Quasi-static fading MAC with many users and finite payload

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***Abstract*—Consider a (multiple-access) wireless communication system where users are connected to a unique base station over a shared-spectrum radio links. Each user has a fixed number k of bits to send to the base station, and his signal gets attenuated by a random channel gain (quasi-static fading). In this paper we consider the many-user asymptotics of Chen-Chen-Guo'2017, where the number of users grows linearly with the blocklength. In addition, we adopt a per-user probability of error criterion of Polyanskiy'2017 (as opposed to classical joint-error probability criterion). Under these two settings we derive bounds on the optimal required energy-per-bit for reliable multi-access communication. We confirm the curious behaviour (previously observed for non-fading MAC) of the possibility of perfect multi-user interference cancellation for user densities below a critical threshold. Further we demonstrate the suboptimality of standard solutions such as orthogonalization (i.e., TDMA/FDMA) and treating interference as noise (i.e. pseudo-random CDMA without multi-user detection).**

I. Introduction

We clearly witness two recent trends in the wireless communication technology: the increasing deployment density and miniaturization of radio-equipped sensors. The first trend results in progressively worsening interference environment, while the second trend puts ever more stringent demands on communication energy efficiency. This suggests a bleak picture for the future networks, where a chaos of packet collisions and interference contamination prevents reliable connectivity.

This paper is part of a series aimed at elucidating the fundamental tradeoffs in this new "dense-networks" regime of communication, and on rigorously demonstrating suboptimality of state-of-the-art radio-access solutions (ALOHA, orthogonalization, or FDMA, and treating interference as noise, or TIN).

Specifically, in this paper we consider a problem of K nodes communicating over a frame-synchronized multiple-access channel. When K is fixed and the frame size n (which we will also call "blocklength" or the "number of degrees of freedom") is taken to infinity we get the classical regime [1], in which the fundamental limits are given by well-known mutual information expressions. A new regime, deemed *many-access*, was put forward by Chen, Chen and Guo [2] (see also [3] for a related massive MIMO MAC analysis). In this regime the number of nodes K grows with blocklength n. It is clear that the most natural scaling is linear: $K = \mu n, n \to \infty$, corresponding to the fact that in time n there are linearly many users that will have updates/traffic to send [4]. That is, if each device wakes up once in every T seconds and transmits over a frame of length t, then in time (proportional to) t there are $K_{tot} \approx t/T$ users where t is large enough for this approximation to hold but small that no device wakes up twice. Further, this linear scaling explains the non-asymptotic plots for $n = 30000$ and $K \le 300$ [4, 5]. The analysis of [2] focused on the regime of infinitely large payloads. In contrast [4] proposed to focus on a model where each of the $K = \mu n$ nodes has only finitely many bits to send. In this regime, it turned out, one gets the relevant engineering tradeoffs. Namely, the communication with finite energy-per-bit is possible as $n \to \infty$ and the optimal energy-per-bit depends on the user density μ. For this to happen, however, a second crucial departure from the classical MAC model was needed: the per-user probability of error, *PUPE*, criterion [4].

These two modifications (the scaling $K = \mu n$ and the PUPE) were investigated in the case of the AWGN channel in [4–6]. We next describe the main discovery of that work. The channel model is:

$$Y^n = \sum_{i=1}^{K} X_i + Z^n\,, \qquad Z^n \sim \mathcal{CN}(0, I_n)\,, \tag{1}$$

and $X_i = f_i(W_i) \in \mathbb{C}^n$ is the codeword of i-th user corresponding to $W_i \in [2^k]$ chosen uniformly at random. The system is said to have PUPE ϵ if there exist decoders $\hat{W}_i = \hat{W}_i(Y^n)$ such that

$$P_{e,u} = \frac{1}{K}\sum_{i=1}^{K} \mathbb{P}\left[W_i \neq \hat{W}_i\right] \le \epsilon\,. \tag{2}$$

The energy-per-bit is defined as $\frac{E_b}{N_0} = \frac{1}{K}\sup_{i\in[K], w\in[2^k]} \|f_i(w)\|^2$. The goal in [4, 6] was to characterize the asymptotic limit

$$\mathcal{E}^*(\mu, k, \epsilon) \triangleq \limsup_{n\to\infty} \inf \frac{E_b}{N_0} \tag{3}$$

where infimum is taken over all possible encoders $\{f_i\}$ and decoders $\{\hat{W}_i\}$ achieving the PUPE ϵ for $K = \mu n$ users.

