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Universal limit linear series and descent of moduli spaces

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Abstract. We introduce a formalism of descent of moduli spaces, and use it to produce limit linear series moduli spaces for families of curves in which the components of geometric fibers may have nontrivial monodromy. We then construct a universal stack of limit linear series over the stack of semistable curves of compact type, and produce new results on existence of real curves with few real linear series.

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1. Introduction

Given a smooth, projective curve C of genus g, recall that a $\mathfrak{g}^r_d$ on C is a linear series of dimension r and degree d; in particular, a $\mathfrak{g}^1_d$ is almost the same as a morphism $C \to \mathbf{P}^1$ of degree d, up to automorphism of the target. On a general complex curve C of even genus g, it is classical that if we set $d = g/2 + 1$, then the number

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of $\mathfrak{g}^1_d$s on C is given by the Catalan number $\frac{1}{d}\binom{2d-2}{d-1}$. It is natural to wonder: if we consider instead C a real curve,[1] what are the possible numbers of real $\mathfrak{g}^1_d$s on C? In [14], the second author showed that there exist real curves C such that all $\frac{1}{d}\binom{2d-2}{d-1}$ of the complex $\mathfrak{g}^1_d$s are in fact real. More recently, Cools and Coppens [2] showed that there also exist real curves C such that only $\binom{d-1}{\lceil \frac{d-1}{2} \rceil}$ of the $\mathfrak{g}^1_d$s on C are real. In this paper, we prove the following as a consequence of general machinery.

Corollary 1.1. *Let g be even, and set $d = g/2 + 1$. Then*

(1) *if d is odd, there exist real smooth projective curves of genus g that carry no real $\mathfrak{g}^1_d$s;*
(2) *if d is even, there exist real smooth projective curves of genus g that carry exactly*

$$\frac{1}{d-1}\binom{d-1}{d/2}$$

real $\mathfrak{g}^1_d$s.

In an apparently completely different direction, the geometry of the moduli spaces $\mathscr{M}_g$ of curves of genus g has been an active subject of research over the past 30 years, focused on questions such as for which g the space $\mathscr{M}_g$ is of general type, and more sharply, what one can say about the codimension-1 subvarieties of $\mathscr{M}_g$ and their associated cohomology classes. In this vein, Khosla [10] has developed machinery for constructing and analyzing families of effective divisors on $\overline{\mathscr{M}}_g$, but Khosla's theory depends on the existence of a suitable partial compactification of the universal moduli space of linear series over $\mathscr{M}_g$. Specifically, a curve of 'compact type' is a nodal curve such that removing any node disconnects the curve; see below for details. Khosla requires an extension of the moduli space over a subset of $\overline{\mathscr{M}}_g$ which is slightly larger than the locus consisting of all curves of compact type. [2] In the 1980's, Eisenbud and Harris [5] introduced their theory of limit linear series for curves of compact type, which suggests what the fibers should be for the universal space Khosla requires. However, while Eisenbud and Harris were able to do a great deal with their theory, constructions of relative moduli spaces of limit linear series in families of curves remained quite limited until recently. In [11,15] the second author and Murray made substantial progress in this direction, and in the present paper, we settle the question completely.

Let $\overline{\mathscr{M}}_g^{\mathrm{ct}}$ denote the open substack of $\overline{\mathscr{M}}_g$ parametrizing semistable curves of compact type; here we may take the base to be Spec $\mathbf{Z}$, or an arbitrary scheme. We then prove the following.

[1] Here we assume that C is 'general' in the sense that it still has $\frac{1}{d}\binom{2d-2}{d-1}$ complex $\mathfrak{g}^1_d$s.

[2] Precisely, Khosla needs a moduli space over the space of 'treelike curves,' which are almost of compact type, except that irreducible components are allowed to have self-nodes. The extension of our results from curves of compact type to arbitrary treelike curves is expected to be routine, but involves constructions using compactified Jacobians which are complementary to what we do in the present paper, and which we do not pursue.

Theorem 1.2. *For any triple g, r, d, there is a proper relative algebraic space*

$$\mathscr{G}^r_d \to \overline{\mathscr{M}}^{\mathrm{ct}}_g$$

whose fiber over a point $[C]$ is the moduli scheme $G^r_d(C)$, which is the classical moduli scheme of linear series in the case that C is smooth, and is a moduli scheme of limit linear series in the case that C is nodal.

In particular, we obtain by pullback moduli spaces of (limit) linear series over arbitrary (flat, proper) families of curves of compact type.

The relationship between Corollary 1.1 and Theorem 1.2 is as follows. First, Corollary 1.1 is proved via degeneration to a curve of compact type, using the theory of limit linear series. But for both results, it is not enough to consider families of curves in which the (geometric) components of every fiber are defined over the base field. On the contrary, we need to consider families where the components have nontrivial monodromy, in either an arithmetic or geometric sense: in the first case, we need to use (geometrically) reducible real curves having components which are exchanged by complex conjugation, while in the second case, even if we work over an algebraically closed field, families arise where components in fibers are exchanged by the monodromy of the family, and hence do not correspond to components of the total space.

In such cases, despite the foundational advances of [11,15], it is not clear how to give a direct definition of limit linear series, because we do not have enough line bundles on the total space to create twists having all the necessary multidegrees. In this paper we develop a general formalism of descent of moduli spaces and apply it to bypass this difficulty and produce the necessary moduli spaces of (limit) linear series. The idea is that any time we have moduli spaces defined only on certain "good" families of varieties, provided the moduli spaces are sufficiently functorial with respect to pullback under a morphism of families, we automatically obtain descent data and thus can descend them under any étale covers. Moreover, when the relevant universal family of varieties admits étale covers which give "good" families, we can carry out this descent universally for all families, obtaining canonical moduli spaces even for families which do not admit "good" étale covers. We thus ultimately construct moduli spaces via descent without giving an intrinsic description of their moduli functors, but we show that the spaces we construct recover the usual moduli functor under any base change where the prior constructions apply. This is carried out in general in Sect. 2. Next, in Sect. 3 we give a direct intrinsic definition of a limit linear series functor as generally as we are able to, we verify that it satisfies the necessary functoriality condition, and we then apply the machinery of Sect. 2 to prove Theorem 1.2, producing the desired moduli spaces of limit linear series over arbitrary families of curves (of compact type). Finally, in Sect. 4 we apply our construction to real and p-adic curves whose components may not be defined over the base field, and conclude the proof of Corollary 1.1. In the process, we develop a more general tool for studying similar enumerative questions on real (or p-adic) linear series. To state the result, we introduce more precise terminology and notation.

The Eisenbud-Harris definition of limit linear series induces a natural scheme structure on an individual curve C of compact type, which we will denote by $G^{r,\mathrm{EH}}_d(C)$. There is a natural bijection between $G^r_d(C)$ and $G^{r,\mathrm{EH}}_d(C)$ which is often an isomorphism; see Theorem 4.6 below for details.

Let K be a field and C_0 a geometrically reduced, geometrically connected, projective curve over K with (at worst) nodal singularities.

- We say that C_0 is *totally split* over K if every irreducible component of C_0 is geometrically irreducible, and every node of C_0 is rational over K.
- If C_0 is totally split, we say that C_0 is *of compact type* over K if its dual graph is a tree, or equivalently, if every node of C_0 is disconnecting, or equivalently, if the Picard variety $\mathrm{Pic}^0(C_0)$ of line bundles on C_0 having degree 0 on every component of C_0 is complete.
- Finally, if C_0 is not necessarily totally split, let L/K be an extension such that C_0 is totally split over L. We say that C_0 is *of compact type* over K if its base extension to C_L is of compact type over L. (One can check that this is independent of the choice of L.)

We will introduce in Definition 3.1.1 below a notion of a "presmoothing family" of curves, which is roughly a family of curves of compact type which admits an étale cover over which we know how to define the functor of limit linear series. We then have the following result.

Theorem 1.3. *Let K be either $\mathbf{R}$ or a p-adic field and C_0 a genus g curve of compact type over K. Let L/K be a finite Galois extension such that the extension $(C_0)_L$ is totally split over L. Given r, d, let*

$$\rho := g - (r+1)(g+r-d)$$

be the Brill-Noether number, and suppose that $\rho = 0$, that the Eisenbud-Harris limit linear series moduli scheme $G^{r,\mathrm{EH}}_d((C_0)_L)$ is finite, and that its set of $\mathrm{Gal}(L/K)$-invariant L-rational points has cardinality n and consists entirely of reduced points.

Given a smooth curve B over K, a point $b_0 \in B(K)$, and a presmoothing family $\pi : C/B$ such that $C_0 = C \times_B b_0$ and the other fibers of π are smooth curves, there exists an open neighborhood U of b_0 in the K-analytic topology on $B(K)$ such that for any $b \in U \smallsetminus \{b_0\}$, the K-scheme $G^r_d(C_b)$ of linear series on C_b is finite, and $G^r_d(C_b)(K)$ has exactly n elements, which are also reduced points.

Using Theorem 1.3, we can consider a degeneration to a curve with a rational component glued to g elliptic tails. Because linear series on $\mathbf{P}^1$ are parametrized by a Grassmannian, and imposing ramification gives a Schubert cycle, the existence of smooth curves with a prescribed number of K-rational linear series can then be reduced to the existence of intersections of Schubert cycles in a suitable Grassmannian with the corresponding number of K-rational points (see Corollary 4.7 below). Applying this together with a family of examples produced by Eremenko and Gabrielov [4] leads to our proof of Corollary 1.1.

2. A descent lemma

In this section we prove a simple lemma that will be useful in constructing the universal limit linear series moduli space. We are motivated by the following question: suppose we have moduli spaces associated to a certain collection of "good" families of varieties in a given class; under what conditions do the moduli spaces extend uniquely to all families in the class? Roughly, our answer is that we obtain the desired extension if the "good" families are closed under étale base change and if the class of varieties admits a 'final object up to étale covers' which contains "good" families. Since the argument is purely formal, we work more generally, replacing our class of families of varieties by an arbitrary site, and the moduli spaces by sheaves of sets on the slice categories of the site.

