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Explosive Development of the Kelvin–Helmholtz Quantum Instability on the He-II Free Surface

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Abstract—We analyze nonlinear dynamics of the Kelvin–Helmholtz quantum instability of the He-II free surface, which evolves during counterpropagation of the normal and superfluid components of liquid helium. It is shown that in the vicinity of the linear stability threshold, the evolution of the boundary is described by the $|\phi|^4$ Klein–Gordon equation for the complex amplitude of the excited wave with cubic nonlinearity. It is important that for any ratio of the densities of the helium component, the nonlinearity plays a destabilizing role, accelerating the linear instability evolution of the boundary. The conditions for explosive growth of perturbations of the free surface are formulated using the integral inequality approach. Analogy between the Kelvin–Helmholtz quantum instability and electrohydrodynamic instability of the free surface of liquid helium charged by electrons is considered.

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1. INTRODUCTION

The tangential discontinuity of velocities in a liquid as well as at the interface between two liquids leads to the emergence of the classical Kelvin–Helmholtz instability (KHI) [1]. In recent years, the KHI in superfluid liquids has been actively studied, including the instability of the interface between different superfluid phases of ^{3}He [2–6] as well as the instability of free ^{4}He surface [7–9] in the superfluid state (the so-called He-II phase appearing at a temperature below 2.17 K [10]). The former case realized for ^{3}He is the closest to the classical KHI (the phases are on different sides of the boundary), while the latter case is principally different since instability appears due to counterpropagation of the normal and superfluid ^{4}He components under the free surface. In this study, we consider the second case that can naturally be referred to as quantum KHI since both components are on the same side of the free surface, and their coexistence is a quantum effect having no classical analog. A typical experimental situation, in which such a relative motion of components is observed, is illustrated in Fig. 1. Experimental investigations of the emergence of instability on the free flat surface of superfluid He-II with the heat flow in the bulk of the liquid have been initiated by I.M. Khalatnikov and are actively performed at the Laboratory of Quantum Crystals, Institute of Solid State Physics, Russian Academy of Sciences (see, for example, [8, 11–14]).

We will use the two-liquid approximation for describing the dynamics of ^{4}He [10] with densities ρ_s and ρ_n of the superfluid and normal components, respectively (total density of the liquid is $\rho = \rho_n + \rho_s$. Both components are treated as incompressible liquids (ρ_n = const and ρ_s = const).

Quantum KHI instability increment for linear perturbations proportional to $e^{i\mathbf{k}\cdot\mathbf{r}_\perp - i\omega t}$ of a flat horizontal surface in the presence of gravity and capillarity is given by the following dispersion relation obtained in [15, 16]:

$$\begin{aligned}\omega_k^2 &= \rho_s(\omega - \mathbf{V}_s\cdot\mathbf{k})^2 \\ &+ \rho_n(\omega - \mathbf{V}_n\cdot\mathbf{k} + i2\nu_n k^2)^2 + 4\rho_n\nu_n^2 k^3 m_n,\end{aligned} \tag{1}$$

where $\mathbf{V}_s$ and $\mathbf{V}_n$ are the mean velocities of the superfluid and normal components, $\mathbf{r}_\perp$ is the horizontal coordinate; t is the time; ν_n is the kinematic viscosity

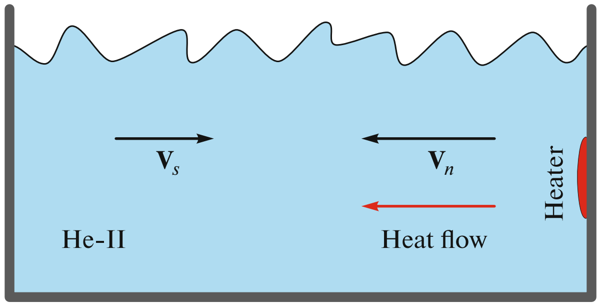

Fig. 1. (Color online) Counterpropagation (with velocities $\mathbf{V}_{s,\,n}$) of the superfluid and normal components of superfluid ^{4}He induced by a heat flow from a heater, which is carried by the normal component. Both components are located in the same volume of ^{4}He and flow along the tangent to the common free surface.

of the normal component, which is defined as the dynamic viscosity normalized to ρ_n; $\mathbf{k}$ and ω are the wavevector and frequency of perturbations, $k = |\mathbf{k}|$, and $m_n = [k^2 - i(\omega - \mathbf{V}_n \cdot \mathbf{k})/\nu_n]^{1/2}$. Also

$$\omega_k^2 \equiv gk + \alpha k^3/\rho \qquad (2)$$

is the dispersion relation for gravity–capillary waves in the absence of average motion of the liquid components, where g is the acceleration due to gravity and α is the surface tension coefficient.

