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Blt Azumaya algebras and moduli of maximal orders

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Abstract

We study moduli spaces of maximal orders in a ramified division algebra over the function field of a smooth projective surface. As in the case of moduli of stable commutative surfaces, we show that there is a Kollár-type condition giving a better moduli problem with the same geometric points: the stack of blt Azumaya algebras. One virtue of this refined moduli problem is that it admits a compactification with a virtual fundamental class.

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1 Introduction

Much recent progress has been made on the structure theory of maximal orders over algebraic surfaces. Several authors have produced a satisfying minimal model program for such orders (a sampling of which is represented by [7–9] and their references). Others have studied the moduli of Azumaya orders in a fixed unramified division algebra and related moduli problems (e.g. [10,15,17,18]).

In this paper we extend the moduli theory to orders in a ramified Brauer class. In so doing we encounter a phenomenon similar to that which occurs in the moduli theory of stable projective surfaces, arising from an analogue of Kollár's condition on the compatibility of the reflexive powers of the dualizing sheaf with base change. Because the global dimension of our orders is 2, things are technically rather simpler than in Kollár's theory, and we arrive at a satisfying moduli space with a natural compactification carrying a virtual fundamental class. We expect a similar story could be told in higher dimension, but it would involve a careful extension of the results of Sect. 5.3.

As in the commutative theory, the naïve moduli problem (given by fixing the properties of the fibers of a family) contains a refined version as a bijective closed substack. This refined moduli problem can be described as a moduli problem of Azumaya algebras on stacks rather than orders on varieties. (One can also interpret this refined problem as a moduli theory of parabolic Azumaya algebras.) These Azumaya algebras have a precise interaction with the ramification divisor arising from the structure of hereditary orders in matrix algebras over discrete valuation rings, first described by Brumer [5], giving them a structure we call *Brumer log terminal*, or *blt*.

We begin in Sect. 2 by studying the local problem, relating hereditary algebras over complete dvrs to Azumaya algebras over root construction stacks. This is globalized in Sect. 2.2. A simple approach to families of maximal orders is described in Sect. 3. The two resulting moduli problems are described in Sect. 4 and compared in Sect. 5 (with a proof that they can differ included in Sect. 5.5). The comparison relies crucially on ideas similar to those introduced by Kollár in his theory of hulls and husks [11] and a local analysis of reflexive Azumaya algebras on families of rational double points. Finally, in Sect. 6 we describe how to compactify the Azumaya problem using algebra-objects of the derived category of a stack (that one might think of as "parabolic generalized Azumaya algebras") along lines familiar from [17].

2 Normal orders and parabolic Azumaya algebras

2.1 Hereditary orders over dvrs

Fix a discrete valuation ring R with uniformizer t and residue field κ. Write R^{hs} for the strict Henselization of R. Fix a separable closure $\kappa \subset \overline{\kappa}$. Fix a positive integer n invertible in R. Given a positive integer r, let $\pi : \mathscr{X}_r \to \operatorname{Spec} R$ be the stack-theoretic quotient of the natural action of $\boldsymbol{\mu}_r$ on $R[s]/(s^r - t)$. The root construction provides an isomorphism $\mathrm{B}\boldsymbol{\mu}_{r,\overline{\kappa}} \xrightarrow{\sim} (\mathscr{X}_r \otimes_R \overline{\kappa})_{\mathrm{red}}$. An Azumaya algebra $\mathscr{A}$ on $\mathscr{X}_r$ thus gives rise to an Azumaya algebra on $\mathrm{B}\boldsymbol{\mu}_{r,\overline{\kappa}}$ by restriction. By §4.1 of [13], any such algebra is isomorphic to the sheaf of endomorphisms of the vector bundle on $\mathrm{B}\boldsymbol{\mu}_{r,\overline{\kappa}}$ associated to a representation of $\boldsymbol{\mu}_r$. Call this the *representation associated to* $\mathscr{A}$; this representation is defined up to tensoring with a character.

Definition 2.1 Say that a hereditary order A over R is *of type* m if $A \otimes R^{hs}$ has exactly m distinct indecomposable projective modules. Given a positive divisor m of n, call an Azumaya algebra $\mathscr{A}$ over $\mathscr{X}$ *of type* m if the representation associated to $\mathscr{A}$ is the restriction of scalars of the regular representation of $\boldsymbol{\mu}_m$ via the natural quotient map $\boldsymbol{\mu}_n \to \boldsymbol{\mu}_m$.

Definition 2.2 The *hereditary site* $\mathcal{F}$ of Spec R is the site whose underlying category consists of faithfully flat quasi-finite étale R-schemes $U \to \mathrm{Spec}\ R$ with U of pure dimension 1, with coverings given by collections of R-maps $U_i \to U$ that are jointly surjective.

Define two stacks on the hereditary site of Spec R as follows.

Definition 2.3 Given an object $U \to \mathrm{Spec}\ R$ of $\mathcal{F}$, an Azumaya algebra $\mathscr{A}$ on $\mathscr{X}_U$ is *n-typed* if for each closed point $u \in U$ the restriction of $\mathscr{A}$ to $\mathscr{X} \otimes_R \mathscr{O}_{U,u}$ has type m for some positive integer m dividing n. A hereditary order A on U is *n-typed* if for every closed point $u \in U$, the restriction of A to $\mathscr{O}_{U,u}$ has type m for some positive integer m dividing n.

Definition 2.4 Given an object $U \to \mathrm{Spec}\ R$ of $\mathcal{F}$, the stack $\mathcal{A}_n$ has as objects over U the groupoid of n-typed Azumaya algebras $\mathscr{A}$ of degree n on $\mathscr{X} \times_{\mathrm{Spec}\ R} U$. The stack $\mathcal{H}_n$ has as objects the groupoid of n-typed hereditary orders on U.

Since the n-typed Azumaya and hereditary properties are étale-local, it is clear that both $\mathcal{A}_n$ and $\mathcal{H}_n$ are stacks.

Proposition 2.5 *Suppose n is invertible in the residue field κ of R. For any object $\mathscr{A} \in \mathcal{A}_n(U)$, the finite $\mathscr{O}_U$-algebra $\pi_*\mathscr{A}$ lies in $\mathcal{H}_n$. The resulting map of stacks $\mathcal{A}_n \to \mathcal{H}_n$ is a 1-isomorphism.*

Proof Since both stacks are limit-preserving and the statements are étale-local on U, it suffices to prove the following: if R above is a strictly Henselian discrete valuation ring then for any locally free sheaf $\mathscr{V}$ of rank n and type m on $\mathscr{X}$, the R-algebra $\pi_*\mathscr{E}nd(\mathscr{V})$ is hereditary of type m, and in fact this gives an equivalence of groupoids between Azumaya algebras of degree n and type m on $\mathscr{X}$ and hereditary R-algebras of degree n and type m. Indeed, since $\mathrm{Br}(K(R))[n] = 0$, the generic fiber of any hereditary R-order and the Brauer class of any Azumaya algebra of degree n over $\mathscr{X}$ are trivial, which reduces us to the case of matrix algebras and orders therein.

We recall Brumer's fundamental description of hereditary orders [5,6] (combined with Artin-de Jong characterization of the number of indecomposable projectives = number of embeddings in maximal orders): given a K-vector space V of dimension n, the hereditary orders in $\mathrm{End}(V)$ of type m are equivalent to collections of R-submodules $\{M_i \subset V\}_{i\in\mathbf{Z}}$ such that for all i we have $M_{i+1} \subsetneq M_i$ and $M_{i+m} =$

tM_i, up to a shift of indices. The equivalence is given by sending $\{M_i\}$ to the ring of endomorphisms f of V such that for all i we have $f(M_i) \subset M_i$; this filtered endomorphism ring is then the hereditary order corresponding to the filtered module $\{M_i\}$.

On the other hand, Azumaya algebras of type m on $\mathscr{X}_n$ are the pullbacks of Azumaya algebras $\mathscr{A}'$ of type m on $\mathscr{X}_m$, and any such algebra $\mathscr{A}'$ is isomorphic to the pushforward of its pullback to $\mathscr{X}_n$ via the natural map $\mathscr{X}_n \to \mathscr{X}_m$.

Thus, it suffices to prove the proposition in case $n = m$. The filtered module $\{M_i\}$ is precisely an object of the category of parabolic vector bundles with denominator n, called $\mathrm{Par}_{\frac{1}{n}}(\mathrm{Spec}\ R, (t))$ in [4], and the corresponding order is nothing other than the endomorphisms of the parabolic sheaf. Just as in [4], we know that there is a locally free sheaf $\mathscr{V}$ on $\mathscr{X}_n$ giving rise to $\{M_i\}$ in such a way that $\mathrm{End}(\mathscr{V})$ equals the endomorphisms of the parabolic sheaf. But the R-module $\mathrm{End}(\mathscr{V})$ is precisely $\pi_*\mathscr{E}nd(\mathscr{V})$.