Precisely, fix a site S. Let $T \subset S$ be a full subcategory. Let $\mathcal{Sh}_S$ denote the stack of sheaves of sets on S. Given a subcategory $U \subset S$, let $\mathcal{Sh}_S(U)$ denote the category of Cartesian functors $U \to \mathcal{Sh}_S$ over S. Concretely, an object of $\mathcal{Sh}_S(U)$ is given by

(1) for each $u \in U$, a sheaf F_u on the slice category $S_{/u}$;
(2) for every arrow $f : u \to v$ between objects of U, an isomorphism of sheaves on $S_{/u}$

$$\beta_f : F_u \xrightarrow{\sim} f^{-1}F_v$$

such that for every further map $g : v \to w$ we have

$$\beta_{g\circ f} = \beta_f \circ f^{-1}\beta_g.$$

Arrows in $\mathcal{Sh}_S(U)$ are given by compatible maps between the sheaves F_u.

Then our main lemma is the following.

Lemma 2.1. *Consider the following conditions.*

(1) *Given an object t of T and a covering $\{s_i \to t\}$ in S, each s_i is in T.*
(2) *There is a small full subcategory $T_0 \subset T$ that is closed under all finite products and finite fiber products in S such that for any object s of S, there is a covering $\{s_i \to s\}$ and maps $s_i \to t_i$ to objects of T_0.*

If T satisfies these conditions then the restriction functor $r : \mathcal{Sh}_S(S) \to \mathcal{Sh}_S(T)$ is an equivalence of categories. In particular, any Cartesian functor $G : T \to \mathcal{Sh}_S$ admits an extension $\widetilde{G} : S \to \mathcal{Sh}_S$, unique up to unique isomorphism.

In other words, we can define sheaves on S if we can define them on a suitable "sieving category". We will apply this in Sect. 3 to describe the universal sheaf of limit linear series. See Remark 2.4 for the key examples of situations in which our conditions (1) and (2) are satisfied.

Proof. The proof is aided by Propositions 2.2 and 2.3 below. We first let T' be the category of all objects of S that admit maps to objects of T_0 and let $T'' = T \cap T'$. The pair $T' \subset S$ satisfies the hypotheses of Proposition 2.2, so the functor

$$\mathcal{Sh}_S(S) \to \mathcal{Sh}_S(T')$$

is an equivalence. The pair $T'' \subset T'$ satisfies the hypotheses of Proposition 2.3, whence the functor

$$\mathcal{Sh}_S(T') = \mathcal{Sh}_{T'}(T') \to \mathcal{Sh}_{T'}(T'') = \mathcal{Sh}_S(T'')$$

is a chain of equivalences. Finally, the pair $T'' \subset T$ also satisfies the hypotheses of Proposition 2.2, so the natural restriction diagram

$$\mathcal{Sh}_S(T) \to \mathcal{Sh}_S(T'')$$

is an equivalence. Composing the diagrams yields the result. □

Proposition 2.2. *Consider the following conditions.*

(1) *Given an object t of T and a covering $\{s_i \to t\}$ in S, each s_i is in T.*
(2) *Any object s of S admits a covering by objects of T.*

If T satisfies these conditions then the restriction functor $r : \mathcal{Sh}_S(S) \to \mathcal{Sh}_S(T)$ is an equivalence of categories. In particular, any Cartesian functor $G : T \to \mathcal{Sh}_S$ admits an extension $\widetilde{G} : S \to \mathcal{Sh}_S$, unique up to unique isomorphism.

Proof. Given an object $s \in S$, choose a covering $\{t_i \to s\}$ by objects of T. By assumption the fiber products $t_{ij} := t_i \times_s t_j$ and $t_{ij\ell} := t_i \times_s t_j \times_s t_\ell$ exist and lie in T. Since $\mathcal{Sh}_S \to S$ is a stack, for any two objects $F, F' \in \mathcal{Sh}_S(S)$ the diagram

$$\mathrm{Hom}(F(s), F'(s)) \longrightarrow \prod \mathrm{Hom}(F(t_i), F(t_j)) \rightrightarrows \prod \mathrm{Hom}(F(t_{ij}), F'(t_{ij}))$$

is exact. This shows that r is fully faithful.

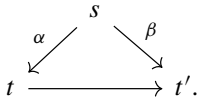
To show that r is essentially surjective, suppose given $G \in \mathcal{Sh}_S(T)$. Given s and the covering $\{t_i \to s\}$ as above, define $\widetilde{G}(s)$ to be objects in $\prod G(t_i)$ with descent data for the covering $\{t_i \to s\}$. Since $\mathcal{Sh}_S$ is a stack, we see that there is a canonical isomorphism $G \to r\widetilde{G}$, and that $\widetilde{G}(s)$ is canonically invariant with respect to the choice of covering $\{t_i \to s\}$. The Cartesian property and uniqueness of $\widetilde{G}$ follow from the Cartesian property of G by the full-faithfulness already established and the stack property of $\mathcal{Sh}_S$. □

Proposition 2.3. *Consider the following conditions.*

(1) *Given an object t of T and a covering $\{s_i \to t\}$ in S, each s_i is in T.*
(2) *There is small full subcategory $T_0 \subset T$ that is closed under all finite products and finite fiber products in S such that every object s of S admits a map to an object of T_0.*

If T satisfies these conditions then the restriction functor $r : \mathcal{Sh}_S(S) \to \mathcal{Sh}_S(T)$ is an equivalence of categories. In particular, any Cartesian functor $G : T \to \mathcal{Sh}_S$ admits an extension $\widetilde{G} : S \to \mathcal{Sh}_S$, unique up to unique isomorphism.

Proof. Given an object $s \in S$, define a cofiltering category F_s as follows. The objects of F_s are arrows $s \to t$ with t in T_0. A morphism from $\alpha : s \to t$ to $\beta : s \to t'$ is a commutative diagram



The category is cofiltering by the assumption that T_0 is closed under products and fiber products.

Suppose given $G \in \mathcal{Sh}_s(T)$. The functor sending $\alpha : s \to t$ in C_t to the sheaf $\alpha^{-1}G(t)$ defines a filtering system of isomorphisms of sheaves on the slice category $S_{/s}$. We define $\widetilde{G}(s)$ to be the colimit of this functor. The Cartesian property of $\widetilde{G}$ follows immediately from the fact that restriction commutes with colimits. Moreover, if s is in T then, since G is Cartesian, we get a canonical isomorphism $\widetilde{G}(s) = G(s)$. Finally, if $\widetilde{G} \in \mathcal{Sh}_S(S)$ is Cartesian, then we must have $\widetilde{G}(s)$ equal to the colimit of the values over F_s, since each of those is canonically isomorphic to $\widetilde{G}(s)$ via the pullback maps. This shows that the construction $G \mapsto \widetilde{G}$ is an essential inverse to r. □

Remark 2.4. One simple way that the hypotheses of Lemma 2.1 can be satisfied is if S contains a final object, T is closed under coverings (condition 1), and T contains a covering of the final object of S (condition 2, with T_0 the collection of all coverings of the final object which lie in T). A similar situation arises if S is the big étale site of a Deligne-Mumford stack $\mathcal{M}$; here we can similarly require that T contains an étale covering of $\mathcal{M}$. In this case, the absolute products required in T_0 are simply fibered products over $\mathcal{M}$.

Remark 2.5. The theory developed here will be relevant to us for the following reason: we will start with a family of curves $X \to B$. The relative moduli space of limit linear series will only be naturally defined over certain base-changed families $X'_B \to B'$, for various $B' \to B$. These B' are diverse: they include an étale covering of B, but also all regular schemes mapping into the smooth locus of the morphism $X \to B$, and all points mapping to B. When we extend the moduli space using the descent theory developed here, we want it to retain its value on the $B' \to B$ where it can already be defined. Thus, we need to interpolate between étale coverings and various other B-schemes. This is what the results of this section accomplish.

Remark 2.6. Concretely, one can realize Proposition 2.2 as follows: suppose given a sheaf G_t for each $t \in T$ and isomorphisms $\phi_f : G_t|_u \to G_u$ for each $f : u \to t$ in T that satisfy the cocycle condition. Then there is a unique sheaf $\widetilde{G}$ on S whose restriction to each $t \in T$ is canonically isomorphic to G_t, up to unique isomorphism.

Remark 2.7. Suppose S is the big étale site of $\operatorname{Spec} k$ and T_0 is the category of iterated fiber products of a single Galois extension $k \subset L$. In this case an object of $\mathcal{Sh}_S(T)$ inherits a Galois descent datum with respect to the extension $k \subset L$, and Lemma 2.1 is solved by Galois descent. (Pedantic note: T itself could be much larger, but since the sheaves are Cartesian functors on the big site, this does not disturb the Galois descent problem.)

As a consequence, under the equivalence

$$r : \mathcal{Sh}_{\operatorname{Spec} k}(\operatorname{Spec} k) \to \mathcal{Sh}_{\operatorname{Spec} k}(T)$$

we have the usual isomorphism

$$G(\operatorname{Spec} k) = r(G)(\operatorname{Spec} L)^{\operatorname{Gal}(L/k)}$$

identifying the global sections of the descended object with the Galois-invariant global sections of the sheaf on T. (The Galois action is induced by the functoriality.)

3. Universal limit linear series by descent

In this section, we recall the fundamental definitions of limit linear series functors (largely following the ideas of §4 of [15]), and verify that they are sufficiently canonical to apply descent theory to construct the universal moduli space. In fact, because we are interested in the universal setting, we will have to address some new technicalities in defining limit linear series functors, which are treated by our definition below of 'consistent' smoothing families.

3.1. Definitions

We begin by defining the families of curves over which we can define a limit linear series functor, and the more general families over which we will be able to descend the resulting moduli spaces.