To predict how $\mathcal{E}^*(\mu, \epsilon)$ behaves, first consider a naive Shannon-theoretic calculation [7]: if K users want to send k bits in n degrees of freedom, then their sum-power P_{tot} should satisfy $n\log(1 + P_{tot}) = kK$. In turn, the sum-power $P_{tot} = \frac{kK}{n}\frac{E_b}{N_0}$. Overall, we get $\mathcal{E}^* \approx \frac{2^{\mu k}-1}{k\mu}$. This turns out to be a correct prediction, but only in the large-μ regime. The true behavior of the fundamental limit is roughly given by

$$\mathcal{E}^*(\mu, k, \epsilon) \approx \max\left(\frac{2^{\mu k}-1}{k\mu}, \mathcal{E}_{\text{s.u.}}\right)\,,$$

where $\mathcal{E}_{\text{s.u.}} = \mathcal{E}_{\text{s.u.}}(k, \epsilon)$ does not depend on μ and corresponds to the single-user minimal energy-per-bit for sending k bits with error ϵ, for which a very tight characterization is given in [8].

In all, results of [4–6] suggest that the minimal energy-per-bit has a certain "inertia": as the user density μ starts to climb from zero up,

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initially the energy-per-bit should stay the same as in the single-user $\mu = 0$ limit. In other words, optimal multiple-access architectures should be able to *perfectly cancel all multi-user interference (MUI)*, achieving an essentially single-user performance for each user, *provided the user density is below a critical threshold*. Note that this is much better than orthogonalization, which achieves the same effect at the expense of shortening the available (to each user) blocklength by a factor of $\frac{1}{K}$. Quite surprisingly, standard approaches to multiple-access such as TDMA and TIN[1], while having an optimal performance at $\mu \to 0$ demonstrated a significant suboptimality for $\mu > 0$ regime. In particular, no "inertia" was observed and the energy-per-bit for those suboptimal architectures is always a monotonically increasing function of the user density μ. This opens the (so far open) quest for finding a future-proof MAC architecture that would achieve $\mathcal{E}_{\text{s.u.}}$ energy-per-bit for a strictly-positive $\mu > 0$.

(In this short summary we omitted another important part of [4]: the issue of random-access – i.e. when the identities/codebooks of active users are apriori unknown. We just mention here that for random access problem, there are a number of low-complexity algorithms that are available [9–15].)

The *contribution* of this paper is in demonstrating the same perfect MUI cancellation effect in a much more practically relevant communication model, in which the ideal unit power-gains of (1) are replaced by random (but static) fading gain coefficients. We consider two cases of the channel state information: known at the receiver (CSIR) and no channel state information (noCSI).

Key technical ideas: For handling the noCSI case we employ the subspace projection decoder similar to the one proposed in [16], which can be seen as a version of the maximum-likelihood decoding (without prior on fading coefficients) – an idea often used in support recovery literature [17–19]. Another key idea is to decode only a subset of users corresponding to the strongest channel gains – a principle originating from Shamai-Bettesh [20]. While the randomness of power-gains increases the energy-per-bit requirements, we also mention that [9] finds an unexpected advantage: the inherent randomization helps the decoder disambiguate different users and improves performance of the belief propagation decoder.

The paper is organized as follows. In Section II we formally define the problem and the fundamental limits. In Section III as a warm-up we discuss the classical regime (K–fixed, $n \to \infty$) under the PUPE criterion. We show that our projection decoder achieves the best known achievability bound in this setting [20]. (We also note that for the quasi-static fading channel model the idea of PUPE is very natural, and implicitly appears in earlier works, e.g. [20, 21], where it is conflated with the outage probability.) After this short warm-up we go to our main Section IV, which contains rigorous achievability and converse bounds for the $K = \mu n, n \to \infty$ scaling regime. Finally, we conclude with numerical evaluations and discussions in Section V, where we also compare our bounds with the TDMA and TIN.

All proofs have been omitted due to space constraints and can be found in [22].