In the framework of the simplest nondissipative two-liquid description [15, 16], the flow of both phases is treated as a potential flow. In this case, the velocities of the phases can be written as $\mathbf{V}_{n,\,s} = \nabla\Phi_{n,\,s}$, where $\Phi_{n,\,s}$ are the velocity potentials satisfying (since $\rho_n = \text{const}$ and $\rho_s = \text{const}$) the Laplace equations

$$\nabla^2\Phi_n = 0, \quad \nabla^2\Phi_s = 0, \qquad (3)$$

and dispersion relation (1) can be reduced to

$$(\omega - \mathbf{V}_m \cdot \mathbf{k})^2 = \omega_k^2 - \frac{\rho_s\rho_n}{\rho}(\mathbf{V} \cdot \mathbf{k})^2, \qquad (4)$$

where $\mathbf{V}_m \equiv (\rho_n V_n + \rho_s V_s)/\rho$ is the mean velocity of the center of mass of the liquid and $\mathbf{V} \equiv \mathbf{V}_s - \mathbf{V}_n$ is the average relative velocity for the liquid components. Without loss of generality, we will henceforth assume that $\mathbf{V}_m = 0$, which implies a transition to the corresponding moving frame of reference.

Dispersion relation (4) makes it possible to find [15, 16] the threshold value of relative velocity

$$V_c = \left(\frac{4\rho^3 g\alpha}{\rho_n^2\rho_s^2}\right)^{1/4}, \qquad (5)$$

which corresponds to wavenumber $k = k_0 \equiv \sqrt{\rho g/\alpha}$. A linear KHI appears at relative velocity $V \equiv |\mathbf{V}| > V_c$.

It should be noted that the dispersion relation for the quantum KHI assumes the same form as the conventional dispersion relation of the KHI for the interface between two ideal immiscible liquids (see, for example, [1], p. 346]) if we use in Eq. (4) classical dispersion relation $\omega_{k,\text{classical}}^2 = (\rho_s - \rho_n)gk/\rho + \alpha k^3/\rho$ for gravity–capillary waves instead of relation (2) for ω_k in the absence of average motion of the liquid components (in this case, we presume that the liquid of density ρ_n is above the interface, while the liquid with density ρ_s is below it). To return from $\omega_{k,\,\text{classical}}$ to relation (2), it is sufficient to perform substitution $\rho_n \to -\rho_n$ (without changing total density ρ) in $\omega_{k,\,\text{classical}}$. The remaining terms in relation (4) are independent of g and, hence, do not change depending on the position of the second liquid with density ρ_n under the free surface (quantum case) or above the interface (classical case). The described difference between the linear dispersion relations for quantum and classical KHIs are only quantitative by nature. A qualitative difference between the classical and quantum KHIs appears at nonlinear stages of instability development. For example, in the limit $V \gg V_c$ for classical KHI, a tendency to the formation of weak root singularities appears at the interface between the liquids, for which the surface remains smooth, but its curvature becomes infinitely large over a finite time interval [17, 18]. Under analogous conditions for the quantum KHI, a tendency to the formation of strong singularities (cusp points) appears [19].

In this study, we consider nonlinear stages of development of the quantum KHI in the vicinity of the stability threshold (i.e., for $|V - V_c|/V_c \ll 1$). In this situation, a narrow packet of surface waves in the Fourier space is excited, which makes it possible to formulate the equation of the envelope of this wave packet. It will be shown that the nonlinearity for any relation between the densities of helium components produces a destabilizing effect, i.e., accelerates the development of the linear instability of the interface and leads to explosive instability with a singularity in the equation of the envelope appearing over a finite time interval. In the context of complete hydrodynamic equations, this means that the solution becomes strongly nonlinear (values of characteristics slopes become of the order of unity) over a finite time.