Definition 3.1.1. A morphism of schemes $\pi : X \to B$ is a *presmoothing family* if:

(I) B is regular and quasicompact;
(II) π is flat and proper;
(III) The fibers of π are curves of compact type;
(IV) Any point in the singular locus of π which is smoothed in the generic fiber is regular in the total space of X.

If the following additional conditions are satisfied, we say that π is a *smoothing family*:

(V) π admits a section;
(VI) every node in every fiber of π is a rational point, and every connected component of the non-smooth locus maps injectively under π;
(VII) every irreducible component of every fiber of π is geometrically irreducible;
(VIII) for any connected component Z of the non-smooth locus of π, if $\pi(Z) \neq B$ then $\pi(Z)$ is a principal closed subscheme, and if $\pi(Z) = B$ and Y, Y' are the closed subschemes of X with $X = Y \cup Y'$ and $Z = Y \cap Y'$, then $\mathscr{O}_Y(Z)|_Z \otimes \mathscr{O}_{Y'}(Z)|_Z \cong \mathscr{O}_Z$.

Conditions (VI) and (VII) require in particular that every fiber is totally split.

It follows from the deformation theory of a nodal curve together with the regularity of B that the image of a connected component of the non-smooth locus is always locally principal (see for instance the deformation theory on p. 82 of [3]), so the condition that it is principal is always satisfied Zariski locally on the base. For the last condition, the existence of the stated Y and Y' depends in an essential way on the fibers being of compact type, and is proved in Proposition 3.1.2 below, which also shows that Z maps isomorphically onto its image in B. In particular, both $\mathscr{O}_Y(Z)|_Z$ and $\mathscr{O}_{Y'}(Z)|_Z$ can be trivialized Zariski locally on B, so we see that both cases of condition (VIII) are satisfied Zariski locally.

Proposition 3.1.2. *If $\pi : X \to B$ satisfies conditions (I)-(VII) of a smoothing family, then every connected component Z of the non-smooth locus of π is regular and maps isomorphically onto its image in B, and $\pi^{-1}(\pi(Z))$ may be written as $Y_Z \cup Y'_Z$, where Y_Z and Y'_Z are closed in $\pi^{-1}(\pi(Z))$, and $Y_Z \cap Y'_Z = Z$.*

Proof. Our hypotheses imply that the map from Z to B is proper, injective and unramified, with trivial residue field extensions. It then follows that it is a closed immersion, using the same argument as in Proposition II.7.3 of [8]. Given this, the regularity is standard; see for instance Proposition 2.1.4 of [15].

Finally, it follows from the regularity that Z (and hence its image) are irreducible, so the existence of Y_Z and Y'_Z in the generic fiber of $\pi(Z)$ follows from condition (VII) of a smoothing family. Taking the closures in $\pi^{-1}(\pi(Z))$, we obtain a decomposition into closed subsets Y_Z and Y'_Z with the correct intersection in the generic fiber. Then Proposition 15.5.3 of [6] implies that Y_Z and Y'_Z are connected in every fiber, which implies because the fibers are of compact type that $Y_Z \cap Y'_Z$ is likewise connected in every fiber. On the other hand, $Y_Z \cap Y'_Z$ is nonregular in $\pi^{-1}(\pi(Z))$, so the regularity of $\pi(Z)$ implies that in any fiber, $Y_Z \cap Y'_Z$ must be contained among the nodes of that fiber, and it must therefore be equal to Z. □

We have the following key observations on construction of (pre)smoothing families:

Proposition 3.1.3. *Any quasicompact family of curves $\pi : X \to B$ which is smooth over $\overline{\mathscr{M}}_g$ and has fibers of compact type is a presmoothing family.*

Given any presmoothing family $\pi : X \to B$, there exists an étale cover of B such that the resulting base change of π is a smoothing family.

Proof. For the first assertion, conditions (II) and (III) of a presmoothing family are immediate, while condition (I) follows from the smoothness of $\overline{\mathscr{M}}_g$, and condition (IV) from the smoothness of the total space of the universal curve over $\overline{\mathscr{M}}_g$ (see Theorem 5.2 of [3]).

For the second assertion, it is standard that étale base change can produce sections through smooth points on every component, and this in turn can be used to ensure that all components of fibers are geometrically irreducible; see the argument for Corollary 4.5.19.3 in **(Err$_{\text{IV}}$, 20)** of [7]. As we have already mentioned, condition (VIII) is satisfied Zariski locally. Finally, since the non-smooth locus of π is finite and unramified over B, if we fix any $b \in B$, Proposition 8(b) of §2.3 of [1] implies that after étale base change, we will have that each connected component of the non-smooth locus has a single (necessarily reduced) point in the fiber X_b, with trivial residue field extension. This implies that after possible further Zariski localization on the base, the same condition will hold on all fibers, which then gives that condition (VI) is satisfied as well. □

We will ultimately show that limit linear series moduli spaces can be descended to any presmoothing family. The smoothing families are almost the families for which we can give a direct definition of the limit linear series functor, but we will need to impose one additional condition, of a more combinatorial nature, which will be satisfied Zariski locally on the base. Before defining the condition, we need to introduce a few preliminaries.

Notation 3.1.4. Given a smoothing family $\pi : X \to B$, let $Z(\pi)$ denote the set of connected components of the non-smooth locus of π, and let $Y(\pi)$ denote the corresponding set of closed subsets arising as Y_Z or Y'_Z in Proposition 3.1.2. Let

$\varpi : Y(\pi) \to Z(\pi)$ denote the resulting surjective two-to-one map, and given $Y \in Y(\pi)$, let Y^c denote the other element of $Y(\pi)$ with $\varpi(Y^c) = \varpi(Y)$.

Thus, for $Y \in Y(\pi)$, if $Z = \varpi(Y)$, we have that $\pi^{-1}(\pi(Z))$ is the union along Z of Y with Y^c.

The following preliminary definition will be of basic importance in the definition of a limit linear series.

Definition 3.1.5. Let $\pi : X \to B$ be a smoothing family. Given $d > 0$, a *multidegree* on π of total degree d is a map $\mathrm{md} : Y(\pi) \to \mathbf{Z}$ such that for each $Y \in Y(\pi)$, we have $\mathrm{md}(Y) + \mathrm{md}(Y^c) = d$.

Thus, we can think of a multidegree as specifying how the total degree d is distributed on each 'side' of every node. This translates into a usual multidegree on every fiber of π as follows.

Proposition 3.1.6. *Let $\pi : X \to B$ be a smoothing family, and* md *a multidegree of total degree d. For every $b \in B$, there is a unique way to assign integers* $\mathrm{md}_b(Y)$ *to each component Y of the fiber X_b such that for every $Z \in Z(\pi)$ with $b \in \pi(Z)$, and each of the two $Y_Z \in Y(\pi)$ with $\varpi(Y_Z) = Z$, we have*

$$\sum_{Y \subseteq Y_Z \cap X_b} \mathrm{md}_b(Y) = \mathrm{md}(Y_Z). \tag{3.1}$$

Proof. The formula (3.1) implies immediately that we must have $\sum_{Y \subseteq X_b} \mathrm{md}_b(Y) = d$, by considering any Z and both choices of Y_Z with $\varpi(Y_Z) = Z$. We then see that (3.1) also determines $\mathrm{md}_b(Y)$ as

$$\mathrm{md}_b(Y) = d - \sum_{Z : Z \cap Y \neq \emptyset} \mathrm{md}(Y_Z), \tag{3.2}$$

where each Y_Z is chosen not to contain Y. Indeed, the sum above is obtained by summing over all nodes on Y of the total degrees on the 'other side' of each node from Y, which means that the degree on every component other than Y occurs exactly once in the sum. Uniqueness follows immediately, and we also claim that if $\mathrm{md}_b(Y)$ is given by (3.2), it necessarily satisfies (3.1). Indeed, the number of components of Y_Z is equal to the number of nodes lying on Y_Z (including Z), and if we sum the formula of (3.2) over all Y in a given Y_Z, we will subtract off $\mathrm{md}(Y_{Z'})$ and $\mathrm{md}(Y^c_{Z'})$ exactly once each for all $Z' \neq Z$ meeting Y_Z, and we will also subtract off $\mathrm{md}(Y^c_Z)$ exactly once, so we can rearrange the terms of the sum to obtain $d - \mathrm{md}(Y^c_Z) = \mathrm{md}(Y_Z)$, together with $d - \mathrm{md}(Y_{Z'}) - \mathrm{md}(Y^c_{Z'}) = 0$ for each $Z' \neq Z$ lying on Y_Z. □

Definition 3.1.7. Let X_0 be a totally split nodal curve. A multidegree on X_0 is *concentrated* on a component Y if for all $Y' \neq Y$, it assigns degree 0 to Y'.

If $\pi : X \to B$ is a smoothing family, and $d > 0$, a multidegree md on π is *uniformly concentrated* if for all $b \in B$, the induced md_b is concentrated on some component Y of X_b.

Our condition on smoothing families is the following, asserting in essence that we have sufficiently many uniformly concentrated multidegrees.

Definition 3.1.8. Let $\pi : X \to B$ be a smoothing family, and $\mathrm{md}_I = \{\mathrm{md}_i\}_{i \in I}$ a finite collection of uniformly concentrated multidegrees on π of some fixed total degree d. We say that md_I *sufficient* if for every $b \in B$ and every component $Y \subseteq X_b$ there exists $i \in I$ such that the restriction of md_i to X_b is concentrated on Y. We say the smoothing family $\pi : X \to B$ is *consistent* if it admits a sufficient collection of uniformly concentrated multidegrees.

Remark 3.1.9. Note that for every $b \in B$ and $Y \subseteq X_b$, there is always a multidegree on π which, when ignoring other fibers of π, is concentrated on Y. But there does not appear to be any reason to expect that there is necessarily a multidegree which is uniformly concentrated and concentrated on Y. In principle, there could be some $b' \in B$ such that if a multidegree is concentrated on Y in X_b, it cannot be concentrated on any component of $X_{b'}$.

