2}), \text{even powers of } z \text{ only} \quad \text{as} \quad |z| \to \infty, \tag{3.20}$$

$$\overline{a}(z) = 1 + O(z^2), \text{even powers of } z \text{ only} \quad \text{as} \quad |z| \to 0, \tag{3.21}$$

where, as mentioned above, $c_n = \prod_{k=n}^{+\infty}(1 - Q_kR_k)$. Notice that the factor c_n explicitly depends on the unknown potentials Q_n, R_n. This fact makes the Riemann–Hilbert (RH) inverse problem a more difficult task. To remedy this issue, it proves convenient to modify the eigenfunctions $M_n(z), N_n(z); \overline{M}_n(z), \overline{N}_n(z)$ and introduce instead a new set of eigenfunctions, $M_n'(z), N_n'(z); \overline{M}_n'(z), \overline{N}_n'(z)$, whose definition and asymptotic behavior (for large and small z) are given by [34]

$$N_n' \equiv \mathcal{A}N_n = \begin{pmatrix} -z^{-1}c_n^{-1}Q_n \\ 1 \end{pmatrix} + O(z^{-2}) \quad \text{as } z \to \infty, \tag{3.22}$$

$$M_n' \equiv \mathcal{A}M_n = \begin{pmatrix} 1 \\ z^{-1}c_nR_{n-1} \end{pmatrix} + O(z^{-2}) \quad \text{as } z \to \infty, \tag{3.23}$$

$$\overline{N}_n' \equiv \mathcal{A}\overline{N}_n = \begin{pmatrix} c_n^{-1} \\ -zR_n \end{pmatrix} + O(z^2) \quad \text{as } z \to 0, \tag{3.24}$$

$$\overline{M}_n' \equiv \mathcal{A}\overline{M}_n = \begin{pmatrix} zQ_{n-1} \\ c_n \end{pmatrix} + O(z^2) \quad \text{as } z \to 0, \tag{3.25}$$

where $\mathcal{A}$ denotes the matrix

$$\mathcal{A} \equiv \begin{pmatrix} 1 & 0 \\ 0 & c_n \end{pmatrix}. \tag{3.26}$$

When solving the inverse problem from the left, the auxiliary functions $\mu_n, \overline{\mu}_n, \mu_n'$ and $\overline{\mu}_n'$ will be used with their asymptotic behavior (in z):

$$\mu_n(z) \equiv \frac{M_n(z)}{a(z)}, \qquad \mu_n'(z) \equiv \frac{M_n'(z)}{a(z)}, \tag{3.27}$$

$$\overline{\mu}_n(z) \equiv \frac{\overline{M}_n(z)}{\overline{a}(z)}, \qquad \overline{\mu}_n'(z) \equiv \frac{\overline{M}_n'(z)}{\overline{a}(z)}, \tag{3.28}$$

$$\mu_n(z) = \begin{pmatrix} 1 + O(z^{-2}) \\ z^{-1}R_{n-1} + O(z^{-3}) \end{pmatrix}, \quad \text{as} \quad |z| \to \infty, \tag{3.29}$$

$$\overline{\mu}_n(z) = \begin{pmatrix} zQ_{n-1} + O(z^3) \\ 1 + O(z^2) \end{pmatrix}, \quad \text{as} \quad z \to 0, \tag{3.30}$$

$$\mu_n'(z) = \begin{pmatrix} 1 \\ z^{-1} c_n R_{n-1} \end{pmatrix} + O(z^{-2}) \qquad \text{as } |z| \to \infty, \tag{3.31}$$

$$\overline{\mu}_n'(z) = \begin{pmatrix} zQ_{n-1} \\ c_n \end{pmatrix} + O(z^2) \qquad \text{as } |z| \to 0. \tag{3.32}$$

So far we have presented generic basic properties of the eigenfunctions and scattering data needed in the analysis of the inverse scattering problem. No symmetry assumption was made on the potentials Q_n and R_n.

4. Ablowitz–Ladik reduction $R_n = \sigma Q_n^*$

For completeness and to make the comparison between all four different integrable symmetry reductions of the AL scattering problem easier, in this section we provide the reader with a brief summary of the major symmetry results between the eigenfunctions, scattering data and norming constants, see [34].

4.1. Symmetries between eigenfunction

$$\begin{pmatrix} \overline{M}_n^{(1)}(z) \\ \overline{M}_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} \sigma M_n^{(2)^*}(1/z^*) \\ M_n^{(1)^*}(1/z^*) \end{pmatrix}, \qquad \begin{pmatrix} \overline{N}_n^{(1)}(z) \\ \overline{N}_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} N_n^{(2)^*}(1/z^*) \\ \sigma N_n^{(1)^*}(1/z^*) \end{pmatrix}. \tag{4.1}$$

4.2. Symmetries between scattering data

$$\overline{a}(z) = a^*(1/z^*), \qquad \overline{b}(z) = \sigma b^*(1/z^*), \tag{4.2}$$

$$\overline{z}_j = 1/z_j^*, \quad \overline{C}_j = -\sigma (z_j^*)^{-2} C_j^*, \quad \overline{\rho}(z) = \sigma \rho^*(1/z^*), \tag{4.3}$$

where here and below $z_j, \overline{z}_j$ mean:

$$a(z_j) = 0, \; \overline{a}(\overline{z}_j) = 0. \tag{4.4}$$

Due to analyticity properties, z_j and $\overline{z}_j$, are finite in number. We will assume that $z_j, \overline{z}_j$ are simple zero's, termed proper zero's, and have same total number outside/inside the unit circle $|z| = 1$: $j = 1, 2, ...J$. These are also the eigenvalues giving rise bound states.

The norming constants are defined by

$$\begin{aligned} &\phi_n(z_j) = b_j \psi_n(z_j), \qquad \overline{\phi}_n(\overline{z}_j) = \overline{b}_j \overline{\psi}_n(\overline{z}_j), \\ &C_j = b_j / a'(z_j), \qquad \overline{C}_j = \overline{b}_j / \overline{a}'(\overline{z}_j). \end{aligned} \tag{4.5}$$

4.3. Symmetries of the modified eigenfunctions

$$\overline{M}_n^{(1)'}(z) = \sigma (c_n^\sigma)^{-1} \, M_n^{(2)'^*}(1/z^*), \tag{4.6}$$

$$\overline{M}_n^{(2)'}(z) = c_n^\sigma \, M_n^{(1)'^*}(1/z^*), \tag{4.7}$$

$$\overline{N}_n^{(1)'}(z) = (c_n^\sigma)^{-1} N_n^{(2)'^*}(1/z^*), \tag{4.8}$$

$$\overline{N}_n^{(2)'}(z) = \sigma c_n^\sigma N_n^{(1)'^*}(1/z^*), \tag{4.9}$$

where we have defined

$$c_n^\sigma = \prod_{k=n}^{\infty} \left(1 - \sigma|Q_k|^2\right). \tag{4.10}$$

5. Reverse-time symmetry reduction $R_n(t) = \sigma Q_n(-t)$, $\sigma = \mp 1$

In this section, we shall establish a crucial integrable symmetry relation that holds between sets of eigenfunctions solutions to the AL scattering problem. This in turn induces an important symmetry restriction between the scattering data.

5.1. Symmetries between the eigenfunctions

To that purpose we write $\phi_n(z,t) \equiv (\phi_n^{(1)}(z,t),\ \phi_n^{(2)}(z,t))^T$ (note that we have now included the variable t in the eigenfunctions), where superscript T denotes matrix transpose, as a solution to system (2.1) with $R_n(t) = \sigma Q_n(-t)$:

$$\phi_{n+1}^{(1)}(z,t) = z\phi_n^{(1)}(z,t) + Q_n(t)\phi_n^{(2)}(z,t), \tag{5.1}$$

$$\phi_{n+1}^{(2)}(z,t) = \sigma Q_n(-t)\phi_n^{(1)}(z,t) + z^{-1}\phi_n^{(2)}(z,t). \tag{5.2}$$

Let $t \to -t,\ z \to 1/z$ in the above equations; rearrange terms to find

$$\phi_{n+1}^{(2)}(1/z,-t) = z\phi_n^{(2)}(1/z,-t) + Q_n(t)\left[\sigma\phi_n^{(1)}(1/z,-t)\right], \tag{5.3}$$

$$\sigma\phi_{n+1}^{(1)}(1/z,-t) = \sigma Q_n(-t)\phi_n^{(2)}(1/z,-t) + z^{-1}\left[\sigma\phi_n^{(1)}(1/z,-t)\right]. \tag{5.4}$$

Now, define the quantities:

$$\phi_n^{(2)}(1/z,-t) \equiv \Phi_n^{(1)}(z,t), \quad \sigma\phi_n^{(1)}(1/z,-t) \equiv \Phi_n^{(2)}(z,t). \tag{5.5}$$

Then equations (5.3) and (5.4) read

$$\Phi_{n+1}^{(1)}(z,t) = z\Phi_n^{(1)}(z,t) + Q_n(t)\Phi_n^{(2)}(z,t), \tag{5.6}$$

$$\Phi_{n+1}^{(2)}(z,t) = \sigma Q_n(-t)\Phi_n^{(1)}(z,t) + z^{-1}\Phi_n^{(2)}(z,t), \tag{5.7}$$

which is exactly the scattering problem (2.1) under the symmetry reduction $R_n(t) = \sigma Q_n(-t)$. Therefore, from (5.6) and (5.7) we have the following symmetry relation

$$\text{If } \begin{pmatrix} \phi_n^{(1)}(z,t) \\ \phi_n^{(2)}(z,t) \end{pmatrix} \text{ solves equation (2.1) with } R_n(t) = \sigma Q_n(-t) \text{ so does } \begin{pmatrix} \phi_n^{(2)}(1/z,-t) \\ \sigma\phi_n^{(1)}(1/z,-t) \end{pmatrix}.$$

Similar symmetry arguments hold for $\overline{\phi}_n(z,t) \equiv (\overline{\phi}_n^{(1)}(z,t),\ \overline{\phi}_n^{(2)}(z,t))^T$.

Next we discuss how the above symmetry and the corresponding boundary conditions (3.1) induce certain symmetry conditions on the eigenfunctions pair and (as we shall se in the next section) on the scattering data. Again let $\phi_n(z,t)$ be a solution to system (2.1) with $R_n(t) = \sigma Q_n(-t)$ obeying the boundary condition given on the left part of equation (3.1). Now, let $z \to 1/z,\ t \to -t$ in (3.1) to obtain

$$\lim_{n\to-\infty} \begin{pmatrix} \sigma\phi_n^{(2)}(1/z,-t) \\ \phi_n^{(1)}(1/z,-t) \end{pmatrix} = z^{-n} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.8}$$

Next, let $\overline{\phi}_n(z,t)$ be another solution to system (2.1) with $R_n(t) = \sigma Q_n(-t)$ satisfying the boundary condition given on the right part of equation (3.1). Therefore, we have the important result:

$$\begin{pmatrix} \overline{\phi}_n^{(1)}(z,t) \\ \overline{\phi}_n^{(2)}(z,t) \end{pmatrix} = \begin{pmatrix} \sigma\phi_n^{(2)}(1/z,-t) \\ \phi_n^{(1)}(1/z,-t) \end{pmatrix}. \tag{5.9}$$

This symmetry relation induces an important symmetry between the eigenfunctions $\overline{M}_n(z,t)$ and $M_n(z,t)$ which reads

$$\begin{pmatrix} \overline{M}_n^{(1)}(z,t) \\ \overline{M}_n^{(2)}(z,t) \end{pmatrix} = \begin{pmatrix} \sigma M_n^{(2)}(1/z,-t) \\ M_n^{(1)}(1/z,-t) \end{pmatrix}. \tag{5.10}$$

Similarly, we assume that $\psi_n(z,t) \equiv (\psi_n^{(1)}(z,t),\ \psi_n^{(2)}(z,t))^T$ is a solution to system (2.1) with $R_n(t) = \sigma Q_n(-t)$ that satisfies the boundary condition (3.2). By letting $z \to 1/z,\ t \to -t$ in (3.2) we obtain

$$\lim_{n\to+\infty} \begin{pmatrix} \psi_n^{(2)}(1/z,-t) \\ \sigma\psi_n^{(1)}(1/z,-t) \end{pmatrix} = z^{n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.11}$$

Clearly, if $\overline{\psi}_n(z,t)$ is a solution to system (2.1), with $R_n(t) = \sigma Q_n(-t)$, satisfying the boundary condition (3.2) we have the result:

$$\begin{pmatrix} \overline{\psi}_n^{(1)}(z,t) \\ \overline{\psi}_n^{(2)}(z,t) \end{pmatrix} = \begin{pmatrix} \psi_n^{(2)}(1/z,-t) \\ \sigma\psi_n^{(1)}(1/z,-t) \end{pmatrix}. \tag{5.12}$$

To obtain a symmetry relation between the corresponding eigenfunctions $N_n(z,t)$ and $\overline{N}_n(z,t)$ we use the result obtained in (5.12) and find

$$\begin{pmatrix} \overline{N}_n^{(1)}(z,t) \\ \overline{N}_n^{(2)}(z,t) \end{pmatrix} = \begin{pmatrix} N_n^{(2)}(1/z,-t) \\ \sigma N_n^{(1)}(1/z,-t) \end{pmatrix}. \tag{5.13}$$

5.2. Symmetries between the modified eigenfunctions

Recall the relation between the two sets of eigenfunctions $\{M_n(z), N_n(z); \overline{M}_n(z), \overline{N}_n(z)\}$ and $\{M_n'(z), N_n'(z); \overline{M}_n'(z), \overline{N}_n'(z)\}$ given by equations (3.22)–(3.25). Use the symmetry relation between the 'unprimed' eigenfunctions established in (5.10) and (5.13), we find, componentwise

$$\overline{M}_n^{(1)'}(z,t) = \sigma(c_n^\sigma(t))^{-1}\, M_n^{(2)'}(1/z,-t), \tag{5.14}$$

$$\overline{M}_n^{(2)'}(z,t) = c_n^\sigma(t)\, M_n^{(1)'}(1/z,-t), \tag{5.15}$$

$$\overline{N}_n^{(1)'}(z,t) = (c_n^\sigma(t))^{-1}\, N_n^{(2)'}(1/z,-t), \tag{5.16}$$

$$\overline{N}_n^{(2)'}(z,t) = \sigma c_n^\sigma(t)\, N_n^{(1)'}(1/z,-t). \tag{5.17}$$

Note that, by definition, $c_n(t) \equiv c_n(Q_k(t), R_k(t)) = \prod_{k=n}^{\infty}(1 - Q_k(t)R_k(t))$. At the symmetry point where $R_n(t) = \sigma Q_n(-t)$ we have

$$c_n^\sigma(t) \equiv c_n(Q_k(t), \sigma Q_k(-t)) = \prod_{k=n}^{\infty}(1 - \sigma Q_k(t)Q_k(-t)) = c_n^\sigma(-t). \tag{5.18}$$

5.3. Symmetry of scattering data

To establish the symmetry relation between the scattering data $a(z)$ and $\overline{a}(z)$ we start from (3.5), (3.6) and the assumption that $R_n(t) = \sigma Q_n(-t)$. Now we have

$$\overline{a}(z,t) = c_n^\sigma(t)\left(\psi_n^{(2)}(1/z,-t)\overline{\phi}_n^{(2)}(z,t) - \sigma\psi_n^{(1)}(1/z,-t)\overline{\phi}_n^{(1)}(z,t)\right) \tag{5.19}$$

$$= c_n^\sigma(t)\left(\psi_n^{(2)}(1/z,-t)\phi_n^{(1)}(1/z,-t) - \sigma^2\psi_n^{(1)}(1/z,-t)\phi_n^{(2)}(1/z,-t)\right). \tag{5.20}$$

However, from (3.5) we get

$$a(1/z,-t) = c_n^\sigma(-t)\left(\psi_n^{(2)}(1/z,-t)\phi_n^{(1)}(1/z,-t) - \psi_n^{(1)}(1/z,-t)\phi_n^{(2)}(1/z,-t)\right). \tag{5.21}$$

Since $c_n^\sigma(t) = c_n^\sigma(-t)$, as s a result we have

$$\overline{a}(z,t) = a(1/z,-t). \tag{5.22}$$

As we shall see later, it turns out that the scattering data $\overline{a}(z,t)$ and $a(z,t)$ are *time-independent* giving rise to the following symmetry between the zeros of $a(z,t)$ and $\overline{a}(z,t)$ (see equation (4.4); these zeros are also termed soliton eigenvalues:

$$\text{If } z_j, \overline{z}_j \in \mathbb{C} \text{ are eigenvalues then } \overline{z}_j = 1/z_j. \tag{5.23}$$

To find the symmetry relation between the scattering data $b(z,t)$ and $\overline{b}(z,t)$ we start from equation (3.8), the symmetry relation (5.9) and the assumption that $R_n(t) = \sigma Q_n(-t)$. We have

$$\begin{aligned}\overline{b}(z,t) &= c_n(t)\left(\overline{\phi}_n^{(1)}(z,t)\psi_n^{(2)}(z,t) - \overline{\phi}_n^{(2)}(z,t)\psi_n^{(1)}(z,t)\right)\\ &= c_n^\sigma(t)\left(\sigma\phi_n^{(2)}(1/z,-t)\psi_n^{(2)}(z,t) - \phi_n^{(1)}(1/z,-t)\psi_n^{(1)}(z,t)\right).\end{aligned} \tag{5.24}$$

Next, from (3.7) we let $z \to 1/z,\ t \to -t$ and make use of the symmetry relation (5.12) to find

$$\begin{aligned}b(1/z,-t) &= c_n(-t)\left(\overline{\psi}_n^{(1)}(1/z,-t)\phi_n^{(2)}(1/z,-t) - \overline{\psi}_n^{(2)}(1/z,-t)\phi_n^{(1)}(1/z,-t)\right)\\ &= c_n^\sigma(-t)\left(\psi_n^{(2)}(z,t)\phi_n^{(2)}(1/z,-t) - \sigma\psi_n^{(1)}(z,t)\phi_n^{(1)}(1/z,-t)\right).\end{aligned} \tag{5.25}$$

Since $c_n^\sigma(-t) = c_n^\sigma(t)$ we have

$$\overline{b}(z,t) = \sigma b(1/z,-t). \tag{5.26}$$

Finally, we derive the symmetry relation between the norming constants, $C_j(t), \overline{C}_j(t)$ and reflection coefficients $\rho(z,t), \overline{\rho}(z,t)$. By definition, we have

$$\overline{\rho}(z,t) = \frac{\overline{b}(z,t)}{\overline{a}(z,t)} = \frac{\sigma b(1/z,-t)}{a(1/z,-t)} = \sigma\rho(1/z,-t), \tag{5.27}$$

where, in obtaining the last result, we used the symmetry relations (5.22) and (7.27). To derive a relation between the norming constants, define a new variable $w = 1/z$. Then $\mathrm{d}/\mathrm{d}z = -w^2\mathrm{d}/\mathrm{d}w$. With this at hand, we next take the derivative of equation (5.22) and find

$$\overline{a}'(\overline{z}_j,t) = -z_j^2 a'(z_j,-t), \tag{5.28}$$

where $a'(\xi) \equiv \mathrm{d}a/\mathrm{d}\xi$. Now, from the definition of the norming constants (see (4.5)), we get

$$\overline{C}_j(t) = \frac{\overline{b}_j(t)}{\overline{a}'(\overline{z}_j,t)} = \frac{\sigma b_j(t)}{-z_j^2 a'(z_j,-t)} = -\sigma z_j^{-2} C_j(-t), \tag{5.29}$$

where z_j and $\overline{z}_j$ are the zeros of the scattering data $a(z,t)$ and $\overline{a}(z,t)$) (see equation (4.4)), i.e. $\overline{a}(\overline{z}_j,0) = 0$; $a(z_j,0) = 0$. Furthermore, z_j and $\overline{z}_j$ are related throughout the symmetry condition (5.23).

6. Reverse space-time reduction $R_n(t) = \sigma Q_{-n}(-t), \sigma = \mp 1$

In this section we obtain all symmetries between the scattering eigenfunctions, reflection coefficients and norming constants for the reverse space-time reduction $R_n(t) = \sigma Q_{-n}(-t)$.

6.1. Symmetries between the eigenfunctions

To that purpose assume that $\phi_n(z) \equiv (\phi_n^{(1)}(z),\ \phi_n^{(2)}(z))^T$, where T denotes matrix transpose, is a solution to system (2.1) with $R_n(t) = \sigma Q_{-n}(-t)$:

$$\phi_{n+1}^{(1)}(t) = z\phi_n^{(1)}(t) + Q_n(t)\phi_n^{(2)}(t), \tag{6.1}$$

$$\phi_{n+1}^{(2)}(t) = \sigma Q_{-n}(-t)\phi_n^{(1)}(t) + z^{-1}\phi_n^{(2)}(t). \tag{6.2}$$

Let $n \to -n,\ t \to -t$ in (6.1) and (6.2); rearrange the result to find

$$\phi_{-(n-1)}^{(2)}(-t) = z^{-1}\phi_{-n}^{(2)}(-t) + \sigma Q_n(t)\phi_{-n}^{(1)}(-t), \tag{6.3}$$

$$\phi_{-(n-1)}^{(1)}(-t) = Q_{-n}(-t)\phi_{-n}^{(2)}(-t) + z\phi_{-n}^{(1)}(-t). \tag{6.4}$$

Define the quantities

$$\Phi_n^{(1)}(t) \equiv \phi_{-n}^{(2)}(-t), \quad \Phi_n^{(2)}(t) \equiv -\sigma\phi_{-n}^{(1)}(-t). \tag{6.5}$$

These auxiliary eigenfunctions satisfy a 'backward' scattering problem

$$\Phi_{n-1}^{(1)}(t) = z^{-1}\Phi_n^{(1)}(t) - Q_n(t)\Phi_n^{(2)}(t), \tag{6.6}$$

$$\Phi^{(2)}_{n-1}(t) = -\sigma Q_{-n}(-t)\Phi^{(1)}_n(t) + z\Phi^{(2)}_n(t), \tag{6.7}$$

which is exactly the scattering problem (2.3) under the symmetry reduction $R_n(t) = \sigma Q_{-n}(-t)$. Thus, we have the following symmetry relation:

If $\begin{pmatrix}\phi^{(1)}_n(z,t)\\ \phi^{(2)}_n(z,t)\end{pmatrix}$ solves (2.1) with $R_n(t) = \sigma Q_{-n}(-t)$ then $\begin{pmatrix}\Phi^{(1)}_n(z,t)\\ \Phi^{(2)}_n(z,t)\end{pmatrix}$ solves (2.3).

Note that if $(\psi^{(1)}_n(z,t),\ \psi^{(2)}_n(z,t))^T$ solves (2.1) so does $\gamma(\psi^{(1)}_n(z,t),\ \psi^{(2)}_n(z,t))^T$ for any non zero constant γ. Thus, using relation (2.4) we have

$$\Phi^{(1)}_n(z,t) = \gamma f^\sigma_n(t)\psi^{(1)}_{n+1}(z,t), \qquad \Phi^{(2)}_n(z,t) = \gamma f^\sigma_n(t)\psi^{(2)}_{n+1}(z,t), \tag{6.8}$$

with

$$f^\sigma_n(t) \equiv \prod_{j=-\infty}^{n} \frac{1}{1-\sigma Q_j(t)Q_{-j}(-t)}. \tag{6.9}$$

In terms of the eigrnfunction $\Phi_n(t)$ defined in equation (6.5), we find

$$\phi^{(2)}_{-n}(z,-t) = \gamma f^\sigma_n(t)\psi^{(1)}_{n+1}(z,t), \qquad -\sigma\phi^{(1)}_{-n}(z,-t) = \gamma f^\sigma_n(t)\psi^{(2)}_{n+1}(z,t). \tag{6.10}$$

Next, we determine the value of γ so that the proper boundary conditions are satisfied. First note that $\lim_{n\to+\infty} f^\sigma_n(t) = 1/c_{-\infty}$ (time-independent). From the boundary conditions (3.1) and (3.2) we find

$$\lim_{n\to+\infty}\left[-\sigma\phi^{(1)}_{-n}(z,-t)\right] = -\sigma z^{-n} = \lim_{n\to+\infty}\left[\gamma f^\sigma_n(t)\psi^{(2)}_{n+1}(z,t)\right] = \frac{\gamma}{c_{-\infty}} z^{-(n+1)}, \tag{6.11}$$

giving rise to $\gamma = -\sigma z c_{-\infty}$. To this end, we have the following relations between the eigenfunctions:

$$\begin{pmatrix}\psi^{(1)}_{n+1}(z,t)\\ \psi^{(2)}_{n+1}(z,t)\end{pmatrix} = \frac{1}{zc^\sigma_{-\infty} f^\sigma_n(t)}\begin{pmatrix}-\sigma\phi^{(2)}_{-n}(z,-t)\\ \phi^{(1)}_{-n}(z,-t)\end{pmatrix}. \tag{6.12}$$

To establish the symmetry relation between the corresponding eigenfunctions $N_n(z,t)$ and $M_n(z,t)$ we use the definition $N_m(z,t) = z^m\psi_m(z,t)$ and $M_m(z,t) = z^{-m}\phi_m(z,t)$. Multiply (6.12) by z^{n+1} and, after some algebra, one concludes that

$$\begin{pmatrix}N^{(1)}_{n+1}(z,t)\\ N^{(2)}_{n+1}(z,t)\end{pmatrix} = \frac{1}{c^\sigma_{-\infty} f^\sigma_n(t)}\begin{pmatrix}-\sigma M^{(2)}_{-n}(z,-t)\\ M^{(1)}_{-n}(z,-t)\end{pmatrix}. \tag{6.13}$$

Note that this is a relation between both analytic eigenfunctions outside the unit circle in the complex z plane. Next, we derive the symmetry relations between the 'bar' eigenfunctions. Following similar steps as before, we have:

If $\begin{pmatrix}\overline{\phi}^{(1)}_n(z,t)\\ \overline{\phi}^{(2)}_n(z,t)\end{pmatrix}$ solves (2.1) with $R_n(t) = \sigma Q_{-n}(-t)$ then $\begin{pmatrix}\overline{\Phi}^{(1)}_n(z,t)\\ \overline{\Phi}^{(2)}_n(z,t)\end{pmatrix}$ solves (2.3),

where $\overline{\phi}^{(2)}_{-n}(t) \equiv \overline{\Phi}^{(1)}_n(-t)$ and $\overline{\phi}^{(1)}_{-n}(t) \equiv -\sigma\overline{\Phi}^{(2)}_n(-t)$. To make sure the boundary conditions are correctly incorporated, we introduce an 'arbitrary' non zero constant $\overline{\gamma}$ so that $\overline{\gamma}\overline{\psi}_n(z,t)$ is a solution to system (2.1). With the help of the transformation (2.4) one has

$$\overline{\phi}^{(2)}_{-n}(z,-t) \;= \overline{\gamma} f^{\sigma}_n(t)\overline{\psi}^{(1)}_{n+1}(z,t), \qquad -\sigma\overline{\phi}^{(1)}_{-n}(z,-t) \;= \overline{\gamma} f^{\sigma}_n(t)\overline{\psi}^{(2)}_{n+1}(z,t). \tag{6.14}$$

From the boundary conditions (3.1) and (3.2) we have

$$\lim_{n\to+\infty}\left[\overline{\phi}^{(2)}_{-n}(z,-t)\right] = z^n \;= \lim_{n\to+\infty}\left[\overline{\gamma} f^{\sigma}_n(t)\overline{\psi}^{(1)}_{n+1}(z,t)\right] = \frac{\overline{\gamma}}{c^{\sigma}_{-\infty}} z^{n+1}, \tag{6.15}$$

leading to $\overline{\gamma} = \frac{c_{-\infty}}{z}$. Thus, the 'bar' eigenfunctions satisfy the following symmetry:

$$\begin{pmatrix}\overline{\psi}^{(1)}_{n+1}(z,t)\\ \overline{\psi}^{(2)}_{n+1}(z,t)\end{pmatrix} = \frac{z}{c^{\sigma}_{-\infty} f^{\sigma}_n(t)}\begin{pmatrix}\overline{\phi}^{(2)}_{-n}(z,-t)\\ -\sigma\overline{\phi}^{(1)}_{-n}(z,-t)\end{pmatrix}. \tag{6.16}$$

To find the corresponding symmetry between the eigenfunctions $\overline{M}_n(z,t)$ and $\overline{N}_n(z,t)$, we multiply (6.16) by $z^{-(n+1)}$; use the definition $\overline{N}_n(z,t) = z^{-n}\overline{\psi}_n(z,t)$ and $\overline{M}_n(z,t) = z^n\overline{\phi}_n(z,t)$ to find

$$\begin{pmatrix}\overline{N}^{(1)}_{n+1}(z,t)\\ \overline{N}^{(2)}_{n+1}(z,t)\end{pmatrix} = \frac{1}{c^{\sigma}_{-\infty} f^{\sigma}_n(t)}\begin{pmatrix}\overline{M}^{(2)}_{-n}(z,-t)\\ -\sigma\overline{M}^{(1)}_{-n}(z,-t)\end{pmatrix}. \tag{6.17}$$

This is a relation between both analytic eigenfunctions inside $|z| < 1$.

