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Poles and Branch Cuts in Free Surface Hydrodynamics

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Abstract

We consider the motion of ideal incompressible fluid with free surface. We analyzed the exact fluid dynamics through the time-dependent conformal mapping $z = x + iy = z(w, t)$ of the lower complex half plane of the conformal variable w into the area occupied by fluid. We established the exact results on the existence vs. nonexistence of the pole and power law branch point solutions for $1/z_w$ and the complex velocity. We also proved the nonexistence of the time-dependent rational solution of that problem for the second- and the first-order moving pole.

1 Introduction

Consider an ideal incompressible fluid with free surface which occupies the infinite region $-\infty < x < \infty$ in the horizontal direction x and extends down to $y \to -\infty$ in the vertical direction y as schematically shown in the left panel of Fig. 1. It is assumed that there is no dependence on the third spatial dimension, i.e., the fluid motion is exactly two dimensional. The bulk of fluid is at the rest, i.e., there is no motion both at $|x| \to \pm\infty$ and $y \to -\infty$. A potential motion of the ideal incompressible fluid with free surface can be addressed by a time-dependent conformal mapping:

$$z(w, t) = x(w, t) + iy(w, t) \tag{1}$$

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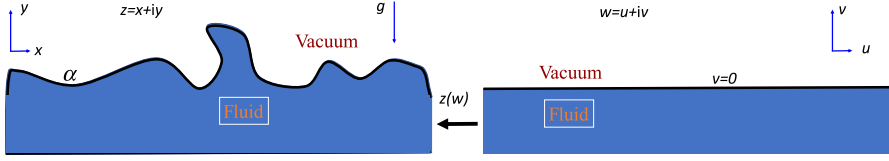
Fig. 1 Shaded area represents the domain occupied by fluid in the physical plane $z = x + iy$ (left) and the same domain in $w = u + iv$ plane (right). Thick solid lines correspond to the fluid's free surface

of the lower complex half-plane $\mathbb{C}^-$ of the auxiliary complex variable $w \equiv u + iv$, $-\infty < u < \infty$, into the area in (x, y) plane occupied by the fluid. Here, the real line $v = 0$ is mapped into the fluid free surface (see Fig. 1) and $\mathbb{C}^-$ is defined by the condition $-\infty < v \leq 0$. The time-dependent fluid free surface is represented in the parametric form as:

$$x = x(u, t), \ y = y(u, t). \tag{2}$$

A decay of perturbation of fluid beyond flat surface at $x(u, t) \to \pm\infty$ and/or $y \to -\infty$ requires that:

$$z(w, t) \to w \ \text{for} \ |w| \to \infty, \ w \in \mathbb{C}^-. \tag{3}$$

The conformal mapping (1) implies that $z(w, t)$ is the analytic function of $w \in \mathbb{C}^-$ and:

$$z_w \neq 0 \ \text{for any} \ w \in \mathbb{C}^-. \tag{4}$$

To account for the fluid motion, one considers a complex velocity potential $\Pi(z, t)$:

$$\Pi = \Phi + i\Theta, \tag{5}$$

where $\Phi(\mathbf{r}, t)$ is the velocity potential determined by the condition that the fluid velocity $\mathbf{v}$ is the potential one, $\mathbf{v} = \nabla\Phi$, and Θ is the stream function Θ defined by:

$$\Theta_x = -\Phi_y \ \text{and} \ \Theta_y = \Phi_x. \tag{6}$$

The incompressibility condition $\nabla \cdot \mathbf{v} = 0$ implies the Laplace equation $\nabla^2 \Phi = 0$ inside fluid, i.e., Φ is the harmonic function inside fluid. Equation (6) represents the Cauchy-Riemann equations ensuring the analyticity of $\Pi(z, t)$ in the domain of z plane occupied by the fluid, so Θ is the harmonic conjugate of Φ. Without loss of generality, we assume a zero Dirichlet boundary condition (BC) for Π as:

$$\Pi \to 0 \ \text{for} \ |x| \to \infty \ \text{or} \ y \to -\infty. \tag{7}$$

The conformal mapping (1) ensures that the function $\Pi(z, t)$ (5) transforms into $\Pi(w, t)$ which is the analytic function of w for $w \in \mathbb{C}^-$ (in the bulk of fluid). Here and below, we abuse the notation and use the same symbols for functions of either w or z (in other words, we assume that, e.g., $\tilde{\Pi}(w, t) = \Pi(z(w, t), t)$ and remove $\tilde{\ }$ sign). The conformal transformation (1) also ensures the Cauchy–Riemann equations $\Theta_u = -\Phi_v$, $\Theta_v = \Phi_u$ in w plane.