For instance, consider the case that B is a surface, with the non-smooth locus of π consisting of two connected components Z_1, Z_2, whose images in B meet at two points b_1, b_2. Thus, the fibers X_b will have three components when $b = b_1$ or b_2. We can always fix choices of $Y_{Z_1}, Y_{Z_1}^c$ and $Y_{Z_2}, Y_{Z_2}^c$ so that in X_{b_1}, both Y_{Z_1} and Y_{Z_2} consist of two components, with $Y_{Z_1} \cap Y_{Z_2}$ then yielding the "middle" component of X_{b_1}. It seems as though it could be possible that we then have Y_{Z_1} and Y_{Z_2} both consisting of only a single component in X_{b_2}, so that $Y_{Z_1} \cap Y_{Z_2}$ is empty in X_{b_2}. If this is the case, we see that the only choice of md which is concentrated on $Y_{Z_1} \cap Y_{Z_2}$ in X_{b_1} is given by setting $\mathrm{md}(Y_{Z_1}) = d$ and $\mathrm{md}(Y_{Z_2}) = d$, but the resulting multidegree in X_{b_2} will not be concentrated on any component (it will have degree d on the extremal components and degree $-d$ in the middle).

Remark 3.1.10. There is not in general a unique minimal sufficient collection of uniformly concentrated multidegrees: for instance, if we have that B is one-dimensional, and there are two points $b_1, b_2 \in B$ such that X_b is smooth if $b \neq b_1, b_2$ and X_{b_1} and X_{b_2} each have one node, then because the behavior at X_{b_1} and X_{b_2} is independent, there are four uniformly concentrated multidegrees, and there are two different ways of choosing two of these four to get a sufficient collection.

On the other hand, while the collection of all uniformly concentrated multidegrees is canonical, it turns out not to be large enough in general to allow for the most transparent treatment of our construction. Indeed, distinct multidegrees may become the same under base change, so when considering base change of limit linear series we will naturally be led to allow the possibility that our collection md_I has repeated entries.

Proposition 3.1.11. *If $\pi : X \to B$ is a smoothing family then for any $b \in B$, there is a Zariski neighborhood U of b on which π is consistent.*

In addition, if $(\mathrm{md}_i)_{i \in I}$ is a sufficient collection of uniformly concentrated multidegrees, and md *is any uniformly concentrated multidegree, then there is a Zariski neighborhood V of b and $i \in I$ such that* $\mathrm{md} = \mathrm{md}_i$ *on V.*

Proof. Given $b \in B$, using the language of [15], we choose U so that X_b meets every node of π over U, and furthermore so that π is 'almost local' over U – see Definition 2.2.2 and Remark 2.2.3 of [15]. Intuitively, this means that every fiber over U is naturally a partial smoothing of the chosen fiber X_b. In this situation, it is clear that a multidegree is uniformly concentrated over U if and only if its restriction to X_b is concentrated on some component, and it follows easily that the smoothing family will be consistent over U.

Similarly, if $(\mathrm{md}_i)_{i\in I}$ is sufficient, then by definition there is some i such that md agrees with md_i on the fiber X_b. But then it is clear that if we set V as above so that π is almost local, then agreement on X_b implies agreement on all of V. □

Notation 3.1.12. Given a smoothing family $\pi : X \to B$, and $Y \in Y(\pi)$ over $Z \in Z(\pi)$, define $\mathscr{O}^Y$ as follows: if $\pi(Z) \neq B$, then $\mathscr{O}^Y = \mathscr{O}_X(Y)$; if $\pi(Z) = B$, then $\mathscr{O}^Y$ is the line bundle on $\mathscr{O}_X$ obtained by gluing $\mathscr{O}_Y(-Z)$ to $\mathscr{O}_{Y^c}(Z)$ along Z.

In the above, we use that when $\pi(Z) = B$ we have $Y \cap Y^c \cong B$, so that the restriction maps induce a canonical isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y) \times_{\mathrm{Pic}(Z)} \mathrm{Pic}(Y^c)$. We also use condition (VIII) of a smoothing family to ensure that the images in $\mathrm{Pic}(Z)$ agree. In this case we also obtain that sections of $\mathscr{O}^Y$ correspond to pairs of sections of $\mathscr{O}_Y(-Z)$ and $\mathscr{O}_{Y^c}(Z)$ which 'agree' on Z; agreement depends in principle on an isomorphism $\mathscr{O}_Y(-Z)|_Z \cong \mathscr{O}_{Y^c}(Z)|_Z$, but is for instance canonical in the cases of sections vanishing along Z.

The purpose of this construction is as follows. Given a line bundle $\mathscr{L}$ on X, its multidegree is defined by looking at the degrees of the restrictions to each $Y \in Y(\pi)$. On fibers, this is equivalent data to a multidegree in the usual sense of the degree on each component, as described by Proposition 3.1.6. Now, given $\mathscr{L}$ of multidegree md, for any $Y \in Y(\pi)$ we have that the multidegree of $\mathscr{L} \otimes \mathscr{O}^Y$ is obtained from md by subtracting 1 from Y and adding 1 to Y^c. Consequently, twisting by the different $\mathscr{O}^Y$, we can change the multidegree of a line bundle from any multidegree to any other (of the same total degree). Given a line bundle $\mathscr{L}$ on X, and a multidegree md, we denote by $\mathscr{L}_{\mathrm{md}}$ the twist of $\mathscr{L}$ having multidegree md, which is unique up to isomorphism.

Data 3.1.13. Suppose $\pi : X \to B$ is a consistent smoothing family. A *limit linear series datum* on π consists of the following:

(1) A sufficient collection md_I of uniformly concentrated multidegrees on π (always assumed of some fixed degree).
(2) For each $Y \in Y(\pi)$ a section s_Y of $\mathscr{O}^Y$ vanishing precisely on Y.
(3) For each $Z \in Z(\pi)$, an isomorphism

$$\theta_Z : \mathscr{O}^{Y_Z} \otimes \mathscr{O}^{Y'_Z} \xrightarrow{\sim} \mathscr{O}_X,$$

where Y_Z and Y'_Z are the elements of $Y(\pi)$ lying over Z.
(4) A distinguished element $i_0 \in I$.

We will denote a limit series datum on π by $\mathscr{D} = (\mathrm{md}_I, \{s_Y\}, \{\theta_Z\}, i_0)$. Given a consistent smoothing family π, we will write $\mathrm{LLSD}(X/B)$ for the set of limit linear series data attached to π.

The purpose of items (2)-(4) of a limit linear series datum is to allow us to pin down the line bundles $\mathscr{L}_{\mathrm{md}}$ precisely, together with maps between them. Specifically, for any given md, there is a unique (up to reordering) minimal collection of twists by the $\mathscr{O}^Y$ in order to get from multidegree md_{i_0} to multidegree md; we then define $\mathscr{L}_{\mathrm{md}}$ to be obtained from $\mathscr{L}$ by the corresponding tensor product. Then, because s_Y induces a map from any line bundle to its twist by $\mathscr{O}^Y$, and $\theta_Z \circ s_{Y^c}$ induces a map in the other direction, for each md, md$'$ we have that a limit linear series datum induces a canonical map $\mathscr{L}_{\mathrm{md}} \to \mathscr{L}_{\mathrm{md}'}$.

Proposition 3.1.14. *If $\pi : X \to B$ is a consistent smoothing family, then we have* $\mathrm{LLSD}(X/B) \neq \emptyset$.

Proof. The existence of the md_I and i_0 are by definition, and likewise the s_Y exist by the construction of $\mathscr{O}^Y$. Finally, the θ_Z exist by construction in the case that $\pi(Z) = B$, and by the hypothesis that $\pi(Z)$ is principal in B in the case that $\pi(Z) \neq B$. □

We now state the general definition of limit linear series for families of curves of compact type.

Definition 3.1.15. Let $f : T \to B$ be a B-scheme, and write $\pi' : X \times_B T \to T$. Suppose $\mathscr{D} := (\mathrm{md}_I, \{s_Y\}, \{\theta_Z\}, i_0)$ is a limit linear series datum. A *T-valued $\mathscr{D}$-limit linear series* of rank r and degree d on π consists of

(1) an invertible sheaf $\mathscr{L}$ of multidegree md_{i_0} on $X \times_B T$, together with
(2) for each $i \in I$ a rank $(r+1)$ subbundle $\mathscr{V}_i \subseteq \pi'_*\mathscr{L}_{\mathrm{md}_i}$

satisfying the following condition: for any multidegree md on π of total degree d, the map

$$\pi'_*\mathscr{L}_{\mathrm{md}} \to \bigoplus_{i \in I} \left(\pi'_*\mathscr{L}_{\mathrm{md}_i}\right)/\mathscr{V}_i \tag{3.3}$$

induced by the s_Y and θ_Z has $(r+1)$st vanishing locus equal to all of T.

In the above, to say that $\mathscr{V}_i$ is a rank-$(r+1)$ subbundle of $\pi'_*\mathscr{L}_{\mathrm{md}_i}$ means that it is locally free of rank $r+1$ and the injection into $\pi'_*\mathscr{L}_{\mathrm{md}_i}$ is preserved under base change (where on $\mathscr{L}_{\mathrm{md}_i}$, base change is applied prior to pushforward). The $(r+1)$st vanishing locus of (3.3) is a canonical scheme structure on the closed subset of points on which the map has kernel of dimension at least $r+1$, defined in terms of perfect representatives of $R\pi'_*\mathscr{L}_{\mathrm{md}}$ and the $R\pi'_*\mathscr{L}_{\mathrm{md}_i}$; see Appendix B.3 of [15] for details.