A. Notations

Let $\mathbb{N}$ denote the set of natural numbers. For $n \in \mathbb{N}$, let $\mathbb{C}^n$ denote the n–dimensional complex Euclidean space. We also let $[n]$ denote the set $\{1, 2, ..., n\}$. Let $S \subset \mathbb{C}^n$. We denote the projection operator or matrix on to the subspace *spanned* by S as P_S and its orthogonal complement as $P_S^{\perp}$. For $0 \le p \le 1$, let $h_2(p) = -p\log_2(p) - (1-p)\log_2(1-p)$ and $h(p) = -p\ln(p) - (1-p)\ln(1-p)$, with $0 \ln 0$ defined to be 0. We denote by $\mathcal{N}(0,1)$ and $\mathcal{CN}(0,1)$ the standard normal and the standard circularly symmetric complex normal distributions, respectively. $\mathbb{P}$ and $\mathbb{E}$ denote probability measure and expectation operator respectively. Q is the complementary CDF function of the standard normal distribution. Lastly, $\|\cdot\|$ represents the standard euclidean norm.

[1]Note that pseudo-random CDMA systems without multi-user detection and large load factor provide an efficient implementation of TIN. So throughout our discussions, conclusions about TIN also pertain to CDMA systems of this kind.

II. Definitions and System Model

In this work, we consider the following quasi-static Rayleigh fading MAC:

$$Y^n = \sum_{i=1}^{K} H_i X_i^n + Z^n \tag{4}$$

where $Z^n \sim \mathcal{CN}(0, I_n)$, and $H_i \overset{iid}{\sim} \mathcal{CN}(0,1)$ are the fading coefficients which are independent of $\{X_i^n\}$ and Z^n. Naturally, we assume that there is a maximum power constraint:

$$\|X_i^n\|^2 \le nP. \tag{5}$$

We consider two cases: 1) no channel state information (no-CSI): neither the transmitters nor the receiver knows the realizations of channel fading coefficients, but they both know the law; 2) channel state information only at the receiver (CSIR): only the receiver knows the realization of channel fading coefficients. The special case of (4) where $H_i = 1, \forall i$ is called the Gaussian MAC (GMAC).

Next, we define a subspace projection based decoder, inspired from [16]. The idea is that in the absence of additive noise the received vector will lie in the subspace spanned by the sent codewords. Formally, let $\mathcal{C}_i, i \in [K]$ and $f_i : [M_i] \to \mathcal{C}_i$, $i \in [K]$ denote the codebook and encoding function of the user i with message set $[M_i]$. Then upon receiving Y from the channel the decoder outputs $g(Y)$ which is given by

$$\begin{aligned} g(Y) &= \left(f_1^{-1}(\hat{c}_1), ..., f_K^{-1}(\hat{c}_K)\right) \\ (\hat{c}_1, ...\hat{c}_K) &= \arg \max_{(c_i \in \mathcal{C}_i)_{i=1}^K} \left\| P_{\{c_i : i \in [K]\}} Y \right\|^2. \end{aligned} \tag{6}$$

III. Classical regime: K fixed, $n \to \infty$

In this section, we focus on the channel under classical asymptotics where K is fixed (and large) and $n \to \infty$. We show that subspace projection decoder achieves a) ϵ–capacity region ($C_{\epsilon,J}$) for the joint error and b) the best known bound for ϵ–capacity region $C_{\epsilon,PU}$ under per-user error. This motivates using projection decoder in the many-user regime.

A. Joint error

A rate tuple $(R_1, ..., R_K)$ is said to be ϵ–achievable [23] for the MAC if there is a sequence of codes whose rates are asymptotically at least R_i such that joint error is asymptotically smaller than ϵ. Then the ϵ–capacity region $C_{\epsilon,J}$ is the closure of the set of ϵ–achievable rates. It can be shown using [23, Theorem 5] that for our channel (4), C_ϵ is given by

$$C_{\epsilon,J} = \{R = (R_1, ..., R_K) : \forall i, R_i \ge 0 \text{ and } P_0(R) \le \epsilon\} \tag{7}$$

where the outage probability $P_0(R)$ is given by

$$P_0(R) = \mathbb{P}\left[\bigcup_{S \subset [K], S \neq \emptyset} \left\{\log\left(1 + P\sum_{i \in S} |H_i|^2\right) \le \sum_{i \in S} R_i\right\}\right] \tag{8}$$

Next we claim that projection decoder achieves $C_{\epsilon,J}$.