BCs at the free surface are time-dependent and consist of kinematic and dynamic BCs. A kinematic BC ensures that free surface moves with the normal velocity component v_n of fluid particles at the free surface. Motion of the free surface is determined by a time derivative of the parameterization (2), while the kinematic BC is given by a projection of **v** into the normal direction as:

$$\mathbf{n} \cdot (x_t, y_t) = v_n \equiv \mathbf{n} \cdot \nabla \Phi |_{x=x(u,t), \ y=y(u,t)}, \tag{8}$$

where $\mathbf{n} = \frac{(-y_u, x_u)}{(x_u^2 + y_u^2)^{1/2}}$ is the outward unit normal vector to the free surface and subscripts here and below mean partial derivatives, $x_t \equiv \frac{\partial x(u,t)}{\partial t}$, etc.

A dynamic BC is given by the time-dependent Bernoulli equation (see, e.g., [25]) at the free surface:

$$\left(\Phi_t + \frac{1}{2} (\nabla \Phi)^2 + gy \right) \Bigg|_{x=x(u,t), \ y=y(u,t)} = -P_\alpha, \tag{9}$$

where g is the acceleration due to gravity and $P_\alpha = -\frac{\alpha(x_u y_{uu} - x_{uu} y_u)}{(x_u^2 + y_u^2)^{3/2}}$ is the pressure jump at the free surface due to the surface tension coefficient α. Here, without loss of generality, we assumed that pressure is zero above the free surface (i.e., in vacuum). Also below in this paper, we assume zero surface tension $\alpha = 0$. All results below apply both to the surface gravity wave case ($g > 0$) and the Rayleigh–Taylor problem ($g < 0$).

Equations (8) and (9) together with the analyticity (with respect to the independent variable w) of both $z(w,t)$ and $\Pi(w,t)$ inside fluid form a closed set of equations which is equivalent to Euler equations for dynamics of ideal fluid with free surface. The idea of using time-dependent conformal transformation like (1) to address free surface dynamics of ideal fluid was exploited by several authors including [8–10,16, 32,34,38,39,43]. We follow the approach of Refs. [16,17,41] to transform from the unknowns $z(w,t)$ and $\Pi(w,t)$ into new equivalent "Dyachenko" variables [13]:

$$R = \frac{1}{z_w}, \tag{10}$$

$$V = i \frac{\partial \Pi}{\partial z} = i R \Pi_w. \tag{11}$$

Then, the dynamical equations at the real line $w = u$ take the following complex form [13]:

$$\frac{\partial R}{\partial t} = i (U R_u - R U_u), \tag{12}$$

$$U = \hat{P}^-(R \bar{V} + \bar{R} V), \quad B = \hat{P}^-(V \bar{V}), \tag{13}$$

$$\frac{\partial V}{\partial t} = i [U V_u - R B_u] + g(R - 1), \tag{14}$$

where

$$\hat{P}^- = \frac{1}{2}(1 + i\hat{H}) \quad \text{and} \quad \hat{P}^+ = \frac{1}{2}(1 - i\hat{H}) \tag{15}$$

are the projector operators of a function $q(u)$ (defined at the real line $w = u$) into functions $q^+(u)$ and $q^-(u)$ analytic in $w \in \mathbb{C}^-$ and $w \in \mathbb{C}^+$, respectively, such that $q = q^+ + q^-$, i.e., $\hat{P}^+(q^+ + q^-) = q^+$ and $\hat{P}^-(q^+ + q^-) = q^-$. Here, we assume that $q(u) \to 0$ for $u \to \pm\infty$. Also the bar means complex conjugation and:

$$\hat{H} f(u) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{f(u')}{u' - u} du' \tag{16}$$

is the Hilbert transform with p.v. meaning a Cauchy principal value of the integral. The nonlocal operator (15) appears in the dynamical equations [Eqs. (12)–(14)], because at each given moment of time, one has to find the relation between the value of Φ at the free surface and its normal derivative to evolve the free surface in the physical plane z using the kinematic BC (8). Such relation is nothing more then the Dirichlet–Neumann operator [12] which can be found in w plane explicitly through the Hilbert transform (16) (see, e.g., Ref. [17] for more discussion on that).

It was found in Ref. [14] that the system (12)–(14) has an arbitrary number of the nontrivial integrals of motion beyond the natural integrals like the Hamiltonian and the horizonal momentum (see Ref. [17]). Many of these integrals commute with each other with respect of the noncanonical Poisson bracket found in Ref. [17]. It was suggested in Ref. [14] that the existence of such commuting integrals of motion might be a sign of the Hamiltonian integrability of the free surface hydrodynamics.