What is $\mathscr{V}$? Since each inclusion $M_{i+1} \subset M_i$ is proper, the eigendecomposition of $\mathscr{V}$ must have n distinct summands, which implies that the representation associated to $\mathscr{V}$ is the regular representation.

What are the automorphisms of $A := \pi_*\mathscr{E}nd(\mathscr{V})$? Any R-automorphism of A localizes to a K-automorphism of $\mathrm{End}(V)$, which by the Skolem–Noether theorem is given by conjugation by an automorphism ϕ of V. If this conjugation is to preserve the set of morphisms stabilizing the filtered module $\{M_i\}$ then ϕ itself must preserve the filtration, which means precisely that ϕ is induced by an automorphism of the parabolic sheaf corresponding to $\{M_i\}$, which in turn is equivalent to ϕ being induced by an automorphism of $\mathscr{V}$. Thus, the induced map $\mathrm{Aut}(\mathscr{E}nd(\mathscr{V})) \to \mathrm{Aut}(A)$ is a bijection, as desired. □

The reader wishing to avoid stacks can also interpret the equivalence purely in terms of parabolic sheaves: the hereditary orders on R are equivalent (as a groupoid) to "parabolic Azumaya algebras": parabolic sheaves of algebras locally isomorphic to the parabolic sheaf of endomorphisms of a parabolic vector bundle with denominator equal to the type of the order. This seems to hold no advantage (when the type is bounded as it is) over the formulation in terms of root stacks.

2.2 Globalization for terminal orders

Let α be a terminal Brauer class over the function field of a smooth surface X in the sense of [8]. The ramification data of α yield a simple normal crossings divisor $D = D_1 + \cdots + D_m \subset X$, and for each component D_i a ramification degree $e_i|n$. Let $\pi : \mathscr{X} \to X$ be the smooth stack that is given by the fiber product (with respect to i) of the root construction of order e_i along D_i. Let η_i be the generic point of D_i. As in Sect. 2.1, an Azumaya algebra over $\mathscr{X}$ has associated representations over each $\mathrm{B}\boldsymbol{\mu}_{e_i,\kappa(D_i)}$; call the representation associated to D_i the *ith representation associated to* $\mathscr{A}$.

Definition 2.6 An Azumaya algebra $\mathscr{A}$ on $\mathscr{X}$ is *Brumer log terminal (blt)* if for every i the local Azumaya algebra $\mathscr{A}_{\eta_i}$ has type e_i.

Recall that a normal order with center X and Brauer class α is an order satisfying the usual conditions (R1) and (S2), see definition (2.3) in [8]. Further a normal order is called *terminal* if its ramification data satisfies additional conditions, see definition (2.5) of [8].

Proposition 2.7 *Pushforward by π defines an equivalence of groupoids between blt Azumaya algebras on $\mathscr{X}$ and terminal orders on X with Brauer class α.*

Remark 2.8 Note that the geometric information about ramification at (singular) points of the ramification divisor as well as ramification along irreducible divisors is incorporated in the construction of $\mathscr{X}$.

Proof The proof is mainly a routine globalization of Proposition 2.5.

First, we have that the pushforward of any such $\mathscr{A}$ is normal, as we can check this locally at any codimension 1 point, where this is an immediate consequence of Proposition 2.5. Thus, the pushforward of a blt Azumaya algebra is normal, as desired. To show that π_* is essentially surjective, note that since any maximal order A is reflexive we have that

$$A = \bigcap_{x \in X^{(1)}} A_x,$$

and similarly for blt Azumaya algebras on $\mathscr{X}$, where $X^{(1)}$ is the set of codimension 1 points of X and $A_x := A \otimes \mathscr{O}_{X,x}$ is the localization. Moreover, π_* commutes with the formation of intersections. It thus suffices to prove the analogous result for localizations at codimension 1 points (keeping track of the embedding in the generic algebras), which is precisely Proposition 2.5.

To show that π_* is fully faithful, it suffices to prove the analogous statement upon replacing X by its localization at η_i. Indeed, since the maximal orders A and the Azumaya algebras $\mathscr{A}$ are reflexive, we have that for any blt Azumaya algebras $\mathscr{A}$ and $\mathscr{B}$ with pushforwards A and B the isomorphisms are given by

$$\mathrm{Isom}(\mathscr{A}, \mathscr{B}) = \bigcap_{x \in X^{(1)}} \mathrm{Isom}(\mathscr{A}_x, \mathscr{B}_x)$$

and

$$\mathrm{Isom}(A, B) = \bigcap_{x \in X^{(1)}} \mathrm{Isom}(A_x, B_x)$$

where $X^{(1)}$ is the set of codimension 1 points of X and the intersection takes place inside the set $\mathrm{Isom}(\mathscr{A}_\eta, \mathscr{B}_\eta)$ of isomorphisms of the generic algebras. Since Proposition 2.5 shows that $\mathrm{Isom}(\mathscr{A}_x, \mathscr{B}_x) = \mathrm{Isom}(A_x, B_x)$, the result follows. □

In more classical terms, terminal orders are parabolic Azumaya algebras with parabolic structure along the ramification divisor.

3 Naïve relative maximal orders

3.1 Definitions and basic geometric properties

Definition 3.1 Let Z be an integral algebraic space. A torsion free coherent sheaf $\mathscr{A}$ of $\mathscr{O}_Z$-algebras is a *maximal order* if any injective morphism $\mathscr{A} \to \mathscr{B}$ of torsion free $\mathscr{O}_Z$-algebras that is an isomorphism over a dense open subspace $U \subset Z$, is an isomorphism.

We will prove that maximality in a family is a fiberwise condition.

Definition 3.2 Given a morphism $X \to S$ with locally Noetherian geometric fibers, an *S-flat family of coherent sheaves* is an S-flat quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite presentation. If X has integral fibers, we will say that a possibly non-flat quasi-coherent $\mathscr{O}_X$-module of finite presentation $\mathscr{G}$ is *torsion free* if its geometric fibers $\mathscr{G}_s$ are torsion free coherent $\mathscr{O}_{X_s}$-modules.

Definition 3.3 Given a flat morphism $X \to S$ with integral fibers, an S-flat family of coherent $\mathscr{O}_X$-algebras $\mathscr{A}$ is

(1) a *relative maximal order* if for any $T \to S$ and any injective morphism $\mathscr{A}_T \to \mathscr{B}$ into a torsion free $\mathscr{O}_{X_T}$-algebra that is an isomorphism over a fiberwise dense open subspace $U \subset X_T$ is an isomorphism;
(12) a *relative normal order* if the geometric fibers $\mathscr{A}$ are R_1 and S_2, in the sense of [8].

While relative normality is defined as a fiberwise condition, relative maximality is not obviously so. Let us prove this.

Lemma 3.4 *Suppose X is a proper integral algebraic space over an algebraically closed field k. A coherent sheaf $\mathscr{A}$ of $\mathscr{O}_X$-algebras is a maximal order on X if and only if it is a relative maximal order on $X/\operatorname{Spec} k$. In particular, for any field extension K/k we have that $\mathscr{A} \otimes K$ is a maximal order on $X \otimes K$.*

Proof Since any relatively maximal order is obviously maximal, it suffices to assume that $\mathscr{A}$ is maximal and prove that it is relatively maximal. Suppose $\mathscr{A}_T \to \mathscr{B}$ is an injective map to a torsion free $\mathscr{O}_{X_T}$-algebra that is an isomorphism over the fiberwise dense open $U \subset X_T$. For any geometric point $\operatorname{Spec} K \to T$, the base change $\mathscr{A}_K \to \mathscr{B}_K$ is thus injective and an isomorphism over a dense open of the scheme X_K. If we can show that this restricted map is always an isomorphism then the result is proven. Thus, we are reduced to the case in which $T = \operatorname{Spec} K$ with K an algebraically closed extension field of k.