Theorem III.1 (Projection decoding achieves $C_{\epsilon,J}$)**.** *Let $R \in C_{\epsilon,J}$ of* (4)*. Then R is ϵ–achievable through a sequence of codes with the decoder being the projection decoder.*

B. Per-user error

The ϵ–capacity region for the channel under per-user error, $C_{\epsilon,PU}$ is defined similarly as $C_{\epsilon,J}$ but with per-user error instead of joint error. $C_{\epsilon,PU}$ is unknown, but the best lower bound is given by the Shamai-Bettesh capacity bound [20]: given a rate tuple $R = (R_1, ..., R_K)$, an upper bound on the per-user probability of error under the channel (4), as $n \to \infty$, is given by

$$\begin{aligned} P_{e,u} &\leq P_e^S(R) \\ &= 1 - \frac{1}{K}\mathbb{E}\sup\left\{|D| : D \subset [K], \forall S \subset D, S \neq \emptyset, \right. \\ &\qquad \left. \sum_{i\in S} R_i < \log\left(1 + \frac{P\sum_{i\in S}|H_i|^2}{1 + P\sum_{i\in D^c}|H_i|^2}\right)\right\} \end{aligned} \tag{9}$$

where the maximizing set, among all those that achieve the maximum, is chosen to contain the users with largest fading coefficients. The corresponding achievability region is

$$C_{\epsilon,PU}^{S.B} = \left\{R : P_e^S(R) \leq \epsilon\right\} \tag{10}$$

and hence it is an inner bound on $C_{\epsilon,PU}$.

We note that, in [20], only the symmetric rate case i.e, $R_i = R_j \; \forall i, j$ is considered. So (9) is the extension of that result to the general non-symmetric case.

Suitable modifying the projection decoder to use CSIR to drop weak users such that the rate tuple of the remaining users is inside the corresponding capacity region (considering dropped users as noise), we can show that the (modified) projection decoding achieves the same asymptotics as (9) for per-user probability of error.

Theorem III.2. *For any $R \in C_{\epsilon,PU}^{S.B}$ there exists a sequence of codes with projection decoder with asymptotic rate R such that the per-user probability of error is asymptotically smaller than ϵ*

In the case of symmetric rate, an outer bound on $C_{\epsilon,PU}$ can be given as follows.

Proposition 1. *If the symmetric rate R is such that $P_{e,u} \leq \epsilon$, then*

$$\begin{aligned} R \leq \min\Bigg\{ &\frac{1}{K(1-\epsilon)}\mathbb{E}\left[\log_2\left(1 + P\sum_{i\in[K]}|H_i|^2\right)\right], \\ &\log_2\left(1 - P\ln(1-\epsilon)\right)\Bigg\} \end{aligned} \tag{11}$$

We refer the readers to [22] for numerical evaluations and discussions in the classical regime. But we just note here that $C_{\epsilon,J}$ (under joint error) tends to $\{0\}$ as $K \to \infty$ because, it can be seen, for the symmetric rate, by considering that order statistics of the fading coefficients that $P_0(R) \to 1$ for $R_i = O(1/K)$. Hence PUPE is more suitable error metric since by virtue of its definition, a constant fraction of users with worst channel gains can be dropped by the decoder.

IV. Many user MAC: $K = \mu n$, $n \to \infty$

This is our main section. We consider the linear scaling regime $K = \mu n$, $n \to \infty$. We are interested in the trade-off of minimum E_b/N_0 required for the PUPE to be smaller than ϵ, with the user density μ ($\mu < 1$). So, we fix the message size k. Let $S = k\mu$ be the spectral efficiency.

We focus on the case of different codebooks, but under symmetric rate. So if M denotes the size of the codebooks, then $S = \frac{K\log M}{n} = \mu \log M$. Hence, given S and μ, M is fixed. Let $P_{tot} = KP$ denote the total power. Therefore denoting by $\mathcal{E}$ the energy-per-bit, $\mathcal{E} = E_b/N_0 = \frac{nP}{\log_2 M} = \frac{P_{tot}}{S}$. For finite E_b/N_0, we need finite P_{tot}, hence we consider the power P decaying as $O(1/n)$.