In this paper, we aim to address the complimentary question (beyond the possible Hamiltonian integrability) which is to study allowed vs. not allowed classes of solutions in the system (12)–(14). To answer that question, we consider analytical continuation of Eqs. (10)–(14) into the complex plane $w \in \mathbb{C}$. In particular, it amounts to straightforward replacing of u by w in the integral representation of $\hat{P}^+q(w)$ and $\hat{P}^-q(w)$ as detailed in Ref. [17]. A complex conjugation $\bar{f}(w)$ of $f(w)$ in Eqs. (12)–(14) and throughout this paper is understood as applied with the assumption that $f(w)$ is the complex-valued function of the real argument w even if w takes the complex values, so that:

$$\bar{f}(w) \equiv \overline{f(\bar{w})}. \tag{17}$$

That definition ensures the analytical continuation of $f(w)$ from the real axis $w = u$ into the complex plane of $w \in \mathbb{C}$. We also notice that in Eqs. (12)–(14) and throughout this paper, we use the partial derivatives over w and u interchangeably by assuming the analyticity in w.

Previous work focused primary on branch point singularities in Refs. [1–7,11,15, 20–24,31,33,35,38,39,47,48] in different versions of fluid dynamic equations. Also pole solutions of z_w and Π_w were found in Ref. [14] which corresponds to zeros of R. The question of extending the allowed classes of exact solution remains extremely to understand the generic dynamics of free surface.

The goal of this paper is to analyze the existence of complex singularities of both R and V in the complex plane w during the nonzero duration of time. The singularities

are not allowed for $w \in \mathbb{C}^-$, because both z and Π are analytic there (inside the fluid domain) and the zeros of z_w are also excluded for $w \in \mathbb{C}^-$, because (1) is the conformal map. However, the singularities are generally allowed for $w \in \mathbb{C}^+$, i.e., outside of the fluid domain. One can trivially have any singularity (including poles, branch points, etc.) for both R and V for $w \in \mathbb{C}^+$ at the initial time $t = 0$. The important question which we analyze if there are singularities that keep their nature in the course of evolution to at least any finite duration of time. We refer to such singularities as "persistent". We found that there are severe restrictions on the existence of persistent poles of arbitrary order. These restriction are given by the following theorems which are proven below in Sect. 2:

Theorem 1 *Assume that R has the pole of the highest order $n_{\max, R} \geq 1$ and V has the pole of the highest order $n_{\max, V} \geq 0$ at $z = a(t)$, $a \in \mathbb{C}^+$ with the corresponding Laurent series:*

$$R = \sum_{j=-n_{\max, R}}^{-1} R_j(t)(w - a)^j + R_{reg}, \quad n_{\max, R} \geq 1 \tag{18}$$

and

$$V = \begin{cases} \sum_{j=-n_{\max, V}}^{-1} V_j(t)(w - a)^j + V_{reg} & \text{for } n_{\max, V} \geq 1, \\ V_{reg} & \text{for } n_{\max, V} = 0, \end{cases} \tag{19}$$

where

$$R_{reg} = \sum_{j=0}^{\infty} R_j(t)(w - a)^j \tag{20}$$

and

$$V_{reg} = \sum_{j=0}^{\infty} V_j(t)(w - a)^j \tag{21}$$

are the regular parts of R and V (these regular parts are the analytic functions at $w = a(t)$). It is assumed that $R_{-n_{\max, R}}(t)$ and $V_{-n_{\max, V}}(t)$ are nonzero. We also define the Taylor series representations at $w = a$ of the functions $\bar{R}$ and $\bar{V}$ (these functions are analytic at $w = a$ from the definition (17), because both R and V are analytic at $w = \bar{a}$) as follows:

$$\bar{R}(w, t) \equiv R_c(t) + \sum_{j=1}^{\infty} R_{c,j}(t)(w - a)^j. \tag{22}$$

and

$$\bar{V}(w, t) \equiv V_c(t) + \sum_{j=1}^{\infty} V_{c,j}(t)(w - a)^j, \tag{23}$$

where $R_c(t) = R_{c,0}(t) \equiv \bar{R}(w, t)|_{w=a}$ and $V_c(t) = V_{c,0}(t) \equiv \bar{V}(w, t)|_{w=a}$ are zero-order terms and $R_{c,j}(t)$, $V_{c,j}(t)$ are the coefficients of the higher order terms of the respective power series. Then, Eqs. (12)–(14) can have persistent in time pole solution (18)–(21), such that both R and V have only simple poles singularities at a moving point $w = a(t)$ only if the following conditions are all satisfied:

(a) $n_{\max,V} < n_{\max,R}$, i.e., the order of the highest poles of V is always lower than the order $n_{\max,R}$ of the highest pole of R.
(b) Moreover, $n_{\max,V} \leq n_{\max,R} - m - 1$, where $m = (n_{\max,R} - 2)/2$ for $n_{\max,R}$ even and $m = (n_{\max,R} - 1)/2$ for $n_{\max,R}$ odd.
(c) The coefficients of Eq. (23) must satisfy the conditions $V_{c,1} = V_{c,2} = \cdots = V_{c,m} = 0$ provided $n_{\max,R} \geq 3$, where m is defined in (b).
(d) The coefficient of the highest nonzero pole of V is given by $V_{-n_{\max,R}+m+1} = -\frac{R_{-n_{\max,R}} V_{c,m+1}}{R_c}$ provided $n_{\max,R} \geq 2$, where m is defined in (b).

Remark 1 For the particular case of $n_{\max,R} = 0$, Theorem 1 recovers Theorem 1 of Ref. [14].

Remark 2 R_c in the denominator in (d) does not create any problem, because the conformal map (1) and the definition (10) imply that $R(w) \neq 0$ for $w \in \mathbb{C}^-$ and, respectively:

$$\bar{R}_c = R(w)|_{w=a} \neq 0 \text{ for } a \in \mathbb{C}^+. \tag{24}$$

This is a fact of essential importance for the proof of Theorem 1.

Remark 3 In addition to the expression in (d) in Theorem 1, it is possible to provide the explicit expressions for the coefficients $V_{-n_{\max,R}+m+2}, \ldots, V_{-1}$ provided $n_{\max,R} \geq 4$. These coefficients are fully determined by the coefficients in Eqs. (18), (22), and (23) only (and depend neither on time derivatives of these coefficients or a_t and g). In particular:

$$V_{-n_{\max,R}+m+2} = \frac{R_{-n_{\max,R}}(R_{c,1}V_{c,m+1} - R_c V_{c,m+2}) - R_{-n_{\max,R}+1}R_c V_{c,m+1}}{(R_c)^2}. \tag{25}$$

However, the other explicit expressions for $V_{-n_{\max,R}+m+3}, \ldots, V_{-1}$ turn increasingly bulky with the increase of $n_{\max,R}$, so we do not provide them here. Equation (25) is derived as the byproduct of the proof of Theorem 1 in Sect. 2.

Remark 4 Theorem 1 provides only the necessary conditions for the existence of the persistent pole solutions. These necessary conditions are quite restrictive and it appears likely that except very rear exceptions, such persistent pole solutions do not exist. The only known exception is the trivial case:

$$g = 0, \quad \frac{\partial R}{\partial t} \equiv 0, \quad \text{and} \quad V \equiv 0, \tag{26}$$

i.e., a stationary solution of fluid at rest without gravity. In Eqs. (12)–(14), the zero velocity $V \equiv 0$ implies that $U = B \equiv 0$. Then, Eq. (12) is satisfied by $\frac{\partial R}{\partial t} \equiv 0$, while

Eq. (14) reduces to $g(R - 1) \equiv 0$. Then, either $R \equiv 1$, i.e., a flat-free surface (which we do not consider as absolutely trivial) or $g = 0$ as in Eq. (26). Any singularity of R for $w \in \mathbb{C}^+$ is allowed for the stationary solution (26). In the sense of the existence of such trivial solution, Theorem 1 cannot be improved at least for $g = 0$ to fully exclude pole solutions in R. If $g \neq 0$, then for the initial conditions with the zero velocity $V \equiv 0$ and pole(s) in R, it is generally expected a formation of logarithmic branch point(s) in infinitely small time in line with the results of Ref. [39]. The detailed study of such logarithmic singularity is, however, beyond the scope of this paper.

Another way to strengthen Theorem 1 is to address the existence of the purely rational time-dependent solutions of Eqs. (12)–(14). It would be generally extremely attractive to find rational solutions containing only pole-type singularities in w. There are examples of different reductions/models of free surface hydrodynamics which allows such rational solutions. They include a free surface dynamics for the quantum Kelvin–Helmholtz instability between two components of superfluid Helium [29,30]; an interface dynamic between ideal fluid and light highly viscous fluid [26], and a motion of the dielectric fluid with a charged and ideally conducting free surface in the vertical electric field [44–46]. The general case of the ideal fluid with free surface considered in this paper, however, appears to resists heavily to the existence of such rational solutions. The following theorem is proven in Sect. 3:

Theorem 2 *Assume the following rational solution of Eqs.* (12)–(14):

$$R = \frac{R_{-2}(t)}{(w - a(t))^2} + \frac{R_{-1}(t)}{(w - a(t))} + 1,$$
$$V = \frac{V_{-1}(t)}{(w - a(t))}. \tag{27}$$

Then, beyond the trivial solution (26), *all possible solutions of Eqs.* (12)–(14) *have one or two zeros of $R(w, t)$ either for $w \in \mathbb{R}$ or for $w \in \mathbb{C}^-$. It implies the singularity of the conformal map* (1) *through the definition* (10) *contradicting the assumption of the mapping of $\mathbb{C}^-$ into the area occupied by fluid. Thus, no non-trivial rational solution* (27) *exists. In other words, the explicit family of nontrivial rational solutions obtained in the proof of this theorem is nonphysical because of the violation of the condition* $R(w) \neq 0$.