Since $\mathscr{B}$ is of finite presentation, we may assume by a standard limit argument that there is a finite type integral k-scheme $T' \to \operatorname{Spec} k$, a torsion free algebra $\mathscr{B}'$ over T' with an injective map $\phi : \mathscr{A}_{T'} \to \mathscr{B}'$, and a dominant morphism $\operatorname{Spec} K \to T'$ such that the base change of ϕ isomorphic to the given inclusion $\mathscr{A}_T \to \mathscr{B}$. The locus over which ϕ is an isomorphism is an open subscheme $U' \subset X_{T'}$ whose restriction to the geometric generic fiber over T' is non-empty. By Chevalley's theorem the image of

U' in T' is constructible, hence contains a dense open, whence shrinking T' we may assume that U' is dense in every fiber. But now T' has a dense set of k-points (as it is of finite type over an algebraically closed field), and we know by assumption that for any such point $t' \in T'$ the restriction $\mathscr{A}_{t'} \hookrightarrow \mathscr{B}_{t'}$ is an isomorphism. We conclude that $U' = T'$, which finishes the proof that $\mathscr{A}$ is a relative maximal order. □

Remark 3.5 Note that if the base field k is not assumed to be algebraically closed, the result of Lemma 3.4 is false. Indeed, there are Brauer classes on varieties X over a field k which are ramified but become unramified over the algebraic closure of k. Any maximal order over k will be geometrically hereditary but non-maximal at the generic points of the preimage of the ramification divisor in $X \otimes \overline{k}$. A simple example is furnished by the quaternion algebra (x, a) over $k(x, y)$, where a is a non-square element of k. This gives a ramified algebra on $\mathbf{P}^2$ whose base change to $\overline{k}$ is trivial, and it follows that no maximal order in this quaternion algebra can be relatively maximal.

Proposition 3.6 *Suppose $X \to S$ is a flat morphism of finite presentation between algebraic spaces whose geometric fibers are integral. An S-flat family of torsion free coherent $\mathscr{O}_X$-algebras $\mathscr{A}$ is a relative maximal order if and only if for every geometric point $s \to S$ the fiber $\mathscr{A}_s$ is a maximal order on the integral $\kappa(s)$-space X_s.*

Proof It follows immediately from the definition that the geometric fibers of a relative maximal order are maximal. To prove the other implication, by Lemma 3.4 it suffices to assume that the geometric fibers are maximal and show that $\mathscr{A}$ is maximal (i.e., we may assume that $T = S$; lifting geometric points to T by taking field extensions does not disturb the hypotheses by Lemma 3.4).

Suppose $\iota : \mathscr{A} \to \mathscr{B}$ is an injection into a torsion free $\mathscr{O}_X$-algebra that is an isomorphism over a fiberwise dense open $U \subset X$. To prove that ι is an isomorphism it suffices to work locally on S, so we can assume that $S = \operatorname{Spec} A$ for A a local ring whose closed point s is the image of a geometric point over which $\mathscr{A}$ is maximal. Since ι is an isomorphism over a fiberwise dense open and $\mathscr{A}$ and $\mathscr{B}$ have torsion free fibers, the reduction $\iota_s : \mathscr{A}_s \to \mathscr{B}_s$ is injective and an isomorphism over a dense open. Since $\mathscr{A}_s$ is maximal (as follows immediately from the same being true of its base change to $\overline{\kappa}(s)$), we conclude that ι_s is an isomorphism. By Nakayama's Lemma, we have that ι is surjective, whence it is an isomorphism, as desired. □

Corollary 3.7 *Suppose X is a smooth projective surface over a field k and D is a central division algebra over its function field. Let $k \to R \twoheadrightarrow k$ be a local Artinian k-algebra with residue field k. Given a maximal order $\mathscr{A} \subset D$, any infinitesimal deformation of $\mathscr{A}$ over $X \otimes_k R$ is a maximal order in the generic algebra $D \otimes_k R$.*

Proof There's only one geometric fiber! □

Proposition 3.8 *Suppose $X \to S = \operatorname{Spec} A$ is an algebraic space of finite presentation with integral fibers over a local ring A with residue field κ. An A-flat family of torsion free $\mathscr{O}_X$-algebras $\mathscr{A}$ is a relative maximal order if and only if its geometric closed fiber is a maximal order on $X \otimes \overline{\kappa}$.*

Proof We may suppose that A is Noetherian and reduced. By Proposition 3.6, it suffices to prove that the geometric fibers are all maximal, which immediately reduces us by a pullback argument and Lemma 3.4 to showing that if A is a discrete valuation ring with algebraically closed residue field then the generic fiber of $\mathscr{A}$ is a maximal order (in the absolute sense).

Let $\mathscr{A}_\eta \hookrightarrow \mathscr{B}_\eta$ by an injection into a torsion free $\mathscr{O}_{X_\eta}$-algebra that is an isomorphism over the generic point of $\mathscr{X}_\eta$. Let $\gamma \in X$ be the generic point of the closed fiber and let $\delta \in X$ be the generic point of X. Considering localizations as quasi-coherent sheaves on X, we can focus on quasi-coherent sheaves of algebras containing $\mathscr{A}$ whose localizations at γ are isomorphic to $\mathscr{A}_\gamma$ via the natural inclusion. A standard argument shows that there is a coherent such algebra $\mathscr{B}$ extending $\mathscr{B}_\eta$; saturating if necessary, we may assume that $\mathscr{B}$ has torsion free fibers. This produces a family $\mathscr{A} \hookrightarrow \mathscr{B}$ over all of X which is an isomorphism over a fiberwise dense open subscheme. Reducing to κ as in the proof of Proposition 3.6, we conclude that $\mathscr{A} \to \mathscr{B}$ is an isomorphism, whence the original map $\mathscr{A}_\eta \hookrightarrow \mathscr{B}_\eta$ is an isomorphism, showing that $\mathscr{A}_\eta$ is maximal. (Applying the same argument to a localization of the normalization in any extension of the fraction field of A shows that the geometric generic fiber of $\mathscr{A}$ is maximal.) $\square$

Let $f : Z \to S$ be a flat morphism of finite presentation between algebraic spaces with integral geometric fibers and $\mathscr{A}$ an S-flat torsion free $\mathscr{O}_Z$-algebra of finite presentation. Define a subfunctor $\mathrm{Az}_{\mathscr{A}} \subset Z$ parametrizing morphisms $T \to Z$ such that $\mathscr{A}_T$ is Azumaya.

Lemma 3.9 *The map of functors* $\mathrm{Az}_{\mathscr{A}} \hookrightarrow Z$ *is a quasi-compact open immersion.*

Proof By absolute Noetherian approximation, there is an algebraic space S_0 of finite type over $\mathbf{Z}$, flat morphism $Z_0 \to S_0$ of finite type with integral geometric fibers, and a morphism $S \to S_0$ such that the pullback of Z_0 to S is isomorphic to Z. Since $\mathscr{A}$ is of finite presentation, we can assume that $\mathscr{A}$ is defined on Z_0. Now, since Z_0 is Noetherian any open subscheme is quasi-compact. Thus, it suffices to prove that $\mathrm{Az}_{\mathscr{A}} \hookrightarrow Z$ is open to conclude that it is quasi-compact.

Since the locus over which $\mathscr{A}$ is locally free is open and contains $\mathrm{Az}_{\mathscr{A}}$, we may shrink Z and assume that $\mathscr{A}$ is locally free. Consider the morphism of locally free sheaves $\mu : \mathscr{A} \otimes \mathscr{A}^\circ \to \mathscr{E}nd(\mathscr{A})$ given by left and right multiplication. We know that $\mathscr{A}_T$ is Azumaya if and only if μ_T is an isomorphism, identifying $\mathrm{Az}_{\mathscr{A}}$ with the functor of points on which μ is an isomorphism. But this is equivalent to the cokernel of μ vanishing, which is clearly an open condition. $\square$

By Chevalley's theorem, the image of $\mathrm{Az}_{\mathscr{A}}$ in S is a constructible set $\mathrm{gAz}_{\mathscr{A}} \subset |S|$.

Definition 3.10 The set $\mathrm{gAz}_{\mathscr{A}}$ will be called the *central simple locus* of $\mathscr{A}$.

The constructible central simple locus has two nice properties. First, it is open.

Proposition 3.11 *Let $Z \to S$ be a proper morphism of finite presentation between algebraic spaces with integral geometric fibers. Given a relative maximal order $\mathscr{A}$ on Z, the central simple locus of $\mathscr{A}$ is open.*

Proof Since the formation of $\mathrm{gAz}_{\mathscr{A}}$ is compatible with base change and $\mathscr{A}$ is of finite presentation, we immediately reduce to the case in which S is Noetherian. Now, since $\mathrm{gAz}_{\mathscr{A}}$ is constructible, to show that it is open it suffices to prove it under the additional assumption that $S = \operatorname{Spec} R$ is the spectrum of a discrete valuation ring and that $\mathrm{gAz}_{\mathscr{A}}$ contains the closed point of S. Let η be the generic point of the closed fiber of Z over S. The localization $\mathscr{A}_\eta$ is a finite flat algebra over the discrete valuation ring $\mathscr{O}_{Z,\eta}$. (The latter is a dvr because the fiber is integral, so the uniformizing parameter on S is also a uniformizer in $\mathscr{O}_{Z,\eta}$.) Moreover, the reduction $\mathscr{A} \otimes \kappa(\eta)$ is a central simple algebra. Thus, the closed fiber of the map $\mathscr{A}_\eta \otimes \mathscr{A}_\eta^\circ \to \mathscr{E}nd(\mathscr{A}_\eta)$ of free $\mathscr{O}_{Z,\eta}$-modules is an isomorphism. By Nakayama's Lemma, the generic fiber is also an isomorphism, which shows that the generic stalk of $\mathscr{A}$ is a central simple algebra over the function field of Z, as desired. □

Second, fixing a Brauer class yields a closed central simple locus, in the following sense.