Let $\mathcal{C}_j = \{c_1^j, ..., c_M^j\}$ be the codebook of user j, of size M. The power constraint is given by $\left\|c_i^j\right\|^2 \leq nP = \mathcal{E}\log_2 M, \forall j \in [K], i \in [M]$. The collection of codebooks $\{\mathcal{C}_j\}$ is called an $(n, M, \epsilon, \mathcal{E}, K)$–code if it satisfies the power constraint described before, and the per-user probability of error is smaller than ϵ. Then, we can define the following fundamental limit for the channel

$$\mathcal{E}^*(M, \mu, \epsilon) = \lim_{n\to\infty} \inf\left\{\mathcal{E} : \exists(n, M, \epsilon, \mathcal{E}, K = \mu n) - code\right\}.$$

A. No-CSI

In this subsection, we focus on the no-CSI case. The difficulty here is that, apriori, we do not know which subset of the users to decode. We have the following theorem.

Theorem IV.1. *Consider the channel* (4) *(no-CSI) with $K = \mu n$ where $\mu < 1$. Fix the spectral efficiency S and target PUPE ϵ. Let $M = 2^{S/\mu}$ denote the size of the codebooks and $P_{tot} = KP$ be the total power. Fix $\nu \in (1-\epsilon, 1]$. Let $\epsilon' = \epsilon - (1-\nu)$. Then if $\mathcal{E} > \mathcal{E}_{no-CSI}^* = \sup_{\frac{\epsilon'}{\nu}<\theta\leq 1}\sup_{\xi\in[0,\nu(1-\theta)]}\frac{P_{tot,\nu}(\theta,\xi)}{S}$, there exists a sequence of $(n, M, \epsilon_n, \mathcal{E}, K = \mu n)$ codes such that $\limsup_{n\to\infty}\epsilon_n \leq \epsilon$, where, for $\frac{\epsilon'}{\nu} < \theta \leq 1$ and $\xi \in [0, \nu(1-\theta)]$,*

$$P_{tot,\nu}(\theta,\xi) = \frac{\hat{f}(\theta,\xi)}{1 - \hat{f}(\theta,\xi)\alpha\left(\xi + \nu\theta, \xi + 1 - \nu(1-\theta)\right)} \tag{12}$$

$$\hat{f}(\theta,\xi) = \frac{f(\theta)}{\alpha(\xi, \xi+\nu\theta)} \tag{13}$$

$$f(\theta) = \frac{\frac{1+\delta_1^*(1-V_\theta)}{V_\theta} - 1}{1 - \delta_2^*} \tag{14}$$

$$V_\theta = e^{-\tilde{V}_\theta} \tag{15}$$

$$\begin{aligned} \tilde{V}_\theta = \delta^* &+ \frac{\theta\mu\nu\ln M}{1-\mu\nu} + \\ &\frac{1-\mu\nu(1-\theta)}{1-\mu\nu}h\left(\frac{\theta\mu\nu}{1-\mu\nu(1-\theta)}\right) + \\ &\frac{\mu(1-\nu(1-\theta))}{1-\mu\nu}h\left(\frac{\theta\nu}{1-\nu(1-\theta)}\right) \end{aligned} \tag{16}$$

$$\delta^* = \frac{\mu h(1-\nu(1-\theta))}{1-\mu\nu} \tag{17}$$

$$c_\theta = \frac{2V_\theta}{1-V_\theta} \tag{18}$$

$$q_\theta = \frac{\mu h(1-\nu(1-\theta))}{1-\mu\nu(1-\theta)} \tag{19}$$

$$\delta_1^* = q_\theta(1+c_\theta) + \sqrt{q_\theta^2(c_\theta^2 + 2c_\theta) + 2q_\theta(1+c_\theta)} \tag{20}$$

$$\begin{aligned} \delta_2^* = \inf\Big\{&x : 0 < x < 1, \\ &-\ln(1-x) - x > \frac{\mu h(1-\nu(1-\theta))}{1-\mu\nu(1-\theta)}\Big\} \end{aligned} \tag{21}$$

$$\alpha(a,b) = a\ln(a) - b\ln(b) + b - a. \tag{22}$$

Hence $\mathcal{E}^* \leq \mathcal{E}_{no-CSI}^*$.

The proof of the above theorem uses a suitably modified projection decoding where only ν fraction of the users are decoded by searching for over all νK tuple of codebooks to find the best projection. In retrospect, our analysis bears similarity to the one in [19], which can be seen (as argued in [4]) as a version of the many-MAC problem with random-access."

B. CSIR

In this subsection, we focus on the CSIR scenario. We could use projection decoding to decode a fraction of users where decoding set is a function of CSIR. But a better bound is obtained by directly using euclidean metric to decode, similar to [4]. Then have the following theorem.