Remark 5 The rational solution (27), however, satisfies Theorem 1 by allowing up to the second-order pole in R and the first-order pole in V. This is the example that Theorem 1 provides only necessary conditions for the existence of the solutions with poles in R and V.

Remark 6 The last term in the right-hand side (r.h.s.) of the first equation of (27) is chosen to satisfy $R \rightarrow 1$ as required from Eq. (3) at $w \rightarrow \infty$, $w \in \mathbb{C}^-$. Also the second equation in (27) satisfies the decaying BC (7).

Remark 7 Theorem 1 is the local results, because we use the Laurent series of solutions of free surface hydrodynamics at any moving point $w = a(t)$, $\text{Im}(a) > 0$. It means

that we are not restricted to rational solutions, because such local analysis does not exclude the existence of branch points for $w \neq a(t)$, $w \in \mathbb{C}^+$. In contrast, Theorem 2 is the global results, because it fully excludes the existence of the rational solution (27) valid for any $w \in \mathbb{C}$.

Remark 8 The exact rational solutions of Eqs. (12)–(14) were obtained in Refs. [14, 40,42,47] for the non-decaying BCs, i.e., for the infinite energy of the fluid.

In contrast to the solution with pole singularities, we show, in Sect. 4, that power law branch points are persistent with Eqs. (12)–(14) which is consistent with the previous results of Refs. [1–7,11,14,20–24,31,33,35,38–40,42,47,48] obtained by various analytic and numerical techniques.

2 Non-persistence of Poles in *R* and *V* Variables

In this section, we prove Theorem 1.

Proof We start the proof by recalling Remark 1 that $R(w) \neq 0$ for $w \in \mathbb{C}^-$, see Eq. (24). Here and below, we often omit the second argument t when we focus on analytical properties in w.

All four functions R, V, U, and B of Eqs. (12)–(14) must have singularities in the upper half-plane $w \in \mathbb{C}^+$ while being analytic for $w \in \mathbb{C}^-$. To understand that, consider the Laurent series (18) and (19) and, similar, the Laurent series:

$$U = \sum_{j=-\max(n_{\max,V},n_{\max,R})}^{-1} U_j(w-a)^j + U_{\text{reg}}, \quad U_{\text{reg}} = \sum_{j=0}^{\infty} U_j(w-a)^j, \quad (28)$$

$$B = \sum_{j=-n_{\max,V}}^{-1} B_j(w-a)^j + B_{\text{reg}}, \qquad B_{\text{reg}} = \sum_{j=0}^{\infty} B_j(w-a)^j. \quad (29)$$

To understand validity of these equations, we notice that using Eq. (15), we can rewrite the definitions (13) as:

$$U = R\bar{V} + \bar{R}V - \hat{P}^+(R\bar{V} + \bar{R}V),$$
$$B = V\bar{V} - \hat{P}^+(V\bar{V}). \quad (30)$$

The functions $\hat{P}^+(R\bar{V} + \bar{R}V)$ and $\hat{P}^+(V\bar{V})$ are analytic at $w = a \in \mathbb{C}^+$; thus, they only contribute to the regular parts U_{reg} and B_{reg}, respectively. The functions $\bar{R}$ and $\bar{V}$ are also analytic at $w = a$ with the Taylor series representations (22) and (23). The sum of two terms $R\bar{V} + \bar{R}V$ in r.h.s. of the first equation in (30) also explains why the summation in r.h.s. of Eq. (28) starts from the most singular term with $j = -\max(n_{\max,V}, n_{\max,R})$. Equations (18),(19),(22),(23), and (28)–(30) imply that generally U and B have the same types of singularities as R and V except special cases when poles of either R or V are canceled out.

If $n_{\max, R} \leq n_{\max, V}$, then the most singular term in Eqs. (12)–(14) is $-i n_{\max, V} R_c V_{-n_{\max, V}}^2 (w - a)^{-2n_{\max, V} - 1}$ in r.h.s of Eq. (14), where we used Eqs. (18)–(23) and (28)–(30). It implies that $V_{-n_{\max, V}} = 0$ and, respectively, we must set that $n_{\max, R} > n_{\max, V}$ which completes the proof of the statement (a) of Theorem 1 as well as it fully covers Theorem 1 for $n_{\max, R} = 1$, so in the remaining part of the proof, we assume that $n_{\max, R} \geq 2$. Also the power of the most singular term in Eq. (28) turns into $j = -max(n_{\max, V}, n_{\max, R}) = -n_{\max, R}$.