Proposition 3.12 *Suppose X is a variety over a field k and S is a k-scheme. Let $\mathscr{A}$ be a relative maximal order on $X \times S$. Suppose there exists a class $\alpha \in \mathrm{Br}(k(X))$ such that for every geometric point $s \in \mathrm{gAz}_{\mathscr{A}}$ the restriction of $\mathscr{A}_s$ to $\kappa(s)(X)$ has Brauer class α. Then the central simple locus $\mathrm{gAz}_{\mathscr{A}}$ is closed in S.*

Proof We immediately reduce to the case in which S is Noetherian. Since $\mathrm{gAz}_{\mathscr{A}}$ is constructible and compatible with base change on S, and relative maximal orders are stable under base change, to show that $\mathrm{gAz}_{\mathscr{A}}$ is closed it suffices to prove it under the additional assumption that $S = \operatorname{Spec} R$ is the spectrum of a dvr and $\mathrm{gAz}_{\mathscr{A}}$ contains the generic point. Let η be the generic point of the closed fiber of $X \times S$. Given an inclusion of finite algebras $\iota : \mathscr{A}_\eta \hookrightarrow B$, there is an S-flat coherent sheaf of $\mathscr{O}_{X\times S}$-algebras $\mathscr{B}$ with an inclusion $\mathscr{A} \hookrightarrow \mathscr{B}$ whose germ over η is isomorphic to ι. Indeed, the subsheaf $B \subset \mathscr{A}_{K(X)}$ is a colimit of the finite algebras that contain $\mathscr{A}$, and some member of the directed system will have stalk B at η.

It follows that $\mathscr{A}_\eta$ is a maximal order in its fraction ring $F := \mathscr{A}_\eta \otimes K(X)$. But we know that F is a central simple algebra with Brauer class restricted from $\mathscr{O}_{X\times S,\eta}$, and therefore that any maximal order over $\mathscr{O}_{X\times S,\eta}$ in F is Azumaya. It follows that $\mathscr{A}_\eta$ is Azumaya, and therefore that $\mathrm{gAz}_{\mathscr{A}}$ contains the closed point of S, as desired. □

Finally, let us define a relative terminal order of relative global dimension 2. Suppose S is an algebraic space and $Z \to S$ is a proper smooth relative surface. Suppose furthermore that $R = R_1 + \cdots + R_m$ is a(n S-flat) relative snc divisor on Z.

Definition 3.13 A Brauer class $\alpha \in \mathrm{Br}(Z \backslash R)$ is *terminal* if its restriction to every geometric fiber Z_s is terminal in the sense of Definition 2.5 of [8] and for each i the ramification index $e_i(s)$ of α along $(R_i)_s$ is independent of s.

A relative maximal order $\mathscr{A}$ on Z with Brauer class α will be called a *relative terminal order*.

When working over a non-algebraically closed field, the pathology of Remark 3.5 remains an issue: given a Brauer class $\alpha \in \mathrm{Br}(k(X))$ that is ramified but such that its base change to $\overline{k}$ is unramified, no maximal order $\mathscr{A}$ with class α will be relatively maximal over k (because it is not geometrically maximal). The order $\mathscr{A}$ is still relatively normal, however. Thus, if one endeavors to study moduli spaces associated to Brauer classes such as α, one should allow certain normal orders. Of course, one would not like to allow arbitrary normal orders in a given division algebra, only those orders whose non-Azumaya locus is related to the ramification locus of α over the base field.

When the base field is algebraically closed this pathology does not happen, as one cannot dissolve ramification with a base extension. We will focus our attention on this case in the present paper.

4 Moduli

4.1 Notation and assumptions

In this section $X \to S$ will denote a proper smooth relative surface of finite presentation and $D = D_1 + \cdots + D_r$ will be a fixed relative snc divisor in X. This means that each D_i is a proper smooth relative curve over S and that for any pair $i \neq j$ the intersection scheme $D_i \cap D_j$ is finite étale over S. We also fix a class $\alpha \in \mathrm{Br}(U)[n]$, where $U = X \setminus D$ and n is invertible on S. In this section we will try to describe moduli of maximal orders with Brauer class locally (on S) equal to α.

Assumption 4.1 There are integers $e_1, \ldots, e_r > 1$ such that for each geometric point $s \to S$, the fiber $\alpha|_{U_s}$ is ramified to order e_i on D_i, and this ramification configuration is terminal in the sense of Definition 2.5 of [8].

Note that the pair (X, Δ) with $\Delta := \sum(1 - \frac{1}{e_i})D_i$ associated to the ramification datum is Kawamata log terminal. This appears to be the genesis of this notation.

A simple example the reader should keep in mind is when S is the spectrum of an algebraically closed field and α is a Brauer class with snc ramification divisor $D = D_1 + \cdots + D_r$. Our more general setup gives us the ability to work with families of such Brauer classes, but a proper theory would allow singular fibers of X/S.

There are two moduli problems that one can associate to the pair $(X/S, \alpha)$, termed as *Naive* and *BLT* families in this article..

4.2 Naïve families

In this section we write $\mathbf{A}$ for the stack of S-flat torsion free coherent algebras on X. As described in [14], $\mathbf{A}$ is an Artin stack locally of finite presentation over S.

Definition 4.2 The stack of *naïve maximal orders* is the stack $\mathbf{NMO}^{\alpha}_{X/S}$ whose objects over an S-scheme T are relative maximal orders $\mathscr{A}$ on $X \times_S T$ such that for every geometric point $t \to T$ the Brauer class of $\mathscr{A}|_{U \times_T t}$ equals $\alpha|_{U \times_S t}$.

Remark 4.3 One might think that in Definition 4.2 one should require that the Brauer class is α étale-locally on the base. As we will see in Sect. 5.5, this does not materially improve the situation.

Lemma 4.4 *Let A be a local Noetherian ring over S, and let $\mathscr{A}$ be a flat family of coherent $\mathscr{O}_X$-algebras over A. If the closed fiber of $\mathscr{A}$ belongs to $\mathbf{NMO}^{\alpha}_{X/S}$ then so does $\mathscr{A}$.*

Proof By Proposition 3.8 $\mathscr{A}$ is a relative maximal order, and the usual characterizations show that $\mathscr{A}$ is Azumaya over U_A. It remains to show that for any geometric fiber of X over A the Brauer class of that fiber of $\mathscr{A}$ is α. It suffices to prove this under the assumption that A is a complete discrete valuation ring. Thus, we may assume that X_A is a regular scheme of dimension 3 and $\mathscr{A}$ is a maximal order which is Azumaya away from a snc divisor $D = D_1 + \cdots + D_r$ and whose Brauer class has order invertible in A. For sufficiently large and divisible N, the Brauer class of $\mathscr{A}_U$ extends to an element of β in the Brauer group of the root construction $X\{D^{1/N}\}$ (in the notation of [12]). By the proper base change theorem for the morphism $X\{D^{1/N}\} \to \operatorname{Spec} A$, the class β is determined by its closed fiber, so it must equal the pullback of α, whence the geometric generic fiber of $\mathscr{A}_U$ has Brauer class α, as desired. □

Corollary 4.5 *Let A be a complete local ring with maximal ideal $\mathfrak{m}$. The functor*

$$\mathbf{NMO}^{\alpha}_{X/S}(A) \to \lim_{n} \mathbf{NMO}^{\alpha}_{X/S}(A/\mathfrak{m}^{n+1})$$

is an equivalence of categories.

Proof This is the classical Grothendieck existence theorem combined with Proposition 3.8 and Lemma 4.4, which says that the effectivization of any formal family lying in $\mathbf{NMO}^{\alpha}_{X/S}$ also lies in $\mathbf{NMO}^{\alpha}_{X/S}$. □

Lemma 4.6 *Suppose $\mathscr{A}$ is a flat family of coherent $\mathscr{O}_X$-algebras over a Noetherian base scheme T that is of finite type over an excellent Dedekind domain or a field. There is an open subscheme $U \subset T$ such that for any geometric point $t \to T$, the geometric fiber $\mathscr{A}_t$ is in $\mathbf{NMO}^{\alpha}_{X/S}$ if and only if t factors through U.*

Proof By Theorem 0.5 of [2], it suffices to prove the result after replacing T by a Dedekind scheme, and now we wish to show that the geometric generic fiber of $\mathscr{A}$ is in $\mathbf{NMO}^{\alpha}_{X/S}$ if and only if all but finitely many geometric fibers lie in $\mathbf{NMO}^{\alpha}_{X/S}$. By Lemma 4.4, if any closed geometric fiber is in $\mathbf{NMO}^{\alpha}_{X/S}$ then the geometric generic fiber is in $\mathbf{NMO}^{\alpha}_{X/S}$. It thus suffices to show that if the geometric generic fiber is in $\mathbf{NMO}^{\alpha}_{X/S}$ then all but finitely many geometric closed fibers are in $\mathbf{NMO}^{\alpha}_{X/S}$.