6.2. Symmetries between the modified eigenfunctions

Since the inverse problem is formulated in terms of the modified eigenfunctions $M'_n(z,t), N'_n(z,t); \overline{M}'_n(z,t), \overline{N}'_n(z,t)$, it proves convenient to derive the corresponding symmetry relations that these 'primed' eigenfunctions satisfy . In fact this symmetry is later used in order to determine the relations between the norming constants and scattering data from the left and right scattering problems. Furthermore, they are needed in order to obtain an alternative reconstruction formula for the potentials that allows one to observe the integrable symmetry at the inverse side. To do so, first note that, by definition, $c_n(t) = \prod_{k=n}^{\infty}(1 - Q_kR_k)$. Thus, at the symmetry point $R_k(t) = \sigma Q_{-k}(-t)$ one can show

$$c^{\sigma}_n(t) = \frac{1}{f^{\sigma}_{-n}(-t)}, \qquad c^{\sigma}_{m+1}(t) = c^{\sigma}_{-\infty} f^{\sigma}_n(t). \tag{6.18}$$

These results are later used to obtain a simplified form for the symmetry relations between the eigenfunctions. With this at hand, we start from the definition of the 'primed' and the standard eigenfunctions given in equations (3.22)–(3.25). Apply the symmetry condition between them, established in (6.13) and (6.17), one finds

$$N^{(1)'}_{n+1}(z,t) = -\sigma(c^{\sigma}_{-\infty})^{-1} M^{(2)'}_{-n}(z,-t), \tag{6.19}$$

$$N^{(2)'}_{n+1}(z,t) = M^{(1)'}_{-n}(z,-t), \tag{6.20}$$

$$\overline{N}^{(1)'}_{n+1}(z,t) = (c^{\sigma}_{-\infty})^{-1}\overline{M}^{(2)'}_{-n}(z,-t), \tag{6.21}$$

$$\overline{N}^{(2)'}_{n+1}(z,t) = -\sigma\overline{M}^{(1)'}_{-n}(z,-t). \tag{6.22}$$

6.3. Symmetry of scattering data

In this section we obtain all symmetry relations between the scattering data and norming constants. Contrary to the RT NLS case, where the scattering data sets, $\{\overline{a}(z,t),\overline{b}(z,t),\overline{C}(t)\}$ and $\{a(z,t),b(z,t),C(t)\}$ are connected through a symmetry condition, here (RST case) the two sets are actually not related. Rather, as we shall see, each scattering data (except $b(z,t),\overline{b}(z,t)$) satisfy its own symmetry requirement.

We start from the definition given in (3.5). Use the symmetry relation between the eigenfunctions obtained in (6.12) to find

$$\begin{aligned} a(z,t) &= c_{n+1}(t)\left(\phi^{(1)}_{n+1}(z,t)\psi^{(2)}_{n+1}(z,t)-\phi^{(2)}_{n+1}(z,t)\psi^{(1)}_{n+1}(z,t)\right)\\ &= \frac{c^{\sigma}_{n+1}(t)}{zc^{\sigma}_{-\infty}f^{\sigma}_{n}(t)}\left(\phi^{(1)}_{n+1}(z,t)\phi^{(1)}_{-n}(z,-t)+\sigma\phi^{(2)}_{n+1}(z,t)\phi^{(2)}_{-n}(z,-t)\right), \end{aligned} \tag{6.23}$$

where using (3.10) we define

$$c^{\sigma}_{m}(t)\ \equiv c_m(Q_k(t),\sigma Q_{-k}(-t)) = \prod_{k=m}^{+\infty}(1-\sigma Q_k(t)Q_{-k}(-t)). \tag{6.24}$$

By letting $n\rightarrow -n-1, t\rightarrow -t$ in (6.23) one arrives at

$$a(z,-t)=\frac{c^{\sigma}_{-n}(-t)}{zc^{\sigma}_{-\infty}f^{\sigma}_{-n-1}(-t)}\left(\phi^{(1)}_{-n}(z,-t)\phi^{(1)}_{n+1}(z,t)+\sigma\phi^{(2)}_{-n}(z,-t)\phi^{(2)}_{n+1}(z,t)\right). \tag{6.25}$$

Recall the definitions of c_n, f_n from (3.10) and (6.9); make use of the identities listed in (6.18) to obtain

$$\frac{c^{\sigma}_{n+1}(t)}{f^{\sigma}_{n}(t)}=\frac{c^{\sigma}_{-n}(-t)}{f^{\sigma}_{-n-1}(-t)}. \tag{6.26}$$

This last result together with (6.23) and (6.25) imply the symmetry condition

$$a(z,t)\ =\ a(z,-t). \tag{6.27}$$

The derivation of the symmetry relation for the scattering data $\overline{a}(z,t)$ follows similar line of arguments as discussed above (hence we omit details). Thus, we have

$$\overline{a}(z,t)=\overline{a}(z,-t). \tag{6.28}$$

Thus $a(z,t)$ and $\overline{a}(z,t)$, which are analytic in different regions of the complex z plane, satisfy their own symmetry relationships. Since, as we shall later see, the scattering data $a(z,t)$ and $\overline{a}(z,t)$ are time-independent, equations (6.27) and (6.28) impose *no restriction* on the soliton eigenvalue z_j and $\overline{z}_j$. As such, they are counted as free parameters (complex in general). Next, we proceed with the derivation of the symmetry between $b(z,t)$ and $\overline{b}(z,t)$. For that purpose, we start from equation (3.8); use the symmetry conditions between the eigenfunctions given in (6.13) and (6.17)

$$\begin{aligned} \overline{b}(z,t) &= c_{n+1}(t)\left(\overline{\phi}^{(1)}_{n+1}(z,t)\psi^{(2)}_{n+1}(z,t)-\overline{\phi}^{(2)}_{n+1}(z,t)\psi^{(1)}_{n+1}(z,t)\right)\\ &= \frac{c^{\sigma}_{n+1}(t)}{zc^{\sigma}_{-\infty}f^{\sigma}_{n}(t)}\left(\overline{\phi}^{(1)}_{n+1}(z)\phi^{(1)}_{-n}(z,-t)+\sigma\overline{\phi}^{(2)}_{n+1}(z,t)\phi^{(2)}_{-n}(z,-t)\right). \end{aligned} \tag{6.29}$$

Next, from (3.7) it follows that

$$
\begin{aligned}
b(z,t) &= c_{n+1}(t)\left(\overline{\psi}^{(1)}_{n+1}(z,t)\phi^{(2)}_{n+1}(z,t) - \overline{\psi}^{(2)}_{n+1}(z,t)\phi^{(1)}_{n+1}(z,t)\right) \\
&= \frac{zc^{\sigma}_{n+1}(t)}{c^{\sigma}_{-\infty}f^{\sigma}_{n}(t)}\left(\overline{\phi}^{(2)}_{-n}(z,-t)\phi^{(2)}_{n+1}(z,t) + \sigma\overline{\phi}^{(1)}_{-n}(z,-t)\phi^{(1)}_{n+1}(z,t)\right). \qquad (6.30)
\end{aligned}
$$

Let $n \to -n-1, t \to -t$ in (6.30) we find

$$
\begin{aligned}
b(z,-t) &= \frac{zc^{\sigma}_{-n}(-t)}{c^{\sigma}_{-\infty}f^{\sigma}_{-n-1}(-t)}\left(\overline{\phi}^{(2)}_{n+1}(z,t)\phi^{(2)}_{-n}(z,-t) + \sigma\overline{\phi}^{(1)}_{n+1}(z,t)\phi^{(1)}_{-n}(z,-t)\right) \\
&= \frac{\sigma zc^{\sigma}_{-n}(-t)}{c^{\sigma}_{-\infty}f^{\sigma}_{-n-1}(-t)}\left(\sigma\overline{\phi}^{(2)}_{n+1}(z,t)\phi^{(2)}_{-n}(z,-t) + \overline{\phi}^{(1)}_{n+1}(z,t)\phi^{(1)}_{-n}(z,-t)\right). \qquad (6.31)
\end{aligned}
$$

Again use the result established in (6.26) to find

$$
\overline{b}(z,t) = \frac{\sigma}{z^2}b(z,-t). \qquad (6.32)
$$

Since the scattering data a and $\overline{a}$ are analytic in different regions of the complex z plane, they are not related through any symmetry (see equations (6.27) and (5.22)), as a result, the corresponding norming constants C_j and $\overline{C_j}$ do not satisfy a symmetry relation by themselves. One needs to go back to the original definitions, see formula (4.5), and find separate symmetry conditions on $b_j, \overline{b}_j$ and derive a trace-type formula to separately determine $a'(z_j), \overline{a}'(\overline{z}_j)$.

7. *PT* symmetric reduction $R_n = \sigma Q^*_{-n}$, $\sigma = \mp 1$

In 2014, Ablowitz and Musslimani [40] discovered a new nonlocal reduction to the AL scattering problem (2.1) that preserves a discrete type of *PT* symmetry, i.e. invariance under the combined transformation of $n \to -n, t \to -t$ and complex conjugation. However, due to size limitation, many significant details pertaining to the direct and inverse problems, particularly the derivation of all implied symmetries were omitted. In this section, we provide detailed analysis of all symmetries that the eigenfunctions, scattering data and norming constants satisfy. We note that the inverse scattering transform for the AL model with *PT* symmetry has been extended by Grahovski, Mohammed and Susanto; one and two-soliton solutions for the nonlocal Ablowitz–Ladik equation were also obtained [51]. This paper goes further by formulating and solving the RST and RT discrete symmetric nonlocal systems and obtains all necessary scattering space symmetries for these and the *PT* symmetric case.

Importantly, as we shall see later, the *PT* symmetric case is fundamentally different from the RST one in the sense that complex conjugation *need not commute* with time reversal symmetry. This means, one cannot simply replace complex conjugation by $-t$.

7.1. Symmetries between the eigenfunctions

Here, we will write down all symmetries of the scattering problem under the reduction $R_n = \sigma Q^*_{-n}$. Since this reduction is *local in time*, we shall omit the explicit time dependence from all dependent variables. To that purpose assume (as before) that $\phi_n(z) \equiv (\phi^{(1)}_n(z),\ \phi^{(2)}_n(z))^T$ is a solution to system (2.1) with $R_n = \sigma Q^*_{-n}$. After complex conjugation is taken combined with the transformation $n \to -n,\ z \to z^*$ one obtains

$$
\phi^{(2)^*}_{-(n-1)}(z^*) = z^{-1}\phi^{(2)^*}_{-n}(z^*) + \sigma Q_n\phi^{(1)^*}_{-n}(z^*), \qquad (7.1)
$$

$$\phi^{(1)^*}_{-(n-1)}(z^*) = Q^*_{-n}\phi^{(2)^*}_{-n}(z^*) + z\phi^{(1)^*}_{-n}(z^*). \tag{7.2}$$

Now, define the quantities $\Phi^{(1)}_n(z) \equiv \phi^{(2)^*}_{-n}(z^*), \Phi^{(2)}_n(z) \equiv -\sigma\phi^{(1)^*}_{-n}(z^*)$. It can been shown that these new functions satisfy the 'reverse' scattering problem (2.3). Therefore, we arrive at the following conclusion

$$\text{If} \quad \begin{pmatrix} \phi^{(1)}_n(z) \\ \phi^{(2)}_n(z) \end{pmatrix} \text{ solves (2.1) with } R_n = \sigma Q^*_{-n} \quad \text{then} \quad \begin{pmatrix} \Phi^{(1)}_n(z) \\ \Phi^{(2)}_n(z) \end{pmatrix} \text{ solves (2.3).}$$

In view of the transformation (2.4) we have

$$\phi^{(2)^*}_{-n}(z^*) = \gamma f^\sigma_n \psi^{(1)}_{n+1}(z), \qquad \phi^{(1)^*}_{-n}(z^*) = -\sigma\gamma f^\sigma_n \psi^{(2)}_{n+1}(z), \tag{7.3}$$

with a nonzero constant γ (to be determined) and

$$f^\sigma_n \equiv \prod_{j=-\infty}^{n} \frac{1}{1-\sigma Q_j Q^*_{-j}}. \tag{7.4}$$

Next, we check boundary conditions (recall that $\lim_{n\to+\infty} f^\sigma_n(t) = 1/c^\sigma_{-\infty}$). From the boundary conditions (3.1) and (3.2) we have

$$\lim_{n\to+\infty} \phi^{(1)^*}_{-n}(z^*) = z^{-n} = -\sigma\gamma \lim_{n\to+\infty}\left[f^\sigma_n \psi^{(2)}_{n+1}(z)\right] = -\frac{\sigma\gamma}{c^\sigma_{-\infty}} z^{-(n+1)}. \tag{7.5}$$

In order for equation (6.11) to hold true we require $\gamma = -\sigma c^\sigma_{-\infty} z$. Thus, we have the following relations between the eigenfunctions:

$$\begin{pmatrix} \psi^{(1)}_{n+1}(z) \\ \psi^{(2)}_{n+1}(z) \end{pmatrix} = \frac{1}{z c^\sigma_{-\infty} f^\sigma_n} \begin{pmatrix} -\sigma\phi^{(2)^*}_{-n}(z^*) \\ \phi^{(1)^*}_{-n}(z^*) \end{pmatrix}. \tag{7.6}$$

To establish the symmetry relation between the corresponding eigenfunctions $N_n(z)$ and $M_n(z)$ we multiply (7.6) by z^{n+1}; use the definition $N_m(z) = z^m\psi_m(z)$ and $M_m(z) = z^{-m}\phi_m(z)$ to obtain

$$\begin{pmatrix} N^{(1)}_{n+1}(z) \\ N^{(2)}_{n+1}(z) \end{pmatrix} = \frac{1}{c^\sigma_{-\infty} f^\sigma_n} \begin{pmatrix} -\sigma M^{(2)^*}_{-n}(z^*) \\ M^{(1)^*}_{-n}(z^*) \end{pmatrix}. \tag{7.7}$$

We next establish the symmetry relations between the 'bar' eigenfunctions. Following similar steps as before, we conclude:

$$\text{If} \quad \begin{pmatrix} \overline{\phi}^{(1)}_n(z) \\ \overline{\phi}^{(2)}_n(z) \end{pmatrix} \text{ solves (2.1) with } R_n(t) = \sigma Q^*_{-n} \quad \text{then} \quad \begin{pmatrix} \overline{\Phi}^{(1)}_n(z) \\ \overline{\Phi}^{(2)}_n(z) \end{pmatrix} \text{ solves (2.3),}$$

where $\overline{\phi}^{(2)^*}_{-n}(z^*) \equiv \overline{\Phi}^{(1)}_n(z)$ and $-\sigma\overline{\phi}^{(1)^*}_{-n}(z^*) \equiv \overline{\Phi}^{(2)}_n(z)$. Note that if $(\overline{\psi}^{(1)}_n(z,t), \overline{\psi}^{(2)}_n(z,t))^T$ solves (2.1), so does $\overline{\gamma}(\overline{\psi}^{(1)}_n(z,t), \overline{\psi}^{(2)}_n(z,t))^T$ for any nonzero constant $\overline{\gamma}$. Thus, using the transformation (2.4) we find

$$\overline{\phi}^{(2)^*}_{-n}(z^*) = \overline{\gamma} f^\sigma_n \overline{\psi}^{(1)}_{n+1}(z), \qquad -\sigma\overline{\phi}^{(1)^*}_{-n}(z^*) = \overline{\gamma} f^\sigma_n \overline{\psi}^{(2)}_{n+1}(z). \tag{7.8}$$

To determine the value of the constant $\overline{\gamma}$ we examine the boundary conditions (3.1) and (3.2). In this case, one sets

$$\lim_{n\to+\infty}\left[\overline{\phi}^{(2)^*}_{-n}(z^*)\right] = z^n = \lim_{n\to+\infty}\left[\overline{\gamma}\, f^{\sigma}_n \overline{\psi}^{(1)}_{n+1}(z)\right] = \frac{\overline{\gamma}}{c^{\sigma}_{-\infty}} z^{n+1}, \tag{7.9}$$

that leads to $\overline{\gamma} = \frac{c^{\sigma}_{-\infty}}{z}$. Thus we have the following relations between the eigenfunctions:

$$\begin{pmatrix} \overline{\psi}^{(1)}_{n+1}(z) \\ \overline{\psi}^{(2)}_{n+1}(z) \end{pmatrix} = \frac{z}{c^{\sigma}_{-\infty} f^{\sigma}_n} \begin{pmatrix} \overline{\phi}^{(2)^*}_{-n}(z^*) \\ -\sigma\overline{\phi}^{(1)^*}_{-n}(z^*) \end{pmatrix}. \tag{7.10}$$

The symmetry between the corresponding eigenfunctions $\overline{M}_n(z)$ and $\overline{N}_n(z)$ are obtained following similar steps as before. One has

$$\begin{pmatrix} \overline{N}^{(1)}_{n+1}(z) \\ \overline{N}^{(2)}_{n+1}(z) \end{pmatrix} = \frac{1}{c^{\sigma}_{-\infty} f^{\sigma}_n} \begin{pmatrix} \overline{M}^{(2)^*}_{-n}(z^*) \\ -\sigma\overline{M}^{(1)^*}_{-n}(z^*) \end{pmatrix}. \tag{7.11}$$

7.2. Symmetry between the modified eigenfunctions

Having determined the symmetries between all eigenfunctions, we turn our attention next to compute the symmetries between the modified eigenfunctions defined by equations (3.22)–(3.30). Since all dependent variables are local in time, we shall suppress the explicit time dependence. In this section, we shall make a frequent use of the identity $c^{\sigma}_n = 1/f^{\sigma^*}_{-n}$ valid whenever $R_n = \sigma Q^*_{-n}$.

We have shown that under the reduction $R_n = \sigma Q^*_{-n}$ the set of eigenfunctions $\{M_n, N_n\}$ and $\{\overline{M}_n, \overline{N}_n\}$ satisfy the symmetry relation given in (7.7) and (7.11). These symmetries, in turn, induce another symmetries between the 'primed' eigenfunctions given by

$$N^{(1)'}_{n+1}(z) = -\frac{\sigma}{c^{\sigma}_{-\infty}}\; M^{(2)'^*}_{-n}(z^*), \tag{7.12}$$

$$N^{(2)'}_{n+1}(z) = M^{(1)'^*}_{-n}(z^*), \tag{7.13}$$

$$\overline{N}^{(1)'}_{n+1}(z) = \frac{1}{c^{\sigma}_{-\infty}}\; \overline{M}^{(2)'^*}_{-n}(z^*), \tag{7.14}$$

$$\overline{N}^{(2)'}_{n+1}(z) = -\sigma\overline{M}^{(1)'^*}_{-n}(z^*). \tag{7.15}$$

7.3. Symmetry between scattering data

To establish the symmetry relation between the scattering data we start from (3.5) combined with the symmetries between the eigenfunctions given in (7.6). We have

$$\begin{aligned} a(z) &= c_{n+1}\left(\phi^{(1)}_{n+1}(z)\psi^{(2)}_{n+1}(z) - \phi^{(2)}_{n+1}(z)\psi^{(1)}_{n+1}(z)\right) \\ &= \frac{c^{\sigma}_{n+1}}{z c^{\sigma}_{-\infty} f^{\sigma}_n}\left(\phi^{(1)}_{n+1}(z)\phi^{(1)^*}_{-n}(z^*) + \sigma\phi^{(2)}_{n+1}(z)\phi^{(2)^*}_{-n}(z^*)\right), \end{aligned} \tag{7.16}$$

where using (3.10) we have defined

$$c^{\sigma}_m \equiv c_m(Q_k, \sigma Q^*_{-k}) = \prod_{k=m}^{+\infty}\left(1 - \sigma Q_k Q^*_{-k}\right). \tag{7.17}$$

Next, let $z \to z^*$, $\quad n \to -n$ in (3.5) to find

$$\begin{aligned} a^*(z^*) &= c^*_{-n}\left(\phi^{(1)^*}_{-n}(z^*)\psi^{(2)^*}_{-n}(z^*) - \phi^{(2)^*}_{-n}(z^*)\psi^{(1)^*}_{-n}(z^*)\right) \\ &= \frac{c^{\sigma^*}_{-n}}{zc^{\sigma^*}_{-\infty}f^{\sigma^*}_{-n-1}}\left(\phi^{(1)^*}_{-n}(z^*)\phi^{(1)}_{n+1}(z) + \sigma\phi^{(2)^*}_{-n}(z^*)\phi^{(2)}_{n+1}(z)\right). \end{aligned} \tag{7.18}$$

With the definitions of f^σ_n and c^σ_n, we conclude

$$\frac{c^\sigma_{n+1}}{f^\sigma_n} = \prod_{k=n+1}^{+\infty}\prod_{j=-\infty}^{n}\left[1-\sigma Q_k Q^*_{-k}\right]\left[1-\sigma Q_j Q^*_{-j}\right] = c^\sigma_{-\infty}. \tag{7.19}$$

On the other hand we have

$$\frac{c^{\sigma^*}_{-n}}{f^{\sigma^*}_{-n-1}} = \prod_{k=-n}^{+\infty}\prod_{j=-\infty}^{-n-1}\left[1-\sigma Q_k Q^*_{-k}\right]^*\left[1-\sigma Q_j Q^*_{-j}\right]^* = c^{\sigma^*}_{-\infty}, \tag{7.20}$$

where $c^{\sigma^*}_{-\infty} \equiv (c^\sigma_{-\infty})^*$. Comparing equations (7.16) with (7.18) gives the symmetry result

$$a(z) \;=\; a^*(z^*). \tag{7.21}$$

Equation (6.27) implies that if z_j is an eigenvalue, i.e. a simple zero of the scattering data $a(z)$ with $|z_j| > 1$ then z^*_j is a simple zero of $a(z)$.