The most singular terms in the left-hand side (l.h.s.) of Eqs. (12) and (14) result from the differentiation of a over t and they have the orders $(w - a)^{-n_{\max, R} - 1}$ and $(w - a)^{-n_{\max, V} - 1}$, respectively. Thus, they can be ignored for the leading order analysis, because they are much less singular than the leading terms in r.h.s. of these equations.

The term of the order $(w - a)^{-2n_{\max, R}}$ is identically zero in Eq. (14) because of $n_{\max, R} > n_{\max, V}$. Now, the most singular term is $-i R_{-n_{\max, R}} [R_c V_{-n_{\max, R} + 1} + R_{-n_{\max, R}} V_{c,1}] (w - a)^{-2n_{\max, R}}$ in r.h.s of Eq. (12) which results in:

$$V_{-n_{\max, R} + 1} = -R_{-n_{\max, R}} V_{c,1} / R_c, \ n_{\max, R} \geq 2, \tag{31}$$

because $R_{-n_{\max, R}} \neq 0$ by the assumptions of Theorem 1. For $n_{\max, R} = 2$, Eq. (31) completes the proof of Theorem 1, so in the remaining part of the proof, we assume that $n_{\max, R} \geq 3$.

Using Eq. (31) to exclude $V_{-n_{\max, R} + 1}$, we obtain the next order term in r.h.s of Eq. (14) as $\frac{-i(n_{\max R} - 2) R_{-n_{\max, R}}^2 V_{c,1}^2}{R_c} (w - a)^{-2n_{\max, R} + 1}$ which implies that:

$$V_{c,1} = 0, \ n_{\max, R} \geq 3. \tag{32}$$

But then, Eq. (31) results in:

$$V_{-n_{\max, R} + 1} = 0, \ n_{\max, R} \geq 3. \tag{33}$$

Thus, we must set:

$$n_{\max, V} \leq n_{\max, R} - 2, \ n_{\max, R} \geq 3, \tag{34}$$

which recovers the statement (b) of Theorem 1 for both $n_{\max, R} = 3$ and $n_{\max, R} = 4$ ($m = 1$ in both these cases as follows from the definition of m in the statement of Theorem 1). From Eqs. (32) and (33), we obtain that the most singular term in r.h.s of Eq. (12) is $-2i R_{-n_{\max, R}} [R_c V_{-n_{\max, R} + 2} + R_{-n_{\max, R}} V_{c,2}] (w - a)^{-2n_{\max, R} + 1}$ which results in:

$$V_{-n_{\max, R} + 2} = -R_{-n_{\max, R}} V_{c,2} / R_c, \ n_{\max, R} \geq 3, \tag{35}$$

because $R_{-n_{\max, R}} \neq 0$ by the assumptions of Theorem 1. For both $n_{\max, R} = 3$ an $n_{\max, R} = 4$, Eqs. (32)–(35) complete the proof of Theorem 1. Therefore, in the remaining part of the proof, we assume that $n_{\max, R} \geq 5$.

Remark 9 For $n_{\max, R} \geq 4$, one can consider at least one next order before reaching terms with A_t in l.h.s. Then, the term of the order $(w - a)^{-2n_{\max, R} + 2}$ is identically zero in Eq. (14) because of Eqs. (31)–(35). Now, the most singular term is $\propto (w - a)^{-2n_{\max, R} + 2}$

in r.h.s of Eq. (12) which results in Eq. (25) from Remark 3 for $n_{\max,R} = 4$ and, respectively, $m = 2$.

Proceeding further by induction for $n_{\max,R} \geq 5$, we complete the proof of Theorem 2 through straightforward calculations by collecting the remaining terms of powers $(w - a)^{-2n_{\max,R}+2}, \ldots, (w - a)^{-n_{\max,R}-2}$ in Eqs. (12) and (14). As it is seen from the previous steps of the induction, the even and odd values of $n_{\max,R}$ need to be treated a little differently. For the odd values, one has to take into account all terms of powers $(w - a)^{-2n_{\max,R}+2}, \ldots, (w - a)^{-n_{\max,R}-2}$. For the even values, it is sufficient to take into account only terms of powers $(w - a)^{-2n_{\max,R}+2}, \ldots, (w - a)^{-n_{\max,R}-3}$. The extra term of the power $(w - a)^{-n_{\max,R}-2}$ is identically zero in Eq. (14), while the term of the same power in Eq. (12) can be used to find the expression for V_{-1}. In contrast, for the odd values of $n_{\max,R}$, it is necessary to take into account all terms of powers $(w - a)^{-2n_{\max,R}+2}, \ldots, (w - a)^{-n_{\max,R}-2}$. Then, the term of the power $(w - a)^{-n_{\max,R}-2}$ in Eq. (14) ensures that $V_{c,m} = 0$ as required in the statement (c) of Theorem 1, while the term of the same power in Eq. (12) can be used to find the expression for V_{-1}. This concludes the proof of Theorem 1. □