Definition 3.1.16. Two T-valued $\mathscr{D}$-limit linear series $(\mathscr{L}, (\mathscr{V}_i)_i)$ and $(\mathscr{L}', (\mathscr{V}'_i)_i)$ are *equivalent* if there exists a line bundle $\mathscr{M}$ on T and an isomorphism $\mathscr{L} \xrightarrow{\sim} \mathscr{L}' \otimes \pi'^*\mathscr{M}$ sending each $\mathscr{V}_i$ to $\mathscr{V}'_i \otimes \mathscr{M}$ under the identification $\pi'_*\left(\mathscr{L}' \otimes \pi'^*\mathscr{M}\right) = \pi'_*\mathscr{L}' \otimes \mathscr{M}$.

Definition 3.1.17. Given a consistent smoothing family $\pi : X \to B$ with a limit linear series datum $\mathscr{D}$, the *functor of $\mathscr{D}$-limit linear series*, denoted $\mathscr{G}^r_d(X/B, \mathscr{D})$, is the functor whose value on a B-scheme $T \to B$ is the set of equivalence classes of T-valued $\mathscr{D}$-limit linear series of rank r and degree d.

3.2. Functoriality

In the remainder of this section, we show that the limit linear series functor is sufficiently canonical to apply our descent machinery. The observation of fundamental importance to us is the following, which we will use to show not only that our definition is independent of the choices made, but also that it is well-behaved under base change.

Proposition 3.2.1. *Suppose $\pi : X \to B$ is a consistent smoothing family with a limit linear series datum $\mathscr{D} = (\mathrm{md}_I, \{s_Y\}, \{\theta_Z\}, i_0)$. Suppose $(\mathscr{L}, (\mathscr{V}_i)_i)$ is a $\mathscr{D}$-limit linear series of rank r, and let* md *be a uniformly concentrated multidegree, not necessarily contained among the chosen* md_i*. Then the map* (3.3) *has empty $(r+2)$nd vanishing locus, so that the kernel is universally a subbundle of rank $r+1$.*

Moreover, if we let $\mathscr{V}$ denote the kernel of (3.3)*, then for any multidegree* md′*, the map*

$$\pi'_*\mathscr{L}_{\mathrm{md}'} \to \left(\pi'_*\mathscr{L}_{\mathrm{md}}\right)/\mathscr{V} \oplus \bigoplus_{i\in I}\left(\pi'_*\mathscr{L}_{\mathrm{md}_i}\right)/\mathscr{V}_i$$

has $(r+1)$st vanishing locus equal to all of T.

Proof. By definition of a limit linear series, (3.3) has vanishing locus equal to all of T, so in order to prove that the kernel is universally a subbundle of rank $r+1$, it suffices to show that the $(r+2)$nd vanishing locus is empty, which is the same as saying that there are no points of t at which the kernel has dimension at least $r+2$. Given $t \in T$, denote by X'_t the corresponding fiber of π'; then the fiber of (3.3) at t is

$$H^0(X'_t, \mathscr{L}_{\mathrm{md}}|_{X'_t}) \to \bigoplus_{i\in I} H^0(X'_t, \mathscr{L}_{\mathrm{md}_i}|_{X'_t})/\mathscr{V}_i|_t.$$

Now, because md is uniformly concentrated, its restriction to X'_t is concentrated on some component of X'_t. On the other hand, because $(\mathrm{md}_i)_i$ is assumed sufficient, this means that md is equal to some md_i after restriction to X'_t, so that $H^0(X'_t, \mathscr{L}_{\mathrm{md}}|_{X'_t}) = H^0(X'_t, \mathscr{L}_{\mathrm{md}_i}|_{X'_t})$. But then the kernel of the above map is contained in $\mathscr{V}_i|_t$, which is $(r+1)$-dimensional, proving the first statement.

Next, the second statement can be checked Zariski locally, so by Proposition 3.1.11, we may assume that in fact $\mathrm{md} = \mathrm{md}_i$ for some i, and then necessarily $\mathscr{V}$ is identified with $\mathscr{V}_i$ under the resulting isomorphism $\mathscr{L}_{\mathrm{md}} \xrightarrow{\sim} \mathscr{L}_{\mathrm{md}_i}$. Thus, the kernel of the given map is (universally) identified with the kernel of (3.3) in multidegree md′, so the $(r+1)$st vanishing loci also agree, as desired (see Proposition B.3.2 of [15]). □

Corollary 3.2.2. *Suppose $\pi : X \to B$ is a consistent smoothing family.*

(1) *Given two limit linear series data $\mathscr{D}$ and $\mathscr{D}'$ on π, there is a canonical isomorphism*

$$\tau^{X/B}_{\mathscr{D},\mathscr{D}'} : \mathscr{G}^r_d(X/B, \mathscr{D}) \xrightarrow{\sim} \mathscr{G}^r_d(X/B, \mathscr{D}').$$

(2) *For any base change $B' \to B$ preserving the smoothing family conditions, there is a canonical isomorphism*

$$\beta^{\mathscr{D}}_{B'/B} : \mathscr{G}^r_d(X_{B'}/B', \mathscr{D}_{B'}) \xrightarrow{\sim} \mathscr{G}^r_d(X/B, \mathscr{D}) \times_B B'.$$

By "canonical" we mean that for any triple $\mathscr{D}, \mathscr{D}', \mathscr{D}''$ of limit linear series data we have that

$$\tau^{X/B}_{\mathscr{D},\mathscr{D}''} = \tau^{X/B}_{\mathscr{D}',\mathscr{D}''} \circ \tau^{X/B}_{\mathscr{D},\mathscr{D}'},$$

and similarly for any triple $B'' \to B' \to B$ preserving the smoothing family conditions, we have that

$$\beta^{\mathscr{D}}_{B''/B} = \beta^{\mathscr{D}}_{B'/B}|_{B''} \circ \beta^{\mathscr{D}_{B'}}_{B''/B'}.$$

Proof. First, if $(\mathrm{md}_i)_i$ and i_0 are fixed, the choices of the s_Y and θ_Z are only used to determine a subfunctor of the functor of all tuples of $(\mathscr{L}, (\mathscr{V}_i)_{i\in I})$, so there is a canonical notion of equality of these subfunctors. We see from the definitions that both the s_Y and θ_Z are unique up to $\mathscr{O}_T^\times$-scalar; for the former, where Y surjects onto B it is crucial that the (fibers of the) support of s_Y is a connected curve, as otherwise s_Y would only be unique up to independent scaling on each connected component. This implies that the rank of (3.3) is independent of the choices of s_Y and θ_Z, as desired. Similarly, because isomorphisms of line bundles are unique up to $\mathscr{O}_T^\times$-scaling, induced identifications of subbundles of global sections are unique, independent of the choice of isomorphism. Because twisting provides a canonical identification between isomorphism classes of line bundles of multidegree md_{i_0} and line bundles of any other given multidegree (of the same total degree), we conclude that the limit linear series functors associated to different choices of i_0 are canonically isomorphic.

Next, in order to show independence of the choice of $(\mathrm{md}_i)_i$, we observe that it is enough to construct canonical isomorphisms when we add a uniformly concentrated multidegree to an existing collection: indeed, we can then compare the functors for any two collections by comparing each to the functor obtained from the union of the collections. Accordingly, suppose we have a collection $(\mathrm{md}_i)_i$, and an additional uniformly concentrated multidegree md'. We define the obvious forgetful map from the functor of limit linear series associated to $(\mathrm{md}_i)_i \cup (\mathrm{md}')$ to the one associated to $(\mathrm{md}_i)_i$, noting that the $(r+1)$st vanishing locus of (3.3) can only increase if we drop a concentrated multidegree from the target. That this forgetful map is an isomorphism of functors then follows immediately from Proposition 3.2.1, as the only possibility is to have $\mathscr{V} \subseteq \pi'_*\mathscr{L}_{\mathrm{md}'}$ be the kernel of (3.3) in multidegree md'.

It remains to address compatibility with base change. Because the Picard functor is compatible with base change, the main complications arise from changes in $Z(\pi)$, which can occur either because the image of B' may be disjoint from the image of some $Z \in Z(\pi)$, or because the preimage in B' of some $Z \in Z(\pi)$ may decompose into two or more connected components. This means in particular that limit linear series data may not have canonical pullbacks. In general, there are natural maps $Z(\pi') \to Z(\pi)$ and $Y(\pi') \to Y(\pi)$ which induce a pullback on multidegrees and on uniformly concentrated multidegrees, and a sufficient collection of

uniformly concentrated multidegrees pulls back to a sufficient collection. However, the pullback map on multidegrees is in general neither injective nor surjective. If we pull back our given collection $(\mathrm{md}_i)_i$ together with i_0, we find that as above, we are comparing two closed subfunctors of a fixed functor: the functor of all tuples of $(\mathscr{L}, (\mathscr{V}_i)_{i\in I})$. Note however that when the map $Z(\pi') \to Z(\pi)$ is not injective, we do not have that each $\mathscr{O}^{Y'}$ can be chosen to be a pullback of some $\mathscr{O}^{Y}$. In addition, when $Z(\pi') \to Z(\pi)$ is not surjective, we will have that different multidegrees for π become the same on π'. Thus, for a given multidegree md and its pullback md' to π', it is not necessarily the case that when we apply our construction to π' to obtain the map (3.3) for md', the result is exactly equal to the pullback of the map for md. However, we claim that each summand of the two maps agree up to $\mathscr{O}_{B'}^{\times}$-scalar, so that the resulting $(r+1)$st vanishing loci conditions are the same. Indeed, the map from multidegree md to some md_i is obtained by a sequence of twists (or inverse twists, using the $\theta_{\varpi(Y)}$) by different $\mathscr{O}^{Y}$. If $\varpi(Y)$ remains nonempty and connected in π', then the pullback of s_Y is a valid choice for $s_{Y'}$, and the claim is clear. Similarly, if $\varpi(Y)$ becomes empty, then the twist by $\mathscr{O}^{Y}$ doesn't change the multidegree for π', and the pullback of s_Y can be used to trivialize the pullback of $\mathscr{O}^{Y}$. Finally, if $\varpi(Y)$ breaks into distinct connected components, denote these by $Y'_1, \ldots, Y'_m$. Then twisting by $\mathscr{O}^{Y}$ pulls back to a composition of twists by the Y'_i, and the pullback of s_Y will agree up to $\mathscr{O}_{B'}^{\times}$-scalar with the product of the $s_{Y'_i}$, so we conclude the claim. It follows that the condition on the $(r+1)$st vanishing locus for the given π' and md' agrees with the pullback of the same condition for π and md.