The derivation of the symmetry relation for the scattering data $\overline{a}(z)$ follows similar steps as for $a(z)$. Indeed, if one starts from the definition (3.6); utilize the symmetry between the eigenfunctions given in (7.10), one arrives at the result

$$\overline{a}(z) = \overline{a}^*(z^*). \tag{7.22}$$

The above equation implies that if $\overline{z}_j$ is a zero of $\overline{a}(z)$ with $|\overline{z}_j| < 1$ then $\overline{z}^*_j$ is a zero of $\overline{a}(z)$ as well, in which case, they are all counted as a (complex) free parameters. It is interesting to note that for the *PT* symmetric case (in fact also for the RST) , all 'bar' quantities (eigenfunctions, scattering data and norming constants) do not 'mix' with their respective 'unbar' quantities. This is in sharp contrast to the RT and AL cases where all eigenfunctions and scattering data in the upper half complex z plane are related throughout a symmetry to their corresponding 'partners' in the lower half complex z plane. In summary, we have the following conclusion:

$$\begin{aligned} &\text{If } \{z_j, \overline{z}_j\} \in \mathbb{C} \text{ are zeros of } a(z), \overline{a}(z), \text{ i.e. eigenvalues, satisfying} \\ &|z_j| > 1 \text{ and } |\overline{z}_j| < 1, \text{ so do } \{z^*_j, \overline{z}^*_j\}. \end{aligned}$$

Finally, we next determine the symmetry of scattering data $b(z)$ and $\overline{b}(z)$. As before, we start from equation (3.8); use the symmetry between the eigenfunctions (7.6) and the identity (7.19) to find

$$\overline{b}(z) = \frac{1}{z}\left(\overline{\phi}^{(1)}_{n+1}(z)\phi^{(1)^*}_{-n}(z^*) + \sigma\overline{\phi}^{(2)}_{n+1}(z)\phi^{(2)^*}_{-n}(z^*)\right). \tag{7.23}$$

With this at hand, it follows from (3.7) that

$$\begin{aligned} b(z) &= c_{n+1}\left(\overline{\psi}^{(1)}_{n+1}(z)\phi^{(2)}_{n+1}(z) - \overline{\psi}^{(2)}_{n+1}(z)\phi^{(1)}_{n+1}(z)\right) \\ &= \frac{zc^\sigma_{n+1}}{c^\sigma_{-\infty}f^\sigma_n}\left(\overline{\phi}^{(2)^*}_{-n}(z^*)\phi^{(2)}_{n+1}(z) + \sigma\overline{\phi}^{(1)^*}_{-n}(z^*)\phi^{(1)}_{n+1}(z)\right). \end{aligned} \tag{7.24}$$

Let $z \to z^*$ and $-n \to n+1$ in (7.24) and complex conjugate the result to find

$$b^*(z^*) = \frac{\sigma z c^{\sigma^*}_{-n}}{c^{\sigma^*}_{-\infty} f^{\sigma^*}_{-n-1}} \left(\sigma \overline{\phi}^{(2)}_{n+1}(z) \phi^{(2)^*}_{-n}(z^*) + \overline{\phi}^{(1)}_{n+1}(z) \phi^{(1)^*}_{-n}(z^*) \right). \tag{7.25}$$

Again use the result established in (7.20), i.e. $\frac{c^{\sigma}_{-n}}{f^{\sigma}_{-n-1}} = c^{\sigma}_{-\infty}$ in which case we get

$$b^*(z^*) = \sigma z \left(\sigma \overline{\phi}^{(2)}_{n+1}(z) \phi^{(2)^*}_{-n}(z^*) + \overline{\phi}^{(1)}_{n+1}(z) \phi^{(1)^*}_{-n}(z^*) \right). \tag{7.26}$$

From equations (7.23) and (7.26) it follows

$$\overline{b}(z) = \frac{\sigma}{z^2} b^*(z^*). \tag{7.27}$$

A summary and highlights of the key symmetry results related to all three nonlocal integrable reductions, including the AL, is given in table 1 (explicit time dependence is indicated only for the RT and RST cases).

8. Inverse problem: Riemann–Hilbert approach

8.1. Preliminaries

In this section we apply the inverse scattering transform to construct an explicit formula for the potentials $Q_n(t)$ and $R_n(t)$. This is accomplished by reformulating the AL scattering problem as a Riemann–Hilbert problem and use projection operators (defined below) to solve for the potentials. Within the framework of AKNS theory, all is needed to solve an integrable evolution equation are symmetries between the eigenfunctions, scattering data and a reconstruction formula for both potentials. While this approach is sufficient to obtain any soliton solution, in this paper (and due to nonlocality), we solve two inverse problems: one from the left and the other from the right, then 'glue' them using the nonlocal symmetries obtained above.

The scattering problem (2.1) can possess discrete eigenvalues (bound states). These occur whenever $a(z)$ has J simple zeros at $\{z_j \text{ s.t. } |z_j| > 1\}_{j=1}^{J}$, i.e. $a(z_j) = 0$ and $\overline{a}(z)$ has $\overline{J} \equiv J$ simple zeros at $\{\overline{z}_j \text{ s.t. } |\overline{z}_j| > 1\}_{j=1}^{\overline{J}}$, i.e. $\overline{a}(\overline{z}_j) = 0$. Indeed, for such values of the spectral parameters $W(\phi_n(z_j), \psi_n(z_j)) = 0$ and $W(\overline{\phi}_n(\overline{z}_j), \overline{\psi}_n(\overline{z}_j)) = 0$. Therefore, from (3.3) and (3.4) we find

$$\phi_n(z_j) = b_j \psi_n(z_j), \tag{8.1}$$

$$\overline{\phi}_n(\overline{z}_j) = \overline{b}_j \overline{\psi}_n(\overline{z}_j). \tag{8.2}$$

In terms of the eigenfunctions $\{M_n, N_n, \overline{M}_n, \overline{N}_n\}$ and $\{M'_n, N'_n, \overline{M}'_n, \overline{N}'_n\}$, equations (8.1) and (8.2) imply

$$M_n(z_j) = b_j z_j^{-2n} N_n(z_j), \qquad M'_n(z_j) = b_j z_j^{-2n} N'_n(z_j), \tag{8.3}$$

$$\overline{M}_n(\overline{z}_\ell) = \overline{b}_\ell \overline{z}_\ell^{2n} \overline{N}_n(\overline{z}_\ell), \qquad \overline{M}'_n(\overline{z}_\ell) = \overline{b}_\ell \overline{z}_\ell^{2n} \overline{N}'_n(\overline{z}_\ell), \tag{8.4}$$

where, $b_j, \overline{b}_j$ is a short notation for $b(z_j), \overline{b}(\overline{z}_j)$ respectively. Furthermore, one can show that the residue of the functions $\mu_n(z), \overline{\mu}_n(z)$ at eigenvalues $z_j, \overline{z}_j$ can be computed using the definition (3.29) and (3.30). Thus we have

$$\mathrm{Res}(\mu_n; z_j) = \frac{M_n(z_j)}{a'(z_j)} = \frac{b_j z_j^{-2n} N_n(z_j)}{a'(z_j)} = z_j^{-2n} C_j N_n(z_j), \tag{8.5}$$

$$\mathrm{Res}(\overline{\mu}_n;\overline{z}_\ell) = \frac{\overline{M}_n(\overline{z}_\ell)}{\overline{a}'(\overline{z}_\ell)} = \frac{\overline{b}_\ell \overline{z}_\ell^{2n}\overline{N}_n(\overline{z}_\ell)}{\overline{a}'(\overline{z}_\ell)} = \overline{z}_\ell^{2n}\overline{C}_\ell \overline{N}_n(\overline{z}_\ell), \tag{8.6}$$

where, we remind the reader, that the norming constants $\overline{C}_\ell \equiv \overline{C}(\overline{z}_\ell), C_\ell \equiv C(z_\ell)$ (see also (4.5)) are defined by

$$C_j = \frac{b_j}{a'(z_j)}, \tag{8.7}$$

$$\overline{C}_\ell = \frac{\overline{b}_\ell}{\overline{a}'(\overline{z}_\ell)}. \tag{8.8}$$

Note that the poles of $\mu_n'(z)$ and $\overline{\mu}_n'(z)$ (see equations (3.23) and (3.25)) are the same as the poles of $\mu_n(z)$ and $\overline{\mu}_n(z)$ respectively. Moreover, the residues of these poles are determined by the relations $\mathrm{Res}(\mu_n';z_j) = z_j^{-2n}C_jN_n'(z_j)$, $\mathrm{Res}(\overline{\mu}_n';\overline{z}_\ell) = \overline{z}_\ell^{2n}\overline{C}_\ell\overline{N}_n'(\overline{z}_\ell)$. As pointed out earlier, the projection operators $P_<$ and $P_>$ will be frequently used to solve a Riemann–Hilbert problems. They are defined as follows: Let $f(w), w \in \mathbb{C}$ be analytic inside the unit circle in the complex z plane. Then

$$P_<(f)(z) = \lim_{\substack{\zeta\to z\\ |\zeta|<1}} \frac{1}{2\pi\mathrm{i}} \oint_{|w|=1} \frac{f(w)}{w-\zeta}\mathrm{d}w. \tag{8.9}$$

Similarly, for any $f(w), w \in \mathbb{C}$ analytic outside the unit circle in the z plane one have

$$P_>(f)(z) = \lim_{\substack{\zeta\to z\\ |\zeta|>1}} \frac{1}{2\pi\mathrm{i}} \oint_{|w|=1} \frac{f(w)}{w-\zeta}\mathrm{d}w. \tag{8.10}$$

Below we list some important properties of the projection operators that we shall use in the formulation of a RH problem. Let $f^{\mathrm{in}}(w)$ ($f^{\mathrm{out}}(w)$) be an analytic function inside (outside) the unit circle. Then we have

(i) $P_<(f^{\mathrm{in}})(z) = f^{\mathrm{in}}(z)$,
(ii) $P_<(f^{\mathrm{out}})(z) = f^{\mathrm{out}}_\infty$,
(iii) $P_>(f^{\mathrm{in}})(z) = 0$,
(iv) $P_>(f^{\mathrm{out}})(z) = f^{\mathrm{out}}_\infty - f^{\mathrm{out}}(z)$.

Next, we apply the inverse scattering transform on the scattering problem from the left.

8.2. Left jump condition

To set up the RH problem from the left, we divide equation (3.3) by $a(z)$, equation (3.4) by $\overline{a}(z)$; use the definition of the eigenfunctions in section 3, then one can show that the new functions $\mu_n'(z)$ and $\overline{\mu}_n'(z)$ satisfy the jump conditions on $|z| = 1$

$$\mu_n'(z) = \overline{N}_n'(z) \; + \; z^{-2n}\rho(z)N_n'(z), \tag{8.11}$$

$$\overline{\mu}_n'(z) = N_n'(z) \; + \; z^{2n}\overline{\rho}(z)\overline{N}_n'(z). \tag{8.12}$$

We subtract from both sides of equation (8.11) the vector $(1\ 0)^T$ and the non analytic parts of $\mu_n'(z)$; apply the projection operator $P_<$ on both sides of the resulting equation to get

$$P_<\left\{\mu_n'(z)-\begin{pmatrix}1\\0\end{pmatrix}-\sum_{j=1}^{J}C_jz_j^{-2n}\left[\frac{N_n'(z_j)}{z-z_j}+\frac{N_n'(-z_j)}{z+z_j}\right]\right\}$$
$$=P_<\left\{\overline{N}_n'(z)-\begin{pmatrix}1\\0\end{pmatrix}-\sum_{j=1}^{J}C_jz_j^{-2n}\left[\frac{N_n'(z_j)}{z-z_j}+\frac{N_n'(-z_j)}{z+z_j}\right]+z^{-2n}\rho(z)N_n'(z)\right\}. \tag{8.13}$$

The function on the left hand side of equation (8.13) is analytic outside the unit circle and approach zero as $|z|\to\infty$ hence, using the properties of the projection operators, we have that the left hand side of equation (8.13) is zero (note: $f_\infty^{\mathrm{out}}=0$.)

$$P_<\left\{\overline{N}_n'(z)-\begin{pmatrix}1\\0\end{pmatrix}-\sum_{j=1}^{J}C_jz_j^{-2n}\left[\frac{N_n'(z_j)}{z-z_j}+\frac{N_n'(-z_j)}{z+z_j}\right]\right\}\;+\;P_<\left\{z^{-2n}\rho(z)N_n'(z)\right\}=0. \tag{8.14}$$

Since $|z_j|>1$ and $\overline{N}_n'(z)$ is analytic inside the unit circle, the function

$$\overline{N}_n'(z)-\begin{pmatrix}1\\0\end{pmatrix}-\sum_{j=1}^{J}C_jz_j^{-2n}\left[\frac{N_n'(z_j)}{z-z_j}+\frac{N_n'(-z_j)}{z+z_j}\right], \tag{8.15}$$

is analytic inside the unit circle and constitute an 'in' function. Using the projections properties, after some algebra, one finds

$$\overline{N}_n'(z)=\begin{pmatrix}1\\0\end{pmatrix}+\sum_{j=1}^{J}C_jz_j^{-2n}\left[\frac{N_n'(z_j)}{z-z_j}+\frac{N_n'(-z_j)}{z+z_j}\right]-\lim_{\substack{\zeta\to z\\|\zeta|<1}}\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{-2n}\rho(w)N_n'(w)}{w-\zeta}\mathrm{d}w. \tag{8.16}$$

Similarly, we subtract from both sides of equation (8.12) all the non analytic parts; identify all functions that are analytic inside and outside the unit circle, one finds (after applying the projection operator)

$$N_n'(z)=\begin{pmatrix}0\\1\end{pmatrix}+\sum_{j=1}^{\bar{J}}\overline{C}_j\bar{z}_j^{2n}\left[\frac{\overline{N}_n'(\bar{z}_j)}{z-\bar{z}_j}+\frac{\overline{N}_n'(-\bar{z}_j)}{z+\bar{z}_j}\right]+\lim_{\substack{\zeta\to z\\|\zeta|>1}}\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{2n}\bar{\rho}(w)\overline{N}_n'(w)}{w-\zeta}\mathrm{d}w. \tag{8.17}$$

8.3. *Closing the system from the left*

To close the system, we evaluate equation (8.16) at the eigenvalues $\pm\bar{z}_j$ and equation (8.17) at $\pm z_j$. This results in a linear algebraic-integral system in the form

$$\overline{N}_n'(\bar{z}_j)=\begin{pmatrix}1\\0\end{pmatrix}+\sum_{k=1}^{J}C_kz_k^{-2n}\left[\frac{N_n'(z_k)}{\bar{z}_j-z_k}+\frac{N_n'(-z_k)}{\bar{z}_j+z_k}\right]-\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{-2n}\rho(w)N_n'(w)}{w-\bar{z}_j}\mathrm{d}w, \tag{8.18}$$

$$\overline{N}_n'(-\bar{z}_j)=\begin{pmatrix}1\\0\end{pmatrix}-\sum_{k=1}^{J}C_kz_k^{-2n}\left[\frac{N_n'(z_k)}{\bar{z}_j+z_k}+\frac{N_n'(-z_k)}{\bar{z}_j-z_k}\right]-\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{-2n}\rho(w)N_n'(w)}{w+\bar{z}_j}\mathrm{d}w, \tag{8.19}$$

$$N_n'(z_j)=\begin{pmatrix}0\\1\end{pmatrix}+\sum_{k=1}^{\bar{J}}\overline{C}_k\bar{z}_k^{2n}\left[\frac{\overline{N}_n'(\bar{z}_k)}{z_j-\bar{z}_k}+\frac{\overline{N}_n'(-\bar{z}_k)}{z_j+\bar{z}_k}\right]+\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{2n}\bar{\rho}(w)\overline{N}_n'(w)}{w-z_j}\mathrm{d}w, \tag{8.20}$$

Table 1. Symmetry relations between scattering data, norming constants and eigenfunctions for the classical integrable Ablowitz–Ladik model, RT, RST and *PT* symmetric nonlocal integrable discrete NLS equations. Here $\sigma = \mp 1$.

Symmetry property	AL model $R_n = \sigma Q_n^*$	RT reduction $R_n(t) = \sigma Q_n(-t)$	RST reduction $R_n(t) = \sigma Q_{-n}(-t)$	*PT* symmetric $R_n = \sigma Q_{-n}^*$
Scattering data	$\overline{a}(z) = a^*(1/z^*)$ $\overline{b}(z) = \sigma b^*(1/z^*)$	$\overline{a}(z,t) = a(1/z,-t)$ $\overline{b}(z,t) = \sigma b(1/z,-t)$	$a(z,t) = a(z,-t)$ $\overline{a}(z,t) = \overline{a}(z,-t)$ $\overline{b}(z,t) = \frac{\sigma}{z^2} b(z,-t)$	$a(z) = a^*(z^*)$ $\overline{a}(z) = \overline{a}^*(z^*)$ $\overline{b}(z) = \frac{\sigma}{z^2} b^*(z^*)$
Eigenvalues	$\overline{z}_j = 1/z_j^*$ free $z_j \in \mathbb{C}$	$\overline{z}_j = 1/z_j$ free $z_j \in \mathbb{C}$	$z_j, \overline{z}_j \in \mathbb{C}$ free parameters	$z_j, z_j^*; \overline{z}_j, \overline{z}_j^*$ free parameters
Norming constants	$\overline{C}_j = -\sigma(z_j^*)^{-2} C_j^*$ free $C_j \in \mathbb{C}$	$\overline{C}_j(t) = -\sigma z_j^{-2} C_j(-t)$ free $C_j \in \mathbb{C}$	$C_j, \overline{C}_j \in \mathbb{C}$ depend on $z_j, \overline{z}_j$	$C_j, \overline{C}_j$ depend on $z_j, \overline{z}_j$
Reflection coefficients	$\overline{\rho}(z) = \sigma \rho^*(1/z^*)$	$\overline{\rho}(z,t) = \sigma \rho(1/z,-t)$	No relation between ρ and $\overline{\rho}$	No relation between ρ and $\overline{\rho}$
Eigenfunctions	$\overline{M}_n(z)$ $= \Lambda M_n^*\left(\frac{1}{z^*}\right)$ $\overline{N}_n(z)$ $= \Lambda^{-1} N_n^*\left(\frac{1}{z^*}\right)$	$\overline{M}_n(z,t)$ $= \Lambda M_n\left(\frac{1}{z}, -t\right)$ $\overline{N}_n(z,t)$ $= \Lambda^{-1} N_n\left(\frac{1}{z}, -t\right)$	$N_{n+1}(z,t)$ $= \frac{\Lambda}{c_{-\infty}^{\sigma} f_n^{\sigma}(t)} M_{-n}(z,-t)$ $\overline{N}_{n+1}(z,t)$ $= \frac{\Lambda^{-1}}{c_{-\infty}^{\sigma} f_n^{\sigma}(t)} \overline{M}_{-n}(z,-t)$	$N_{n+1}(z)$ $= \frac{\Lambda}{c_{-\infty}^{\sigma} f_n^{\sigma}} M_{-n}^*(z^*)$ $\overline{N}_{n+1}(z)$ $= \frac{\Lambda^{-1}}{c_{-\infty}^{\sigma} f_n^{\sigma}} \overline{M}_{-n}^*(z^*)$
Modified eigenfunctions	$\overline{M}_n'(z)$ $= \Lambda_c M_n'^*\left(\frac{1}{z^*}\right)$ $\overline{N}_n'(z)$ $= \Lambda_c^{-1} N_n'^*\left(\frac{1}{z^*}\right)$	$\overline{M}_n'(z,t)$ $= \Lambda_c M_n'\left(\frac{1}{z}, -t\right)$ $\overline{N}_n'(z,t)$ $= \Lambda_c^{-1} N_n'\left(\frac{1}{z}, -t\right)$	$N_{n+1}'(z,t)$ $= \Lambda_c M_{-n}'(z,-t)$ $\overline{N}_{n+1}'(z,t)$ $= \frac{\Lambda_c^{-1}}{c_{-\infty}^{\sigma}} \overline{M}_{-n}'(z,-t)$	$N_{n+1}'(z)$ $= \Lambda_c M_{-n}'^*(z^*)$ $\overline{N}_{n+1}'(z)$ $= \frac{\Lambda_c^{-1}}{c_{-\infty}^{\sigma}} \overline{M}_{-n}'^*(z^*)$
Λ	$\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix}$
Λ_c	$\begin{pmatrix} 0 & \sigma c_{-\infty}^{\sigma^{-1}} \\ c_{-\infty}^{\sigma} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sigma c_{-\infty}^{\sigma^{-1}} \\ c_{-\infty}^{\sigma} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sigma c_{-\infty}^{\sigma^{-1}} \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sigma c_{-\infty}^{\sigma^{-1}} \\ 1 & 0 \end{pmatrix}$

$$N_n'(-z_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{k=1}^{\bar{J}} \overline{C}_k \bar{z}_k^{2n} \left[\frac{\overline{N}_n'(\bar{z}_k)}{z_j + \bar{z}_k} + \frac{\overline{N}_n'(-\bar{z}_k)}{z_j - \bar{z}_k} \right] + \frac{1}{2\pi\mathrm{i}} \oint_{|w|=1} \frac{w^{2n}\overline{\rho}(w)\overline{N}_n'(w)}{w + z_j} \mathrm{d}w. \tag{8.21}$$

8.4. Time-evolution: left scattering problem

In this section, we provide the time evolution of all scattering data and norming constants. Following similar lines of derivation as detailed in [34] one finds the time-evolution of the scattering data to be

$$\partial_\tau a(z,\tau) = 0, \qquad \partial_\tau b(z,\tau) = 2\mathrm{i}\omega b(z,\tau), \tag{8.22}$$

in which case, the explicit time-dependence is give by

$$a(z,\tau) = a(z,0), \qquad b(z,\tau) = \mathrm{e}^{2\mathrm{i}\omega\tau} b(z,0). \tag{8.23}$$

Similarly, the other set of scattering data satisfy the evolution equations

$$\partial_\tau \overline{a}(z,\tau) = 0, \qquad \partial_\tau \overline{b}(k) = -2\mathrm{i}\omega \overline{b}(z,\tau). \tag{8.24}$$

Thus, we have

$$\overline{a}(z,\tau) = \overline{a}(z,0), \qquad \overline{b}(z,\tau) = \mathrm{e}^{-2\mathrm{i}\omega\tau} \overline{b}(z,0). \tag{8.25}$$

The evolution of the norming constants C_j and $\overline{C}_j$ defined in equations (8.7) and (8.8) is given by

$$C_\ell = C_\ell(0)\mathrm{e}^{2\mathrm{i}\omega_\ell\tau}, \qquad C_\ell(0) = \frac{b_\ell(0)}{a'(z_\ell,0)}, \tag{8.26}$$

$$\overline{C}_\ell = \overline{C}_\ell(0)\mathrm{e}^{-2\mathrm{i}\overline{\omega}_\ell\tau}, \qquad \overline{C}_\ell(0) = \frac{\overline{b}_\ell(0)}{\overline{a}'(\bar{z}_\ell,0)}, \tag{8.27}$$

where

$$\omega_\ell = \frac{1}{2}\left(z_\ell - z_\ell^{-1}\right)^2, \qquad \overline{\omega}_\ell = \frac{1}{2}\left(\bar{z}_\ell - \bar{z}_\ell^{-1}\right)^2. \tag{8.28}$$

8.5. Reconstruction of the potentials: left scattering problem

To reconstruct the potential we subtract from both sides of (8.12) all the non analytic parts of $\overline{\mu}_n'(z)$ and apply $P_<$ to both sides of equation (8.12) to find

$$\begin{aligned} &P_< \left\{ \overline{\mu}_n'(z) - \sum_{j=1}^{\bar{J}} \overline{C}_j \bar{z}_j^{2n} \left[\frac{\overline{N}_n'(\bar{z}_j)}{z - \bar{z}_j} + \frac{\overline{N}_n'(-\bar{z}_j)}{z + \bar{z}_j} \right] \right\} \\ &= P_< \{N_n'(z)\} + P_< \left\{ z^{2n}\overline{\rho}(z)\overline{N}_n'(z) \right\} - P_< \left\{ \sum_{j=1}^{\bar{J}} \overline{C}_j \bar{z}_j^{2n} \left[\frac{\overline{N}_n'(\bar{z}_j)}{z - \bar{z}_j} + \frac{\overline{N}_n'(-\bar{z}_j)}{z + \bar{z}_j} \right] \right\}. \end{aligned} \tag{8.29}$$

The function inside the brackets, on the right hand side of equation (8.29), is an analytic function inside the unit circle therefore, using properties of projection operators, we find

$$P_{<}\left\{\overline{\mu}_n'(z)-\sum_{j=1}^{\bar{J}}\overline{C}_j\bar{z}_j^{2n}\left[\frac{\overline{N}_n'(\bar{z}_j)}{z-\bar{z}_j}+\frac{\overline{N}_n'(-\bar{z}_j)}{z+\bar{z}_j}\right]\right\}=\overline{\mu}_n'(z)-\sum_{j=1}^{\bar{J}}\overline{C}_j\bar{z}_j^{2n}\left[\frac{\overline{N}_n'(\bar{z}_j)}{z-\bar{z}_j}+\frac{\overline{N}_n'(-\bar{z}_j)}{z+\bar{z}_j}\right].$$

Moreover, the function $N_n'(z)$ is analytic outside the unite circle and approach $(0\ 1)^T$ as $|z|\to\infty$ hence $P_{<}\{N_n'(z)\}=(0\ 1)^T$. Similarly, the function $\sum_{j=1}^{\bar{J}}\overline{C}_j\bar{z}_j^{2n}\left[\frac{\overline{N}_n'(\bar{z}_j)}{z-\bar{z}_j}+\frac{\overline{N}_n'(-\bar{z}_j)}{z+\bar{z}_j}\right]$ is analytic outside the unit circle and tend to zero as $|z|\to\infty$, thus,

$$P_{<}\left\{\sum_{j=1}^{\bar{J}}\overline{C}_j\bar{z}_j^{2n}\left[\frac{\overline{N}_n'(\bar{z}_j)}{z-\bar{z}_j}+\frac{\overline{N}_n'(-\bar{z}_j)}{z+\bar{z}_j}\right]\right\}=0. \tag{8.30}$$

Putting all things together we obtain

$$\overline{\mu}_n'(z)=\begin{pmatrix}0\\1\end{pmatrix}+\sum_{j=1}^{\bar{J}}\overline{C}_j\bar{z}_j^{2n}\left[\frac{\overline{N}_n'(\bar{z}_j)}{z-\bar{z}_j}+\frac{\overline{N}_n'(-\bar{z}_j)}{z+\bar{z}_j}\right]+\lim_{\substack{\zeta\to z\\|\zeta|<1}}\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{2n}\bar{\rho}(w)\overline{N}_n'(w)}{w-\zeta}\mathrm{d}w. \tag{8.31}$$

Note that the eigenfunctions N_n and $\overline{N}_n$ satisfy the parity relation [34]

$$N_n'^{(1)}(-z)=-N_n'^{(1)}(z),\qquad N_n'^{(2)}(-z)=N_n'^{(2)}(z), \tag{8.32}$$

$$\overline{N}_n'^{(1)}(-z)=\overline{N}_n'^{(1)}(z),\qquad \overline{N}_n'^{(2)}(-z)=-\overline{N}_n'^{(2)}(z). \tag{8.33}$$

From equations (8.16), (8.32) and the expansion $\frac{1}{w-z}\approx\frac{1}{w}+\frac{z}{w^2}+\cdots$ as $z\to0$ we find

$$\begin{aligned}\overline{N}_n'^{(2)}(z)&=2z\sum_{j=1}^{J}\frac{C_jz_j^{-2n}N_n'^{(2)}(z_j)}{z^2-z_j^2}-\lim_{\substack{\zeta\to z\\|\zeta|<1}}\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{-2n}\rho(w)N_n'^{(2)}(w)}{w-\zeta}\mathrm{d}w\\&\approx-2z\sum_{j=1}^{J}C_jz_j^{-2(n+1)}N_n'^{(2)}(z_j)-\frac{z}{2\pi\mathrm{i}}\oint_{|w|=1}w^{-2(n+1)}\rho(w)N_n'^{(2)}(w)\mathrm{d}w\\&\quad-\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}w^{-(2n+1)}\rho(w)N_n'^{(2)}(w)\mathrm{d}w+\cdots,\quad\text{as}\quad z\to0.\end{aligned} \tag{8.34}$$