Remark 10 One can immediately count that the total number of conditions obtained from the powers $(w - a)^{-2n_{\max,R}-1}, \ldots, (w - a)^{-n_{\max,R}-2}$ in Eqs. (12) and (14) is $2n_{\max,R}$. However, the number of the nontrivial conditions is only $n_{\max,R} + m$. These nontrivial conditions result in $V_{-n_{\max,R}} = V_{-n_{\max,R}+1} = \ldots = V_{-n_{\max,R}+m} = 0$, $V_{c,1} = \ldots = V_{c,m} = 0$, and explicit expressions for $V_{-n_{\max,R}+m+1}, \ldots, V_{-1}$. Remaining trivially satisfied $2n_{\max,R} - m$ conditions (trivial zeros) occur in Eq. (14) for the terms with the powers $(w - a)^{-2n_{\max,R}}, (w - a)^{-2n_{\max,R}+2}, (w - a)^{-2n_{\max,R}+4}, \ldots, (w - a)^{-n_{\max,R}-2}$ for the even $n_{\max,R}$ and for the powers $(w - a)^{-2n_{\max,R}}, (w - a)^{-2n_{\max,R}+2}, (w - a)^{-2n_{\max,R}+4}, \ldots, (w - a)^{-n_{\max,R}-3}$ for the odd $n_{\max,R}$. Another trivial zero is for the power $(w - a)^{-2n_{\max,R}-1}$ in Eq. (12). Terms of lower orders $(w - a)^{-n_{\max,R}-1}, \ldots$ can be additionally used to provide conditions for time derivative of different coefficients. Roughly, we can summarize Theorem 1 that the order of poles in V is at least twice smaller than the order of poles in R

3 Nonexistence of the Rational Solution with the First- or the Second-Order Poles

In this section, we prove Theorem 2. We first obtain the exact rational solution of Eqs. (12)–(14), but then show that it is not physical.

Proof We look for all possible functions $R_{-1}(t)$, $R_{-2}(t)$, $V_{-1}(t)$, and $a(t)$, such that Eq. (27) is the exact solution of Eqs. (12)–(14). We plug in (27) into Eqs. (12)–(14) and look for the exact solutions. The projectors in Eq. (13) are easy to evaluate using partial fractions over w if we notice that the complex conjugation of Eq. (27) is given by:

$$\bar{R} = \frac{\bar{R}_{-2}(t)}{(w - \bar{a}(t))^2} + \frac{\bar{R}_{-1}(t)}{(w - \bar{a}(t))} + 1,$$

$$\bar{V} = \frac{\bar{V}_{-1}(t)}{(w - \bar{a}(t))}, \tag{36}$$

where we recall that we do not conjugate w to obtain the analytical continuation from the real line $w = u$ as explained in Sect. 1.

We collect terms with all possible powers of $(w - a)$ in both Eqs. (12) and (14). The order $(w - a)^{-5}$ is trivially satisfied, because we set $V_{-2} = 0$ in Eq. (27) as required by Theorem 1. The order $(w - a)^{-4}$ needs that:

$$V_{-1} = \frac{R_{-2}\bar{V}_{-1}}{\bar{R}_{-2} + (a - \bar{a})(\bar{R}_{-1} + a - \bar{a})}, \tag{37}$$

where we assumed that $V_{-1} \neq 0$. In the opposite case of $V_{-1} = 0$, we immediately obtain that the only possible solution is $g = 0$ and both R_{-2}, R_{-1} are time independent thus recovering the trivial case (26). Thus, below, we assume $V_{-1} \neq 0$.