By Proposition 2.7, the generic fiber $\mathscr{A}_\eta$ is the pushforward of a blt Azumaya algebra $\mathfrak{A}_\eta$ on $\mathscr{X}_\eta$ along the morphism $\pi_\eta : \mathscr{X}_\eta \to X_\eta$. By spreading out, we may assume after removing finitely many points from T that $\mathfrak{A}$ extends to all of $\mathscr{X}$. Moreover, the isomorphism $\mathscr{A}_\eta \xrightarrow{\sim} \pi_*\mathfrak{A}_\eta$ extends to an isomorphism over some dense open $U \subset X$ that contains the generic fiber. The complement of U will have finite image in T, whereupon we have identified the remaining fibers with pushforwards of blt Azumaya algebras with Brauer class α, rendering them elements of $\mathbf{NMO}^{\alpha}_{X/S}$, as desired. □

Proposition 4.7 *The stack $\mathbf{NMO}^{\alpha}_{X/S}$ is an Artin stack locally of finite presentation over S, and the morphism $\mathbf{NMO}^{\alpha}_{X/S} \to \mathbf{A}$ is an open immersion.* (*Recall that* $\mathbf{A}$ *stands for the stack of S-flat torsion free coherent algebras on X.*)

Proof It suffices to prove the latter statement by Tag 01TQ of [20]. Since belonging to $\mathbf{NMO}^{\alpha}_{X/S}$ is a fiberwise statement, this follows immediately from Lemma 4.6. □

We arrive at the somewhat surprising conclusion that maximal orders with Brauer class α form an open substack of the stack of all coherent algebras. However, the deformation theory is "arbitrarily bad" in the sense that it is identical to the deformation theory of maximal orders. We will describe a refinement of the moduli problem with the same closed points but different infinitesimal properties that has a natural compactification admitting a virtual fundamental class.

Remark 4.8 Without the Assumption 4.1, the openness of the locus of naïve families is undoubtedly false.

4.3 Blt Azumaya families

Write $\pi : \widetilde{X} \to X$ for the stack $X\langle D^{1/n}\rangle$ in the notation of Section 3.B of [12]; the stack $\widetilde{X}$ is a product of root constructions on each D_i and is a smooth proper Deligne–Mumford relative surface over S.

Definition 4.9 The stack of *blt Azumaya algebras* is the stack $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$ whose objects over T are Azumaya algebras $\mathscr{A}$ on $\widetilde{X}_T$ such that for every geometric point $t \to T$ the fiber $\mathscr{A}_t$ is a blt Azumaya algebra with Brauer class α_t.

Proposition 4.10 *The stack $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$ is an Artin stack locally of finite presentation over S.*

Proof It is a standard result that for an Azumaya algebra A on a stack Z, the deformation group is given by $\mathrm{H}^1(Z, A/\mathscr{O}_Z)$ and the obstruction group is given by $\mathrm{H}^2(Z, A/\mathscr{O}_Z)$. Indeed, the sheaf $A/\mathscr{O}_Z$ is the sheaf of derivations of A, and this is precisely the sheaf of infinitesimal automorphisms of A (by the Skolem–Noether theorem). Since any deformation of A is locally trivial, the cohomology of the sheaf of infinitesimal automorphisms determines the deformation in the standard way. (In slightly different language: deformations of A form a gerbe with structure group $A/\mathscr{O}_Z$. See [10,17] for explicit calculations.) These groups satisfy Artin's axioms for a deformation and obstruction theory [1]. Moreover, if Z is proper over the base, Olsson's Grothendieck Existence Theorem for stacks [19, Section 11] shows that formal families of Azumaya algebras on Z algebraize. It follows from the main results of [1] that the stack of Azumaya algebras on $\widetilde{X}$ is an Artin stack locally of finite presentation on S. The locus where the type at each x_i is e_i is an open substack. Finally, the proper and smooth base change theorem in étale cohomology shows that the locus on which the fibers have Brauer class α is open and closed. □

Remark 4.11 While it is easy to write down the deformation theory associated to $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$, it is mysterious what the deformation theory is for $\mathbf{NMO}^{\alpha}_{X/S}$. This is one indication that the former moduli problem is likely better behaved.

5 Relations among the moduli problems

5.1 Pushforwards of Azumaya families are naïve families

Let $\mathscr{A}$ be a family in $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$ over a base T. The pushforward morphism $\pi : \widetilde{X} \to X$ yields a sheaf of algebras $A := \pi_* \mathscr{A}$.

Proposition 5.1 *The algebra A described above is a family in $\mathbf{NMO}^{\alpha}_{X/S}$.*

Proof First, since $\widetilde{X}$ is tame and $\mathscr{A}$ is T-flat and coherent, we know that A is also T-flat and coherent, and that the formation of A is compatible with base change on T. Thus, to show that A is a family in $\mathbf{NMO}^{\alpha}_{X/S}$, it suffices to assume that T is the spectrum of an algebraically closed field K. Since $\widetilde{X} \to X$ is an isomorphism over a dense open subset, we know that A is generically Azumaya with Brauer class α. By Proposition 2.7 we have that A is terminal, and Assumption 4.1 implies that any terminal order is maximal, completing the proof. □

Pushforward along π thus defines a 1-morphism of stacks

$$\Phi : \mathbf{BLT}^{\alpha}_{\widetilde{X}/S} \to \mathbf{NMO}^{\alpha}_{X/S} .$$

This morphism will be the object of study for the rest of this section. In particular, we will show that it is a proper bijection that is not in general surjective on tangent spaces. Thus this realizes $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$ as something between $\mathbf{NMO}^{\alpha}_{X/S}$ and its normalization. We are not sure what normality properties $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$ enjoys, but it is likely that it can be arbitrarily bad (although one might hope for stabilization as one varies discrete parameters like the second Chern class).

5.2 Naïve families over complete dvrs and reflexive blt Azumaya algebras

Let R be a complete dvr over S with uniformizer t and algebraically closed residue field k and let $A \in \mathbf{NMO}^{\alpha}_{X/S}(R)$. In this section we will show that locally on X_R the family A comes from a reflexive Azumaya algebra over a stack with A_{n-1}-singularities and coarse moduli space X_R. We will use this in Sect. 5.4 to show that Φ satisfies the valuative criterion of properness.

Write $\overline{X} = X[D^{1/n}]$, in the notation of Section 3.B of [12]. This is a stack with coarse moduli space X that may be locally described as follows: at a crossing section of two components D_1 and D_2 of X with local equations $t_1 = 0$ and $t_2 = 0$, the stack $\overline{X}$ is given by taking the stack-theoretic quotient for the action of $\boldsymbol{\mu}_n$ on $\mathscr{O}[w_1, w_2]/(w_1^n - t_1, w_2^n - t_2)$ given by $\zeta \cdot (w_1, w_2) = (\zeta w_1, \zeta^{-1} w_2)$. Since D has relative normal crossings, we see that $\overline{X}$ has flat families of A_{n-1}-singularities in fibers.

As in Sect. 2.2, we have a smooth stack $\widetilde{X}$ dominating $\overline{X}$.

We will prove the following local structure theorem in this section, and then study reflexive Azumaya algebras on $\overline{X}$ in the following section.

Proposition 5.2 *Let* Spec $R \to S$ *be a dvr over* S. *Any algebra in* $\mathbf{NMO}^{\alpha}_{X/S}(R)$ *is the pushforward from* $\overline{X}$ *of a unique reflexive blt Azumaya algebra on* $\widetilde{X}_R$ *with Brauer class* α.

Proof Let $A \in \mathbf{NMO}^{\alpha}_{X/S}(R)$. By Proposition 2.7, the generic fiber A_η is the pushforward of an Azumaya algebra $\mathscr{A}_\eta$ on $\widetilde{X}_\eta$. Since $\widetilde{X} \to \overline{X}$ is relatively tame, we see that the pushforward of $\mathscr{A}_\eta$ to $\overline{X}$ is a reflexive blt Azumaya algebra $\overline{A}_\eta$ that pushes forward to A_η.