The article is organized as follows. In Section 2, we consider basic equations in two-liquid hydrodynamics with kinematic and dynamic boundary conditions on the free surface. In Section 3, a transformation to the effective one-liquid description is made for 2D flows using harmonically conjugate potentials (stream functions). In Section 4, we demonstrate that as a result of our transformations, the initial problem of description of nonlinear development of the quantum KHI, which appears due to relative motion of the normal and superconducting phases, becomes equivalent (up to trivial removal of constants) to the problem of dynamics of the electron-charged boundary of liquid helium in an electric field (the limit when the charge is completely screens the fields over the liquid). This analogy has allowed us to use a number of results obtained earlier from analysis of the behavior of liquid helium in an electric field for the problem considered here. In Section 5, using the results obtained in [20, 21], we demonstrate that the evolution of the interface in the vicinity of the stability threshold can be described by the $|\phi|^4$ relativistically invariant Klein–Gordon equation with nonlinear attraction for the complex envelope of the wave being excited. It is important that for any relation between the densities of the helium components, nonlinearity plays a destabilizing role, accelerating the linear instability development. In Section 6, using the integral inequality approach in the Klein–Gordon equations and analogy with the motion of an effective Newtonian particle in a certain potential, we formulate sufficient conditions for explosive buildup of perturbations of the free surface. In concluding Section 7, we consider the hard excitation of strongly nonlinear solutions and their applicability in full two-liquid hydrodynamics.

2. BASIC EQUATIONS

We limit our analysis to 2D flows for which all quantities depend on pair of variables $\mathbf{r} = (x, y)$, where x and y are the horizontal and vertical coordinate, respectively. In this case, we have $\nabla = (\partial/\partial x, \partial/\partial y)$.

The helium surface in the unperturbed state is plane $y = 0$, and the motion of helium components along the x axis is uniform (i.e., equality $\Phi_{n,s} = V_{n,s}x$ holds for the velocity potentials, where $V_{n,s}$ are horizontal velocity components). As mentioned above, we can assume without loss of generality that $\rho_n V_n + \rho_s V_s = 0$, which corresponds to analysis of the problem in the center-of-mass system. Then the velocity components can be expressed in terms of average relative velocity $V = V_s - V_n > 0$ of the components as $V_{n,s} = \mp\rho_{s,n}V/\rho$.

We assume that the perturbed free surface of liquid helium is defined by equation $y = \eta(x, t)$; i.e., the liquid occupies domain

$$-\infty < x < \infty, \quad -\infty < y < \eta(x,t).$$

Perturbations of velocity potentials, which appear because of deformation of the boundary, decay in the bulk:

$$\Phi_{n,s} \to V_{n,s}x, \quad y \to -\infty. \tag{6}$$

The motion of the boundary is determined by the dynamic and kinematic boundary conditions. The dynamic condition (time-dependent Bernoulli equation for a two-component liquid) has form

$$\begin{aligned}\rho_n\left(\frac{\partial\Phi_n}{\partial t} + \frac{(\nabla\Phi_n)^2}{2}\right) + \rho_s\left(\frac{\partial\Phi_s}{\partial t} + \frac{(\nabla\Phi_s)^2}{2}\right) \\ = -\rho g\eta + \frac{\alpha\eta_{xx}}{(1+\eta_x^2)^{3/2}} + \Gamma, \quad y = \eta,\end{aligned} \tag{7}$$

where $\eta_x \equiv \partial\eta/\partial x$, $\eta_{xx} \equiv \partial^2\eta/\partial x^2$, the first term on the right-hand side is responsible for the force of gravity, and the second term, for capillary forces. These forces tend to return the perturbed boundary of the liquid to the initial planar state. Quantity Γ is the Bernoulli constant; its value that ensures the fulfillment of condition (7) in the unperturbed state $\Phi_{n,s} = V_{n,s}x$ and $\eta = 0$ is given by

$$\Gamma = \frac{\rho_n V_n^2 + \rho_s V_s^2}{2} = \frac{\rho_n\rho_s V^2}{2\rho}.$$

Finally, in accordance with the kinematic condition, normal velocity of the boundary must coincide with the normal component of the velocity in each phase,

$$\frac{\eta_t}{\sqrt{1+\eta_x^2}} = \partial_n\Phi_n = \partial_n\Phi_s, \quad y = \eta(x,t), \tag{8}$$

where $\eta_t \equiv \partial\eta/\partial t$ and $\partial_n \equiv \mathbf{n}\cdot\nabla$ indicates the derivative with respect to the outward normal

$$\mathbf{n} = (-\eta_x, 1)\frac{1}{\sqrt{1+\eta_x^2}}$$

to the boundary of the liquid.