Theorem IV.2. *Consider the channel* (4) *(with CSIR) with $K = \mu n$ where $\mu < 1$. Fix the spectral efficiency S and target PUPE ϵ. Let $M = 2^{S/\mu}$ denote the size of the codebooks and $P_{tot} = KP$ be the total power. Fix $\nu \in (1-\epsilon, 1]$. Let $\epsilon' = \epsilon - (1-\nu)$. Then if $\mathcal{E} > \mathcal{E}^*_{CSIR} = \sup_{\frac{\epsilon'}{\nu} < \theta \leq 1} \inf_{0 \leq \rho \leq 1} \frac{P_{tot,\nu}(\theta,\rho)}{S}$, there exists a sequence of $(n, M, \epsilon_n, \mathcal{E}, K = \mu n)$ codes such that $\limsup_{n\to\infty} \epsilon_n \leq \epsilon$, where, for $\frac{\epsilon'}{\nu} < \theta \leq 1$,*

$$P_{tot,\nu}(\theta,\rho) = \frac{(1+\rho)\left(e^{\mu\nu\left(\frac{h(\theta)}{\rho}+\theta\ln M\right)}-1\right)}{\alpha(\nu(1-\theta),\nu) - \left(e^{\mu\nu\left(\frac{h(\theta)}{\rho}+\theta\ln M\right)}-1\right)\alpha(\nu,1)(1+\rho)} \quad (23)$$

and $\alpha(\cdot)$ is given by (22)*. Hence $\mathcal{E}^* \leq \mathcal{E}^*_{CSIR}$.*

The analysis uses the Gallager's rho trick employed in [4].

C. Converse

In this section we present a converse for $\mathcal{E}^*$, based on the Fano inequality and the results from [24]. Let $\{H_i\}$ be the iid fading coefficients – distribution denoted by random variable H.

Theorem IV.3. *Let M be the codebook size. Given $\epsilon \leq 1 - \frac{1}{M}$ and μ, let $S = \mu \log M$. Then, assuming that $\mathbb{E}\left[|H|^4\right] < \infty$ and $\mathbb{E}\left[|H|^2\right] = 1$, we have*

$$\mathcal{E}^*(M,\mu,\epsilon) \geq \inf \frac{P_{tot}}{S} \quad (24)$$

where infimum is taken over all $P_{tot} > 0$ that satisfies both

$$\theta S - \epsilon\mu\log\left(2^{S/\mu}-1\right) - \mu h_2(\epsilon) \leq \log\left(1 + P_{tot}\alpha\left(1-\theta,1\right)\right), \forall\theta \in [0,1] \quad (25)$$

$$\epsilon \geq 1 - \mathbb{E}\left[Q\left(Q^{-1}\left(\frac{1}{M}\right) - \sqrt{\frac{2P_{tot}}{\mu}|H|^2}\right)\right] \quad (26)$$

where $\alpha(\cdot)$ is given by (22)*.*

Bounds tighter than (25) can be obtained if further assumptions are made on the codebook. For instance, if we assume that each codebook consists of iid entries of the form $\frac{C}{K}$ where C is sampled from a distribution with zero mean and finite variance (P_{tot}), then using ideas similar to [25, Theorem 3] we have that in order for the iid codebook to achieve PUPE ϵ the energy-per-bit $\mathcal{E}$ should satisfy $\mathcal{E} \geq \inf \frac{P_{tot}}{\mu \log M}$ where infimum is taken over all $P_{tot} > 0$ that satisfies

$$\ln M - \epsilon\ln(M-1) - h(\epsilon) \leq M\mathcal{V}\left(\frac{1}{\mu M}, P_{tot}\right) - \mathcal{V}\left(\frac{1}{\mu}, P_{tot}\right) \quad (27)$$

where $\mathcal{V}$ is given by [25]

$$\mathcal{V}(r,\gamma) = r\ln\left(1+\gamma-\mathcal{F}(r,\gamma)\right) + \ln\left(1+r\gamma-\mathcal{F}(r,\gamma)\right) - \frac{\mathcal{F}(r,\gamma)}{\gamma} \quad (28)$$

$$\mathcal{F}(r,\gamma) = \frac{1}{4}\left(\sqrt{\gamma\left(\sqrt{r}+1\right)^2+1} - \sqrt{\gamma\left(\sqrt{r}-1\right)^2+1}\right)^2 \quad (29)$$

V. Numerical evaluation and discussion

In this section, we provide the results of numerical evaluation of the bounds in the paper. We focus on the trade-off of user density μ with the minimum energy-per-bit $\mathcal{E}^*$ for a given message size k and target probability of error P_e.