Comparing (8.34) with the asymptotic form ($z\to0$) of the eigenfunction $\overline{N}_n'^{(2)}(z)$ given by equation (3.24) one finds

$$R_n(t)=2\sum_{j=1}^{J}C_j(t)z_j^{-2(n+1)}N_n'^{(2)}(z_j,t)+\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}w^{-2(n+1)}\rho(w,t)N_n'^{(2)}(w,t)\mathrm{d}w. \tag{8.35}$$

Remark. In deriving the expression for R_n we used the fact that

$$\oint_{|w|=1}w^{-(2n+1)}\rho(w)N_n'^{(2)}(w)\mathrm{d}w=0, \tag{8.36}$$

which can be proved using the symmetry property between the eigenfunctions given in (8.32) and the fact that $\rho(w)$ is an odd function. To obtain the expression for the potential Q_n, start from (8.31); use (8.33) and the expansion $\frac{1}{w-z}\approx\frac{1}{w}+\frac{z}{w^2}+\cdots$ as $z\to0$ to find

$$\overline{\mu}_n^{\prime(1)}(z) = 2z\sum_{j=1}^{\overline{J}} \frac{\overline{C}_j\overline{z}_j^{2n}\overline{N}_n^{\prime(1)}(\overline{z}_j)}{z^2-\overline{z}_j^2} + \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{2n}\overline{\rho}(w)\overline{N}_n^{\prime(1)}(w)}{w-z}\mathrm{d}w$$
$$\approx -2z\sum_{j=1}^{\overline{J}}\overline{C}_j\overline{z}_j^{2(n-1)}\overline{N}_n^{\prime(1)}(\overline{z}_j) + \frac{z}{2\pi\mathrm{i}}\oint_{|w|=1} w^{2(n-1)}\overline{\rho}(w)\overline{N}_n^{\prime(1)}(w)\mathrm{d}w$$
$$+\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1} w^{2n-1}\overline{\rho}(w)\overline{N}_n^{\prime(1)}(w)\mathrm{d}w + \cdots, \quad \text{as} \quad z\to 0. \tag{8.37}$$

Comparing (8.37) with its asymptotic (in z) expansion given in (3.25) one finds

$$Q_{n-1} \approx -2\sum_{j=1}^{\overline{J}}\overline{C}_j\overline{z}_j^{2(n-1)}\overline{N}_n^{\prime(1)}(\overline{z}_j) + \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1} w^{2(n-1)}\overline{\rho}(w)\overline{N}_n^{\prime(1)}(w)\mathrm{d}w$$
$$+\frac{1}{2\pi iz}\oint_{|w|=1} w^{2n-1}\overline{\rho}(w)\overline{N}_n^{\prime(1)}(w)\mathrm{d}w + \cdots, \quad \text{as} \quad z\to 0. \tag{8.38}$$

Use the result

$$\oint_{|w|=1} w^{2n-1}\overline{\rho}(w)\overline{N}_n^{\prime(1)}(w)\mathrm{d}w = 0, \tag{8.39}$$

which can be proved using the symmetry property between the eigenfunctions given in (8.32) and the fact that $\overline{\rho}(w)$ is an odd function, gives

$$Q_{n-1}(t) = -2\sum_{j=1}^{\overline{J}}\overline{C}_j(t)\overline{z}_j^{2(n-1)}\overline{N}_n^{\prime(1)}(\overline{z}_j,t) + \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1} w^{2(n-1)}\overline{\rho}(w,t)\overline{N}_n^{\prime(1)}(w,t)\mathrm{d}w. \tag{8.40}$$

8.6. Right scattering problem

One could instead work out the entire inverse problem using a 'right' scattering problem:

$$\psi_n(z) = \alpha(z)\overline{\phi}_n(z) + \beta(z)\phi_n(z), \tag{8.41}$$

$$\overline{\psi}_n(z) = \overline{\alpha}(z)\phi_n(z) + \overline{\beta}(z)\overline{\phi}_n(z). \tag{8.42}$$

Clearly, the left and right scattering problems are related by $S_R(z) = S_L^{-1}(z)$, where S_R, S_L are the scattering matrices from the right and left respectively. Component-wise, we have

$$\begin{pmatrix}\overline{\alpha}(z) & \overline{\beta}(z)\\ \beta(z) & \alpha(z)\end{pmatrix} = \frac{1}{c_{-\infty}}\begin{pmatrix}\overline{a}(z) & -b(z)\\ -\overline{b}(z) & a(z)\end{pmatrix}. \tag{8.43}$$

To setup a jump condition for a RH problem, divide equation (8.42) by $\overline{\alpha}(z)$; use the definition of the eigenfunctions $M_n(z), \overline{M}_n(z), N_n(z), \overline{N}_n(z)$ to get

$$\overline{\nu}_n(z) = M_n(z) + \overline{R}(z)z^{-2n}\overline{M}_n(z), \tag{8.44}$$

where $\overline{\nu}_n(z) \equiv \frac{\overline{N}_n(z)}{\overline{\alpha}(z)}$ and $\overline{R}(z) \equiv \frac{\overline{\beta}(z)}{\overline{\alpha}(z)}$. Next, divide equation (8.41) by $\alpha(z)$ and make use of the modified eigenfunctions to find

$$\nu_n(z) = \overline{M}_n(z) + R(z)z^{2n}M_n(z), \tag{8.45}$$

where, $\nu_n(z) \equiv \frac{N_n(z)}{\alpha(z)}$. The eigenfunctions have the following definitions and asymptotic behavior

$$\nu_n(z) = \frac{N_n(z)}{\alpha(z)} = \begin{pmatrix} -z^{-1}c_{-\infty}c_n^{-1}Q_n + O(z^{-3}) \\ c_{-\infty}c_n^{-1} + O(z^{-2}) \end{pmatrix} \quad |z| \to \infty, \tag{8.46}$$

$$\overline{\nu}_n(z) = \frac{\overline{N}_n(z)}{\overline{\alpha}(z)} = \begin{pmatrix} c_{-\infty}c_n^{-1} + O(z^2) \\ -zc_{-\infty}c_n^{-1}R_n + O(z^3) \end{pmatrix}, \quad \text{as } z \to 0. \tag{8.47}$$

The scattering problem from the right can possess discrete eigenvalues (bound states). These occur whenever $\alpha(z)$ has J simple zeros at $\{z_j \text{ s.t. } |z_j| > 1\}_{j=1}^{J}$, i.e. $a(z_j) = 0$ and $\overline{\alpha}(z)$ has $\overline{J}$ simple zeros at $\{\overline{z}_j \text{ s.t. } |\overline{z}_j| > 1\}_{j=1}^{\overline{J}}$, i.e. $\overline{\alpha}(\overline{z}_j) = 0$. Indeed, for such values of the spectral parameters $W(\phi_n(z_j), \psi_n(z_j)) = 0$ and $W(\overline{\phi}_n(\overline{z}_j), \overline{\psi}_n(\overline{z}_j)) = 0$. Therefore, from (8.41) and (8.42) we find

$$\psi_n(z_j) = \beta_j\phi_n(z_j), \quad \Rightarrow \quad N_n(z_j) = \beta_j z_j^{2n} M_n(z_j), \tag{8.48}$$

$$\overline{\psi}_n(\overline{z}_\ell) = \overline{\beta}_\ell\overline{\phi}_n(\overline{z}_\ell), \quad \Rightarrow \quad \overline{N}_n(\overline{z}_\ell) = \overline{\beta}_\ell \overline{z}_\ell^{-2n}\overline{M}_n(\overline{z}_\ell). \tag{8.49}$$

The residues of the functions $\nu_n(z)$ and $\overline{\nu}_n(z)$ at the eigenvalues are given by

$$\mathrm{Res}(\nu_n; z_j) = \frac{\beta_j z_j^{2n} M_n(z_j)}{\alpha'(z_j)} = z_j^{2n} B_j M_n(z_j), \tag{8.50}$$

$$\mathrm{Res}(\overline{\nu}_n; \overline{z}_\ell) = \frac{\overline{\beta}_\ell \overline{z}_\ell^{-2n}\overline{M}_n(\overline{z}_\ell)}{\overline{\alpha}'(\overline{z}_\ell)} = \overline{z}_\ell^{-2n}\overline{B}_\ell\overline{M}_n(\overline{z}_\ell), \tag{8.51}$$

where we have defined $\beta_j \equiv \beta(z_j)$; $\overline{\beta}_\ell \equiv \overline{\beta}(\overline{z}_\ell)$ and

$$B_j = \frac{\beta_j}{\alpha'(z_j)}, \tag{8.52}$$

$$\overline{B}_\ell = \frac{\overline{\beta}_\ell}{\overline{\alpha}'(\overline{z}_\ell)}. \tag{8.53}$$

8.7. Right jump condition

As was done with the left scattering problem, here we shall use the modified eigenfunction to formulate and solve a right Riemann–Hilbert problem. First, define the functions $\nu_n'(z)$ and $\overline{\nu}_n'(z)$

$$\nu_n'(z) \equiv \frac{N_n'(z)}{\alpha(z)}, \qquad \overline{\nu}_n'(z) \equiv \frac{\overline{N}_n'(z)}{\overline{\alpha}(z)}, \tag{8.54}$$

whose asymptotic behavior in z is given by

$$\nu_n'(z) = \begin{pmatrix} -z^{-1}c_{-\infty}c_n^{-1}Q_n \\ c_{-\infty} \end{pmatrix} + O(z^{-2}) \quad \text{as } |z| \to \infty, \tag{8.55}$$

$$\overline{\nu}'_n(z) = \begin{pmatrix} c_{-\infty}c_n^{-1} \\ -zc_{-\infty}R_n \end{pmatrix} + O(z^2) \qquad \text{as} \quad |z| \to 0. \tag{8.56}$$

Note that the functions $\nu'_n(z)$ and $\overline{\nu}'_n(z)$ also satisfy the jump conditions on $|z| = 1$

$$\nu'_n(z) = \overline{M}'_n(z) + R(z)z^{2n}M'_n(z), \tag{8.57}$$

$$\overline{\nu}'_n(z) = M'_n(z) + \overline{R}(z)z^{-2n}\overline{M}'_n(z). \tag{8.58}$$

Remark. The poles of $\nu'_n(z)$ and $\overline{\nu}'_n(z)$ are the same as the poles of $\nu_n(z)$ and $\overline{\nu}_n(z)$ respectively. Moreover, the residues of these poles are determined by the relations $\mathrm{Res}(\nu'_n; z_j) = z_j^{2n}B_jM'_n(z_j)$, $\mathrm{Res}(\overline{\nu}'_n; \overline{z}_\ell) = \overline{z}_\ell^{-2n}\overline{B}_\ell\overline{M}'_n(\overline{z}_\ell)$. We subtract from both sides of equation (8.57) the value of $\nu'_n(z)$ at infinity, i.e. $(0\ c_{-\infty})^T$ and all the non analytic parts of $\nu'_n(z)$; apply the projection operator $P_<$ on both sides of the resulting system; one gets

$$P_<\left\{\nu'_n(z) - \begin{pmatrix} 0 \\ c_{-\infty} \end{pmatrix} - \sum_{j=1}^{J} B_jz_j^{2n}\left[\frac{M'_n(z_j)}{z - z_j} + \frac{M'_n(-z_j)}{z + z_j}\right]\right\}$$
$$= P_<\left\{\overline{M}'_n(z) - \begin{pmatrix} 0 \\ c_{-\infty} \end{pmatrix} - \sum_{j=1}^{J} B_jz_j^{2n}\left[\frac{M'_n(z_j)}{z - z_j} + \frac{M'_n(-z_j)}{z + z_j}\right] + z^{2n}R(z)M'_n(z)\right\}. \tag{8.59}$$

The function inside the brackets on the left hand side of equation (8.59) is analytic outside the unit circle and approach zero as $|z| \to \infty$ hence, by properties of the projectors, the left hand side of equation (8.59) vanishes. Since $|z_j| > 1$ and $\overline{M}'_n(z)$ is analytic inside the unit circle, the function

$$\overline{M}'_n(z) - \begin{pmatrix} 0 \\ c_{-\infty} \end{pmatrix} - \sum_{j=1}^{J} B_jz_j^{2n}\left[\frac{M'_n(z_j)}{z - z_j} + \frac{M'_n(-z_j)}{z + z_j}\right], \tag{8.60}$$

is analytic inside the unit circle and constitute an 'in' function. Now use the definition of the projection $P_<$ from (8.9) to find

$$\overline{M}'_n(z) = \begin{pmatrix} 0 \\ c_{-\infty} \end{pmatrix} + \sum_{j=1}^{J} B_jz_j^{2n}\left[\frac{M'_n(z_j)}{z - z_j} + \frac{M'_n(-z_j)}{z + z_j}\right] - \lim_{\substack{\zeta\to z \\ |\zeta|<1}} \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2n}R(w)M'_n(w)}{w - \zeta}\mathrm{d}w. \tag{8.61}$$

Similarly, we subtract from both sides of equation (8.58) all the non analytic parts; apply the projection operator $P_>$ on both sides of the resulting equation; identify parts of the equation that are analytic inside and/or outside the unite circle to finally find

$$M'_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{\overline{J}} \overline{B}_j\overline{z}_j^{-2n}\left[\frac{\overline{M}'_n(\overline{z}_j)}{z - \overline{z}_j} + \frac{\overline{M}'_n(-\overline{z}_j)}{z + \overline{z}_j}\right] + \lim_{\substack{\zeta\to z \\ |\zeta|>1}} \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n}\overline{R}(w)\overline{M}'_n(w)}{w - \zeta}\mathrm{d}w. \tag{8.62}$$

8.8. Closing the system from right

To close the system, we evaluate equation (8.61) at the eigenvalues $\pm\overline{z}_j$ and (8.62) at $\pm z_j$. This results in a linear algebraic-integral system composed of

$$\overline{M}'_n(\bar{z}_j) = \begin{pmatrix} 0 \\ c_{-\infty} \end{pmatrix} + \sum_{k=1}^{J} B_k z_k^{2n} \left[\frac{M'_n(z_k)}{\bar{z}_j - z_k} + \frac{M'_n(-z_k)}{\bar{z}_j + z_k} \right] - \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2n} R(w) M'_n(w)}{w - \bar{z}_j} \mathrm{d}w, \quad (8.63)$$

$$\overline{M}'_n(-\bar{z}_j) = \begin{pmatrix} 0 \\ c_{-\infty} \end{pmatrix} - \sum_{k=1}^{J} B_k z_k^{2n} \left[\frac{M'_n(z_k)}{\bar{z}_j + z_k} + \frac{M'_n(-z_k)}{\bar{z}_j - z_k} \right] - \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2n} R(w) M'_n(w)}{w + \bar{z}_j} \mathrm{d}w, \quad (8.64)$$

$$M'_n(z_j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=1}^{\bar{J}} \overline{B}_k \bar{z}_k^{-2n} \left[\frac{\overline{M}'_n(\bar{z}_k)}{z_j - \bar{z}_k} + \frac{\overline{M}'_n(-\bar{z}_k)}{z_j + \bar{z}_k} \right] + \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n} \overline{R}(w) \overline{M}'_n(w)}{w - z_j} \mathrm{d}w, \quad (8.65)$$

$$M'_n(-z_j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{k=1}^{\bar{J}} \overline{B}_k \bar{z}_k^{-2n} \left[\frac{\overline{M}'_n(\bar{z}_k)}{z_j + \bar{z}_k} + \frac{\overline{M}'_n(-\bar{z}_k)}{z_j - \bar{z}_k} \right] + \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n} \overline{R}(w) \overline{M}'_n(w)}{w + z_j} \mathrm{d}w. \quad (8.66)$$

8.9. Reconstruction of the potentials: right scattering problem

To reconstruct the potential we subtract from both sides of (8.58) all the non analytic parts of $\overline{\nu}'_n(z)$ and apply $P_<$ to both sides of equation (8.58) to find

$$P_< \left\{ \overline{\nu}'_n(z) - \sum_{j=1}^{\bar{J}} \overline{B}_j \bar{z}_j^{-2n} \left[\frac{\overline{M}'_n(\bar{z}_j)}{z - \bar{z}_j} + \frac{\overline{M}'_n(-\bar{z}_j)}{z + \bar{z}_j} \right] \right\}$$

$$= P_< \{M'_n(z)\} + P_< \left\{ z^{-2n} \overline{R}(z) \overline{M}'_n(z) \right\} - P_< \left\{ \sum_{j=1}^{\bar{J}} \overline{B}_j \bar{z}_j^{-2n} \left[\frac{\overline{M}'_n(\bar{z}_j)}{z - \bar{z}_j} + \frac{\overline{M}'_n(-\bar{z}_j)}{z + \bar{z}_j} \right] \right\}. \quad (8.67)$$

The function inside the brackets in the right hand side of equation (8.67) is an analytic function inside the unit circle. Moreover, the function $M'_n(z)$ is analytic outside the unite circle and approach $(1\ 0)^T$ as $|z| \to \infty$ hence $P_< \{M'_n(z)\} = (1\ 0)^T$. Similarly, the function $\sum_{j=1}^{\bar{J}} \overline{B}_j \bar{z}_j^{-2n} \left[\frac{\overline{M}'_n(\bar{z}_j)}{z - \bar{z}_j} + \frac{\overline{M}'_n(-\bar{z}_j)}{z + \bar{z}_j} \right]$ is analytic outside the unit circle and tend to zero as $|z| \to \infty$. Using properties of the projectors, we obtain

$$\overline{\nu}'_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \overline{B}_j \bar{z}_j^{-2n} \left[\frac{\overline{M}'_n(\bar{z}_j)}{z - \bar{z}_j} + \frac{\overline{M}'_n(-\bar{z}_j)}{z + \bar{z}_j} \right] + \lim_{\substack{\zeta \to z \\ |\zeta| < 1}} \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n} \overline{R}(w) \overline{M}'_n(w)}{w - \zeta} \mathrm{d}w. \quad (8.68)$$

Note that the eigenfunctions M'_n and $\overline{M}'_n$ satisfy the parity relation $M'^{(1)}_n(-z) = M'^{(1)}_n(z)$, $M'^{(2)}_n(-z) = -M'^{(2)}_n(z), \overline{M}'^{(1)}_n(-z) = -\overline{M}'^{(1)}_n(z), \overline{M}'^{(2)}_n(-z) = \overline{M}'^{(2)}_n(z)$. From (8.61); the parity symmetries and the expansion $\frac{1}{w-z} \approx \frac{1}{w} + \frac{z}{w^2} + \cdots$ as $z \to 0$ one finds

$$\begin{aligned} \overline{M}'^{(1)}_n(z) &= 2z \sum_{j=1}^{J} \frac{B_j z_j^{2n} M'^{(1)}_n(z_j)}{z^2 - z_j^2} - \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2n} R(w) M'^{(1)}_n(w)}{w - z} \mathrm{d}w \\ &\approx -2z \sum_{j=1}^{J} B_j z_j^{2(n-1)} M'^{(1)}_n(z_j) - \frac{z}{2\pi \mathrm{i}} \oint_{|w|=1} w^{2(n-1)} R(w) M'^{(1)}_n(w) \mathrm{d}w \\ &\quad - \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} w^{2n-1} R(w) M'^{(1)}_n(w) \mathrm{d}w + \cdots, \quad \text{as} \quad z \to 0. \end{aligned} \quad (8.69)$$

Since R and $M_n'^{(1)}$ are respectively odd and even functions of their argument, we have

$$\oint_{|w|=1} w^{2n-1}R(w)M_n'^{(1)}(w)\mathrm{d}w = 0. \tag{8.70}$$

Comparing (8.69) with the asymptotic expansion $\overline{M}_n'^{(1)}(z) = zQ_{n-1} + O(z^2)$, as $|z| \to 0$ one finds

$$Q_{n-1}(t) = -2\sum_{j=1}^{J} B_j(t)z_j^{2(n-1)}M_n'^{(1)}(z_j,t) - \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1} w^{2(n-1)}R(w,t)M_n'^{(1)}(w,t)\mathrm{d}w. \tag{8.71}$$

To obtain the expression for the potential R_n we start from equation (8.68); use the parity in z of the eigenfunctions and the expansion $\frac{1}{w-z} \approx \frac{1}{w} + \frac{z}{w^2} + \cdots$ as $z \to 0$ to find

$$\begin{aligned}\overline{\nu}_n'^{(2)}(z) &= 2z\sum_{j=1}^{\overline{J}} \frac{\overline{B}_j\overline{z}_j^{-2n}\overline{M}_n'^{(2)}(\overline{z}_j)}{z^2-\overline{z}_j^2} + \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{-2n}\overline{R}(w)\overline{M}_n'^{(2)}(w)}{w-z}\mathrm{d}w \\ &\approx -2z\sum_{j=1}^{\overline{J}}\overline{B}_j\overline{z}_j^{-2(n+1)}\overline{M}_n'^{(2)}(\overline{z}_j) + \frac{z}{2\pi\mathrm{i}}\oint_{|w|=1} w^{-2(n+1)}\overline{R}(w)\overline{M}_n'^{(2)}(w)\mathrm{d}w \\ &\quad + \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1} w^{-2n-1}\overline{R}(w)\overline{M}_n'^{(2)}(w)\mathrm{d}w + \cdots, \quad \text{as} \quad z \to 0.\end{aligned} \tag{8.72}$$

To that end, use the result

$$\oint_{|w|=1} w^{-(2n+1)}\overline{R}(w)\overline{M}_n^{(2)'}(w)\mathrm{d}w = 0, \tag{8.73}$$

along with the asymptotic expression $\overline{\nu}_n'^{(2)}(z) = -zc_{-\infty}R_n + O(z^2)$ as $|z| \to 0$ to get

$$R_n(t) = 2c_{-\infty}^{-1}\sum_{j=1}^{\overline{J}}\overline{B}_j(t)\overline{z}_j^{-2(n+1)}\overline{M}_n'^{(2)}(\overline{z}_j,t) - \frac{c_{-\infty}^{-1}}{2\pi\mathrm{i}}\oint_{|w|=1} w^{-2(n+1)}\overline{R}(w,t)\overline{M}_n'^{(2)}(w,t)\mathrm{d}w. \tag{8.74}$$

Remark. To solve for R_n one of course needs to find $c_{-\infty}$ which formally depends on both Q_n and R_n (which are formally unknown.) However, since $c_{-\infty}$ is constant in time (see equation (3.11)), it is therefore determined by the initial conditions.

8.10. Time-evolution: right scattering problem

In this section we give the time evolution of the scattering problem from the right side. Following the ideas presented in [34] one gets

$$\partial_\tau\alpha(z,\tau) = 0, \qquad \partial_\tau\beta(z,\tau) = -2\mathrm{i}\omega\beta(z,\tau), \tag{8.75}$$

$$\partial_\tau\overline{\alpha}(z,\tau) = 0, \qquad \partial_\tau\overline{\beta}(k) = 2\mathrm{i}\omega\overline{\beta}(z,\tau). \tag{8.76}$$

The explicit time-dependence is found to be

$$\alpha(z,\tau) = \alpha(z,0), \qquad \beta(z,\tau) = \mathrm{e}^{-2\mathrm{i}\omega\tau}\beta(z,0), \tag{8.77}$$

$$\overline{\alpha}(z,\tau) = \overline{\alpha}(z,0), \qquad \overline{\beta}(z,\tau) = \mathrm{e}^{2\mathrm{i}\omega\tau}\overline{\beta}(z,0). \tag{8.78}$$

With this at hand, the evolution of the norming constants B_ℓ and $\overline{B}_\ell$ define by equations (8.52) and (8.53) is readily obtained from

$$B_\ell = B_\ell(0)\mathrm{e}^{-2\mathrm{i}\omega_\ell\tau}, \qquad B_\ell(0) = \frac{\beta_\ell(0)}{\alpha'(z_\ell,0)}, \tag{8.79}$$

$$\overline{B}_\ell = \overline{B}_\ell(0)\mathrm{e}^{2\mathrm{i}\overline{\omega}_\ell\tau}, \qquad \overline{B}_\ell(0) = \frac{\overline{\beta}_\ell(0)}{\overline{\alpha}'(\overline{z}_\ell,0)}, \tag{8.80}$$

with ω_ℓ and $\overline{\omega}_\ell$ defined by equation (8.28).