3. TRANSITION TO EFFECTIVE ONE-LIQUID DESCRIPTION

Let us introduce the average velocity of the medium as

$$\mathbf{v} \equiv \frac{\rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s}{\rho}. \tag{9}$$

The equations of motion can be written in terms of a single effective liquid of density ρ, which flows at velocity $\mathbf{v}$. Velocity (9) corresponds to potential

$$\Phi = \frac{\rho_n\Phi_n + \rho_s\Phi_s}{\rho}, \tag{10}$$

i.e., $\mathbf{v} = \nabla\Phi$. We also introduce auxiliary velocity potential

$$\phi = \sqrt{\rho_s\rho_n}(\Phi_n - \Phi_s)/\rho. \tag{11}$$

Potentials Φ and ϕ are linear combinations of harmonic potentials $\Phi_{s,n}$; therefore, these potentials satisfy Laplace equations

$$\nabla^2\Phi = 0, \quad \nabla^2\phi = 0.$$

Conditions (6) deep inside fluid can be written as

$$\Phi \to 0, \quad \phi \to -Vx\sqrt{\rho_s\rho_n}/\rho, \quad y \to -\infty. \tag{12}$$

Dynamic boundary condition (7) assumes the form

$$\begin{aligned}\rho\left(\frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2}\right) = -\rho g\eta + \frac{\alpha\eta_{xx}}{(1+\eta_x^2)^{3/2}} + \Gamma - \frac{\rho(\nabla\phi)^2}{2}, \\ y = \eta.\end{aligned} \tag{13}$$

It can easily be seen from expression (8) that the kinematic condition for potential Φ becomes

$$\frac{\eta_t}{\sqrt{1+\eta_x^2}} = \partial_n\Phi, \quad y = \eta. \tag{14}$$

Finally, the kinematic condition for potential ϕ is obviously trivial:

$$\partial_n\phi = 0, \quad y = \eta. \tag{15}$$

Thus, the initial equations of motion for the two components of liquid helium in the nondissipative approximation can be reduced to classical equations for a potential flow of a single incompressible liquid with a free surface, which takes into account the capillary and gravity forces, with additional term $\rho(\nabla\phi)^2/2$ on the right-hand side of time-dependent Bernoulli equation (13). This term is responsible for the effect of counterpropagation of the liquid helium components and, hence, for the evolution of the Kelvin–Helmholtz instability. Let us consider this term in greater detail.

We introduce auxiliary function ψ which is the harmonic conjugate with potential ϕ; i.e., ψ and ϕ are connected by the Cauchy–Riemann relations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$

Let us clarify the physical meaning of quantity ψ. Similarly to relation (11), it can be written as

$$\psi = \sqrt{\rho_s\rho_n}(\Psi_n - \Psi_s)/\rho, \tag{16}$$

where $\Psi_{n,\,s}$ is the stream function for the normal and superconducting He-II components, which are connected with potentials $\Phi_{n,\,s}$ by relations

$$\frac{\partial\Phi_{n,s}}{\partial x} = \frac{\partial\Psi_{n,s}}{\partial y}, \quad \frac{\partial\Phi_{n,s}}{\partial y} = -\frac{\partial\Psi_{n,s}}{\partial x}.$$

Therefore, harmonic function ψ is (to within a constant factor) the difference between the stream functions for different helium components.

As a consequence of the Cauchy–Riemann relations, we obtain

$$\partial_n\phi|_{y=\eta} = -\partial_\tau\psi|_{y=\eta},$$

where

$$\partial_\tau \equiv \frac{1}{\sqrt{1+\eta_x^2}}(1,\eta_x)\cdot\nabla$$

is the tangential derivative. Then boundary condition (15) can be written as

$$\partial_\tau\psi = 0, \quad y = \eta,$$

i.e., quantity ψ does not change along the boundary. Without loss of generality, we can set $\psi|_{y\,=\,\eta} = 0$.

After the introduction of ψ, the equations of motion assume final form

$$\nabla^2\Phi = 0, \quad \nabla^2\psi = 0, \tag{17}$$

$$\Phi \to 0, \quad \psi \to -Vy\sqrt{\rho_s\rho_n}/\rho, \quad y \to -\infty, \tag{18}$$

$$\begin{aligned}&\rho\left(\frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2}\right)\\ &= -\rho g\eta + \frac{\alpha\eta_{xx}}{(1+\eta_x^2)^{3/2}} + \Gamma - \frac{\rho(\nabla\psi)^2}{2}, \quad y = \eta,\end{aligned} \tag{19}$$

$$\frac{\eta_t}{\sqrt{1+\eta_x^2}} = \partial_n\Phi, \quad \psi = 0, \quad y = \eta. \tag{20}$$

It is important that the problem of determining function ψ and, as a consequence, of key term $\rho(\nabla\psi)^2/2$ in the dynamic boundary condition decouples from the general problem of motion of the boundary. Indeed, it can be seen from these equations that quantity ψ is completely determined by shape η of the boundary and is independent of the form of its motion (i.e., of potential Φ). It is exactly this circumstance that determines the possibility of transition to the one-liquid description for the given problem.