For $k = 100$ bits, we evaluate the trade-off from the bounds in this paper for $P_e = 10^{-3}$ and $P_e = 10^{-1}$ in figures 1 and 2 respectively. For TDMA, we split the frame of length n equally among K users, and compute the smallest P_{tot} the ensures the existence of a single user quasi-static AWGN code of rate S, blocklength $\frac{1}{\mu}$ and probability of error ϵ using the bound from [16]. TIN is computed using a method similar to theorem IV.2.

From these figures, we clearly observe the *perfect MUI cancellation* effect mentioned in the introduction. As μ increases from 0, the $\mathcal{E}^*$ is almost a constant (slightly increasing for the achievability bounds) but then undergoes a "phase transition" where $\mathcal{E}^*$ increases sharply. Hence this suggests there is a certain $\mathcal{E}_{\text{s.u.}} = \mathcal{E}_{\text{s.u.}}(k,\epsilon)$ and $\mu_{\text{s.u.}} > 0$ such that $\mathcal{E}^* = \mathcal{E}_{\text{s.u.}}$ for all $\mu < \mu_{\text{s.u.}}$. Further, standard schemes for multiple-access like TDMA and TIN do not have this behavior. Moreover, although these suboptimal schemes have an optimal trade-off at $\mu \to 0$ they show a significant suboptimality at higher μ. We note again that this perfect MUI cancellation which was observed in standard GMAC [4, 5] is also present in the more practically relevant quasi-static fading model. So, we suspect that this effect is a characteristic of the many-user MAC.

The fact that orthogonalization is not optimal is one of the key practical implications of our work. It was observed before in the GMAC [4, 5] and here we again witness it in the more relevant QS-MAC. To give another intuition for this suboptimality we consider a $K = \mu n$ user binary adder MAC $Y = \sum_{i=1}^{K} X_i$ where $X_i \in \{0,1\}$ and addition is over $\mathbb{Z}$. It can be shown that using TDMA or TIN the message size is bounded by a constant. But there exist explicit codes that achieve a message size of around $\frac{\log \mu n}{2\mu} \to \infty$ with low complexity [26–28]. (See [22] for more elaborate discussion).Hence we see that TDMA and TIN are severely suboptimal for the many-user adder MAC as well.

We remark here the no-CSI bound on $\mathcal{E}^*$ increases sharply in the neighborhood as $\mu \to 0$. In fact, it can be seen from expressions in theorem IV.1 that $\mathcal{E}^* = O(\sqrt{-\ln\mu})$ as $\mu \to 0$. Hence the bound is not optimal for small μ.

There are interesting directions for future work. A natural one is to extend the results to massive MIMO. Another direction is to come up with better achievability bounds using either a different decoding technique or performing better analysis, for example, using results on Gaussian processes (see [6] where it has been employed for the GMAC). From a practical standpoint, there is also a question of finding MAC architectures that would achieve $\mathcal{E}_{\text{s.u.}}$ for $\mu > 0$.

References

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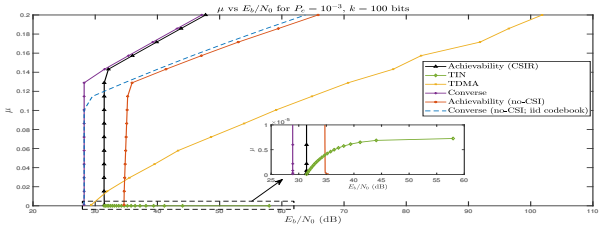

Fig. 1: μ vs E_b/N_0 for $\epsilon \leq 10^{-3}$, $k = 100$

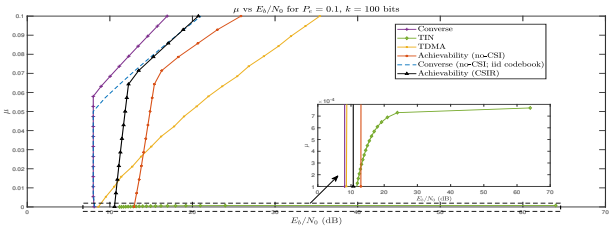

Fig. 2: μ vs E_b/N_0 for $\epsilon \leq 10^{-1}$, $k = 100$

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