The morphisms $\widetilde{X}_R \to \overline{X}_R \to X_R$ are isomorphisms over the generic point of the closed fiber of X_R. Moreover, the order A is Azumaya in a neighborhood of that point, and all of the orders and Azumaya algebras described so far are contained in the localization B of A at this point. □

Lemma 5.3 *Let* Z *be an integral* S_2 *Noetherian Deligne–Mumford stack and* A *a finite-dimensional* $\kappa(Z)$*-algebra. Suppose for each codimension* 1 *point* z *there is given a maximal order* $B_z \subset A$ *over the local ring* $\mathscr{O}_{Z,z}$. *Then there is at most one maximal order* B *over* Z *such that* $B \otimes \mathscr{O}_{Z,z} = B_z \subset A$.

Proof Given two such maximal orders B and B', consider the algebra $B'' := B \cap B'$. Since B and B' are S_2, we have that B'' is also S_2. Since B'' is S_2 and maximal in codimension 1 it is maximal. By hypothesis, the inclusions $B'' \subset B$ and $B'' \subset B'$ are isomorphisms are all codimension 1 points. Thus, $B'' \to B$ and $B'' \to B'$ are isomorphisms, as desired. □

Now let $\overline{A}$ be any reflexive extension of $\overline{A}_\eta$ that localizes to B. We see that the pushforward of $\overline{A}$ is a maximal order agreeing with A in the generic fiber and at the generic point of the closed fiber, and thus at all codimension 1 points. Applying Lemma 5.3, we conclude that $\overline{A}$ pushes forward to A, as desired. □

5.3 Local structure of reflexive Azumaya algebras on families of rational double points

In this section we will analyze the local structure of reflexive Azumaya algebras on $\overline{X}$.

Let R be a complete dvr with uniformizer t and algebraically closed residue field k of characteristic 0. Let $Z :=$ Spec $B \to$ Spec R be a smooth relative affine surface and $D_1, D_2 \subset Z$ smooth relative curves whose intersection $S := D_1 \cap D_2$ is isomorphic to the scheme-theoretic image of a section of Z/R. Replacing Z with an open subscheme containing S if necessary, we may assume that D_i is the vanishing locus of a global function $t_i \in \Gamma(Z, \mathscr{O}_Z)$, $i = 1, 2$. Let $Z' =$ Spec $B[w]/(w^n - t_1 t_2)$ be the cyclic cover branched along $D_1 \cup D_2$; there is a section $\sigma : R \xrightarrow{\sim} S' \subset Z'$ lifting S. There is a stack $\mathscr{Z}$ with coarse moduli space Z' given by taking the quotient of Spec $B[w_1, w_2]/(w_1^n - t_1, w_2^n - t_2)$ by the action of $\boldsymbol{\mu}_n$ in which $\zeta \cdot (w_1, w_2) = (\zeta w_1, \zeta^{-1} w_2)$. The natural map $\mathscr{Z} \to Z'$ is an isomorphism away from the singular locus S'.

Write $z \in Z'$ for the closed point of S', and let $Y' =$ Spec $\mathscr{O}^{\mathrm{hs}}_{Z',z}$ and $\mathscr{Y}' = Y \times_{Z'} \mathscr{Z}$ be the Henselizations of Z' and $\mathscr{Z}$ at z. Because R is strictly Henselian, there is a section $T \subset Y \to \operatorname{Spec} R$ lying over S'. Finally, let Y be the Henselization of Y' along T and let $\mathscr{Y} = \mathscr{Y}' \times_{Y'} Y$, with $\pi : \mathscr{Y} \to Y$ the natural map. We have that $(\mathscr{Y} \times_Y T)_{\mathrm{red}}$ is isomorphic to $\mathrm{B}\boldsymbol{\mu}_{n,T}$. Write $U = Y \setminus T$; this is in fact the regular locus of Y, and it has regular geometric fibers over R. Note that, as a limit of Henselian local schemes, Y is itself still a Henselian local scheme.

Lemma 5.4 *The Brauer group* $\mathrm{Br}(U)$ *is trivial.*

Proof By purity, we have that $\mathrm{Br}(U) = \mathrm{Br}(\mathscr{Y})$, so it suffices to show that the latter vanishes. Since Y is Henselian along T, we have by the usual deformation arguments that $\mathrm{Br}(\mathscr{Y}) = \mathrm{Br}(\mathrm{B}\boldsymbol{\mu}_{n,T})$, so it suffices to show that this last group is trivial.

Consider the projection $\pi : \mathrm{B}\boldsymbol{\mu}_{n,T} \to T$. The Leray spectral sequence yields

$$\mathrm{H}^p(T, \mathbf{R}^q\pi_*\mathbf{G}_m) \Rightarrow \mathrm{H}^{p+q}(\mathrm{B}\boldsymbol{\mu}_{n,T}, \mathbf{G}_m).$$

We know by §4.2 of [13] that $\mathbf{R}^2\pi_*\boldsymbol{\mu}_n = 0$ and $\mathbf{R}^1\pi_*\mathbf{G}_m = \mathbf{Z}/n\mathbf{Z}$. Since R is Henselian with algebraically closed residue field we have that $\mathrm{H}^1(T, \mathbf{Z}/n\mathbf{Z}) = 0$. The sequence of low degree terms then shows that the pullback map $\mathrm{H}^2(T, \mathbf{G}_m) \to \mathrm{H}^2(\mathrm{B}\boldsymbol{\mu}_{n,T}, \mathbf{G}_m)$ is an isomorphism. But, again because R is Henselian with algebraically closed residue field, we know that $\mathrm{H}^2(T, \mathbf{G}_m) = \mathrm{Br}(T) = 0$. □

Corollary 5.5 *A reflexive Azumaya algebra on Y has the form $\mathscr{E}nd(M)$, where M is a reflexive $\mathscr{O}_Y$-module.*

Proof Let $\mathscr{A}$ be a reflexive Azumaya algebra. By Lemma 5.4 we know that $\mathscr{A}|_U \cong \mathscr{E}nd(V)$ with V a locally free coherent sheaf on U. If M is the unique reflexive coherent extension of V then $\mathscr{E}nd(M)$ is reflexive and isomorphic to $\mathscr{A}$ in codimension 1, whence $\mathscr{A} \cong \mathscr{E}nd(M)$. □

Proposition 5.6 *Suppose $\mathscr{A}$ is a reflexive Azumaya algebra of degree r on Y such that the restriction $\mathscr{A} \otimes k$ is a reflexive Azumaya algebra on $Y \otimes k$. Then*

(1) *$\mathscr{A} \cong \mathscr{E}nd(M)$ with M a direct sum of indecomposable reflexive $\mathscr{O}_Y$-modules of rank* 1*;*
(2) *there is a blt Azumaya algebra $\mathscr{B}$ on $\mathscr{Y}$ such that $\mathscr{A} = \pi_*\mathscr{B}$.*

Proof By assumption we have that $\mathscr{A} \otimes k \cong \mathscr{E}nd(V)$ with V a reflexive $\mathscr{O}_{Y \otimes k}$-module. But $Y \otimes k$ is the Henselization of an A_{n-1}-singularity, so we know that V decomposes as a direct sum of reflexive modules of rank 1 by the McKay correspondence [3]. This gives rise to a full set of idempotents $e_j \in \mathscr{A}(Y \otimes k)$, $j = 1, \ldots, r$. Since Y is Henselian, these idempotents lift to global sections $\widetilde{e}_j$ of $\mathscr{A}$. By Corollary 5.5 we have that $\mathscr{A} \cong \mathscr{E}nd(M)$. The idempotents $\widetilde{e}_j$ decompose M as a direct sum of submodules of rank 1. Since M is reflexive, each of these summands is reflexive, proving the first statement.

To prove the second statement, note that a reflexive sheaf of rank 1 on Y is the pushforward along π of a unique invertible sheaf on $\mathscr{Y}$. Thus, M is isomorphic to π_*N for some locally free sheaf $\mathscr{V}$ on $\mathscr{Y}$. The Azumaya algebra $\mathscr{B} = \mathscr{E}nd(N)$ has reflexive pushforward that is canonically isomorphic to $\mathscr{A}$ over U, whence $\mathscr{A} \cong \pi_*\mathscr{B}$, as desired. □

5.4 Proof that Φ is a proper bijection

In this section we show that $\Phi : \mathbf{BLT}^{\alpha}_{\widetilde{X}/S} \to \mathbf{NMO}^{\alpha}_{X/S}$ is a proper morphism. Since it is already locally of finite presentation and bijective, it suffices to show the following valuative criterion.

Proposition 5.7 *If R is a complete dvr over S then any naïve family A on X_R has the form $\pi_*\mathscr{A}$, where $\mathscr{A}$ is an Azumaya family on $\widetilde{X}_R$.*

Proof By Proposition 5.2, we know that A is the pushforward of a family $\mathscr{B}$ of reflexive Azumaya algebras on $\overline{X}$. It thus suffices to show that $\mathscr{B} \cong p_*\mathscr{A}$, where $p : \widetilde{X} \to \overline{X}$ is the natural morphism.