Finally, although not every md' for π' arises by pullback from an md – because a $Z \in Z(\pi)$ may break into multiple connected components after base change – we can check equality of the subfunctors after Zariski localization. If we restrict to almost local open subsets of B' (as described in the proof of Proposition 3.1.11), then the maps $Z(\pi') \to Z(\pi)$ and $Y(\pi') \to Y(\pi)$ become injective, so the induced map on multidegrees is surjective, and we obtain the desired equality of functors. □

Remark 3.2.3. Note that nearly all the hypotheses of a smoothing family are automatically preserved under base change (to a suitable base): the only one which is not necessarily preserved is the regularity hypothesis on X, and of course this is still preserved if the base change is étale, or if B' is a point. The only other condition which is not obviously preserved under base change is condition (VIII). Although scheme-theoretic image is not in general preserved under base change, in our situation Proposition 3.1.2 says that the map to B from each connected component of the non-smooth locus of π is a closed immersion, so the condition that the image is principal is indeed preserved under base change. On the other hand, if we have $Z \in Z(\pi)$ such that $\pi(Z) \neq B$, but the image of B' is contained in $\pi(Z)$, then we observe that the first case of condition (VIII) for Z implies that $\mathscr{O}_X(Y_Z) \otimes \mathscr{O}_X(Y_Z^c) \cong \mathscr{O}_X$, which implies in turn that the second case of condition (VIII) is satisfied for Z'.

Additionally, it is clear from the definition that the consistent condition is satisfied under base change of a smoothing family. In particular, the corollary implicitly

includes the statement that in fibers corresponding to smooth curves we recover the usual functor of linear series.

Conceptually, we should think of independence of base change as being a consequence of independence of limit linear series data. Indeed, in simple cases (for instance, in a family with a unique singular fiber), there may be a natural minimal choice of a sufficient collection of concentrated multidegrees, but this will not be preserved under base change: for instance, any collection of more than one concentrated multidegree will become redundant under restriction to a smooth fiber.

Given the first statement of Corollary 3.2.2, we will henceforth drop the $\mathscr{D}$ from $\mathscr{G}^r_d(X/B, \mathscr{D})$. We can formalize the functor $\mathscr{G}^r_d(X/B)$ either by fixing a choice of $\mathscr{D}$ for each X/B, or by defining $\mathscr{G}^r_d(X/B)$ to be the limit over all choices of $\mathscr{D}$ of the spaces $\mathscr{G}^r_d(X/B, \mathscr{D})$ under the isomorphisms provided by Corollary 3.2.2. From either of these points of view, the second statement of Corollary 3.2.2 then tells us that if X/B is a consistent smoothing family, and $B' \to B$ is any morphism such that $X_{B'} \to B'$ is a (necessarily consistent) smoothing family, we have a canonical isomorphism

$$\beta_{B'/B} : \mathscr{G}^r_d(X'/B') \xrightarrow{\sim} \mathscr{G}^r_d(X/B) \times_B B'.$$

Moreover, for any further morphism $B'' \to B'$ such that $X_{B''} \to B''$ is also a smoothing family, we have that

$$\beta_{B''/B} = \beta_{B'/B}|_{B''} \circ \beta_{B''/B'}.$$

Notation 3.2.4. Given a presmoothing family $\pi : X \to B$ with a section $s : B \to X$, let $\mathrm{Pic}^{d,s}(X/B)$ denote the Picard scheme of line bundles having degree d on the component of each fiber of π containing s, and degree 0 on every other component.

Proposition 3.2.5. *If $\pi : X \to B$ is a consistent smoothing family, then the functor $\mathscr{G}^r_d(X/B)$ is representable by a scheme $G^r_d(X/B)$ that is proper over B, and canonically compatible with base changes that preserve the consistent smoothing family condition.*

Moreover, we have that $G^r_d(X/B)$ has universal relative dimension at least the Brill-Noether number ρ. If we further have that for some $b \in B$, the space $G^r_d(X_b)$ has dimension ρ, then we have that $G^r_d(X/B)$ is Cohen-Macaulay and flat over B at every point over b.

The terminology of "universal relative dimension at least ρ" is as introduced in Definition 3.1 of [16].

Proof. Given the definitions, representability is rather standard. First, our functor is visibly a Zariski sheaf, so it suffices to show representability locally on the base. Now, we have already mentioned twisting by the $\mathscr{O}^Y$ allows us to move between any two multidegrees of fixed total degree, and the hypothesized section means that line bundles of every degree exist. Thus, if we choose any multidegree which is positive on every component of every fiber – for instance, by summing all our uniformly concentrated multidegrees – we can produce a relatively ample

line bundle. Passing to an open cover of B if necessary we may assume we have a relatively ample divisor D. The existence of the section also implies that the relative Picard functor $\mathrm{Pic}^{d,s}(X/B)$ is representable, and carries a Poincare line bundle $\widetilde{\mathscr{L}}$. Twisting by a sufficiently high multiple of D, we can then construct $G^r_d(X/B)$ as a closed subscheme of a product of relative Grassmannians over $\mathrm{Pic}^{d,s}(X/B)$, with the closed conditions given by vanishing along the given multiple of D, intersected with the $(r+1)$st vanishing loci occurring in the definition of limit linear series.

Again passing to an open cover of the base, we can assume our smoothing family is "almost local" in the sense of Definition 2.2.2 of [15] (see also the proof of Proposition 3.1.11). We briefly sketch the ingredients of the remaining argument before giving the details. In [11], a theory of 'linked determinantal loci' is developed which is precisely tailored to capture the conditions on $(r+1)$st vanishing loci for the maps (3.3) which arise in the definition of limit linear series for a curve with two components. The main result of [11] is that these linked determinantal loci have the desired behavior in terms of codimension and Cohen-Macaulayness. In [11,12] it is described how the linked determinantal locus construction can be applied to deduce the desired results on limit linear series for arbitrary curves of compact type (and in fact more generally). Thus, the desired statements are essentially already contained in these papers. However, there are some differences in hypotheses and details of constructions which should be addressed. The more basic is that the dimensionality statement, proved as Theorem 6.1 of [12], places more restrictions on its families of curves, due to the desire to consider curves not of compact type. However, the argument goes through verbatim in our case. The idea is that we can realize the limit linear series spaces for arbitrary curves of compact type as being cut out by conditions coming from pairs of adjacent components.

We then deduce Cohen-Macaulayness and flatness from Theorem 3.1 (see also Remark 3.5) of [11]. The complication here is that there are several variants of our definition of limit linear series used for different purposes, and while they all agree set-theoretically, we have not previously shown that their scheme structures agree. Specifically, the scheme structure used in [11] imposes the condition on the $(r+1)$st vanishing locus of (3.3) on a smaller collection of multidegrees; thus, what we know is that our scheme structure is a closed subscheme of the one in [11], with the same support, and in order to complete the proof of the proposition we want to show that it is in fact the same scheme. While we consider all multidegrees md in our definition, the definition used in [11] considers only multidegrees which are nonnegative on all components and equal to zero on all but two adjacent components. We will show that the schemes agree on fibers; it will then follow from Nakayama's lemma and the flatness of the larger scheme that the two scheme structures must agree. Now, the argument of the penultimate paragraph of the proof of Theorem 4.3.4 of [15] shows that imposing the vanishing conditions on the multidegrees considered in [11] then implies that the same conditions are satisfied on all multidegrees which are "nonnegative" in the sense of being nonnegative on every $Y \in Y(\pi)$. Then, the proof of Proposition 3.4.12 of [15] shows that for any multidegree md, there is some such nonnegative multidegree md' such that the kernel of (3.3) for md' injects into the kernel for md; it then follows from Corollary B.3.5 of [15] that the vanishing

condition for md′ implies the same vanishing condition for md. We thus obtain the desired agreement of scheme structures. □

Corollary 3.2.6. *Suppose $\pi : X \to B$ is a presmoothing family, and we also fix r, d. Then there exists a proper algebraic space $G^r_d(X/B)$ over B such that for any base change $B' \to B$ making the resulting $\pi' : X' \to B'$ into a consistent smoothing family, we have a functorial identification $G^r_d(X/B) \times_B B' \cong G^r_d(X'/B')$.*

If moreover π admits a section s, then $G^r_d(X/B)$ maps to $\mathrm{Pic}^{d,s}(X/B)$, compatibly with the above identification.

In general, $G^r_d(X/B)$ has universal relative dimension at least the Brill-Noether number ρ, and if for some b we have that the fiber of $G^r_d(X/B)$ has dimension ρ over b, then $G^r_d(X/B)$ is Cohen-Macaulay and flat over b.

Proof. Let T be the full subcategory of the big étale site of Sch_B consisting of arrows $B'' \to B$ such that $X_{B''}/B''$ is a consistent smoothing family. By Propositions 3.1.3 and 3.1.11, there is an étale cover $B' \to B$ such that $X' := X \times_B B'$ is a consistent smoothing family. It follows from Remark 2.4 that T satisfies the conditions of Lemma 2.1. Applying that Lemma, there is a sheaf $\mathscr{G}^r_d(X/B)$ on the big étale site of Sch_B whose value on any $B'' \to B$ over which $X_{B''}/B''$ is a consistent smoothing family is the sheaf $\mathscr{G}^r_d(X_{B''}/B'')$ of Definition 3.1.17. Since $\mathscr{G}^r_d(X_{B'}/B')$ is a proper scheme, we have that $\mathscr{G}^r_d(X/B)$ is itself a proper algebraic space over B (see Tags 083R, 0410, 03KG and 03KM of [17]).