4. ANALOGY WITH THE DYNAMICS OF LIQUID HELIUM IN AN ELECTRIC FIELD

Let us demonstrate that Eqs. (17)–(20) are identical to the equations appearing in the description of instability of the electron-charged free boundary of liquid helium in an external electric field. We assume that liquid helium is at a low temperature so that the normal phase is absent ($\rho_n \approx 0$ and $\rho \approx \rho_s$). The unperturbed (planar) boundary is charged by electrons with surface charge density σ. It is well known that electrons can freely move over the boundary, thus ensuring that the boundary is equipotential one [22, 23]. In an applied vertical uniform electric field, the field strengths over (E_o) and inside (E_i) the liquid are connected by relation $E_o - E_i = 4\pi\sigma$.

Velocity potential Φ of the (single) liquid and electric field potentials over (φ_o) and inside (φ_i) the liquid satisfy Laplace equations

$$\nabla^2\Phi = 0, \quad \nabla^2\varphi_{i,o} = 0.$$

These equations must be solved together with the following conditions in the bulk and on the free boundary:

$$\Phi \to 0, \quad \varphi_{i,o} \to -E_{i,o}y, \quad y \to -\infty,$$

$$\begin{aligned}\rho\left(\frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2}\right) &= -\rho g\eta + \frac{\alpha\eta_{xx}}{(1+\eta_x^2)^{3/2}}\\ &+ \gamma - \frac{(\nabla\varphi_i)^2 - (\nabla\varphi_o)^2}{8\pi}, \quad y = \eta,\end{aligned}$$

$$\frac{\eta_t}{\sqrt{1+\eta_x^2}} = \partial_n\Phi, \quad \varphi_i = \varphi_o = 0, \quad y = \eta,$$

where the Bernoulli constant is $\gamma = (E_i^2 - E_o^2)/8\pi$ and the potential of boundary $y = \eta$ is assumed to be zero. The last term in the time-dependent Bernoulli equation is responsible for the electrostatic pressure at the free boundary; it includes the pressures over and under the surface.

The above equations turn out to be identical to Eqs. (17)–(20) for the quantum KHI (derived above in a particular case when $E_o = 0$ and, accordingly, $\varphi_o = 0$ and $E_i = -4\pi\sigma$). This case (when the surface charge completely screens the field over the liquid) was realized, for example, in experiments [22, 24]. In the framework of this analogy, auxiliary flow function ψ and electric field potential φ_i in the liquid, as well as velocity difference V and field strength E_i, are connected by relations

$$\psi\sqrt{4\pi\rho} \equiv \varphi_i, \quad V\sqrt{4\pi\rho_n\rho_s/\rho} \equiv E_i.$$

It should be noted that the Bernoulli constants also coincide in this case ($\Gamma \equiv \gamma$).

The revealed analogy makes it possible to use the results obtained earlier from analysis of the electrohydrodynamic instability of the charged surface of liquid helium [20, 21, 25–27] for analyzing the quantum Kelvin–Helmholtz instability.

Concluding this section, we note that this analogy cannot be extended to the general 3D case. In the case of KHI, there exists a preferred direction, viz., the direction of flow of the liquids (x-axis in our case). There is no preferred direction for electrohydrodynamic instability—the problem is invariant to rotation about the vertical y axis along which the external electric field is directed.

5. AMPLITUDE EQUATION FOR THE DYNAMICS OF THE FREE BOUNDARY

Analysis performed in [20] revealed that the liquid helium boundary in an electric field becomes unstable when the following condition holds:

$$E_i^2 + E_o^2 > E_c^2,$$

where

$$E_c^2 = 8\pi\sqrt{\rho g \alpha}.$$

In the vicinity of the instability threshold, harmonics with wavenumbers close to $k_0 = \sqrt{\rho g/\alpha}$ grow. Let us introduce the supercriticality parameter as

$$\delta = \frac{E_i^2 + E_o^2 - E_c^2}{E_c^2}.$$

If $|\delta| \ll 1$, it is natural to construct the equation for the envelope for describing the dynamics of the boundary. For the 2D case (in the 3D case, it is necessary to consider the interaction of three plane waves with wavevectors turned through $2\pi/3$ [28, 29]), the shape of the boundary is sought in form

$$\eta(x,t) = \frac{1}{2k_0}[u(x,t)e^{ik_0x} + u^*(x,t)e^{-ik_0}] + O(|u|^2),$$

where u is the dimensionless complex amplitude (envelope) of the wave, and asterisk marks complex conjugation. It was shown in [21] that the evolution of the boundary is described by the Klein–Gordon equation with cubic nonlinearity:

$$\frac{1}{2gk_0}\frac{\partial^2 u}{\partial t^2} = \delta u + \frac{1}{2k_0^2}\frac{\partial^2 u}{\partial x^2} + \left(\Delta^2 - \frac{5}{16}\right)u|u|^2,$$

where we have introduced notation

$$\Delta = \frac{E_i^2 - E_o^2}{E_c^2}.$$

It can be seen that in the linear approximation (in the spatially homogeneous case), the amplitude increases exponentially for $\delta > 0$. In this case, the nonlinearity hampers the instability development for $0 < \Delta^2 < 5/16$ and accelerates it for $\Delta^2 > 5/16$.