Let $V \subset \overline{X}$ be the smooth locus of $\overline{X}/S$. By construction there is a natural diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dd] \\ V \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & \overline{X} \end{tikzcd}$$

in which the diagonal arrows are fiberwise dense open immersions whose complements have codimension two in each geometric fiber. By the theory of hulls [11], we have that the adjunction map $\mathscr{B} \to j_*\mathscr{B}_V$ is an isomorphism. Since $pi = j$, to prove the result it suffices to prove that $i_*\mathscr{B}_V$ is an Azumaya algebra on $\widetilde{X}$.

This latter statement is étale local on X, so we may replace X by the local Henselian scheme Y of section 5.3. In this case we have that $\mathscr{B}$ is isomorphic to $p_*\mathscr{A}$ for some Azumaya algebra $\mathscr{A}$. The algebra $i_*\mathscr{B}_V$ is thus isomorphic to $i_*\mathscr{A}_V$, and so it suffices to show that the adjunction map $a : \mathscr{A} \to i_*\mathscr{A}_V$ is an isomorphism. But the stack $\widetilde{X}$ is regular and $\mathscr{A}$ is locally free, so a is an isomorphism if and only if it is an isomorphism in codimension 1. Since V has codimension 2, we know that a is an isomorphism at every codimension 1 point, and the result follows. □

5.5 Proof that Φ need not be an isomorphism

In this section we prove that the map $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S} \to \mathbf{NMO}^{\alpha}_{X/S}$ need not be an isomorphism by exhibiting examples for which the map on tangent spaces is not surjective.

Let $D \subset X$ be a smooth divisor in a projective surface such that

(1) there is an infinitesimal deformation $\mathcal{D} \subset X_{k[\varepsilon]}$ for which $\mathscr{O}_{X_{k[\varepsilon]}}(\mathcal{D} - D_{k[\varepsilon]})$ is non-torsion in $\mathrm{Pic}(X_{k[\varepsilon]})$;
(2) there is a blt Azumaya algebra $\mathscr{A}$ on $\widetilde{X}$ for which the pushforward $\pi_*\mathscr{A}$ is a maximal order on X of period n and $\mathrm{H}^2(\widetilde{X}, \mathscr{A}/\mathscr{O}_{\widetilde{X}}) = 0$.

Write $\widetilde{X}' = X_{k[\varepsilon]}\{\mathcal{D}^{1/n}\}$ and (by abuse of notation) $\pi : \widetilde{X}' \to X_{k[\varepsilon]}$ for the projection to the coarse moduli space. By deforming $\mathscr{A}$ to $\widetilde{X}'$ we will make a tangent vector to $\mathbf{NMO}_{X/S}$ that does not lie in the image of the tangent map to $\mathbf{BLT}_{\widetilde{X}/S}$.

Proposition 5.8 *There is a deformation $\mathscr{A}'$ of $\mathscr{A}$ to a blt Azumaya algebra on $\widetilde{X}'$ such that the resulting object $A' = \pi_*\mathscr{A}'$ of $\mathbf{NMO}_{X/S}(k[\varepsilon])$ is not in the image of $\mathbf{BLT}_{\widetilde{X}/S}$.*

Proof The key to the proof is to relate the dualizing bimodule of A' to the divisor class $\mathcal{D}$.

Lemma 5.9 *Given $\widetilde{X}'$, $\mathscr{A}'$ and A' as above, there is a natural isomorphism*

$$\omega_{A'}^{\otimes n} \cong A' \otimes_{\mathscr{O}_{X_{k[\varepsilon]}}} \omega^n_{X_{k[\varepsilon]}/k[\varepsilon]}((n-1)\mathcal{D}).$$

Let us briefly accept Lemma 5.9 and see how to complete the proof of Proposition 5.8. We need the following lemma:

Lemma 5.10 *The pullback map* $\mathrm{Pic}(X_{k[\varepsilon]}) \to \mathrm{H}^1(X_{k[\varepsilon]}, A'^{\times})$ *is injective modulo torsion as a map of pointed sets.*

Proof Recall that the pointed set $\mathrm{H}^1(X_{k[\varepsilon]}, A'^{\times})$ classifies right modules locally isomorphic to A_A. The reduced norm defines a sequence $\mathscr{O}^{\times} \to (A')^{\times} \to \mathscr{O}^{\times}$ such that the composition is raising to the nth power (and thus surjective in the étale topology). Applying the étale H^1 functor we see that the natural map

$$\mathrm{H}^1\left(X_{k[\varepsilon]}, \mathscr{O}^{\times}_{X_{k[\varepsilon]}}\right) \to \mathrm{H}^1\left(X_{k[\varepsilon]}, (A')^{\times}\right)$$

is injective modulo n-torsion. This proves the lemma. □

If A' is in the image of $\mathbf{BLT}_{\widetilde{X}/S}(k[\varepsilon])$, the analogous computation with $D_{k[\varepsilon]}$ in place of $\mathcal{D}$ would yield an isomorphism between the bimodules $\omega_{A'}^{\otimes n}$ and $A' \otimes_{\mathscr{O}_{X_{k[\varepsilon]}}} \omega^n_{X_{k[\varepsilon]}/k[\varepsilon]}((n-1)D_{k[\varepsilon]})$. Applying Lemma 5.10, we conclude that $\mathscr{O}(\mathcal{D} - D_{k[\varepsilon]})$ is torsion in $\mathrm{Pic}(X_{k[\varepsilon]})$, contrary to our original hypothesis. □

It remains to prove Lemma 5.9.

Proof of Lemma 5.9 To simplify notation, write $X' = X_{k[\varepsilon]}$ and write $\omega_{X'}$ for the relative dualizing sheaf over $k[\varepsilon]$. Recall that the dualizing bimodule is given by the sheaf $\mathscr{H}om_{\mathscr{O}_{X'}}(A', \omega_{X'})$. Writing $A' = \pi_*\mathscr{A}'$ and using duality, we have isomorphisms of bimodules

$$\begin{aligned}\omega_{A'} = \mathscr{H}om_{\mathscr{O}_{X'}}\left(\pi_*\mathscr{A}', \omega_{X'}\right) &= \pi_*\mathscr{H}om\left(\mathscr{A}', \pi^!\omega_{X'}\right)\\ &= \pi_*\mathscr{H}om\left(\mathscr{A}', \omega_{\widetilde{X}'}\right) = \pi_*\left(\mathscr{A}' \otimes \omega_{\widetilde{X}'}\right).\end{aligned}$$

As an $\mathscr{A}'$-bimodule, we have that $(\mathscr{A}' \otimes \omega_{\widetilde{X}'})^{\otimes n} \cong \mathscr{A}' \otimes \omega_{X'}^{\otimes n}((n-1)\mathcal{D})$. To see this, note that we can locally write $\widetilde{X}' = [\mathrm{Spec}\ \mathscr{O}_{X'}[z]/(z^n - t)/\boldsymbol{\mu}_n]$, where t is a local equation for $\mathcal{D}$; computing the relative differentials immediately yields the result.

There is a natural map

$$\left(\pi_*\left(\mathscr{A}' \otimes \omega_{\widetilde{X}'}\right)\right)^{\otimes n} \to \pi_*\left(\left(\mathscr{A}' \otimes \omega_{\widetilde{X}'}\right)^{\otimes n}\right)$$

giving rise (using the computation of the preceding paragraph) to a map

$$\phi : \omega_{A'} \to A' \otimes \omega_{X'}^{\otimes n}((n-1)D)$$

that we wish to show is an isomorphism.

Note that an étale-local model for X', A', $\mathscr{A}'$ around a closed point of X' is given by the trivial family whose fiber is the standard cyclic algebra. Thus, to prove that ϕ is an isomorphism it suffices to prove it for the local constant family, and thus (by compatibility with pullback) for the local family over a smooth surface over k. But this is Proposition 5 of [9]. $\square$

To give a concrete example, let $X = E \times E$ for a smooth projective curve of genus 1 over an algebraically closed field of characteristic 0 and let $D = D_1 + D_2$ be the sum of two disjoint closed fibers of the second projection. Let $E' \subset E$ be the complement of the image of D under pr_2. There is a finite covering $C \to E'$ of degree 2 that is totally ramified at both points of $E \backslash E'$, giving a class $\alpha \in \mathrm{H}^1(E', \mathbf{Z}/2\mathbf{Z})$. Choosing any $\beta \in \mathrm{H}^1(E, \mu_2)$ we can form the class $\mathrm{pr}_2^* \alpha \cup \mathrm{pr}_1^* \beta \in \mathrm{H}^2(E \times E', \boldsymbol{\mu}_2)$, giving a Brauer class $\gamma \in \mathrm{Br}(k(E \times E))$. Elementary computations show that the ramification extension of this Brauer class on each component of D is given by the class of β, so that maximal orders must be hereditary along D.