9. Relation between the reflection coefficients

In this section we establish the symmetry relation between the left and right reflections coefficients. The connection is possible only for the *PT* symmetric and RST cases where symmetries in scattering space do *not* mix the 'bar' quantities with their respective 'unbar' ones. This is in sharp contrast to the AL and RT cases where scattering data outside and inside the unit circle are related. By definition, we have

$$R(z) = \frac{\beta(z)}{\alpha(z)} = -\frac{\overline{b}(z)}{a(z)}, \qquad \overline{R}(z) = \frac{\overline{\beta}(z)}{\overline{\alpha}(z)} = -\frac{b(z)}{\overline{a}(z)}, \tag{9.1}$$

$$\rho(z) = \frac{b(z)}{a(z)}, \qquad \overline{\rho}(z) = \frac{\overline{b}(z)}{\overline{a}(z)}. \tag{9.2}$$

9.1. AL reduction $R_n(t) = \sigma Q_n^*(t)$

For the Ablowitz–Ladik case (and RT), the symmetries given in section 4 (for AL) and section 5 (for RT) were obtained from the left scattering problem and relate the reflection coefficient $\rho(z)$ (defined outside the unit circle) to $\overline{\rho}(z)$ (defined inside the unit circle). For the AL model, they are given by

$$\overline{a}(z) = a^*(1/z^*), \qquad \overline{b}(z) = \sigma b^*(1/z^*), \qquad \overline{\rho}(z) = \sigma\rho^*(1/z^*). \tag{9.3}$$

Thus, it is expected that the symmetry between the reflection coefficients $\overline{R}(z)$ and $R(z)$ (defined for the right scattering problem) to be related. Indeed, from (9.1)–(9.3) we find

$$\overline{R}(z) = \sigma R^*(1/z^*). \tag{9.4}$$

9.2. RT reduction $R_n(t) = \sigma Q_n(-t)$

Here, a similar situation happens as for the AL case. The 'left' symmetries in scattering space were derived in section 5 and are given by

$$\overline{a}(z,t) = a(1/z,-t), \qquad \overline{b}(z,t) = \sigma b(1/z,-t), \qquad \overline{\rho}(z,t) = \sigma\rho(1/z,-t). \tag{9.5}$$

Using the definitions (9.1) and (9.2) we find, after some algebra,

$$\overline{R}(z,t) = \sigma R(1/z,-t). \tag{9.6}$$

9.3. RST reduction $R_n(t) = \sigma Q_{-n}(-t)$

In this case we have

$$a(z,t) = a(z,-t), \qquad \overline{a}(z,t) = \overline{a}(z,-t), \qquad \overline{b}(z,t) = \frac{\sigma}{z^2} b(z,-t). \tag{9.7}$$

Using the definition of the reflection coefficients (given above), one finds

$$R(z,t) = -\frac{\sigma}{z^2}\rho(z,-t), \qquad \overline{R}(z,t) = -\sigma z^2 \overline{\rho}(z,-t). \tag{9.8}$$

9.4. PT symmetric reduction $R_n = \sigma Q^*_{-n}$

For the *PT* symmetric case, the scattering data satisfy the following symmetries

$$a(z) = a^*(z^*), \qquad \overline{a}(z) = \overline{a}^*(z^*), \qquad \overline{b}(z) = \frac{\sigma}{z^2} b^*(z^*). \tag{9.9}$$

Now from the definition of the reflection coefficients we have

$$R(z) = -\frac{\sigma}{z^2}\rho^*(z^*), \qquad \overline{R}(z) = -\sigma z^2 \overline{\rho}^*(z^*). \tag{9.10}$$

10. Norming constants: symmetries and time evolution

In this section, we connect the norming constants pair $\{C_j(t), \overline{C}_j(t)\}$ (defined in the left scattering problem) to their respective one $\{B_j(t), \overline{B}_j(t)\}$ (defined in the right scattering) for the *PT* symmetric and RST cases only. For the AL and RT cases (where $C_j(t)$ and $\overline{C}_j(t)$ already related via a symmetry), we derive the symmetry condition between $B_j(t)$ and $\overline{B}_j(t)$. Furthermore, we give the time evolution of each norming constant, which later, is used in determining the time evolution of the potentials.

10.1. AL reduction $R_n = \sigma Q^*_n$

Since the AL case is local in time, we shall omit all explicit time-dependence. Our starting point is equation (8.63), which together with the fact that $M_n^{(1)'}(z_k)$ is an even function of z_k takes the form

$$\overline{M}_n^{(1)'}(\overline{z}_j) = 2\overline{z}_j \sum_{k=1}^{J} B_k z_k^{2n} \frac{M_n^{(1)'}(z_k)}{\overline{z}_j^2 - z_k^2} - \frac{1}{2\pi\mathrm{i}} \oint_{|w|=1} \frac{w^{2n} R(w) M_n^{(1)'}(w)}{w - \overline{z}_j} \mathrm{d}w. \tag{10.1}$$

Now use the symmetries between the corresponding eigenfunctions as well as scattering data established in section 4, i.e. $\overline{M}_n^{(1)'}(z) = \sigma (c_n^\sigma)^{-1}\, M_n^{(2)'*}(1/z^*), \overline{M}_n^{(2)'}(z) = c_n^\sigma\, M_n^{(1)'*}(1/z^*)$ together with (9.4) to rewrite equation (10.1):

$$\begin{aligned} M_n^{(2)'*}(1/\overline{z}_j^*) &= 2\sigma \overline{z}_j \sum_{k=1}^{J} B_k z_k^{2n} \frac{\overline{M}_n^{(2)'*}(1/z_k^*)}{\overline{z}_j^2 - z_k^2} \\ &\quad - \frac{1}{2\pi\mathrm{i}} \oint_{|w|=1} \frac{w^{2n} \overline{R}^*(1/w^*) \overline{M}_n^{(2)'*}(1/w^*)}{w - \overline{z}_j} \mathrm{d}w. \end{aligned} \tag{10.2}$$

Upon using the symmetry between the eigenvalues, $\bar{z}_j = 1/z_j^*$, in (10.2) together with complex conjugation we find

$$M_n^{(2)'}(z_j) = -2\sigma z_j \sum_{k=1}^{J} (z_k^*)^{-2} B_k^* (z_k^*)^{2n} \frac{\overline{M}_n^{(2)'}(1/z_k^*)}{z_j^2 - (z_k^*)^{-2}} + \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{(w^*)^{2n} \overline{R}(1/w^*) \overline{M}_n^{(2)'}(1/w^*)}{w^* - z_j^{-1}} \mathrm{d}w^*. \tag{10.3}$$

Next, we consider the integral term in equation (10.3). With the change of variables $\zeta = 1/w^*$, we have

$$\begin{aligned}\oint_{|w|=1} \frac{(w^*)^{2n} \overline{R}(1/w^*) \overline{M}_n^{(2)'}(1/w^*)}{w^* - z_j^{-1}} \mathrm{d}w^* &= - \oint_{|\zeta|=1} \frac{z_j \zeta^{-2n} \overline{R}(\zeta) \overline{M}_n^{(2)'}(\zeta)}{\zeta(z_j - \zeta)} \mathrm{d}\zeta \\ &= + \oint_{|\zeta|=1} \frac{\zeta^{-2n} \overline{R}(\zeta) \overline{M}_n^{(2)'}(\zeta)}{\zeta - z_j} \mathrm{d}\zeta \\ &\quad - \oint_{|\zeta|=1} \zeta^{-(2n+1)} \overline{R}(\zeta) \overline{M}_n^{(2)'}(\zeta) \mathrm{d}\zeta. \end{aligned} \tag{10.4}$$

The last term in equation (10.4) vanishes by the residue theorem since $\overline{R}(\zeta)$ and $\overline{M}_n^{(2)'}(\zeta)$ are odd and even functions of ζ respectively. On the other hand, from equation (8.65) which, after using $\overline{M}_n^{(2)'}(-\bar{z}_k) = \overline{M}_n^{(2)'}(\bar{z}_k)$, reduces to

$$M_n^{(2)'}(z_j) = 2z_j \sum_{k=1}^{\bar{J}} \overline{B}_k (z_k^*)^{2n} \frac{\overline{M}_n^{(2)'}(1/z_k^*)}{z_j^2 - (z_k^*)^{-2}} + \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n} \overline{R}(w) \overline{M}_n^{(2)'}(w)}{w - z_j} \mathrm{d}w. \tag{10.5}$$

When equation (10.5) is compared with the result of (10.4) and (10.3), we arrive at the symmetry condition:

$$\overline{B}_k = -\sigma (z_k^*)^{-2} B_k^*. \tag{10.6}$$

Alternatively, one could reach the same conclusion by directly working with the definitions of the norming constants given by (8.52) and (8.53) and the symmetry properties of the scattering data $\beta_\ell, \overline{\beta}_\ell, \alpha'(z_\ell)$ and $\overline{\alpha}'(\bar{z}_\ell)$ in a manner similar to what was done for the left scattering problem. Our approach we adopted in this section, shows the 'stability and robustness' of the symmetries of the AL model.

10.2. RT reduction $R_n(t) = \sigma Q_n(-t)$

Next, we derive the symmetry relation between the norming constants $B_j(t)$ and $\overline{B}_j(t)$. Notice that since the RT reduction is nonlocal in time, one has to keep explicit time dependence for all variables. The symmetries between the eigenvalues and reflection coefficients are given by (5.23) and (9.6) respectively, i.e. $\bar{z}_j = 1/z_j$ (complex z_j) and $\overline{R}(z,t) = \sigma R(1/z, -t)$. From (8.63) we have

$$\overline{M}_n^{(1)'}(\bar{z}_j, t) = 2\bar{z}_j \sum_{k=1}^{J} B_k(t) z_k^{2n} \frac{M_n^{(1)'}(z_k, t)}{\bar{z}_j^2 - z_k^2} - \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2n} R(w,t) M_n^{(1)'}(w,t)}{w - \bar{z}_j} \mathrm{d}w. \tag{10.7}$$

The symmetries between the modified eigenfunctions, for the RT case, has been established in section 5.2 and are given by $\overline{M}_n^{(1)'}(z,t) = \sigma(c_n^\sigma(t))^{-1}\, M_n^{(2)'}(1/z,-t)$ and $\overline{M}_n^{(2)'}(z,t) = c_n^\sigma(t)\, M_n^{(1)'}(1/z,-t)$. Substituting these relations together with $\overline{z}_j = 1/z_j$ and $\overline{R}(z,t) = \sigma R(1/z,-t)$ back into (10.7) gives

$$M_n^{(2)'}(z_j,t) = -2\sigma z_j \sum_{k=1}^{J}(z_k)^{-2}B_k(-t)(z_k)^{2n}\frac{\overline{M}_n^{(2)'}(1/z_k,t)}{z_j^2-(z_k)^{-2}} - \frac{1}{2\pi \mathrm{i}}\oint_{|w|=1}\frac{w^{2n}\overline{R}(1/w,t)\overline{M}_n^{(2)'}(1/w,t)}{w-z_j^{-1}}\mathrm{d}w. \tag{10.8}$$

To simplify the integral in equation (10.8), we make the change of variables $\zeta = 1/w$ and get

$$\begin{aligned}\oint_{|w|=1}\frac{w^{2n}\overline{R}(1/w,t)\overline{M}_n^{(2)'}(1/w,t)}{w-z_j^{-1}}\mathrm{d}w &= \oint_{|\zeta|=1}\frac{z_j\zeta^{-2n}\overline{R}(\zeta,t)\overline{M}_n^{(2)'}(\zeta,t)}{\zeta(z_j-\zeta)}\mathrm{d}\zeta\\ &= -\oint_{|\zeta|=1}\frac{\zeta^{-2n}\overline{R}(\zeta)\overline{M}_n^{(2)'}(\zeta)}{\zeta-z_j}\mathrm{d}\zeta\\ &\quad+\oint_{|\zeta|=1}\zeta^{-(2n+1)}\overline{R}(\zeta)\overline{M}_n^{(2)'}(\zeta)\mathrm{d}\zeta.\end{aligned} \tag{10.9}$$

As with the AL case the last term in equation (10.9) vanishes by the residue theorem since $\overline{R}(\zeta)$ and $\overline{M}_n^{(2)'}(\zeta)$ are odd and even functions of ζ respectively. On the other hand, from equation (8.65) which, after using $\overline{M}_n^{(2)'}(-\overline{z}_k) = \overline{M}_n^{(2)'}(\overline{z}_k)$, reduces to

$$M_n^{(2)'}(z_j,t) = 2z_j\sum_{k=1}^{\overline{J}}\overline{B}_k(z_k)^{2n}\frac{\overline{M}_n^{(2)'}(1/z_k,t)}{z_j^2-(z_k)^{-2}} + \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{-2n}\overline{R}(w,t)\overline{M}_n^{(2)'}(w,t)}{w-z_j}\mathrm{d}w. \tag{10.10}$$

To this end, contrast equations (10.9) and (10.8) with (10.10) to find the symmetry result

$$\overline{B}_k(t) = -\sigma(z_k)^{-2}B_k(-t). \tag{10.11}$$

Again, we can obtain this symmetry by directly working with the definitions of the norming constants (defined for the right scattering problem) and all necessary symmetries that $\beta_\ell, \overline{\beta}_\ell, \alpha'(z_\ell)$ and $\overline{\alpha}'(\overline{z}_\ell)$ satisfy.

10.3. RST reduction $R_n(t) = \sigma Q_{-n}(-t)$

To do so, start from equation (8.20); use the parity property $\overline{N}_n^{(1)'}(-\overline{z}_k) = \overline{N}_n^{(1)'}(\overline{z}_k)$ and by letting $n \to n+1$ in the resulting equation, we find

$$N_{n+1}^{(1)'}(z_j,t) = 2z_j\sum_{k=1}^{\overline{J}}\overline{C}_k(t)\overline{z}_k^2\overline{z}_k^{2n}\frac{\overline{N}_{n+1}^{(1)'}(\overline{z}_k,t)}{z_j^2-\overline{z}_k^2} + \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{w^{2(n+1)}\overline{\rho}(w,t)\overline{N}_{n+1}^{(1)'}(w,t)}{w-z_j}\mathrm{d}w. \tag{10.12}$$

Now substitute the symmetry relations,

$$N_{m+1}^{(1)'}(z,t) = -\frac{\sigma}{c_{-\infty}^\sigma}\, M_{-m}^{(2)'}(z,-t), \qquad \overline{N}_{m+1}^{(1)'}(z,t) = \frac{1}{c_{-\infty}^\sigma}\, \overline{M}_{-m}^{(2)'}(z,-t), \tag{10.13}$$

(found in (5.14) and (5.16)), into (10.12); make the change of variables $n \to -n, \quad t \to -t$ and find

$$M_n^{(2)'}(z_j, t) = -2\sigma z_j \sum_{k=1}^{\bar{J}} \overline{C}_k(-t)\bar{z}_k^2 \bar{z}_k^{-2n} \frac{\overline{M}_n^{(2)'}(\bar{z}_k, t)}{z_j^2 - \bar{z}_k^2} - \frac{\sigma}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n} w^2 \overline{\rho}(w, -t) \overline{M}_n^{(2)'}(w, t)}{w - z_j} \mathrm{d}w. \tag{10.14}$$

On the other hand, substituting $\overline{M}_n^{(2)'}(-\bar{z}_k) = \overline{M}_n^{(2)'}(\bar{z}_k)$ into equation (8.65) [which was obtained from solving the right scattering problem], we find

$$M_n^{(2)'}(z_j, t) = 2z_j \sum_{k=1}^{\bar{J}} \overline{B}_k(t) \bar{z}_k^{-2n} \frac{\overline{M}_n^{(2)'}(\bar{z}_k, t)}{z_j^2 - \bar{z}_k^2} + \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n} \overline{R}(w, t) \overline{M}_n^{(2)'}(w, t)}{w - z_j} \mathrm{d}w. \tag{10.15}$$

Comparing equations (10.15) to (10.14) and use the relation between the reflection coefficients $\overline{R}(z, t) = -\sigma z^2 \overline{\rho}(z, -t)$ established in equation (9.8) to find the following results

$$-\sigma(\bar{z}_j)^2 \overline{C}_j(-t) = \overline{B}_j(t). \tag{10.16}$$

The derivation of symmetry relation between the norming constants $C_j(t)$ and $B_j(t)$ follows similar step. Start from equation (8.18); make the change of variables $n \to n+1$ and use the fact that $N_n^{(2)'}(z_k)$ is an even function of z_k to find

$$\overline{N}_{n+1}^{(2)'}(\bar{z}_j, t) = 2\bar{z}_j \sum_{k=1}^{J} C_k(t) z_k^{-2} z_k^{-2n} \frac{N_{n+1}^{(2)'}(z_k, t)}{\bar{z}_j^2 - z_k^2} - \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n} w^{-2} \rho(w, t) N_{n+1}^{(2)'}(w, t)}{w - \bar{z}_j} \mathrm{d}w. \tag{10.17}$$

Next, we use the symmetry conditions (5.15) and (5.17), i.e. $N_{m+1}^{(2)'}(z, t) = M_{-m}^{(1)'}(z, -t)$ and $\overline{N}_{m+1}^{(2)'}(z, t) = -\sigma \overline{M}_{-m}^{(1)'}(z, -t)$ in equation (10.17); let $n \to -n, \quad t \to -t$ in the resulting equation to find

$$\overline{M}_n^{(1)'}(\bar{z}_j, t) = -2\sigma \bar{z}_j \sum_{k=1}^{J} z_k^{-2} C_k(-t) z_k^{2n} \frac{M_n^{(1)'}(z_k, t)}{\bar{z}_j^2 - z_k^2} + \frac{\sigma}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2n} w^{-2} \rho(w, -t) M_n^{(1)'}(w, t)}{w - \bar{z}_j} \mathrm{d}w. \tag{10.18}$$

On the other hand, we have from equation (8.63) together with the fact that $M_n^{(1)'}(z_k)$ is an even function of z_k

$$\overline{M}_n^{(1)'}(\bar{z}_j, t) = 2\bar{z}_j \sum_{k=1}^{J} B_k(t) z_k^{2n} \frac{M_n^{(1)'}(z_k, t)}{\bar{z}_j^2 - z_k^2} - \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2n} R(w, t) M_n^{(1)'}(w, t)}{w - \bar{z}_j} \mathrm{d}w. \tag{10.19}$$

Comparing equations (10.19) to (10.18); use the relation between the reflection coefficients $R(z, t) = -\frac{\sigma}{z^2} \rho(z, -t)$ given in equation (9.8) we find

$$-\sigma z_j^{-2} C_j(-t) = B_j(t). \tag{10.20}$$

10.4. PT symmetric reduction $R_n(t) = \sigma Q^*_{-n}(t)$

In this section we derive the symmetry between the norming constants pairs $\{C_j, \overline{C}_j\}$ and $\{B_j, \overline{B}_j\}$ when $R_n = \sigma Q^*_{-n}$. To do so, we start from equation (8.20); use the fact that $\overline{N}_n^{(1)'}(-\overline{z}_k) = \overline{N}_n^{(1)'}(\overline{z}_k)$ along with $n \to n+1$ to get

$$N_{n+1}^{(1)'}(z_j) = 2z_j \sum_{k=1}^{\overline{J}} \overline{C}_k \overline{z}_k^{2(n+1)} \frac{\overline{N}_{n+1}^{(1)'}(\overline{z}_k)}{z_j^2 - \overline{z}_k^2} + \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2(n+1)}\overline{\rho}(w)\overline{N}_{n+1}^{(1)'}(w)}{w - z_j} \mathrm{d}w. \tag{10.21}$$

Now use the symmetry $N_{n+1}^{(1)'}(z) = -\frac{\sigma}{c_{-\infty}^{\sigma}} M_{-n}^{(2)'^*}(z^*), \overline{N}_{n+1}^{(1)'}(z) = \frac{1}{c_{-\infty}} \overline{M}_{-n}^{(2)'^*}(z^*)$, given in (5.14) and (5.16) to find

$$M_n^{(2)'^*}(z_j^*) = -2\sigma z_j \sum_{k=1}^{\overline{J}} \overline{z}_k^2 \overline{C}_k \overline{z}_k^{-2n} \frac{\overline{M}_n^{(2)'^*}(\overline{z}_k^*)}{z_j^2 - \overline{z}_k^2} - \frac{\sigma}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{-2n}w^2\overline{\rho}(w)\overline{M}_n^{(2)'^*}(w^*)}{w - z_j} \mathrm{d}w. \tag{10.22}$$

On the other hand, start from equation (8.65) derived from the right scattering problem apply the parity symmetry $\overline{M}_n^{(2)'}(-\overline{z}_k) = \overline{M}_n^{(2)'}(\overline{z}_k)$ combined with complex conjugation to get

$$M_n^{(2)^{*'}}(z_j) = 2z_j^* \sum_{k=1}^{\overline{J}} \overline{B}_k^* \overline{z}_k^{*-2n} \frac{\overline{M}_n^{(2)^{*'}}(\overline{z}_k)}{z_j^{*2} - \overline{z}_k^{*2}} - \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{*-2n}\overline{R}^*(w)\overline{M}_n^{(2)^{*'}}(w)}{w^* - z_j^*} \mathrm{d}w^*. \tag{10.23}$$

Let $w^* = u$. Then $\mathrm{d}w^* = \mathrm{d}u$ and the orientation on the unit circle is the opposite of what we had before. Substituting the result (9.10), i.e. $\overline{R}(z) = -\sigma z^2 \overline{\rho}^*(z^*)$; make the transformation $z_j \to z_j^*, \overline{z}_k \to \overline{z}_k^*$ to get

$$M_n^{(2)^{*'}}(z_j^*) = 2z_j \sum_{k=1}^{\overline{J}} \overline{B}_k^* \overline{z}_k^{-2n} \frac{\overline{M}_n^{(2)^{*'}}(\overline{z}_k^*)}{z_j^2 - \overline{z}_k^2} - \frac{\sigma}{2\pi \mathrm{i}} \oint_{|u|=1} \frac{u^{-2n}u^2\overline{\rho}(u)\overline{M}_n^{(2)^{*'}}(u^*)}{u - z_j} \mathrm{d}u. \tag{10.24}$$

Comparing equations (10.24) to (10.22) we have

$$-\sigma \overline{z}_j^2 \overline{C}_j = \overline{B}_j^*. \tag{10.25}$$

To find the relation between the norming constants C_j and B_j we start from equation (8.18); use the fact that $N_n^{(2)'}(z_k)$ is an even function of z_k; use the symmetry conditions (5.15) and (5.17), i.e. $N_{n+1}^{(2)'}(z) = M_{-n}^{(1)'^*}(z^*)$, $\overline{N}_{n-1}^{(2)'}(z) = -\sigma \overline{M}_{-n}^{(1)'^*}(z^*)$, to get

$$\overline{M}_n^{(1)'^*}(\overline{z}_j^*) = -2\sigma \overline{z}_j \sum_{k=1}^{J} z_k^{-2} C_k z_k^{2n} \frac{M_n^{(1)'^*}(z_k^*)}{\overline{z}_j^2 - z_k^2} + \frac{\sigma}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{w^{2n}w^{-2}\rho(w)M_n^{(1)'^*}(w^*)}{w - \overline{z}_j} \mathrm{d}w. \tag{10.26}$$

Alternatively, equation (8.63) combined with the parity property of $M_n^{(1)'}(z_k)$ as well as $R(z) = -\sigma z^{-2}\rho^*(z^*)$ to find (after making the transformation $\overline{z}_j \to \overline{z}_j^*$ and $z_j^* \to z_j$)

$$\overline{M}_n^{(1)^{*'}}(\overline{z}_j^*) = 2\overline{z}_j \sum_{k=1}^{J} B_k^* z_k^{2n} \frac{M_n^{(1)^{*'}}(z_k^*)}{\overline{z}_j^2 - z_k^2} + \frac{\sigma}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{u^{2n}u^{-2}\rho(u)M_n^{(1)^{*'}}(u^*)}{u - \overline{z}_j} \mathrm{d}u. \tag{10.27}$$

From equations (10.27) and (10.26) we conclude

$$-\sigma z_j^{-2} C_j = B_j^*. \tag{10.28}$$

10.5.Evolution of norming constants

The time evolution of the eigenfunctions given by equation (2.2) determines the evolution of the scattering data, and hence, the norming constants. Following similar line of derivation detailed in [34] applied to the right and left scattering problems, one find the following time-evolution of the norming constants $C_1(t), \overline{C}_1(t), B_1(t)$ and $\overline{B}_1(t)$ defined in equations (8.7), (8.8), (8.52) and (8.53):

$$\begin{aligned} C_1(\tau) &= C_1(0)\mathrm{e}^{2\mathrm{i}\omega_1\tau}, \quad & \overline{C}_1(\tau) &= \overline{C}_1(0)\mathrm{e}^{-2\mathrm{i}\overline{\omega}_1\tau}, \\ B_1(\tau) &= B_1(0)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}, \quad & \overline{B}_1(\tau) &= \overline{B}_1(0)\mathrm{e}^{2\mathrm{i}\overline{\omega}_1\tau}, \end{aligned} \tag{10.29}$$

where

$$\omega_1 = \frac{1}{2}\left(z_1 - z_1^{-1}\right)^2, \qquad \overline{\omega}_1 = \frac{1}{2}\left(\bar{z}_1 - \bar{z}_1^{-1}\right)^2. \tag{10.30}$$

11. Alternative reconstruction formula for Q_n and R_n

One of the corner stones of the AL theory is the integrable symmetry reduction between the two potentials R_n and Q_n which in turn leads to an integrable equation for one potential: Q_n or R_n. Examples include the AL reduction (1.6), *PT* symmetric reduction (1.7) and the RST, RT reductions, respectively given in (1.9) and (1.10). As such, it is desirable to have a reconstruction formula for the potentials that preserves this kind of reduction symmetries in a *straightforward* way, i.e. one should be able to observe the symmetry on the inverse side simply by looking at the functional form of both potentials. However, this is not the case with the potentials R_n and Q_{n-1} derived in equations (8.35), (8.40), (8.71) and (8.74). To remedy this issue, we derive in this section an alternative formula for the potentials by inserting the symmetries between the eigenfunction in such a way that one can 'see' the integrable symmetry reduction in an obvious way.

11.1. RST reduction $R_n(t) = \sigma Q_{-n}(-t)$

In this section we obtain an alternative reconstruction formula for the potential Q_n under the assumption that $R_n(t) = \sigma Q_{-n}(-t)$. We start from the symmetry relation (5.15), i.e.

$$N_{n+1}^{(2)'}(z, t) = M_{-n}^{(1)'}(z, -t), \tag{11.1}$$

and the potential $Q_{n-1}(t)$ given by equation (8.71) (after making the transformation $n \to n+1$)

$$Q_n(t) = -2\sum_{j=1}^{J} B_j(t) z_j^{2n} M_{n+1}^{\prime(1)}(z_j, t) - \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1} w^{2n} R(w, t) M_{n+1}^{\prime(1)}(w, t)\mathrm{d}w. \tag{11.2}$$

Substituting equation (11.1) into equation (11.2); use the symmetry relation between the reflection coefficients given in equation (9.8), i.e. $R(z, t) = -\frac{\sigma}{z^2}\rho(z, -t)$ as well as $-\sigma z_j^{-2} C_j(-t) = B_j(t)$ obtained in (10.20), we find

$$Q_n(t) = 2\sigma\sum_{j=1}^{J} C_j(-t) z_j^{2(n-1)} N_{-n}^{\prime(2)}(z_j, -t) + \frac{\sigma}{2\pi\mathrm{i}}\oint_{|w|=1} w^{2(n-1)}\rho(w, -t) N_{-n}^{\prime(2)}(w, -t)\mathrm{d}w. \tag{11.3}$$

Scrutinizing the expression for the potential $R_n(t)$ given by equation (8.35) shows that the symmetry condition $R_n(t) = \sigma Q_{-n}(-t)$ is indeed obvious.