In the case of our interest (when $E_o = 0$), for small supercriticality $\delta \approx 0$, we have $E_i \approx E_c$ and, hence, $\Delta \approx 1$. In such a case, the amplitude equation takes the form

$$\frac{1}{2gk_0}\frac{\partial^2 u}{\partial t^2} = \delta u + \frac{1}{2k_0^2}\frac{\partial^2 u}{\partial x^2} + \frac{11}{16}u|u|^2, \tag{21}$$

i.e., the nonlinearity is destabilizing. Analysis performed in Section 4 shows that this equation also describes the evolution of the KHI of the He-II boundary. In this case, supercriticality is given by

$$\delta = \frac{V - V_c}{V_c}, \tag{22}$$

where the critical velocity is defined by Eq. (5).

Due to the destabilizing effect of the nonlinearity, a tendency to an explosive increase in the amplitude of the He-II boundary appears during the KHI evolution in the spatially homogeneous solution; it increases unlimitedly over a finite time as

$$u \propto \frac{1}{t - t_c}, \quad t \to t_c,$$

where t_c is the instant of “explosion.” It is interesting that this result is independent of the ratio of the densities of the normal and superfluid helium components. The coefficient of the nonlinear term in Eq. (21) turns out to be universal. In the model developed here, the only quantity depending on the ratio of densities is the difference in velocities of the components (5), which is a threshold for the KHI development and appears in supercriticality condition (22).

After scaling

$$t \to \frac{t}{\sqrt{2gk_0}}, \quad x \to \frac{x}{\sqrt{2}k_0}, \quad u \to \frac{4u}{\sqrt{11}}$$

amplitude equation (21) for envelope u assumes the following compact form:

$$\frac{\partial^2 u}{\partial t^2} = \delta u + \frac{\partial^2 u}{\partial x^2} + |u|^2 u. \tag{23}$$

It corresponds to Hamiltonian

$$H = \int \left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2 - \delta |u|^2 - \frac{|u|^4}{2} \right) dx, \tag{24}$$

which is an integral of motion.

6. CONDITIONS FOR EXPLOSIVE DEVELOPMENT OF THE QUANTUM KELVIN–HELMHOLTZ INSTABILITY

Thus, we have established that during the development of quantum KHI, wave packet envelope u obeys the Klein–Gordon complex nonlinear equation with cubic nonlinearity, which is known as the $|\phi|^4$ model. The specific feature of Eq. (23) is that the nonlinearity does not stabilize linear instability, but on the contrary, enhances it, leading to an explosive increase in amplitudes under certain conditions. Indeed, assuming that perturbation of boundary $\eta(x, t)$ and, hence, amplitude $u(x, t)$ is localized in space, we consider, analogously to [30], the temporal evolution of square of L^2 norm

$$B(t) \equiv \int |u|^2 dx. \tag{25}$$

Equations (23)–(25) allow us to write

$$\begin{aligned} B_{tt} &= \int [2|u_t|^2 + u_{tt}u^* + uu_{tt}^*] dx \\ &= -4H + \int [6|u_t|^2 - 2\delta|u|^2 + 2|u_x|^2] dx, \end{aligned} \tag{26}$$

where we have used integration by parts with respect to x with allowance for decreasing boundary conditions for $|x| \to \infty$. It should be noted that contribution from $2\int |u|^4 dx$ was completely absorbed by term $-4H$. The subscripts in Eq. (26) and below indicate differentiation: $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $B_{tt} = \partial^2 B/\partial t^2 = d^2B/dt^2$, and so on.