Suppose π admits a section s. For any $B'' \to B$ such that $X_{B''}/B''$ is a consistent smoothing family, there is a forgetful morphism $\mathscr{G}^r_d(X_{B''}/B'') \to \mathrm{Pic}^{d,s}(X_{B''}/B'')$ defined by choosing the (unique) twist of the underlying line bundle that has multidegree concentrated along the image of s, as described in Notation 3.2.4. This defines a morphism of sheaves on T. By Lemma 2.1, these extend uniquely to a morphism $\mathscr{G}^r_d(X/B) \to \mathrm{Pic}^{d,s}(X/B)$ of sheaves on Sch_B.

Universal relative dimension for stacks is defined (Definition 7.1 of [16]) in terms of descent from a smooth cover, so the desired statement follows from Proposition 3.2.5. Similarly, since fiber dimension is invariant under base extension and flatness and Cohen-Macaulayness descend, the final statements also follow from Proposition 3.2.5. □

More generally, we have the following.

Corollary 3.2.7. *Suppose $\mathscr{B}$ is a Deligne-Mumford stack and $\widetilde{\pi} : \mathscr{X} \to \mathscr{B}$ is a curve, and there exists an étale cover $B \to \mathscr{B}$ by a scheme such that the induced $\pi' : X \to B$ is a presmoothing family. Then for any fixed r, d, there exists a Deligne-Mumford stack $\mathscr{G}^r_d(\mathscr{X}/\mathscr{B})$ which is proper and a relative algebraic space over $\mathscr{B}$ such that for any $B' \to \mathscr{B}$ making the resulting $\pi' : X' \to B'$ into a consistent smoothing family, we have a functorial identification $\mathscr{G}^r_d(\mathscr{X}/\mathscr{B}) \times_{\mathscr{B}} B' \cong G^r_d(X'/B')$.*

If moreover π admits a section s, then $\mathscr{G}^r_d(\mathscr{X}/\mathscr{B})$ maps to $\mathrm{Pic}^{d,s}(\mathscr{X}/\mathscr{B})$, compatibly with the above identification.

In general, $\mathscr{G}^r_d(X/B)$ has universal relative dimension at least the Brill-Noether number ρ, and if for some b we have that the fiber of $\mathscr{G}^r_d(X/B)$ has dimension ρ over b, then $\mathscr{G}^r_d(X/B)$ is Cohen-Macaulay and flat over b.

Proof. The proof is identical to the proof of Corollary 3.2.6, using the big étale site of $\mathscr{B}$ in place of Sch_B. □

In view of Proposition 3.1.3, the following is an immediate consequence of Corollary 3.2.7.

Corollary 3.2.8. *There is a proper relative algebraic space*

$$\mathscr{G}^r_d \to \overline{\mathscr{M}}^{\mathrm{ct}}_g$$

with the property that for any morphism $B \to \overline{\mathscr{M}}^{\mathrm{ct}}_g$ such that the pullback of the universal family to B is a consistent smoothing family $X \to B$, there is a natural isomorphism

$$\mathscr{G}^r_d \times_{\overline{\mathscr{M}}^{\mathrm{ct}}_g} B \cong G^r_d(X/B)$$

of B-spaces.

We have thus constructed universal moduli stacks of limit linear series, proving Theorem 1.2.

Finally, we want a good description of the fibers of our construction. In particular, we have the following.

Corollary 3.2.9. *Suppose that k is a field, X_0 a curve of compact type over k, and k'/k is a finite Galois extension over which X_0 is totally split. Given r, d, we have the scheme $G^r_d(X'_0/k')$ and the algebraic space $G^r_d(X_0/k)$, and the k-points of $G^r_d(X_0/k)$ are canonically identified with the k'-points of $G^r_d(X'_0/k')$ which are invariant under the natural* $\mathrm{Gal}(k'/k)$*-action.*

Proof. This follows from Remark 2.7, since the category T_0 (as described in Corollary 3.2.6) can be chosen to consist of the iterated fiber products of $\operatorname{Spec} k'$ over $\operatorname{Spec} k$. □

4. Applications over real and p-adic fields

We now apply our results to study rational points over real and p-adic fields, in particular proving Theorem 1.3 and Corollary 1.1 in the real case. We first prove the following basic result. (While we believe this should be in the literature somewhere with an actual proof, we have failed to find it.)

Proposition 4.1. *Suppose given a field K which is either* $\mathbf{R}$ *or a p-adic field, a smooth curve B over K, a point $b \in B(K)$, and a finite flat morphism $\pi : Y \to B$ of K-schemes. Suppose further that no ramification point of π over b is a K-point. Then there is an analytic neighborhood U of b such that for all $b' \in U(K)$, none of the K-rational points in the fibers $Y_{b'}$ are ramified, and the number of K-rational points over U is constant.*

In the above, $B(K)$ and $Y(K)$ denote sets of K-rational points.

Proof. Let $\nu : (U, u) \to (B, b)$ be an étale neighborhood of b, with $u \in U(K)$, over which Y splits as a disjoint union $Y \times_B U = \bigsqcup_{i=1}^{n} Y_i$, with $Y_i \to U$ an isomorphism for $i = 1, \ldots, m$ and $Y_j \to U$ a morphism whose fiber over b contains no K-points for $j = m+1, \ldots, n$. Since ν is an analytic-local isomorphism, it suffices to prove the result for $Y_U \to U$ and thus we may assume that Y breaks up in the manner described. We will thus replace B by U and work under the additional assumption that Y decomposes.

The result follows from the following key property: the induced map $Y_i(K) \to B(K)$ is a proper map of Hausdorff topological spaces. To prove this it suffices (by choosing embeddings and using the fact that the analytic topology is finer than the Zariski topology) to prove the same statement under the assumption that $B = \mathbf{A}^n$ and $Y_i = \mathbf{P}^m \times \mathbf{A}^n$, the map $Y_i \to B$ being replaced by the second projection map.

We wish to show that the preimage of a compact set S in $\mathbf{A}^n(K)$ is compact. But this preimage is just $S \times \mathbf{P}^m(K)$. Since a product of two compact spaces is compact, it thus suffices to show that $\mathbf{P}^n(K)$ is itself compact. To prove this, it suffices to show that $\mathbf{P}^n(K)$ is the image of a compact set under a continuous map. If $K = \mathbf{R}$, we know that $\mathbf{P}^n(K)$ is the image of the unit sphere in $\mathbf{A}^{n+1}(K)$. If K is non-archimedean, write $\mathscr{O}$ for the ring of integers. Let $C_i \subset \mathscr{O}^{n+1}$ be the subset for which the ith coordinate is 1. Since $\mathscr{O}$ is compact, C_i is itself compact. Scaling a point of $\mathbf{P}^n(K)$ by the inverse of the coordinate with the largest absolute value, we see that there is a surjection

$$\bigsqcup_{i=1}^{n} C_i \to \mathbf{P}^n(K).$$

This establishes the assertion.

With this property in hand, we can conclude the proof. Since the preimage y_j of b in Y_j for $j = m+1, \ldots, n$ is not a K-point, each Y_j for j in this range has the property that $\pi(Y_j(K)) \subset B(K)$ does not contain b, so that $B(K) \backslash \pi(Y_j(K))$ is an open subset not containing b, which means we can shrink B so that $Y_j(K)$ is empty for each $j = m+1, \ldots, n$. Since each other map $Y_j \to B$, $j = 1, \ldots, m$ is an isomorphism, we get the desired result. □

We next recall the Eisenbud-Harris definitions of limit linear series.

Definition 4.2. Let X_0 be a totally split curve of compact type, with dual graph Γ. For $v \in V(\Gamma)$, let Y_v be the corresponding component of X_0, and for $e \in E(\Gamma)$, let Z_e be the corresponding node. An *Eisenbud-Harris limit* $\mathfrak{g}^r_d$ on X_0 is a tuple $(\mathscr{L}^v, V^v)_{v \in V(\Gamma)}$ of $\mathfrak{g}^r_d$s on the Y_v, satisfying the condition that for each node Z_e, if Y_v, $Y_{v'}$ are the components containing Z_e, then for $j = 0, \ldots, r$, we have

$$a_j + a'_{r-j} \geqslant d, \tag{4.1}$$

where $a_0, \ldots, a_r$ and $a'_0, \ldots, a'_r$ are the vanishing sequences at Z_e of $(\mathscr{L}^v, V^v)$ and $(\mathscr{L}^{v'}, V^{v'})$ respectively.

The limit $\mathfrak{g}^r_d$ given by $(\mathscr{L}^v, V^v)_v$ is said to be *refined* if (4.1) is an equality for all e and j.

By definition, the set of Eisenbud-Harris limit linear series is a subset of the product of the spaces $G^r_d(Y_v)$ of usual linear series on the Y_v. If we choose for each $e \in E(\Gamma)$ sequences $a_\bullet, a'_\bullet$ for which (4.1) is satisfied with equality, we obtain a product of spaces of linear series on the smooth curves Y_v with imposed ramification at the Z_e, and taking the union over varying choices of the $a_\bullet, a'_\bullet$, one sees that we obtain every Eisenbud-Harris limit linear series on X_0. This then induces a natural scheme structure:

Notation 4.3. In the situation of Definition 4.2, let $G^{r,\mathrm{EH}}_d(X_0)$ denote the moduli scheme of Eisenbud-Harris limit linear series, with scheme structure as described above.

In order to state the strongest comparison results, the following preliminary proposition will be helpful.