11.2. PT symmetric case $R_n = \sigma Q^*_{-n}$

Here, we provide a different expression for the potential Q_n that preserves the *PT* symmetry. Since this symmetry is local in time, we omit the explicit time dependence. We start from the symmetry relation (5.15), i.e. $N^{(2)'}_{n+1}(z) = M^{(1)'^*}_{-n}(z^*)$, and the potential $Q_{n-1}(t)$ given by equation (8.71)

$$Q_{n-1} = -2\sum_{j=1}^{J} B_j z_j^{2(n-1)} M_n'^{(1)}(z_j) - \frac{1}{2\pi \mathrm{i}}\oint_{|w|=1} w^{2(n-1)} R(w) M_n'^{(1)}(w)\mathrm{d}w. \tag{11.4}$$

Substitute the above symmetry between the eigenfunctions, reflection coefficients and norming constants $R(z) = -\frac{\sigma}{z^2}\rho^*(z^*)$, $-\sigma z_j^{-2} C_j = B_j^*$ (respectively obtained in equation (9.10) and (10.28)), back into (11.4), we find

$$Q_n = 2\sigma\sum_{j=1}^{J} C_j^* (z_j^*)^{2(n-1)} N^{(2)'^*}_{-n}(z_j) - \frac{\sigma}{2\pi \mathrm{i}}\oint_{|u|=1} (u^*)^{2(n-1)}\rho^*(u) N^{(2)'^*}_{-n}(u)\mathrm{d}u^*. \tag{11.5}$$

For the convenience of the reader and to make the comparison with R_n easier, we again write down the formula for the potential R_n given in equation (8.35):

$$R_n = 2\sum_{j=1}^{J} C_j z_j^{-2(n+1)} N_n'^{(2)}(z_j) + \frac{1}{2\pi \mathrm{i}}\oint_{|w|=1} w^{-2(n+1)}\rho(w) N_n'^{(2)}(w)\mathrm{d}w. \tag{11.6}$$

As expected, comparing equations (11.5) and (11.6), we see that the *PT* symmetry, $R_n = \sigma Q^*_{-n}$ is indeed preserved.

11.3. RT reduction $R_n(t) = \sigma Q_n(-t)$

To obtain an alternative expression for the potential $Q_n(t)$ we start from equation (8.17) for the function $N_n'^{(1)}(z,t)$ whose large z asymptotics is given by (use the parity property, (8.33))

$$N_n'^{(1)}(z,t) = \frac{1}{z}\left[2\sum_{j=1}^{\bar{J}} \overline{C}_j(t)\bar{z}_j^{2n}\overline{N}_n'^{(1)}(\bar{z}_j,t) - \frac{1}{2\pi \mathrm{i}}\oint_{|w|=1} w^{2n}\bar{\rho}(w,t)\overline{N}_n'^{(1)}(w,t)\mathrm{d}w\right] + O(1/z^2). \tag{11.7}$$

Substituting the symmetry relation $\overline{N}^{(1)'}_n(z,t) = (c_n^-(t))^{-1}\, N^{(2)'}_n(1/z,-t)$ established in (5.16) into equation (11.7) gives

$$\begin{aligned} N_n'^{(1)}(z,t) = {} & \frac{2(c_n^-(t))^{-1}}{z}\sum_{j=1}^{\bar{J}} \overline{C}_j\bar{z}_j^{2n} N^{(2)'}_n(1/\bar{z}_j,-t) \\ & - \frac{(c_n^-(t))^{-1}}{2\pi i z}\oint_{|w|=1} w^{2n}\bar{\rho}(w,t) N^{(2)'}_n(1/w,-t)\mathrm{d}w + O(1/z^2). \end{aligned} \tag{11.8}$$

Comparing (11.8) with (3.22) we find

$$Q_n(t) = -2\sum_{j=1}^{\overline{J}} \overline{C}_j(t)\overline{z}_j^{2n} N_n^{(2)'}(1/\overline{z}_j, -t) + \frac{1}{2\pi \mathrm{i}}\oint_{|w|=1} w^{2n}\overline{\rho}(w,t)N_n^{(2)'}(1/w,-t)\mathrm{d}w. \tag{11.9}$$

We have already established the symmetry relation between the eigenvalues, norming constants and the reflection coefficients:

$$\overline{z}_j = 1/z_j, \quad \overline{C}_j(t) = -\sigma z_j^{-2}C_j(-t), \quad \overline{\rho}(z,t) = \sigma\rho(1/z,-t).$$

Substituting these quantities into equation (11.9) along with the change of variable $\zeta = 1/w$ we find

$$Q_n(t) = 2\sigma\sum_{j=1}^{J} C_j(-t)z_j^{-2(n+1)}N_n'^{(2)}(z_j,-t) + \frac{\sigma}{2\pi\mathrm{i}}\oint_{|\zeta|=1}\zeta^{-2(n+1)}\rho(\zeta,-t)N_n'^{(2)}(\zeta,-t)\mathrm{d}\zeta. \tag{11.10}$$

Scrutinizing the expression for the potential $R_n(t)$ obtained in equation (8.35) shows that the RT symmetry $R_n(t) = \sigma Q_n(-t)$ is indeed preserved.

11.4. AL reduction $R_n(t) = \sigma Q_n^*(t)$

To obtain the alternative expression for the potential Q_n we again start from equation (8.17) with its large z asymptotics equation (11.7). Substituting the symmetry relation $\overline{N}_n^{(1)'}(z,t) = (c_n^{\sigma})^{-1}\, N_n^{(2)'^*}(1/z^*,t)$ established in (5.16) into equation (11.7); compare the result with the asymptotic formula given in (3.22) we find

$$Q_n(t) = -2\sum_{j=1}^{\overline{J}}\overline{C}_j(t)\overline{z}_j^{2n}N_n^{(2)'^*}(1/\overline{z}_j^*,t) + \frac{1}{2\pi\mathrm{i}}\oint_{|w|=1} w^{2n}\overline{\rho}(w,t)N_n^{(2)'^*}(1/w^*,t)\mathrm{d}w. \tag{11.11}$$

Next, insert all symmetries between the eigenvalues, norming constants and the reflection coefficients $\overline{z}_j = 1/z_j^*$, $\overline{C}_j = -\sigma(z_j^*)^{-2}C_j^*$, $\overline{\rho}(z) = \sigma\rho^*(1/z^*)$ back in (11.9); use the change of variable $\zeta = 1/w^*$ in the contour integral (note that the orientation on ζ space is the same as on w) to finally obtain

$$Q_n(t) = 2\sigma\sum_{j=1}^{J} C_j^*(t)(z_j^*)^{-2(n+1)}N_n'^{(2)^*}(z_j,t) - \frac{\sigma}{2\pi\mathrm{i}}\oint_{|\zeta|=1}(\zeta^*)^{-2(n+1)}\rho^*(\zeta,t)N_n'^{(2)^*}(\zeta,t)\mathrm{d}\zeta^*. \tag{11.12}$$

Comparing the above expression for $Q_n(t)$ with the one we have already derived for $R_n(t)$ (see equation (8.35)) shows that the symmetry condition $R_n(t) = \sigma Q_n^*(t)$ is indeed satisfied.

12. Trace formulae and symmetries for $b_j, \overline{b}_j$

For both integrable discrete Ablowitz–Ladik and the reverse-time reduction case (RT NLS) the symmetries connect the scattering data and norming constants in the upper half complex z plane to their corresponding quantities defined in the lower half z plane. For example, the eigenvalues and norming constants are related through equations (5.23) and (5.29), for the RT NLS, and equation (4.3) for the AL lattice. This implies that the eigenvalues z_j and norming constants C_j are counted as free parameters and the values of $\overline{z}_j$ and $\overline{C}_j$ are uniquely determined by the underlying up-down symmetries. As discovered in [41, 43] the situation with the RST and *PT* symmetric cases is very different. Firstly, the symmetries of the scattering data and norming constants do *not* relate their respective values in the upper half complex z plane to those in the lower half plane. This can be seen as follows. In the RST case, see equations (6.27) and (6.28) where now the eigenvalues z_j and $\bar{z}_j$ are counted as free parameters. In the *PT* case see equations (7.21) and (7.22) where again the eigenvalues z_j and $\bar{z}_j$ are counted as free parameters. Secondly, in order to understand the underlying symmetries of the norming constants C_j and $\overline{C}_j$ we need to separate their numerators/denominators, i.e. $C_j, = b_j/a'(z_j), \overline{C}_j = \bar{b}_j/\bar{a}'(z_j)$. For the denominators $a'(z_j), \bar{a}'(z_j)$ we resort to trace formulas for discrete systems (see section 12) see [34] which were used to show that the data $a(z), \bar{a}(z)$ can be calculated in terms of eigenvalues and data $\rho(z), \bar{\rho}(z)$ (or $b(z), \bar{b}(z)$). From $a'(z_j), \bar{a}'(z_j)$ we find that C_j and $\overline{C}_j$ depend on the eigenvalues $z_j, \bar{z}_j$. For the numerator we need to find symmetries involving $b_j, \bar{b}_j$. In the RST case, see equations (12.31) and (12.33) from section 12.2; for the *PT* case see equations (12.38) and (12.42). In both cases the coefficients $b_j, \bar{b}_j$ count as additional parameters. Using these symmetries is critical when finding soliton solutions–e.g. see section 13. It turns out that whenever eigenfunctions in the same plane are related, such as occurs with the PT and RST NLS equations this leads to symmetry conditions on $b_j, \bar{b}_j$.

12.1. Trace formulae and computing a' and $\bar{a}'$

12.1.1. RST reduction $R_n(t) = \sigma Q_{-n}(-t)$. In this section we develop a trace formula for the RST case and use it to compute the norming constants as a function of the eigenvalues. To that purpose, we assume that $a(z)$ and $\bar{a}(z)$ have simple zeros $\{\pm z_j : |z_j| > 1\}_{j=1}^{J}$ and $\{\pm \bar{z}_j : |\bar{z}_j| < 1\}_{j=1}^{J}$ respectively. Define the following quantities:

$$\widetilde{a}(z) = \prod_{j=1}^{J}\left(\frac{z^2-\bar{z}_j^2}{z^2-z_j^2}\right)a(z), \qquad \widetilde{\bar{a}}(z) = \prod_{j=1}^{J}\left(\frac{z^2-z_j^2}{z^2-\bar{z}_j^2}\right)\bar{a}(z). \tag{12.1}$$

According to these definitions and the analytic properties of the scattering data we have: (i) $\widetilde{a}(z)$ is analytic outside the unit circle (where it has no zeros) and $\widetilde{a}(z) \to 1$ as $|z| \to \infty$, (ii) $\widetilde{\bar{a}}(z)$ is analytic inside the unit circle and has no zeros. Taking into account that both $a(z)$ and $\bar{a}(z)$ are even functions of z, we find

$$\log \widetilde{a}(z) = -\frac{1}{2\pi \mathrm{i}}\oint_{|\zeta|=1}\frac{\zeta \log \widetilde{a}(\zeta)}{\zeta^2-z^2}\mathrm{d}\zeta, \qquad \oint_{|\zeta|=1}\frac{\zeta \log \widetilde{\bar{a}}(\zeta)}{\zeta^2-z^2}\mathrm{d}\zeta = 0, \qquad |z|>1, \tag{12.2}$$

$$\log \widetilde{\bar{a}}(z) = \frac{1}{2\pi \mathrm{i}}\oint_{|\zeta|=1}\frac{\zeta \log \widetilde{\bar{a}}(\zeta)}{\zeta^2-z^2}\mathrm{d}\zeta, \qquad \oint_{|\zeta|=1}\frac{\zeta \log \widetilde{a}(\zeta)}{\zeta^2-z^2}\mathrm{d}\zeta = 0, \qquad |z|<1. \tag{12.3}$$

Taking the logarithm of $\widetilde{a}(z)$ and $\widetilde{\bar{a}}(z)$; use their properties outlined in equations (12.2) and (12.3) to find

$$\log a(z) = \sum_{j=1}^{J}\log\left(\frac{z^2-z_j^2}{z^2-\bar{z}_j^2}\right) - \frac{1}{2\pi \mathrm{i}}\oint_{|\zeta|=1}\frac{\zeta \log[\widetilde{a}(\zeta)\widetilde{\bar{a}}(\zeta)]}{\zeta^2-z^2}\mathrm{d}\zeta, \quad |z|>1, \tag{12.4}$$

$$\log \bar{a}(z) = \sum_{j=1}^{J}\log\left(\frac{z^2-\bar{z}_j^2}{z^2-z_j^2}\right) + \frac{1}{2\pi \mathrm{i}}\oint_{|\zeta|=1}\frac{\zeta \log[\widetilde{a}(\zeta)\widetilde{\bar{a}}(\zeta)]}{\zeta^2-z^2}\mathrm{d}\zeta, \quad |z|<1. \tag{12.5}$$

As mentioned earlier, the scattering data satisfy a unitarity condition given in equation (3.11), which in our case, is given by

$$a(z)\overline{a}(z) - b(z)\overline{b}(z) = c_{-\infty}^{\sigma} \equiv \prod_{k=-\infty}^{\infty} (1 - \sigma Q_k(-t)Q_k(t)). \tag{12.6}$$

Equation (12.6) together with the identity $\widetilde{a}(z)\widetilde{\overline{a}}(z) = a(z)\overline{a}(z)$ and the symmetry (9.7) give

$$\log a(z) = \sum_{j=1}^{J} \log\left(\frac{z^2 - z_j^2}{z^2 - \overline{z}_j^2}\right) - \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta \log[c_{-\infty}^{\sigma} + \sigma\zeta^{-2} b(\zeta,t)b(\zeta,-t)]}{\zeta^2 - z^2} \mathrm{d}\zeta, \qquad |z| > 1, \tag{12.7}$$

$$\log \overline{a}(z) = \sum_{j=1}^{J} \log\left(\frac{z^2 - \overline{z}_j^2}{z^2 - z_j^2}\right) + \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta \log[c_{-\infty}^{\sigma} + \sigma\zeta^{-2} b(\zeta,t)b(\zeta,-t)]}{\zeta^2 - z^2} \mathrm{d}\zeta, \qquad |z| < 1. \tag{12.8}$$

Equations (12.7) and (12.8) imply that one can reconstruct the scattering data a and $\overline{a}$ *only* from knowledge of their own simple zeros $z_j, \overline{z}_j$ and *one* function b (recall that $\overline{b}$ is related to b.). While one can develop a general formula for the norming constants, we shall restrict the discussion to reflectionless potentials for which $b = \overline{b} = 0$. The second term in equation (12.7) vanishes since the function $\zeta/(\zeta^2 - z^2)$ is analytic inside the unite circle ($|z| > 1$). Taking the derivative of equation (12.7) with respect to z gives

$$a'(z) = 2z \prod_{j=1}^{J} \left(\frac{z^2 - z_j^2}{z^2 - \overline{z}_j^2}\right) \sum_{j=1}^{J} \left[\frac{1}{z^2 - z_j^2} - \frac{1}{z^2 - \overline{z}_j^2}\right]. \tag{12.9}$$

Following similar steps as above, we find

$$\overline{a}'(z) = 2c_{-\infty}^{\sigma} z \prod_{j=1}^{J} \left(\frac{z^2 - \overline{z}_j^2}{z^2 - z_j^2}\right) \sum_{j=1}^{J} \left[\frac{1}{z^2 - \overline{z}_j^2} - \frac{1}{z^2 - z_j^2}\right]. \tag{12.10}$$

At a single eigenvalue $z = z_1, \overline{z}_1 \in \mathbb{C}$ we have

$$a'(z_1) = \frac{2z_1}{z_1^2 - \overline{z}_1^2}, \qquad \overline{a}'(\overline{z}_1) = \frac{2c_{-\infty}^{\sigma}\overline{z}_1}{\overline{z}_1^2 - z_1^2}. \tag{12.11}$$

12.1.2. PT symmetric reduction $R_n(t) = \sigma Q_n^*(t)$. Next, we repeat the calculations outlines above, but now for the *PT* symmetric case where $R_n = \sigma Q_n^*$. The main difference here, is the existence of more zeros of the scattering data $a(z)$ and $\overline{a}(z)$, i.e. they appear in quartets (see see equations (7.21) and (7.22)). Denote by $\{\pm z_j, \pm z_j^* : |z_j| > 1\}_{j=1}^{J}$ and $\{\pm \overline{z}_j, \pm \overline{z}_j^* : |\overline{z}_j| < 1\}_{j=1}^{J}$, the (simple) zeros of $a(z)$ and $\overline{a}(z)$ respectively. Furthermore, as before, it is assumed that all eigenvalues are *complex* and simple. The case of real eigenvalues is considered later. As before, define

$$\widetilde{a}(z) = \prod_{j=1}^{J} \frac{(z^2 - \overline{z}_j^2)(z^2 - \overline{z}_j^{*2})}{(z^2 - z_j^2)(z^2 - z_j^{*2})} a(z), \quad \widetilde{\overline{a}}(z) = \prod_{j=1}^{J} \frac{(z^2 - z_j^2)(z^2 - z_j^{*2})}{(z^2 - \overline{z}_j^2)(z^2 - \overline{z}_j^{*2})} \overline{a}(z). \tag{12.12}$$

According to these definitions, $\widetilde{a}(z)$ is analytic outside the unit circle, where it has no zeros, while $\widetilde{\overline{a}}(z)$ is analytic inside the unit circle, and it has no zeros. Moreover, $\widetilde{a}(z) \to 1$ as $|z| \to \infty$. Taking into account that both $a(z)$ and $\overline{a}(z)$ are even functions of z, we have

$$\log \widetilde{a}(z) = -\frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta \log \widetilde{a}(\zeta)}{\zeta^2 - z^2} \mathrm{d}\zeta, \qquad \oint_{|\zeta|=1} \frac{\zeta \log \widetilde{\overline{a}}(\zeta)}{\zeta^2 - z^2} \mathrm{d}\zeta = 0, \quad |z| > 1, \tag{12.13}$$

$$\log \widetilde{\overline{a}}(z) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta \log \widetilde{\overline{a}}(\zeta)}{\zeta^2 - z^2} \mathrm{d}\zeta, \qquad \oint_{|\zeta|=1} \frac{\zeta \log \widetilde{a}(\zeta)}{\zeta^2 - z^2} \mathrm{d}\zeta = 0, \quad |z| < 1. \tag{12.14}$$

After some algebra, we find

$$\log a(z) = \sum_{j=1}^{J} \log \left[\frac{(z^2 - z_j^2)(z^2 - z_j^{*2})}{(z^2 - \overline{z}_j^2)(z^2 - \overline{z}_j^{*2})} \right] - \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta \log(\widetilde{a}(\zeta)\widetilde{\overline{a}}(\zeta))}{\zeta^2 - z^2} \mathrm{d}\zeta, \quad |z| > 1, \tag{12.15}$$

$$\log \overline{a}(z) = \sum_{j=1}^{J} \log \left[\frac{(z^2 - \overline{z}_j^2)(z^2 - \overline{z}_j^{*2})}{(z^2 - z_j^2)(z^2 - z_j^{*2})} \right] + \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta \log(\widetilde{a}(\zeta)\widetilde{\overline{a}}(\zeta))}{\zeta^2 - z^2} \mathrm{d}\zeta, \quad |z| < 1. \tag{12.16}$$

From the unitarity relation (3.11) with $R_k = \sigma Q_{-k}^*$ we have

$$a(z)\overline{a}(z) - b(z)\overline{b}(z) = c_{-\infty}^{\sigma} \equiv \prod_{k=-\infty}^{\infty} (1 - \sigma Q_k Q_{-k}^*). \tag{12.17}$$

Using the symmetry between the scattering data $\overline{b}(z) = \frac{\sigma}{z^2} b^*(z^*)$ (see equation (7.27)) the evenness of $a(z), \overline{a}(z)$ and the identity $\widetilde{a}(z)\widetilde{\overline{a}}(z) = a(z)\overline{a}(z)$ we arrive at the general trace formulae

$$\begin{aligned} \log a(z) = & \sum_{j=1}^{J} \log \left[\frac{(z^2 - z_j^2)(z^2 - z_j^{*2})}{(z^2 - \overline{z}_j^2)(z^2 - \overline{z}_j^{*2})} \right] \\ & - \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta \log(c_{-\infty}^{\sigma} + \sigma \zeta^{-2} b(\zeta) b^*(\zeta^*))}{\zeta^2 - z^2} \mathrm{d}\zeta, \qquad |z| > 1, \end{aligned} \tag{12.18}$$

$$\begin{aligned} \log \overline{a}(z) = & \sum_{j=1}^{J} \log \left[\frac{(z^2 - \overline{z}_j^2)(z^2 - \overline{z}_j^{*2})}{(z^2 - z_j^2)(z^2 - z_j^{*2})} \right] \\ & + \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta \log(c_{-\infty}^{\sigma} + \sigma \zeta^{-2} b(\zeta) b^*(\zeta^*))}{\zeta^2 - z^2} \mathrm{d}\zeta, \qquad |z| < 1. \end{aligned} \tag{12.19}$$

For simplicity, we consider the reflectionless potentials case again and find

$$\begin{aligned} a'(z) = 2z & \prod_{j=1}^{J} \frac{(z^2 - z_j^2)(z^2 - z_j^{*2})}{(z^2 - \overline{z}_j^2)(z^2 - \overline{z}_j^{*2})} \\ & \times \sum_{j=1}^{J} \left[\frac{1}{z^2 - z_j^2} + \frac{1}{z^2 - z_j^{*2}} - \frac{1}{z^2 - \overline{z}_j^2} - \frac{1}{z^2 - \overline{z}_j^{*2}} \right]. \end{aligned} \tag{12.20}$$

The computation of $\overline{a}'(z)$ follows analogous lines of derivation. The result is

$$\overline{a}'(z) = 2c^{\sigma}_{-\infty} z \prod_{j=1}^{J} \frac{(z^2 - \overline{z}_j^2)(z^2 - \overline{z}_j^{*2})}{(z^2 - z_j^2)(z^2 - z_j^{*2})}$$
$$\sum_{j=1}^{J} \left[\frac{1}{z^2 - \overline{z}_j^2} + \frac{1}{z^2 - \overline{z}_j^{*2}} - \frac{1}{z^2 - z_j^2} - \frac{1}{z^2 - z_j^{*2}} \right]. \tag{12.21}$$

At a single $(J = 1)$ *complex* eigenvalue $z = z_1, \overline{z}_1$ we find

$$a'(z_1) = \frac{2z_1(z_1^2 - z_1^{*2})}{(z_1^2 - \overline{z}_1^2)(z_1^2 - \overline{z}_1^{*2})}, \tag{12.22}$$

$$a'(z_1^*) = \frac{2z_1^*(z_1^{*2} - z_1^2)}{(z_1^{*2} - \overline{z}_1^2)(z_1^{*2} - \overline{z}_1^{*2})}, \tag{12.23}$$

$$\overline{a}'(\overline{z}_1) = \frac{2c^{\sigma}_{-\infty}\overline{z}_1(\overline{z}_1^2 - \overline{z}_1^{*2})}{(\overline{z}_1^2 - z_1^2)(\overline{z}_1^2 - z_1^{*2})}, \tag{12.24}$$

$$\overline{a}'(\overline{z}_1^*) = \frac{2c^{\sigma}_{-\infty}\overline{z}_1^*(\overline{z}_1^{*2} - \overline{z}_1^2)}{(\overline{z}_1^{*2} - z_1^2)(\overline{z}_1^{*2} - z_1^{*2})}. \tag{12.25}$$

A reconstruction formula for the scattering data $a, \overline{a}$ in terms of their simple zeros when all eigenvalues are *real* and different is nearly identical to what we have done for the RST case with the only exception being in the integral terms appearing in equations (12.7) and (12.8) where $c^{\sigma}_{-\infty} + \sigma\zeta^{-2}b(\zeta, t)b(\zeta, -t)$ is replaced by $c^{\sigma}_{-\infty} + \sigma\zeta^{-2}b(\zeta, t)b^*(\zeta^*, t)$. Since we are interested in the case where the scattering data $b, \overline{b}$ identically vanish (reflectionless potentials), the formula for a' and $\overline{a}'$ given in (12.11) thus still hold whenever the eigenvalues are *real*.

12.2. Computing symmetries of b_j and $\overline{b}_j$

In this section we obtain the symmetries that b_j and $\overline{b}_j$ satisfy at discrete eigenvalues z_j and $\overline{z}_j$. This in turn will be later used to determine the dependence of the norming constants C_j and $\overline{C}_j$ on these eigenvalues. We shall distinguish between two cases: RST and *PT* symmetric. Note that the (time-independent) quantity $c^{\sigma}_{-\infty}$ frequently appears (directly or implicitly) in the eigenfunctions, scattering data, norming constants and potentials. Thus, we first need to find its value. The computation of $c^{\sigma}_{-\infty}$ for the general soliton case (arbitrary J, assuming reflectionless potentials) is cumbersome. Since in this paper, we explicitly compute only one soliton solutions, we set $J = 1$ and assume $\rho = \overline{\rho} = 0$; in the general reconstruction formula for c_n (cf. [34]):

$$c_n = 1 - 2\overline{C}_1\overline{z}_1^{2n-1}\overline{N}_n^{\prime(2)}(\overline{z}_1). \tag{12.26}$$

Solving for $\overline{N}_n^{(2)'}(\overline{z}_1)$ from equations (8.18) and (8.20) to find

$$\overline{N}_n^{(2)'}(\overline{z}_1) = \frac{2\overline{z}_1\left(\overline{z}_1^2 - z_1^2\right)C_1 z_1^{-2n}}{\left(z_1^2 - \overline{z}_1^2\right)^2 + 4\overline{C}_1 C_1 \overline{z}_1^{2n+2} z_1^{-2n}}. \tag{12.27}$$

Substituting equation (12.27) into (12.26); use the fact that $|z_1| > 1$ and $|\bar{z}_1| < 1$ we find $c^{\sigma}_{-\infty} = z_1^2/\bar{z}_1^2$. Note that the dependence of $c^{\sigma}_{-\infty}$ on σ is 'hidden' in the choice of the eigenvalues.