To obtain the lower estimate of term

$$\int 6|u_t|^2 dx = 6\int R_t^2 dx + 6\int \phi_t^2 R^2 dx \tag{27}$$

in Eq. (26), we write complex amplitude u in form $u \equiv Re^{i\phi}$, where $R = |u|$ is the amplitude and ϕ is the phase. Using the Cauchy–Bunyakovsky inequality

$$\left|\int fg dx\right| \le \left(\int |f|^2 dx\right)^{1/2} \left(\int |g|^2 dx\right)^{1/2},$$

which is valid for complex-valued functions f and g, we obtain inequalities

$$|B_t|^2 = 2\left|\int RR_t dx\right| \le 2B^{1/2}\left(\int R_t^2 dx\right)^{1/2} \tag{28}$$

and

$$|Q| = 2\left|\int \phi_t R^2 dx\right| \le 2B^{1/2}\left(\int \phi_t^2 R^2 dx\right)^{1/2}, \tag{29}$$

where $Q \equiv i\int [u_t u^* - uu_t^*] dx$ is the integral of motion ($Q_t \equiv 0$) of Eq. (23). When Eq. (23) is used in the quantum field theory and the theory of solitons, this integral is sometimes referred to as the charge (see, for example, [31, 32]); however, we will not use this term below since the concept of charge has already been used in Sections 4 and 5 in another context.

Using inequalities (28) and (29), we obtain the following inequality from (27):

$$\int 6|u_t|^2 dx \ge \frac{3B_t^2}{2B} + \frac{3Q^2}{2B}.$$

Substituting this expression into (26) and omitting term $\int 2|u_x|^2 dx$ (the disregard of this nonnegative term is compatible with the sign of the inequality), we arrive at differential inequality

$$B_{tt} \ge \frac{3}{2}\frac{B_t^2}{B} + \frac{3}{2}\frac{Q^2}{B} - 4H - 2\delta B. \tag{30}$$

Change of variables $B = A^{-2}$ allows us to write inequality (30) in form

$$A_{tt} \le -\frac{\partial U(A)}{\partial A}, \tag{31}$$

where

$$U(A) = -\frac{HA^2}{2} - \delta\frac{A^2}{2} + \frac{A^6 Q^2}{8}. \tag{32}$$

Differential inequality (31) can be written in the equivalent form of ordinary differential equation

$$A_{tt} = -\frac{\partial U(A)}{\partial A} - h^2(t), \tag{33}$$

where $-h^2(t)$ is an unknown nonpositive force.

Analysis of the formation of singularity in Eq. (23) can be performed based on the method proposed in [33] (see [34–36] for the further development of this method). The method is based on analogy of Eq. (33) with the equation of motion of an effective Newtonian "particle" with coordinate A in potential (32) under the action of an additional (generally, nonpotential) force $-h(t)^2$, which pulls the particle to the origin of coordinates. When this particle achieves zero ($A = 0$), singularity $B = \infty$ is formed in Eq. (23). The form of potential $U(A)$ (32) is shown qualitatively in Fig. 2 for $Q \neq 0$ depending on values of H and δ. (The particular case of $Q = 0$ can be considered analogously; see also [30].)

It is convenient to introduce particle energy

$$W(t) \equiv \frac{A_t^2}{2} + U(A), \tag{34}$$

which depends on time due to the presence of force $-h(t)^2$ as

$$\frac{dW(t)}{dt} = A_t\left[A_{tt} + \frac{\partial U(A)}{\partial A}\right] = -h(t)^2 A_t. \tag{35}$$

The complete classification of sufficient conditions for the formation of the singularity over a finite time (also known as the wave collapse or just collapse [37]) can be obtained. The corresponding exact theorem can easily be formulated (when needed) based on the following considerations that should be analyzed separately for $A_t(0) > 0$ and $A_t(0) \le 0$.

(a) If for the given initial conditions $A(0)$ and $A_t(0) \le 0$ the particle reached the origin ($A = 0$ in Eq. (33)) under the action of conservative force $-\partial U(A)/\partial A$ alone, it would definitely reach the origin of coordinates in the same or shorter time if force $-h(t)^2$ were taken into account. This is due to the fact that in accordance with relation (35), in the case with $A_t \le 0$ considered here, we have inequality $W(t) \ge W(0)$ for energy. In the case depicted in Fig. 2a, collapse occurs when $W(0) > 0$, i.e., when the particle has sufficient initial energy for reaching zero during a finite time. In the case shown in Fig. 2b, it is necessary that either $A(0)$ be on the left of the barrier (for any $W(0)$), or the value of $W(0)$ be over the barrier (for $A(0)$ on the right of the barrier). The case illustrated in Fig. 2c is the simplest because the collapse occurs here for any values of $A(0)$, $A_t(0)$, and $W(0)$.