Any non-constant infinitesimal deformation of D (e.g., that induced by moving along E) will give a $\mathcal{D}$ as in the statement of Proposition 5.8. It remains to show that there is an unobstructed Azumaya algebra on the stack $\widetilde{X} \to E \times E$ branched over D. Since $\mathrm{Br}(\widetilde{X}) = \mathrm{Br}'(\widetilde{X})$, there is certainly some Azumaya algebra in that class. Producing one that is unobstructed is a standard argument that can be found written out for projective surfaces in Proposition 3.2 of [10]. We omit the details.

Remark 5.11 The construction given here also shows that fixing the Brauer class to be α étale-locally on the base of families in Definition 4.2 does not ameliorate the situation, as any infinitesimal deformation of the class of α on $E \times E'$ is constant.

6 Generalized Azumaya algebras on $\widetilde{X}$

In this section we will suppose that $S = \mathrm{Spec}\ k$ is the spectrum of an algebraically closed field. We will compactify the stack $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$. The constructions described here are almost identical to those in [17]. By the proper and smooth base change theorems in étale cohomology, any family in $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$ defines a section of the finite constant sheaf (scheme!) $\mathbf{R}^2 f_* \boldsymbol{\mu}_n$, where $f : \widetilde{X} \to S$ is the structural morphism, giving a morphism of stacks

$$c : \mathbf{BLT}^{\alpha}_{\widetilde{X}/S} \to \mathbf{R}^2 f_* \boldsymbol{\mu}_n.$$

Thus, to compactify $\mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$ we will compactify each fiber.

Write $\overline{\alpha} \in \mathrm{H}^2(\widetilde{X}, \boldsymbol{\mu}_n)$ for a lift of α via the Kummer sequence. The fiber of c over $\overline{\alpha}$ will be denoted $\mathbf{BLT}^{\overline{\alpha}}_{\widetilde{X}/S}$. Let $p : \mathscr{X} \to \widetilde{X}$ be a $\boldsymbol{\mu}_n$-gerbe representing the class

$\overline{\alpha}$. That there is such an Artin stack is discussed in Section 2.4 of [13]. In Sections 2.2 and 2.3 of [13] or in [16] the reader will also find a discussion of the theory of $\mathscr{X}$-twisted sheaves in connection with the Brauer group.

Definition 6.1 A torsion free $\mathscr{X}$-twisted sheaf $\mathscr{F}$ is *blt* if the $\mathscr{O}_{\widetilde{X}}$-algebra $p_*\mathscr{E}nd(\mathscr{F})$ is blt on the Azumaya locus.

Let $\mathbf{Sh}_{\mathscr{X}}$ denote the stack of torsion free blt $\mathscr{X}$-twisted sheaves of rank n with trivial determinant. The basic result on the stack $\mathbf{Sh}_{\mathscr{X}}$ is the following.

Proposition 6.2 *The stack $\mathbf{Sh}_{\mathscr{X}}$ is an Artin stack locally of finite presentation over S. Moreover, $\mathbf{Sh}_{\mathscr{X}}$ is a $\mathbf{G}_m$-gerbe over an algebraic space $\mathrm{Sh}_{\mathscr{X}}$ with proper connected components.*

Proof This is proven in Sections 3 and 4 of [18], once we note that any torsion free $\mathscr{X}$-twisted sheaf of rank n is automatically stable when the Brauer class has period n. □

Let $\mathbf{Sh}^f_{\mathscr{X}}$ denote the locus of locally free $\mathscr{X}$-twisted sheaves. The morphism $\mathscr{V} \mapsto p_*\mathscr{E}nd(\mathscr{V})$ defines a morphism of stacks $e : \mathbf{Sh}^f_{\mathscr{X}} \to \mathbf{BLT}^{\overline{\alpha}}_{\widetilde{X}/S}$.

Lemma 6.3 *The morphism e is an epimorphism of stacks.*

Proof Since both stacks are locally of finite presentation, it suffices to prove that if S is strictly Henselian and A is an Azumaya on $\widetilde{X}_S$ with Brauer class α, then A is of the form $p_*\mathscr{E}nd(\mathscr{V})$ for $\mathscr{V}$ a blt locally free $\mathscr{X}_S$-twisted sheaf.

This follows immediately from Giraud's description of the cohomology class in $\mathrm{H}^2(\widetilde{X}, \boldsymbol{\mu}_n)$ associated to A: one takes the stack of isomorphisms $\mathscr{E}nd(V) \xrightarrow{\sim} A$ with V locally free with trivialized determinant $\det V \xrightarrow{\sim} \mathscr{O}$. This is a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$, and the sheaves V glue to give an $\mathscr{X}$-twisted sheaf of rank n with trivial determinant. □

Let $G = \mathrm{Pic}_{\widetilde{X}/S}[n]$ be the (finite) n-torsion subgroupscheme of the relative Picard scheme. Given an invertible sheaf $\mathscr{L}$ with a trivialization $\mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{O}_{\widetilde{X}}$ there is an induced 1-morphism $\otimes\mathscr{L} : \mathbf{Sh}_{\mathscr{X}} \to \mathbf{Sh}_{\mathscr{X}}$.

Lemma 6.4 *The morphisms $\otimes\mathscr{L}$ defined above as $\mathscr{L}$ ranges over a set of representatives for G define an action $G \times \mathrm{Sh}_{\mathscr{X}} \to \mathrm{Sh}_{\mathscr{X}}$.*

Proof Given an invertible sheaf $\mathscr{L}$ with a trivialization $\mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{O}$ and a torsion free sheaf $\mathscr{F}$ of rank n with a trivialization $\det \mathscr{F} \xrightarrow{\sim} \mathscr{O}$, we get a trivialization

$$\det(\mathscr{F} \otimes \mathscr{L}) \xrightarrow{\sim} \det(\mathscr{F}) \otimes \mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{O} \otimes \mathscr{O} \xrightarrow{\sim} \mathscr{O}.$$

This map induces the action. □

Proposition 6.5 *The morphism $e : \mathscr{V} \mapsto p_*\mathscr{E}nd(\mathscr{V})$ induces an isomorphism of stacks*

$$[\mathrm{Sh}^f_{\mathscr{X}}/G] \xrightarrow{\sim} \mathbf{BLT}^{\overline{\alpha}}_{\widetilde{X}/S}.$$

Proof Via the morphism e the scalar multiplication action on $\mathscr{V}$ is sent to the trivial action on $p_*\mathscr{E}nd(\mathscr{V})$ so that e factors through an epimorphism of stacks $\varepsilon : \mathrm{Sh}^f_{\mathscr{X}} \to \mathbf{BLT}^{\alpha}_{\widetilde{X}/S}$. It follows from the Skolem–Noether theorem that any isomorphism $p_*\mathscr{E}nd(\mathscr{V}) \xrightarrow{\sim} p_*\mathscr{E}nd(\mathscr{V}')$ comes from an isomorphism $\mathscr{V} \xrightarrow{\sim} \mathscr{V}' \otimes L$ for some invertible sheaf L, and that any invertible sheaf L induces a canonical isomorphism $p_*\mathscr{E}nd(\mathscr{V}) \xrightarrow{\sim} p_*\mathscr{E}nd(\mathscr{V} \otimes L)$.

Since G acts by twisting by invertible sheaves, the morphism ε factors through the quotient as $\overline{\varepsilon} : [\mathrm{Sh}^f_{\mathscr{X}}/G] \to \mathbf{BLT}^{\overline{\alpha}}_{\widetilde{X}/S}$. On the other hand, suppose given an isomorphism $p_*\mathscr{E}nd(\mathscr{V}) \xrightarrow{\sim} p_*\mathscr{E}nd(\mathscr{W})$. By the above remark, we have that there is an invertible sheaf M and an isomorphism $\mathscr{V} \xrightarrow{\sim} \mathscr{W} \otimes M$. Taking determinants gives an isomorphism $\det V \xrightarrow{\sim} \det W \otimes M^{\otimes n}$. Via the isomorphisms $\det V \xrightarrow{\sim} \mathscr{O}$ and $\det W \xrightarrow{\sim} \mathscr{O}$ we get a canonical isomorphism $M^{\otimes n} \xrightarrow{\sim} \mathscr{O}$, displaying $\mathscr{W}$ as the image of $\mathscr{V}$ under $\otimes M$. This shows that $\overline{\varepsilon}$ is a monomorphism, showing that it is an isomorphism. □

Remark 6.6 The stack $[\mathrm{Sh}_{\mathscr{X}}/G]$ gives the desired compactification of $\mathbf{BLT}^{\widetilde{\alpha}}_{\widetilde{X}/S}$. Arguing as in Proposition 6.5.1.1 of [17] and Section 6.5.2 of [17], one can also show that $[\mathrm{Sh}_{\mathscr{X}}/G]$ carries a virtual fundamental class. We will not discuss this in further detail here.

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