12.2.1. RST reduction $R_n(t) = \sigma Q_{-n}(-t)$. Here, we assume that $R_n(t) = \sigma Q_{-n}(-t)$ holds together with all implied symmetries. At an eigenvalue, equation (8.3) give

$$M^{(2)'}_{-n}(z_j,t) = b_j(t) z_j^{2n} N^{(2)'}_{-n}(z_j,t), \qquad M^{(1)'}_{-n}(z_j,t) = b_j(t) z_j^{2n} N^{(1)'}_{-n}(z_j,t). \quad (12.28)$$

Using the symmetry relation $N^{(1)'}_{m+1}(z_j,t) = -\sigma \left(c^{\sigma}_{-\infty}\right)^{-1} M^{(2)'}_{-m}(z_j,-t)$ between the eigenfunctions (see equation (6.19)) together with the left part of equation (12.28) results in

$$N^{(1)'}_{n+1}(z_j,t) = -\sigma \left(c^{\sigma}_{-\infty}\right)^{-1} b_j(-t) z_j^{2n} N^{(2)'}_{-n}(z_j,-t). \quad (12.29)$$

On the other hand, the eigenfunctions $N^{(2)'}_{n}(z_j,t)$ and $M^{(1)'}_{n}(z_j,t)$ are connected via the symmetry relation (6.20), i.e. $N^{(2)'}_{n+1}(z,t) = M^{(1)'}_{-n}(z,-t)$. This symmetry, when combined with the right equation from (12.28), gives rise to

$$N^{(2)'}_{-n}(z_j,-t) = b_j(t) z_j^{-2} z_j^{-2n} N^{(1)'}_{n+1}(z_j,t). \quad (12.30)$$

Substituting equation (12.30) into (12.29) one obtains the symmetry recall that $c^{\sigma}_{-\infty}$ is time independent)

$$b_j(-t)\, b_j(t) = -\sigma c^{\sigma}_{-\infty} z_j^2, \qquad \Longrightarrow \qquad b_j^2(0) = -\sigma c^{\sigma}_{-\infty} z_j^2, \quad (12.31)$$

where we used the time evolution of $b(t)$ given by equation (8.23). For a one soliton solution we find

$$b_1^2(0) = -\sigma z_1^4/\bar{z}_1^2, \quad \sigma = \mp 1 \quad \Longrightarrow \quad b_1(0) = \frac{s z_1^2}{\bar{z}_1} \mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}, \quad s = \pm 1. \quad (12.32)$$

To determine the value of $\bar{b}_j(t)$, one follows similar steps: start from equation (8.4); apply the symmetries (6.21) and (6.22) and end up with

$$\bar{b}_j(-t)\bar{b}_j(t) = -\sigma c^{\sigma}_{-\infty} (\bar{z}_j)^{-2} \qquad \Longrightarrow \qquad \bar{b}_j^2(0) = -\sigma c^{\sigma}_{-\infty} (\bar{z}_j)^{-2}. \quad (12.33)$$

In obtaining the expression for $\bar{b}_j(0)$ we made use of the evolution of the scattering data $\bar{b}_j(t)$ given by equation (8.25). For the one soliton case we find

$$\bar{b}_1(0) = \frac{s z_1}{\bar{z}_1^2} \mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}, \quad z_1, \bar{z}_1 \in \mathbb{C}, \quad \sigma = \mp 1, \; s = \pm 1. \quad (12.34)$$

With the help of equation (12.11) we are now ready to compute the norming constants C_1 and $\bar{C}_1$. We thus have ($\sigma = \mp 1$ and $s = \pm 1$)

$$C_1(0) = \frac{s z_1 (z_1^2 - \bar{z}_1^2)}{2\bar{z}_1} \mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}, \quad \bar{C}_1(0) = \frac{s(\bar{z}_1^2 - z_1^2)}{2 z_1 \bar{z}_1} \mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}, \; z_1, \bar{z}_1 \in \mathbb{C}. \quad (12.35)$$

12.2.2. PT symmetric reduction $R_n = \sigma Q^*_{-n}$. Here, we obtain the symmetries that $b_j, \bar{b}_j$ satisfy at discrete eigenvalues z_j and $\bar{z}_j$ for the *PT* symmetric case. Since this symmetry is local in time, we shall not explicitly indicate any time dependence of the eigenfunctions and scattering data. Our starting point is again equation (12.28). Substituting the symmetry condition

$N^{(1)'}_{m+1}(z_j) = -\sigma\left(c^{\sigma}_{-\infty}\right)^{-1} M^{(2)'^*}_{-m}(z^*_j)$, given by equation (7.12), into the left part of equation (12.28) one finds

$$N^{(1)'^*}_{n+1}(z^*_j) = -\sigma\left(c^{\sigma^*}_{-\infty}\right)^{-1} b_j z_j^{2n} N^{(2)'}_{-n}(z_j), \tag{12.36}$$

where $c^{\sigma^*}_{-\infty} \equiv (c^{\sigma}_{-\infty})^*$. Next, we use the symmetry (7.13), i.e. $N^{(2)'}_{n+1}(z_j) = M^{(1)'^*}_{-n}(z^*_j)$ to rewrite the right part of equation (12.28) in the form

$$N^{(2)'^*}_{n+1}(z^*_j) = b_j z_j^{2n} N^{(1)'}_{-n}(z_j), \quad \Longrightarrow \quad N^{(2)'}_{-n}(z_j) = z_j^{-2} b^*_j z_j^{-2n} N^{(1)'^*}_{n+1}(z^*_j). \tag{12.37}$$

Substituting equation (12.37) back into (12.36) gives

$$|b_j|^2 = -\sigma z_j^2 (c^{\sigma}_{-\infty})^*. \tag{12.38}$$

Since for the *PT* symmetric case the eigenvalues z_j and $\bar{z}_j$ are *not* related through any symmetry we find, for a one soliton solution (recall $c^{\sigma}_{-\infty} = z_1^2/\bar{z}_1^2$)

$$|b_1|^2 = -\frac{\sigma|z_1|^4}{(\bar{z}^*_1)^2} \quad \Longrightarrow \quad b_1 = \frac{z_1^2}{\bar{z}_1}\mathrm{e}^{\mathrm{i}\theta_1}, \quad \sigma = -1, \quad z_1, \bar{z}_1 \in \mathbb{R}, \tag{12.39}$$

with arbitrary and real constant θ_1. Similar expression can be obtained for $\bar{b}_j$. Indeed, starting from equation (8.4) we have

$$\overline{M}^{(1)'}_{-n}(\bar{z}_\ell) = \bar{b}_\ell(\bar{z}_\ell)^{-2n}\overline{N}^{(1)'}_{-n}(\bar{z}_\ell), \qquad \overline{M}^{(2)'}_{-n}(\bar{z}_\ell) = \bar{b}_\ell(\bar{z}_\ell)^{-2n}\overline{N}^{(2)'}_{-n}(\bar{z}_\ell). \tag{12.40}$$

Now use the symmetries $\overline{M}^{(2)'}_{-n}(\bar{z}_j) = c^{\sigma^*}_{-\infty}\overline{N}^{(1)'^*}_{n+1}(\bar{z}^*_j)$ and $\overline{M}^{(1)'}_{-n}(\bar{z}_j) = -\sigma\overline{N}^{(2)'^*}_{n+1}(\bar{z}^*_j)$ respectively obtained in (7.14) and (7.15) to rewrite equation (12.40) in the form

$$\overline{N}^{(2)'^*}_{n+1}(\bar{z}^*_\ell) = -\sigma\bar{b}_\ell(\bar{z}_\ell)^{-2n}\overline{N}^{(1)'}_{-n}(\bar{z}_\ell),\ \overline{N}^{(1)'}_{-n}(\bar{z}_\ell) = (c^{\sigma}_{-\infty})^{-1}(\bar{z}_\ell)^2\bar{b}^*_\ell(\bar{z}_\ell)^{2n}\overline{N}^{(2)'^*}_{n+1}(\bar{z}^*_\ell). \tag{12.41}$$

In order for the system of equations in (12.41) to be consistent one finds

$$|\bar{b}_\ell|^2 = -\sigma(\bar{z}_\ell)^{-2}c^{\sigma}_{-\infty}. \tag{12.42}$$

For a one soliton solution with $c^{\sigma}_{-\infty} = z_1^2/\bar{z}_1^2$ (with real $z_1, \bar{z}_1$) we find that solution exists only when $\sigma = -1$, in which case we have

$$\bar{b}_1 = \frac{z_1}{\bar{z}_1^2}\mathrm{e}^{\mathrm{i}\bar{\theta}_1}, \quad z_1, \bar{z}_1 \in \mathbb{R}, \tag{12.43}$$

with arbitrary and real constant $\bar{\theta}_1$. To write down a closed form expression for the norming constants C_1 and $\overline{C}_1$, we use the above formulae, the definition of norming constants equations (4.5) and (12.11) with $\sigma = -1$:

$$C_1(0) = \frac{z_1(z_1^2 - \bar{z}_1^2)\mathrm{e}^{\mathrm{i}\theta_1}}{2\bar{z}_1}, \qquad \overline{C}_1(0) = \frac{(\bar{z}_1^2 - z_1^2)\mathrm{e}^{\mathrm{i}\bar{\theta}_1}}{2z_1\bar{z}_1}, \ z_1, \bar{z}_1 \in \mathbb{R}. \tag{12.44}$$

13. One soliton solution

In this section we compute the one soliton solution for the AL, RT, RST, *PT* NLS equations given in (1.3), (1.12), (1.11), (1.8) respectively, as well as the AL model (1.3). They all correspond to the case where $J = 1$ with eigenvalues $z_1, \bar{z}_1$ ($|z_1| > 1, |\bar{z}_1| < 1$) being (in general) complex and different (depending on the symmetries at hand). Moreover we will assume that the potentials are reflectionless i.e. $\rho = 0, \overline{\rho} = 0,\ R = 0, \overline{R} = 0$.

In order to write down a closed form formula for the soliton we need: (i) all symmetries (which we already obtained) and (ii) a reconstruction formula for the potential R_n and/or Q_n which can be chosen either from equations (8.35), (8.40), (8.71) or (8.74). We shall provide those eigenfunctions that are necessary for the construction of the potentials.

13.1. Computing eigenfunctions and potentials: left scattering problem

Starting from equation (8.20) we find (after some algebra) that the relevant eigenfunctions $\overline{N}_n^{(1)'}(\bar{z}_1)$ and $N_n^{(2)'}(z_1)$ are given by

$$\overline{N}_n^{(1)'}(\bar{z}_1) = \frac{\left(z_1^2 - \bar{z}_1^2\right)^2}{\left(z_1^2 - \bar{z}_1^2\right)^2 + 4z_1^2\overline{C}_1 C_1 \bar{z}_1^{2n} z_1^{-2n}}, \tag{13.1}$$

$$N_n^{(2)'}(z_1) = \frac{\left(z_1^2 - \bar{z}_1^2\right)^2}{\left(z_1^2 - \bar{z}_1^2\right)^2 + 4\overline{C}_1 C_1 \bar{z}_1^{2n+2} z_1^{-2n}}. \tag{13.2}$$

From equations (8.35) and (8.40) with $\overline{J} = 1, \overline{\rho} = 0$, the potentials $R_n(\tau)$ and $Q_n(\tau)$ are given by

$$R_n(\tau) = 2C_1(\tau) z_1^{-2(n+1)} N_n'^{(2)}(z_1) \implies R_n(\tau) = \frac{2C_1(\tau) z_1^{-2(n+1)} \left(z_1^2 - \bar{z}_1^2\right)^2}{\left(z_1^2 - \bar{z}_1^2\right)^2 + 4\overline{C}_1(\tau) C_1(\tau) \bar{z}_1^{2n+2} z_1^{-2n}}. \tag{13.3}$$

$$Q_{n-1}(\tau) = -2\overline{C}_1(\tau)\bar{z}_1^{2(n-1)}\overline{N}_n'^{(1)}(\bar{z}_1) \implies Q_n(\tau) = -\frac{2\overline{C}_1(\tau)\bar{z}_1^{2n}\left(z_1^2 - \bar{z}_1^2\right)^2}{\left(z_1^2 - \bar{z}_1^2\right)^2 + 4\overline{C}_1(\tau) C_1(\tau) \bar{z}_1^{2n+2} z_1^{-2n}} \tag{13.4}$$

where

$$C_1(\tau) = C_1(0)\mathrm{e}^{2\mathrm{i}\omega_1 t},\ \overline{C}_1(\tau) = \overline{C}_1(0)\mathrm{e}^{-2\mathrm{i}\overline{\omega}_1 t},\ \omega_1 = \frac{1}{2}\left(z_1 - z_1^{-1}\right)^2,\ \overline{\omega}_1 = \frac{1}{2}\left(\bar{z}_1 - \bar{z}_1^{-1}\right)^2.$$

We use these formulae below.

13.2. Ablowitz–Ladik solitons

The classical integrable AL model is characterized by soliton eigenvalues and norming constants obeying the relation given in (4.3), i.e. $\bar{z}_1 = 1/z_1^*$ and $\overline{C}_1(\tau) = -\sigma(z_1^*)^{-2}C_1^*(\tau) = -\sigma(z_1^*)^{-2}C_1^*(0)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}$ with with $z_1, C_1(0)$ being arbitrary complex parameters. Substituting these parameters in (13.3) gives the well-known AL NLS one-soliton solution

$$R_n(\tau) = \frac{2\left(|z_1|^4 - 1\right)^2 C_1(0)\mathrm{e}^{2\mathrm{i}\omega_1\tau} z_1^{-2(n+1)}}{\left(|z_1|^4 - 1\right)^2 - 4\sigma|C_1(0)|^2|z_1|^{-4n}}. \tag{13.5}$$

To compute the potential $Q_n(\tau)$, we substitute the eigenvalues and norming constants in equation (13.4) to find

$$Q_n(\tau) = \frac{2\sigma\left(|z_1|^4 - 1\right)^2 C_1^*(0)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}(z_1^*)^{-2(n+1)}}{(|z_1|^4 - 1)^2 - 4\sigma|C_1(0)|^2|z_1|^{-4n}}. \tag{13.6}$$

Clearly, the symmetry $R_n(\tau) = \sigma Q_n^*(\tau)$ is preserved. To avoid singularity, one chooses $\sigma = -1$. To put this soliton in a more 'familiar' form, we let $z_1 = \exp(\xi + \mathrm{i}\eta)$, with $\xi > 0$ (since $|z_1| > 1$) and constant real η. Furthermore, we represent the norming constant $C_1(0)$ in the form $C_1(0) = \frac{1}{2}(|z_1|^4 - 1)\exp(\chi_1 + \mathrm{i}\chi_2)$. With this we have

$$Q_n(\tau) = -\sinh(2\xi)\mathrm{e}^{-\mathrm{i}\chi_2}\mathrm{e}^{2\mathrm{i}\eta(n+1)}\mathrm{sech}(2\xi n - \chi_1)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}. \tag{13.7}$$

This one-soliton solution contains four free real parameters: ξ, η, χ_1 and χ_2.

13.3. RT NLS solitons

From (5.23), (5.29) and (8.26) we have $\bar{z}_1 = 1/z_1$ with $z_1 \in \mathbb{C}$ and $\overline{C}_1(\tau) = -\sigma z_1^{-2}C_1(-\tau) = -\sigma z_1^{-2}C_1(0)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}$. To this end, the RT NLS one-soliton solution is given by

$$R_n(\tau) = \frac{2C_1(0)\mathrm{e}^{2\mathrm{i}\omega_1\tau}z_1^{-2(n+1)}\left(z_1^2 - z_1^{-2}\right)^2}{\left(z_1^2 - z_1^{-2}\right)^2 - 4\sigma C_1^2(0)z_1^{-4(n+1)}}. \tag{13.8}$$

This is a four parameter family solution for which z_1 and $C_1(0)$ are arbitrary *complex* constants. The potential $Q_n(\tau)$ can be found from equation (13.4):

$$Q_n(\tau) = \frac{2\sigma C_1(0)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}z_1^{-2(n+1)}\left(z_1^2 - z_1^{-2}\right)^2}{\left(z_1^2 - z_1^{-2}\right)^2 - 4\sigma C_1^2(0)z_1^{-4(n+1)}}. \tag{13.9}$$

Clearly, the symmetry $R_n(\tau) = \sigma Q_n(-\tau)$ is preserved. Define the complex parameter $\tilde{C}_1$ by $4C_1^2(0)z_1^{-4} \equiv (z_1^2 - z_1^{-2})^2\tilde{C}_1^2$. Then equation (13.9) is rewritten as

$$Q_n(\tau) = \frac{2\sigma C_1(0)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}z_1^{-2(n+1)}}{1 - \sigma\tilde{C}_1^2 z_1^{-4n}}. \tag{13.10}$$

Notice that this soliton can be singular, even at time zero. Indeed, consider an arbitrary point on the integers $n = n_0$. Then singularity occurs whenever the initial condition satisfies

$$\tilde{C}_1^2 = \sigma z_1^{4n_0}. \tag{13.11}$$

Thus, we exclude this from the initial data. As we stated below equation (3.14), the AL scattering theory is mathematically well grounded for potentials that satisfy the conditions $\|Q\|_1 = \sum_{-\infty}^{\infty}|Q_n| < \infty$ and $\|R\|_1 = \sum_{-\infty}^{\infty}|R_n| < \infty$. As such, choosing initial conditions satisfying equation (13.11) would *not* be consistent with the analysis presented in this paper.

We next examine the time evolution of the soliton. Recall that $\omega_1 = \frac{1}{2}\left(z_1 - z_1^{-1}\right)^2$. With the definition $z_1 = \mathrm{e}^{\xi+\mathrm{i}\eta}$ we get $\omega_1 = \sinh(\xi + \mathrm{i}\eta) = \sinh\xi\cos\eta + i\cosh\xi\sin\eta$. Thus, as far as the time-evolution is concerned, we have

$$Q_n(\tau) \sim \mathrm{e}^{-2\mathrm{i}\omega_1\tau} = \mathrm{e}^{2\mathrm{i}[1-\cosh(2\xi)\cos(2\eta)]\tau}\,\mathrm{e}^{2\sinh(2\xi)\sin(2\eta)\tau}. \tag{13.12}$$

The first exponent on the right hand side of equation (13.12) is bounded in time, whereas the other one, $\exp[2\tau\sinh 2\xi\sin 2\eta]$, either grows or decays in time.

13.4. RST NLS solitons

Recall in this case that the eigenvalues $z_1, \bar{z}_1$ are (in general) complex and counted as free parameters. Substituting the norming constants (and their respecting time-dependent) given by equation (12.35) and (10.29) into (13.3) to find

$$R_n(\tau) = \frac{s(z_1\bar{z}_1)^{-1}(z_1^2 - \bar{z}_1^2)\mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}\mathrm{e}^{2\mathrm{i}\omega_1\tau}z_1^{-2n}}{1 - \mathrm{e}^{\mathrm{i}(1+\sigma)\pi/2}\mathrm{e}^{2\mathrm{i}(\omega_1-\overline{\omega}_1)\tau}\bar{z}_1^{2n}z_1^{-2n}}. \tag{13.13}$$

On the other hand, from equation (13.4) we obtain the formula for the potential $Q_n(\tau)$ given by

$$Q_n(\tau) = -\frac{s(z_1\bar{z}_1)^{-1}(z_1^2 - \bar{z}_1^2)\mathrm{e}^{-\mathrm{i}(1+\sigma)\pi/4}\mathrm{e}^{-2\mathrm{i}\omega_1\tau}z_1^{2n}}{1 - \mathrm{e}^{-\mathrm{i}(1+\sigma)\pi/2}\mathrm{e}^{-2\mathrm{i}(\omega_1-\overline{\omega}_1)\tau}\bar{z}_1^{-2n}z_1^{2n}}. \tag{13.14}$$

Note that when $\sigma = \mp 1$ we have the two identities: $\mathrm{e}^{-\mathrm{i}(1+\sigma)\pi/2} = \mathrm{e}^{\mathrm{i}(1+\sigma)\pi/2}$ and $\mathrm{e}^{-\mathrm{i}(1+\sigma)\pi/4} = -\sigma\mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}$. With this at hand, equation (13.14) is rewritten as

$$Q_n(\tau) = \frac{\sigma s(z_1\bar{z}_1)^{-1}(z_1^2 - \bar{z}_1^2)\mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}\mathrm{e}^{-2\mathrm{i}\omega_1\tau}z_1^{2n}}{1 - \mathrm{e}^{\mathrm{i}(1+\sigma)\pi/2}\mathrm{e}^{-2\mathrm{i}(\omega_1-\overline{\omega}_1)\tau}\bar{z}_1^{-2n}z_1^{2n}}. \tag{13.15}$$

Since $z_1, \bar{z}_1$ are free complex constants defined outside/inside the unit circle $|z| = 1$ respectively, this is a four parameter family of solutions. And as expected, the two potentials $R_n(\tau)$ and $Q_n(\tau)$ do satisfy the RST symmetry $R_n(\tau) = \sigma Q_{-n}(-\tau)$. It is evident from equation (13.15) that $Q_n^*(\tau) \neq Q_n(-\tau)$; thus clearly demonstrating the non commutativity between time reversal and complex conjugation. Finally, we remark that the RST soliton can develop a singularity in finite time. This can happen when the eigenvalues z_1 and $\bar{z}_1$ are real (the general complex case does not develop singularity due to the dependence of ω_1 and $\overline{w}_1$ on the eigenvalues). As an example, at a grid point $n = 0$, the denominator in (13.15) vanish when

$$\tau_s = \frac{(1+\sigma)\pi - 4\pi}{4(\omega_1 - \overline{\omega}_1)}, \quad z_1, \bar{z}_1 \in \mathbb{R}. \tag{13.16}$$

13.5. PTNLS solitons

The *PT* symmetric one soliton is characterized by two arbitrary *real* eigenvalues $z_1, \bar{z}_1$ with norming constants (and their time evolution) given by equations (12.44) and (10.29) i.e. $C_1(\tau) = \frac{z_1(z_1^2-\bar{z}_1^2)\mathrm{e}^{\mathrm{i}\theta_1}}{2\bar{z}_1}\mathrm{e}^{2\mathrm{i}\omega_1\tau}$, $\overline{C}_1(\tau) = \frac{(\bar{z}_1^2-z_1^2)\mathrm{e}^{\mathrm{i}\overline{\theta}_1}}{2z_1\bar{z}_1}\mathrm{e}^{-2\mathrm{i}\overline{\omega}_1\tau}$ and $\omega_1 = \left(z_1 - z_1^{-1}\right)^2/2$; $\overline{\omega}_1 = \left(\bar{z}_1 - \bar{z}_1^{-1}\right)^2/2$. With this at hand, the *PT* NLS one-soliton solution is given by ($\sigma = -1$)

$$R_n(\tau) = \frac{(z_1\bar{z}_1)^{-1}(z_1^2 - \bar{z}_1^2)\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{2\mathrm{i}\omega_1\tau}z_1^{-2n}}{1 - \mathrm{e}^{\mathrm{i}(\theta_1+\overline{\theta}_1)}\mathrm{e}^{2\mathrm{i}(\omega_1-\overline{\omega}_1)\tau}\bar{z}_1^{2n}z_1^{-2n}}. \tag{13.17}$$

Next, we compute $Q_n(\tau)$. After some algebra, we find

$$Q_n(\tau) = -\frac{(z_1\bar{z}_1)^{-1}(z_1^2 - \bar{z}_1^2)\mathrm{e}^{-\mathrm{i}\theta_1}\mathrm{e}^{-2\mathrm{i}\omega_1\tau}z_1^{2n}}{1 - \mathrm{e}^{-\mathrm{i}(\theta_1+\overline{\theta}_1)}\mathrm{e}^{-2\mathrm{i}(\omega_1-\overline{\omega}_1)\tau}\bar{z}_1^{-2n}z_1^{2n}}. \tag{13.18}$$

Clearly, the *PT* symmetry $R_n(\tau) = -Q_{-n}^*$ is preserved. Since $z_1, \bar{z}_1, \theta_1, \overline{\theta}_1$ are all free real constants, the soliton constitute a four parameter family of solutions. Equations (13.18) or (13.17) reproduces the one-soliton result first reported in [40] under the transformation $\theta_1 \to \theta_1 + \pi$.

The soliton given in (13.18) also develops a singularity in finite time. Indeed, when $n = 0$ (as an example), we find the blow-up time to be

$$\tau_s = \frac{2\pi - (\theta_1 + \overline{\theta}_1)}{2(\omega_1 - \overline{\omega}_1)}. \tag{13.19}$$

13.6. Computing eigenfunctions and potentials: right scattering problem

As was mentioned in section 6, all one needs to compute a soliton solution for an integrable system (that originates from the AKNS scattering problem) are symmetries between the eigenfunctions, scattering data and a reconstruction formula for R_n or Q_n. However, due to nonlocality (in space), we extended the analysis by studying a 'right' scattering problem and connected it with the left one using proper symmetry relations. For completeness of presentation, we use the results from the right scattering problem to compute a one soliton solution and show that they coincide with the one we obtained in section 13.1. Solving equations (8.63) and (8.65) for the eigenfunctions gives

$$M_n^{(1)'}(z_1) = \frac{\left(z_1^2 - \overline{z}_1^2\right)^2}{\left(z_1^2 - \overline{z}_1^2\right)^2 + 4\overline{B}_1 B_1 \overline{z}_1^{2-2n} z_1^{2n}}, \tag{13.20}$$

$$\overline{M}_n^{(2)'}(\overline{z}_1) = \frac{c_{-\infty}\left(z_1^2 - \overline{z}_1^2\right)^2}{\left(z_1^2 - \overline{z}_1^2\right)^2 + 4B_1\overline{B}_1 z_1^{2n+2}\overline{z}_1^{-2n}}. \tag{13.21}$$

The potentials are recovered from equations (8.71) and (8.74):

$$R_n = 2c_{-\infty}^{-1}\overline{B}_1\overline{z}_1^{-2(n+1)}\overline{M}_n^{\prime(2)}(\overline{z}_1) \quad \Longrightarrow \quad R_n = \frac{2\overline{B}_1\overline{z}_1^{-2(n+1)}\left(z_1^2 - \overline{z}_1^2\right)^2}{\left(z_1^2 - \overline{z}_1^2\right)^2 + 4B_1\overline{B}_1 z_1^{2n+2}\overline{z}_1^{-2n}}, \tag{13.22}$$

$$Q_{n-1} = -2B_1 z_1^{2(n-1)} M_n^{\prime(1)}(z_1) \quad \Longrightarrow \quad Q_n = -\frac{2B_1 z_1^{2n}\left(z_1^2 - \overline{z}_1^2\right)^2}{\left(z_1^2 - \overline{z}_1^2\right)^2 + 4\overline{B}_1 B_1 \overline{z}_1^{-2n} z_1^{2n+2}}. \tag{13.23}$$

With this at hand, we can write down the one-soliton solution for the RST and *PT* symmetric NLS equations.