(b) For $A_t(0) > 0$, the sufficient conditions for the collapse can be formulated for the cases shown in Figs. 2b and 2c. In the case illustrated in Fig. 2c, the collapse occurs for any values of $A(0)$, $A_t(0)$, and $W(0)$, because the monotonicity of potential $U(A)$ stops the motion of the particle to the right over a finite time (nonzero force $-h(t)^2$ only accelerated this process), after which the particle falls to zero during a finite time (in this case also, nonzero force $-h(t)^2$ only accelerates this process). To ensure, for example, the fulfillment of condition $H < 0$ for $\delta < 0$, in is necessary in this case that nonlinearity be quite strong for the negative contribution from term $-\int (1/2)|u|^4 dx$ in Hamiltonian (24) to exceed the contributions from all remaining positive terms. In the case shown in Fig. 2b with $A_t(0) > 0$, the collapse appears a fortiori when $A(0)$ is on the left of the barrier, and initial energy $W(0)$ is insufficient for overcoming the barrier even when force $-h^2(t)$ is ignored. With allowance for force $-h^2(t)$, the particle necessarily stops on the left of the barrier and then falls to zero over a finite time. In the remaining cases shown

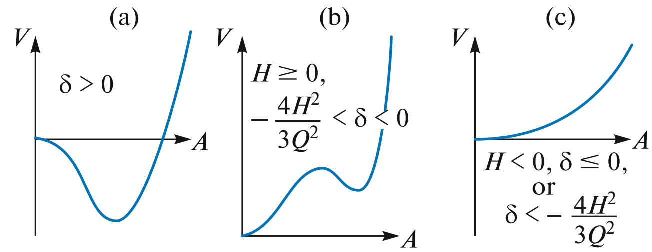

Fig. 2. (Color online) Qualitative form of potential $U(A)$ (32) as a function of H and δ for $Q \neq 0$.

in Figs. 2a and 2b ($A(0)$ is on the right of the barrier), the particle can stuck in the vicinity of the potential minimum because, as follows from Eq. (35) for $A_t(0) > 0$, we have $W(t) \le W(0)$; i.e., the particle loses energy, and after the reflection from the wall, the energy may turn out to be insufficient for overcoming the barrier. For this reason, the sufficient condition for the collapse in these case cannot be formulated (although collapse is still possible, but its full description requires detailed knowledge of the $-h(t)^2$ dependence).

If one of the above sufficient conditions for the explosive KHI evolution holds and $A_t(0) \le 0$, time t_c for the emergence of the singularity satisfies inequality

$$t_c \le \int_0^{A(0)} \frac{dA}{\sqrt{2[W(0) - U(A)]}},$$

following from Eqs. (33) and (34).

It should also be noted that for spatially homogeneous initial conditions $-h^2(t) \equiv 0$ (all integrals in this case must be considered in the sense of their values per unit length along the x axis), all inequalities of this section become equalities; among other things, inequality (31) becomes an ordinary differential equation for a Newtonian particle. Therefore, the sufficient criteria for the collapse on the class of homogeneous solution in this section become sufficient and necessary conditions for the collapse, which generalizes the criteria for collapse from [30], where the contribution from integral of motion Q was not taken into account. The asymptotic form of falling of the particle to zero ($A = 0$) corresponds to constant velocity A_t; therefore, for $B = A^{-2}$ and, accordingly, for squared amplitude $|u|^2$ of the envelope of a surface wave, the asymptotic dynamics of the collapse corresponds to law $(t_c - t)^{-2}$.

7. CONCLUSIONS

It should be noted that some of sufficient criteria for the explosive increase of the amplitudes, which were formulated in Section 6, are applicable in the case when a plane surface is stable to small perturbations ($\delta < 0$). This means that the excitation of instability is hard, and a large initial perturbation of a linearly stable regime may lead to the emergence of a singularity over a finite time. In all cases, Eq. (23) in the vicinity of the singularity becomes inapplicable: next orders of perturbation theory now make a contribution of the same order of magnitude as the nonlinearity in Eq. (23). It can then be concluded that the solution becomes strongly nonlinear in finite time t_c; i.e., the characteristic slopes of the surface become of the order of unity due to the action of the leading nonlinearity of the Klein–Gordon equation. After this, we generally expect the breaking of waves. In this region, it is necessary to consider complete hydrodynamic equations of Section 3, which is beyond the scope of this article.

It should be noted that indefinitely strongly nonlinear stages of the quantum KHI were analyzed in [19] disregarding gravity and capillarity; complete integrability of the equations of motion was demonstrated in the sense of reduction of exact dynamics to the Laplace growth equation that has an infinitely large number of integrals of motion and is associated with the dispersionless limit of the Toda hierarchy [38]. Analysis of the possible integrability of complete hydrodynamic equations of Section 3 is also an interesting subject for future investigations.

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