Proposition 4.4. *Let X_0 be a totally split curve of compact type, and given r, d, suppose that the moduli space $G^{r,\mathrm{EH}}_d(X_0)$ of limit $\mathfrak{g}^r_d$s has dimension equal to the Brill-Noether number ρ. Then the refined limit linear series are dense in $G^{r,\mathrm{EH}}_d(X_0)$.*

Proof. We will prove the following more precise statement: Suppose that $G^{r,\mathrm{EH}}_d(X_0)$ has dimension ρ, and we are given, for each pair (e, v) of an edge and adjacent vertex in Γ, a sequence

$$0 \leqslant a_0^{(e,v)} < a_1^{(e,v)} < \cdots < a_r^{(e,v)} \leqslant d,$$

such that for every edge e, connecting v to v', we have

$$a_j^{(e,v)} + a_{r-j}^{(e,v')} \geqslant d \text{ for } j = 0, \ldots, r. \tag{4.2}$$

Then the closed subset $S_{(a^{(e,v)})}$ of $G^{r,\mathrm{EH}}_d(X_0)$ consisting of limit linear series $(\mathscr{L}^v, V^v)_v$ with V^v having vanishing sequence at Z_e at least equal to $a^{(e,v)}$ for all (e, v) has dimension

$$\rho - \sum_{e,j} \left(a_j^{(e,v)} + a_{r-j}^{(e,v')} - d \right) \tag{4.3}$$

if it is nonempty. From this, it will follow immediately that the refined limit linear series are dense.

Arguing the contrapositive, suppose there exist sequences $a^{(e,v)}$ as above such that $S_{(a^{(e,v)})}$ has dimension strictly bigger than asserted; we claim that if (4.2) is strict for any (e, v) and j, then we can decrease some $a_j^{(e,v)}$ while preserving the condition that the resulting $S_{(a^{(e,v)})}$ has too large a dimension. Now, the subset $S_{(a^{(e,v)})}$ is by definition a product of spaces of linear series with imposed ramification on the Y_v; if we denote by ρ_v and d_v the expected and actual dimensions of these spaces respectively, we always have $d_v \geqslant \rho_v$ for all v, and $\sum_v \rho_v$ is equal to the expression in (4.3). We therefore have that the condition that $S_{(a^{(e,v)})}$ has too large dimension is equivalent to saying that $d_v > \rho_v$ for some v. Now, suppose that we have (e, v) and j with strict inequality in (4.2), and let v' be the other vertex adjacent

to e. If $d_{v''} > \rho_{v''}$ for some $v'' \neq v$, then we can replace $a_j^{(e,v)}$ with $d - a_{r-j}^{(e,v')}$ for $j = 0, \ldots, r$, which only changes d_v and $\rho_{v''}$, and in particular preserves that $d_{v''} > \rho_{v''}$. On the other hand, if $d_{v''} = \rho_{v''}$ for all $v'' \neq v$, then necessarily $d_v > \rho_v$, and we can similarly replace $a_{r-j}^{(e,v')}$ with $d - a_j^{(e,v)}$ for $j = 0, \ldots, r$, preserving the hypothesis that $d_v > \rho_v$. This proves the claim, and iterating this procedure, we eventually produce a subset of $G_d^{r,\mathrm{EH}}(X_0)$ of dimension greater than ρ. This proves the desired contrapositive statement. □

The most basic comparison result is then the following.

Lemma 4.5. *In the situation of Definition 4.2, choose the sufficient collection of concentrated multidegrees with $I = V(\Gamma)$, and md_v having degree d on Y_v and degree 0 on the other components. Then restriction from X_0 to Y_v induces a bijection from $G_d^r(X_0)$ to $G_d^{r,\mathrm{EH}}(X_0)$.*

Indeed, this is the rank-1 case of Lemma 4.1.6 and Proposition 4.2.9 of [15].

In general, it is not clear that the scheme structures on $G_d^r(X_0)$ and $G_d^{r,\mathrm{EH}}(X_0)$ agree, but it follows from [11,15] that under the most common circumstances we will have agreement. The improved statement using Proposition 4.4 is as follows.

Theorem 4.6. *The map $G_d^r(X_0) \to G_d^{r,\mathrm{EH}}(X_0)$ constructed in Lemma 4.5 is an isomorphism of schemes when the following conditions are satisfied:*

(I) $G_d^{r,\mathrm{EH}}(X_0)$ has dimension equal to the Brill-Noether number ρ;
(II) either $\rho = 0$, or $G_d^{r,\mathrm{EH}}(X_0)$ is reduced.

Proof. Proposition 4.4 implies that under condition (I), we also have that the refined limit linear series are dense. The case that $G_d^{r,\mathrm{EH}}(X_0)$ is reduced is then Corollary 3.3 of [11]. In the case $\rho = 0$, density implies that in fact every limit linear series is refined, in which case the desired statement is Proposition 4.2.6 of [15]. □

We can now give the proof of our main theorem on limit linear series over local fields.

Proof of Theorem 1.3. First, by Corollary 3.2.6, we may restrict to a neighborhood of b_0 over which $G_d^r(X/B)$ is quasi-finite and flat; we then have that it is finite (see Tag 0A4X of [17]), and then it immediately follows that it is also a scheme. By Theorem 4.6, our hypotheses imply that $G_d^r((X_0)_L) \cong G_d^{r,\mathrm{EH}}((X_0)_L)$, and we can then invoke Corollary 3.2.9 to conclude that $G_d^r(X_0)$ is finite, that $G_d^r(X_0)(K)$ has size exactly n, and that the the scheme structure at each element of $G_d^r(X_0)(K)$ is reduced. The desired statement then follows from Proposition 4.1. □

Following the argument of [13,14], we can now reduce the question of existence of real curves with given numbers of real linear series to statements on real points of intersections of Schubert cycles. Specifically, we have the following.

Corollary 4.7. *Fix $r < d$, and suppose K is either $\mathbf{R}$ or a p-adic field. Given a complete flag*

$$(0) = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{d+1} = K^{d+1},$$

let $\Sigma(F_\bullet)$ be the Schubert cycle in $G(r+1, d+1)$ consisting of $(r+1)$-dimensional subspaces meeting F_{d-1} in codimension 1 (inside the subspace). Given $P \in \mathbf{P}^1$, let $F_\bullet^P$ denote the flag in $\Gamma(\mathbf{P}^1, \mathscr{O}(d))$ determined by order of vanishing at P. Given g with $g = (r+1)(r+g-d)$, suppose that there exist $P_1, \ldots, P_g \in \mathbf{P}^1(\bar{K})$ such that the divisor $P_1 + \cdots + P_g$ is $\mathrm{Gal}(\bar{K}/K)$-invariant, and with

$$\bigcap_{i=1}^{g} \Sigma(F_\bullet^{P_i})$$

finite, having n K-rational points, all of which are reduced.

Then there exist smooth projective curves X of genus g over K such $G^r_d(X)$ is finite, with exactly n K-rational points, all of which are reduced.

Proof. Making use of our new machinery, the proof is closely based on the proof of Theorem 2.5 of [13]. Given the Galois invariance of $P_1 + \cdots + P_g$, we can construct a (not necessarily totally split) curve X_0 over K of compact type by attaching suitable elliptic tails to the P_i. Letting L/K be a finite Galois extension over which all the P_i are rational, we will have that $(X_0)_L$ is a totally split curve of compact type. Now, as described in the proof of Theorem 2.5 of [13], the only possible vanishing sequences at the nodes in the $\rho = 0$ case are $d-r-1, d-r, \ldots, d-2, d$ on the elliptic tails, with complementary sequence $0, 2, 3, \ldots, r+1$ on the rational "main component." Moreover, there is a unique linear series on each elliptic tail with the desired vanishing at P_i, even scheme-theoretically, so we find that $G^{r,\mathrm{EH}}_d((X_0)_L)$ consists entirely of refined limit linear series, and is isomorphic to the space of $\mathfrak{g}^r_d$s on $\mathbf{P}^1$ with vanishing sequence $0, 2, 3, \ldots, r+1$ at each P_i. This latter space is precisely $\bigcap_{i=1}^g \Sigma(F_\bullet^{P_i})$ (considered over L), since $F^{P_i}_{d-1}$ is the space of polynomials vanishing to order at least 2 at P_i. By construction, $\mathrm{Gal}(L/K)$ acts on $G^{r,\mathrm{EH}}_d((X_0)_L)$ via its action on X_0 itself, which means that our isomorphism is $\mathrm{Gal}(L/K)$-equivariant, and the invariant points on both sides are identified. Thus, the hypotheses of Theorem 1.3 are satisfied, and taking any presmoothing family with special fiber X_0 (see §2 of [14] for justification that this is possible), by considering smooth fibers X sufficiently near X_0, we obtain the desired properties for $G^r_d(X)$. □

We can now apply Corollary 4.7 to examples of Eremenko and Gabrielov in the real case.

Proof of Corollary 1.1. Using Corollary 4.7, this is immediate from Examples 1.3 and 2.5 of [4], where their m is our $d-1$. □

Remark 4.8. In terms of producing real curves with few real $\mathfrak{g}^1_d$s, the result of Corollary 1.1 is the best possible using our given degeneration, since Eremenko and Gabrielov prove that their examples yield the smallest possible number of real points in the corresponding intersection of Schubert cycles. Nonetheless, it is *a priori* possible (in the case d is even) that there exist real curves with fewer real $\mathfrak{g}^1_d$s, but that these are not "close enough" to the degenerate curves we consider to be studied via our techniques.

Remark 4.9. It is an interesting phenomenon that the examples of Eremenko and Gabrielov achieving the minimum possible number of real solutions (in the Schubert cycle setting) occur not with all ramification points non-real, but with exactly two real. Computations of Hein et al. [9] in small degree appear to show that on the other hand, the numbers obtained by Cools and Coppens are the minimum achievable when all the ramification points are non-real.

In fact, every such explicit Schubert calculus example also gives an example where we can produce higher-genus curves having the same number of real $\mathfrak{g}^1_d$s, using Corollary 4.7. However, we are not aware of infinite families of examples other than those of [4].

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