13.6.1. RST NLS soliton. In this case, we have two *complex* eigenvalues $z_1, \overline{z}_1$ that are *not* related, i.e. they are counted as free parameters and $\sigma = \mp 1$. From equations (10.16), (10.20), (10.29) and (10.30) we find

$$\overline{B}_1(\tau) = -\sigma(\overline{z}_1)^2\overline{C}_1(-\tau) = -\sigma(\overline{z}_1)^2\overline{C}_1(0)\mathrm{e}^{2\mathrm{i}\overline{\omega}_1\tau}, \tag{13.24}$$

$$B_1(\tau) = -\sigma z_1^{-2} C_1(-\tau) = -\sigma z_1^{-2} C_1(0)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}, \tag{13.25}$$

where $\omega_1 = \frac{1}{2}\left(z_1 - z_1^{-1}\right)^2, \overline{\omega}_1 = \frac{1}{2}\left(\overline{z}_1 - \overline{z}_1^{-1}\right)^2$. Thus we have

$$R_n(\tau) = -\frac{2\sigma\overline{C}_1(0)\mathrm{e}^{2\mathrm{i}\overline{\omega}_1\tau}\left(\overline{z}_1\right)^{-2n}}{1 + \frac{4\overline{z}_1^2 C_1(0)\overline{C}_1(0)}{\left(z_1^2 - \overline{z}_1^2\right)^2}\exp\left[2\mathrm{i}(\overline{\omega}_1 - \omega_1)\tau\right] z_1^{2n}\overline{z}_1^{-2n}}. \tag{13.26}$$

Note that from (12.35) $\frac{4\overline{z}_1^2 C_1(0)\overline{C}_1(0)}{(z_1^2-\overline{z}_1^2)^2} = -\mathrm{e}^{\mathrm{i}(1+\sigma)\pi/2}$. Substituting the norming constants (12.35) back into (13.26) results in the one-soliton solution for the RST NLS equation (1.11)

$$R_n(\tau) = \frac{s\sigma(z_1\overline{z}_1)^{-1}(z_1^2-\overline{z}_1^2)\mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}\mathrm{e}^{2\mathrm{i}\overline{\omega}_1\tau}\,(\overline{z}_1)^{-2n}}{1-\mathrm{e}^{\mathrm{i}(1+\sigma)\pi/2}\exp\left[2\mathrm{i}(\overline{\omega}_1-\omega_1)\tau\right]z_1^{2n}\overline{z}_1^{-2n}}, \tag{13.27}$$

where $\sigma = \mp 1$ and $s = \pm 1$. After some algebra, one can put equation (13.27) in a form that precisely matches the one-soliton solution given in (13.13), showing a consistency between the left and right scattering problems. Next we compute the potential $Q_n(\tau)$. From (13.23) we find

$$Q_n(\tau) = \frac{s(z_1\overline{z}_1)^{-1}(z_1^2-\overline{z}_1^2)\mathrm{e}^{\mathrm{i}(1+\sigma)\pi/4}\mathrm{e}^{-2\mathrm{i}\overline{\omega}_1\tau}\,(\overline{z}_1)^{2n}}{1-\mathrm{e}^{\mathrm{i}(1+\sigma)\pi/2}\exp\left[-2\mathrm{i}(\overline{\omega}_1-\omega_1)\tau\right]z_1^{-2n}\overline{z}_1^{2n}}. \tag{13.28}$$

Comparing the two potentials from (13.27) and (13.28) clearly shows that the integrable symmetry $R_n(t) = \sigma Q_{-n}(-t)$ is indeed satisfied.

13.6.2. PT NLS soliton. In this case, we have two *real* eigenvalues $z_1, \overline{z}_1$ that are *not* related with $\sigma = -1$. From equations (10.25), (10.28), (10.29) and (10.30) we find

$$\overline{B}_1(\tau) = -\sigma(\overline{z}_1)^2\overline{C}_1^*(\tau) = -\sigma(\overline{z}_1)^2\overline{C}_1^*(0)\mathrm{e}^{2\mathrm{i}\overline{\omega}_1\tau}, \tag{13.29}$$

$$B_1(\tau) = -\sigma z_1^{-2}C_1^*(\tau) = -\sigma z_1^{-2}C_1^*(0)\mathrm{e}^{-2\mathrm{i}\omega_1\tau}, \tag{13.30}$$

where $\omega_1 = \frac{1}{2}\left(z_1 - z_1^{-1}\right)^2, \overline{\omega}_1 = \frac{1}{2}\left(\overline{z}_1 - \overline{z}_1^{-1}\right)^2$, giving rise to

$$R_n(\tau) = -\frac{2\sigma\overline{C}_1^*(0)\mathrm{e}^{2\mathrm{i}\overline{\omega}_1\tau}\,(\overline{z}_1)^{-2n}}{1+\frac{4\overline{z}_1^2C_1^*(0)\overline{C}_1^*(0)}{\left(z_1^2-\overline{z}_1^2\right)^2}\exp\left[2\mathrm{i}(\overline{\omega}_1-\omega_1)\tau\right]z_1^{2n}\overline{z}_1^{-2n}}. \tag{13.31}$$

Note that from (12.35), $\frac{4\overline{z}_1^2C_1^*(0)\overline{C}_1^*(0)}{(z_1^2-\overline{z}_1^2)^2} = -\mathrm{e}^{-\mathrm{i}(\theta+\overline{\theta}_1)}$. Substituting the norming constants (12.44) into (13.31) results in the one-soliton solution for the *PT* symmetric NLS equation (1.8)

$$R_n(\tau) = -\frac{(z_1\overline{z}_1)^{-1}(z_1^2-\overline{z}_1^2)\mathrm{e}^{-\mathrm{i}\theta_1}\mathrm{e}^{2\mathrm{i}\overline{\omega}_1\tau}\,(\overline{z}_1)^{-2n}}{1-\mathrm{e}^{-\mathrm{i}(\theta+\overline{\theta}_1)}\exp\left[2\mathrm{i}(\overline{\omega}_1-\omega_1)\tau\right]z_1^{2n}\overline{z}_1^{-2n}}. \tag{13.32}$$

To this end, the potential $Q_n(\tau)$ is found from (13.23) as

$$Q_n = \frac{(z_1\overline{z}_1)^{-1}(z_1^2-\overline{z}_1^2)\mathrm{e}^{\mathrm{i}\overline{\theta}_1}\mathrm{e}^{-2\mathrm{i}\overline{\omega}_1\tau}\,(\overline{z}_1)^{2n}}{1-\mathrm{e}^{\mathrm{i}(\theta+\overline{\theta}_1)}\exp\left[-2\mathrm{i}(\overline{\omega}_1-\omega_1)\tau\right]z_1^{-2n}\overline{z}_1^{2n}}. \tag{13.33}$$

Comparing the two potentials from (13.32) and (13.28) clearly shows that the integrable symmetry $R_n = -Q_{-n}^*$ is indeed satisfied. By letting $\overline{\theta}_1 \to \overline{\theta}_1 + \pi$ we recover the *PT* symmetric one-soliton solution first reported in [40].

14. Remarks on nonlocal Painlevé equations

The Painlevé equations are special nonlinear ordinary differential equations which have no moveable branch points in the complex plane see [52]. Remarkably, they frequently arise from a self-similar reductions of integrable nonlinear evolution equations see [7, 53]. They are particularly interesting due to their rich structure in the complex plane as well as their deep connections to integrable systems. While originally developed in the context of continuous integrable evolution equations, discrete Painlevé equations have also been proposed and extensively studied see [54].

Until recently, most of the work on the subject has focused on the mathematical properties of local 'classical' continuous and discrete Painlevé equations [55–63]. Recently however, Ablowitz and Musslimani proposed new nonlocal Painlevé-type equations that arise from a self similar reductions of the *PT*, RST and RT nonlocal integrable evolution equations, see [40–42]. Below we summarize some of the those already mentioned and add some new ones.

$$PT \text{ Painlevé A1 [42]:} \quad f_{zz}(z) + \lambda f(z) - 2\sigma f^2(z) f^*(-z) = 0, \tag{14.1}$$

$$PT \text{ Painlevé A2 [42]:} \quad f_{zz}(z) + \mathrm{i}zf_z(z) + (\nu_0 + i) f(z) - 2\sigma f^2(z) f^*(-z) = 0, \tag{14.2}$$

$$RST \text{ Painlevé [41]:} \quad f_{zz}(z) + \mathrm{i}zf_z(z) + \mathrm{i}f(z) - 2\sigma\kappa f^2(z) f(\kappa z) = 0, \tag{14.3}$$

$$RT \text{ Painlevé [41]:} \quad f_{zz}(z) + \mathrm{i}zf_z(z) + \mathrm{i}f(z) - 2\sigma\kappa f^2(z) f(-\kappa z) = 0, \tag{14.4}$$

$$PT \text{ discrete Painlevé [40]:} \quad u_{n+1} + u_{n-1} + \frac{\delta u_n}{1 + \sigma u_n u^*_{-n}} = 0, \tag{14.5}$$

where $\sigma = \mp 1, \lambda, \nu_0, \delta \in \mathbb{R}, \kappa = (-1)^{-1/2}$ and $f(z)$,u_n are (in general) complex valued functions of the real variables z and n. We note that equation (14.1) is a nonlocal generalization of an elliptic function and equation (14.5) is a nonlocal generalization of an addition formula of an elliptic function.

In this paper, we propose some new continuous and discrete nonlocal Painlevé equations. First note that the above *PT* discrete Painlevé equation can be modified. Looking for a solution of the form $Q_n(t) = \mathrm{e}^{\mathrm{i}\lambda t} u_n,\ \lambda \in \mathbb{R}$ in the RST discrete NLS equation (1.11) we find

$$u_{n+1} + u_{n-1} + \frac{\delta_1 u_n}{1 + \sigma u_n u_{-n}} = 0,\ \delta_1 = \lambda - 2, \tag{14.6}$$

where $\delta_1 \in \mathbb{R}$. Writing out first few terms, it appears that numerically, we can find nontrivial solutions/ non even solutions to equation (14.6). That is to say, given u_0 and u_1, we can find all other u_n for all $n \geqslant 2$. An interesting class of similarity reductions are accelerated waves see [53]. For example for the standard NLS equation (1.1) the similarity reduction

$$q(x,t) = f(z)\mathrm{e}^{-\mathrm{i}\beta_0 t(x - \gamma_0 t^2)},\ z = x - \beta_0 t^2,\ \gamma_0 = -\frac{2\beta_0}{3},\ \beta_0 \in \mathbb{R}, \tag{14.7}$$

leads to the following Painlevé-type equation

$$f_{zz} - \beta_0 z f + 2\sigma f^2 f^* = 0. \tag{14.8}$$

This is the second Painlevé equation when f^* is replaced by f.

Motivated by Galilean invariance of the nonlocal PTNLS equation we can look for accelerated type similarity reductions of nonlocal equations. In the PTNLS equation (1.13) we introduce the following reduction

$$q(x,t) = f(z)\mathrm{e}^{-\xi_0 t(x - \mu t^2)},\ z = x + \mathrm{i}\xi_0 t^2,\ \mu = -\frac{2\xi_0}{3},\ \xi_0 \in \mathbb{R}, \tag{14.9}$$

which leads to the nonlocal Painlevé-type equation

$$f_{zz}(z) - \mathrm{i}\xi_0 z f(z) + 2\sigma f^2(z) f^*(-z) = 0. \tag{14.10}$$

This is a nonlocal generalization of the second Painlevé equation. In [41] the following nonlocal reverse space-time generalization of the modified KdV (RST mKdV) equation was found

$$w_t(x,t) + 6\sigma w(x,t) w(-x,-t) w_x(x,t) + w_{xxx}(x,t) = 0. \tag{14.11}$$

The traveling wave similarity reduction $u(x,t) = f(z), z = x - ct, c \in \mathbb{R}$ leads to the following nonlocal differential equation

$$f_{zzz}(z) + 6\sigma f(z) f(-z) f_z(z) - c f_z(z) = 0. \tag{14.12}$$

It should be noted that the one-soliton solution for the nonlocal RST mKdV equation does not satisfy equation (14.12), unless taken to be even (in which case it reduces back to the classical mKdV equation). It is left for a future research to understand the behavior of the solutions to this equation.

We also remark that looking for a self-similar solution to the nonlocal mKdV equation (14.11) of the form

$$w(x,t) = \frac{1}{3t^{1/3}} f(z), \qquad z = \frac{x}{3t^{1/3}}, \tag{14.13}$$

we find

$$f_{zzz} - (zf)_z - 6\sigma f^2 f_z = 0, \qquad \Longrightarrow \qquad f_{zz} - zf - 2\sigma f^3 = \alpha, \tag{14.14}$$

where α is constant. Interestingly enough, even though we began with a *nonlocal* equation (14.11), the result is still *local*; it is the second Painlevé transcendent. It is likely that this equation, i.e. (14.14), plays an important role in the long time asymptotic solution to the nonlocal mKdV equation (14.11).

We believe that it is important to study the behavior of these nonlocal Painlevé type equations. It is an important topic for future study.

15. Conclusion

In 1975/76 Ablowitz and Ladik formulated a theory for discrete integrable systems whose core idea is a discrete compatibility condition between two linear problems: the first being a second order discrete Schrödinger-type scattering problem while the other is a time-evolution system. The outcome of this theory is a nonlinear evolution equation in time (continuous or discrete) and second order discrete in space that is guaranteed to be (i) integrable, in the sense of existence of an infinite number of conservation laws and (ii) solvable by the inverse scattering transform. For the specific case given by (2.1) and (2.2) this compatibility condition results in a coupled evolution equations for the 'potentials' R_n and Q_n, equations (1.4) and (1.5). Ablowitz and Ladik found that this system is compatible under the integrable symmetry reduction ($\sigma = \mp 1$)

$$R_n = \sigma Q_n^*, \tag{15.1}$$

and leads to the well-known Ablowitz–Ladik model equation (1.3).

More than four decades have passed before new integrable symmetry reductions of the Ablowitz–Ladik scattering problem (2.1) were discovered. Indeed, in 2014, Ablowitz and Musslimani noted (for the first time) that the coupled system of evolution equations (1.4) and (1.5) admit the so-called *PT* symmetric reduction

$$R_n(t) = \sigma Q^*_{-n}(t), \tag{15.2}$$

giving rise to the *PT* symmetric nonlinear Schrödinger equation (1.8). Notably, that equation preserves the 'discrete' *PT* symmetry, i.e. invariance of the evolution equation under the combined transformation of $n \to -n, t \to -t$ and complex conjugation. In recent years, there has been an intense research interest in the physics and mathematics of linear and nonlinear systems that admits *PT* symmetry, with the main focus being in quantum physics and optics see [64–92].

In 2016, Ablowitz and Musslimani discovered two new integrable symmetry reductions for the AL scattering problem. Those are the reverse space-time and reverse time only reduction respectively given by

$$R_n(t) = \sigma Q_{-n}(-t), \tag{15.3}$$

$$R_n(t) = \sigma Q_n(-t), \tag{15.4}$$

giving rise to the so-called RST and RT NLS equations (1.11) and (1.12).

The inverse scattering theory and soliton solution for the *PT* symmetric NLS (1.8) has been briefly outlined (due to page limitation) in [40] whereas in [41] the RST and RT NLS equations were proposed and shown to be integrable discrete system (few conserved quantities were also given).

In this paper, we provide a full account of the scattering and inverse scattering transform for all three cases: *PT* symmetric, RST and RT NLS equations. In particular, we derived all symmetries between the eigenfunctions, scattering data as well as for the modified eigenfunctions. The inverse scattering problem is solved using a left-right Riemann–Hilbert formulation. A trace formula is obtained for the RST and *PT* symmetric cases that is later used to express the norming constants as a function of the eigenvalues (zeros of the scattering data a and $\overline{a}$.) An alternative reconstruction formula for the potentials that allows one to easily 'observe' the integrable symmetry reduction at hand is derived. Soliton solutions for all three cases are obtained and their properties are discussed. New Painlevé type equations are also proposed.

Finally, we outline a number of interesting research directions pertaining to integrable nonlocal RT, RST and *PT* symmetric discrete NLS systems:

- In earlier papers, we have extended the IST with rapidly decaying data for the continuous nonlocal NLS, sine/sinh-Gordon equations to the IST with nonzero constant amplitude background [82–84]. This analysis should be carried out for the discrete nonlocal equations discussed in this paper. We remark that IST with nonzero background for the classical integrable AL model has been studied [47, 93].
- Solutions and properties of the above mentioned continuous and discrete nonlocal Painleve-type equations should be investigated.
- Explicit multi-soliton, multi-pole (i.e. non simple pole) solutions for the RT, RST and *PT* symmetric NLS equations should be obtained.
- The theory associated with periodic/quasi-periodic solutions of the RT, RST and *PT* symmetric NLS equations should be developed.

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References

[1] Chen D N, Christodoulides Z and Segev M 2012 *Rep. Prog. Phys.* **75** 086401
[2] Segev M 1998 Optical spatial solitons *Opt. Quantum Electron.* **30** 503
[3] Agrawal G P 1989 *Nonlinear Fiber Optics* (San Diego, CA: Academic)
[4] Kivshar Y S and Agrawal G P 2003 *Optical Solitons: from Fibers to Photonic Crystals* (New York: Academic)
[5] Pethick C J and Smith H 2008 *Bose–Einstein Condensates in Dilute Gases* (Cambridge: Cambridge University Press)
[6] Ablowitz M J 2011 *Nonlinear Dispersive Waves, Asymptotic Analysis, Solitons* (Cambridge: Cambridge University Press)
[7] Ablowitz M J and Segur H 1981 *Solitons and Inverse Scattering Transform* (*SIAM Studies in Applied Mathematics* vol 4) (Philadelphia, PA: SIAM)
[8] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons. The inverse Scattering Method* (New York: Plenum)
[9] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
[10] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* I (Amsterdam: North-Holland)
[11] Yang J 2010 *Nonlinear Waves in Integrable and Nonintegrable Systems* (Philadelphia, PA: SIAM)
[12] Pelloni B 2015 *Nonlinearity* **28** R1–38
[13] Christodoulides D N, Lederer F and Silberberg Y 2003 *Nature* **424** 817
[14] Lederer F, Stegeman G I, Christodoulides D N, Assanto G, Segev M and Silberberg Y 2008 *Phys. Rep.* **463** 1
[15] Henning D and Tsironis G P 1999 *Phys. Rep.* **307** 333
[16] Kevrekidis P G 2009 *The Discrete Nonlinear Schrödinger Equation* (New York: Springer)
[17] Braun O M and Kivshar Y S 1998 *Phys. Rep.* **306** 1
[18] Flach S and Willis C R 1998 *Phys. Rep.* **295** 181
[19] Christodoulides D N and Joseph R J 1988 *Opt. Lett.* **13** 794
[20] Eisenberg H, Silberberg Y, Morandotti R, Boyd A and Aitchison J 1998 *Phys. Rev. Lett.* **81** 3383
[21] Morandotti R, Peschel U, Aitchison J, Eisenberg H and Silberberg Y 1999 *Phys. Rev. Lett.* **83** 2726
[22] Ablowitz M J and Musslimani Z H 2003 *Physica* D **184** 276
[23] Ablowitz M J and Musslimani Z H 2002 *Phys. Rev.* E **65** 056618
[24] Ablowitz M J, Musslimani Z H and Biondini G 2002 *Phys. Rev.* E **65** 026602
[25] Ablowitz M J and Musslimani Z H 2001 *Phys. Rev. Lett.* **87** 254102
[26] Davydov A S 1973 *J. Theor. Biol.* **38** 559
[27] Su W P, Schieffer J R and Heeger A J 1979 *Phys. Rev. Lett.* **42** 1698
[28] Sievers A J and Takeno S 1988 *Phys. Rev. Lett.* **61** 970
[29] Ablowitz M J and Ladik J F 1975 *J. Math. Phys.* **16** 598
[30] Ablowitz M J and Ladik J F 1976 *J. Math. Phys.* **17** 1011
[31] Xia B and Fokas A S 2018 *Physica* D **364** 27
[32] Biondini G and Bui A 2015 *J. Phys. A: Math. Theor.* **48** 375202
[33] Biondini G and Wang Q 2019 *Stud. Appl. Math.* **142** 139
[34] Ablowitz M J, Prinari B and Trubatch A D 2004 *Discrete, Continuous Nonlinear Schrödinger Systems* (Cambridge: Cambridge University Press)
[35] Kulish P P 1981 *Lett. Math. Phys.* **5** 191
[36] Kundu A and Ragnisco O 1994 *J. Phys. A: Math. Gen.* **27** 6335
[37] Kuznetsov V B, Salerno M and Sklyanin E K 2000 *J. Phys. A: Math. Gen.* **33** 171
[38] Suris Y B 1997 *Inverse Problems* **3** 1121
[39] Doikou A and Sklaveniti S 2019 *J. Phys.* A (https://doi.org/10.1088/1751-8121/ab8c18)
[40] Ablowitz M J and Musslimani Z H 2014 *Phys. Rev.* E **90** 032912
[41] Ablowitz M J and Musslimani Z H 2017 *Stud. Appl. Math.* **139** 7

[42] Ablowitz M J and Musslimani Z H 2013 *Phys. Rev. Lett.* **110** 064105
[43] Ablowitz M J and Musslimani Z H 2016 *Nonlinearity* **29** 915
[44] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
[45] Ablowitz M J and Musslimani Z H 2019 *J. Phys. A: Math. Theor.* **52** 15LT02
[46] Caudrelier V and Crampe N 2019 *Nucl. Phys.* B **946** 114720
[47] Ablowitz M J, Biondini G and Prinari B 2007 *Inverse Problems* **23** 1711
[48] Ortiz A K and Prinari B 2019 *Stud. Appl. Math.* **143** 373
[49] Biondini G and Hwang G 2008 *Inverse Problems* **24** 065011
[50] Fibich G 2015 *The Nonlinear Schrödinger Equation: Singular Solutions and Optical Collapse* (New York: Springer)
[51] Grahovski G G, Mohammed A J and Susanto H 2018 *Theor. Math. Phys.* **197** 1412
[52] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
[53] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[54] Hietarinta J, Joshi N and Nijhoff F W 2016 *Discrete Systems and Integrability* (Cambridge: Cambridge University Press)
[55] Quispel G R W, Roberts J A G and Thompson C J 1988 *Phys. Lett.* A **126** 419
[56] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* D **34** 183
[57] Ramani A, Grammaticos B and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825
[58] Ramani A, Grammaticos B and Hietarinta J 1991 *Phys. Rev. Lett.* **67** 1829
[59] Sakai H 2001 *Commun. Math. Phys.* **220** 165
[60] Ablowitz M J and Segur H 1975 *J. Math. Phys.* **16** 1054
[61] Grammaticos B, Halburd R G, Ramani A and Viallet C-M 2009 *J. Phys. A: Math. Theor.* **42** 454002
[62] Clarkson P A 2003 *J. Comput. Appl. Math.* **153** 127
[63] Clarkson P A and Mansfield E L 2003 *Nonlinearity* **16** R1–26
[64] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
[65] Makris K G, El Ganainy R, Christodoulides D N and Musslimani Z H 2008 *Phys. Rev. Lett.* **100** 103904
[66] Musslimani Z H, Makris K G, El Ganainy R and Christodoulides D N 2008 *Phys. Rev. Lett.* **100** 030402
[67] El Ganainy R, Makris K G, Christodoulides D N and Musslimani Z H 2007 *Opt. Lett.* **32** 2632
[68] Konotop V, Yang J and Zezyulin D 2016 *Rev. Mod. Phys.* **88** 035002
[69] Christodoulides D N and Yang J (eds) 2018 *Parity-time Symmetry and its Applications* (New York: Springer)
[70] Gbur G and Makris K G 2018 Introduction to non-Hermitian photonics in complex media: PT-symmetry and beyond *Photonics Res.* **6** PTS1–3
[71] El Ganainy R, Makris K G, Khajavikhan M, Musslimani Z H, Rotter S and Christodoulides D N 2018 *Nat. Phys.* **14** 11
[72] Lou S Y and Huang F 2017 *Sci. Rep.* **7** 869
[73] Yang J 2018 *Phys. Rev.* E **98** 042202
[74] Yang J 2018 *Phys. Lett.* A **383** 328
[75] Yang B and Yang J 2018 *Stud. Appl. Math.* **140** 178
[76] Yang B and Yang J 2019 *Lett. Math. Phys.* **109** 945–73
[77] Xu Z and Chow K 2016 *Appl. Math. Lett.* **56** 72
[78] Wen X, Yan Z and Yang Y 2016 *Chaos* **26** 063123
[79] Fokas A S 2016 *Nonlinearity* **29** 319
[80] Ma L, Shen S and Zhu Z 2017 *J. Math. Phys.* **58** 103501
[81] Gadzhimuradov T A and Agalarov A M 2016 *Phys. Rev.* A **93** 062124
[82] Ablowitz M J, Luo X-D and Musslimani Z H 2018 *J. Math. Phys.* **59** 0011501
[83] Ablowitz M J, Feng B-F, Luo X-D and Musslimani Z H 2018 *Theor. Math. Phys.* **59** 1241
[84] Feng B F, Luo X D, Ablowitz M J and Musslimani Z H 2018 *Nonlinearity* **31** 5385
[85] Ablowitz M J, Feng B-F, Luo X-D and Musslimani Z H 2018 *Stud. Appl. Math.* **141** 267
[86] Chen K, Deng X, Lou S Y and Zhang D J 2018 *Stud. App. Math.* **141** 113
[87] Li M and Xu T 2015 *Phys. Rev.* E **91** 033202
[88] Gurses M and Pekcan A 2018 *Symb. Differ. Equ. Appl.* **266** 27
[89] Gurses M and Pekcan A 2018 *J. Math. Phys.* **59** 051501
[90] Ji J L and Zhu Z N 2017 *Commun. Nonlinear Sci.* **42** 699
[91] Yang B and Chen Y 2018 *Nonlinear Dyn.* **94** 489–502
[92] Rao J G, Cheng Y and He J S 2017 *Stud. Appl. Math.* **139** 568
[93] Prinari B and Vitale F 2016 *Stud. Appl. Math.* **137** 28