The order $(w - a)^{-3}$ in Eq. (14) is satisfied by Eq. (37), while Eq. (12) requires that:

$$a_t = \frac{i\bar{V}_{-1}}{a - \bar{a}}. \tag{38}$$

The order $(w - a)^{-2}$ in Eq. (12) together with the condition (37) requires a time independence of R_{-2}, that is:

$$R_{-2} = \text{const} \tag{39}$$

(provided that $R_{-2} \neq 0$), while Eq. (14) at that order is valid only for

$$g = 0. \tag{40}$$

The order $(w - a)^{-1}$ in Eq. (12) requires a time independence of R_{-1}, that is:

$$R_{-1} = \text{const}, \tag{41}$$

while Eq. (14) needs a time independence of V_{-1}, that is:

$$V_{-1} = \text{const}. \tag{42}$$

Solving Eq. (37) for $(a - \bar{a})$ together with Eqs. (39), (41) and (42) show that $(a - \bar{a})$ must be constant in time, i.e., the imaginary part of a must be constant. Then, Eq. (38) requires that $V_{-1} = \text{Re}(V_{-1})$ and, moreover:

$$a = a_{r,1}t + a_{r,0} + ia_i, \quad V_{-1} = 2a_{r,1}a_i, \tag{43}$$

where $a_{r,1}$, $a_{r,0}$, a_i are the arbitrary real constants. It remains to satisfy Eq. (37) which together with Eq. (43) gives that either $a_{r,1} = V_{-1} = 0$ (which recovers the trivial case (26)) or:

$$R_{-1} = \frac{\text{Im}(R_{-2})}{a_i} + 2ia_i. \tag{44}$$

The exact solution (27), (43), (44) is valid for the arbitrary complex constant value of R_{-2} and zero gravity $g = 0$. It means that solution propagates with the constant velocity in the horizontal direction with all residues being time independent.

The analyticity of R for $w \in \mathbb{C}^-$ requires that $a_i > 0$. We now check locations of zeros of R which are poles of z_u. Using Eq. (27), we obtain that $R = 0$ for:

$$w = a_{r,1}t + a_{r,0} - \frac{\mathrm{Im}(R_{-2})}{2a_i} \pm \left(\frac{\mathrm{Im}(R_{-2})^2}{4a_i^2} - a_i^2 - \mathrm{Re}(R_{-2})\right)^{1/2}. \qquad (45)$$

Equation (45) either has two real roots which implies a singularity at fluid's free surface with mapping of $z(w)$ into infinity or it has two complex conjugated roots, one is in $\mathbb{C}^-$ thus violating the analyticity of $z(w)$ for $w \in \mathbb{C}^-$. Thus, we conclude that the rational solution (27) is not compatible with the condition (4) that the mapping (1) is conformal for $w \in \mathbb{C}^-$ which completes the proof. □

4 Persistence of Branch Cuts

We show in this section that, contrary to poles analyzed in Sect. 2, power law branch cuts are persistent in time for free surface dynamics. Assume that in the small neighborhood of $w = a$, the following expansions hold:

$$
\begin{aligned}
V &= V_0 + V_\gamma (w - a)^\gamma + \ldots, \\
R &= R_0 + R_\gamma (w - a)^\gamma + \ldots, \\
U &= U_0 + U_\gamma (w - a)^\gamma + \ldots, \\
B &= B_0 + B_\gamma (w - a)^\gamma + \ldots,
\end{aligned}
\qquad (46)
$$

where γ is the complex number and "..." designates terms with less singular powers (i.e., with powers γ_1, such that $\mathrm{Re}(\gamma) < \mathrm{Re}(\gamma_1)$). Similar to Sect. 2, we perform local analysis at $w = a$ on the persistence of singularities but this time with the expansion (46).

Equations (30) and (46) imply that:

$$U_\gamma = R_c V_\gamma + V_c R_\gamma, \quad \text{and} \quad B_\gamma = V_c V_\gamma, \qquad (47)$$

where we collected terms with the power $(w - a)^\gamma$ and used definitions of R_c and V_c from Eqs. (22) and (23).

Plugging expansions (46) into Eqs. (12)–(14) above and collecting the most singular terms of the order $(w - a)^{\gamma - 1}$, we obtain that:

$$-R_\gamma \frac{\partial a}{\partial t} = \mathrm{i}\left(U_0 R_\gamma - R_0 U_\gamma\right), \qquad (48)$$

$$-V_\gamma \frac{\partial a}{\partial t} = \mathrm{i}\left(U_0 V_\gamma - R_0 B_\gamma\right). \qquad (49)$$

Multiplying Eq. (48) by V_γ and subtracting from it Eq. (49) multiplied by R_γ, we obtain the compatibility condition:

$$R_0(U_\gamma V_\gamma - B_\gamma R_\gamma) = 0. \tag{50}$$

Using Eqs. (50) and (47), we find the compatibility condition:

$$R_0 R_c V_\gamma = 0. \tag{51}$$

According to our assumptions, $R_c \neq 0$ as explained in Sect. 2. Then, the remaining possibilities in Eq. (51) are that either $R_0 = 0$ or $V_\gamma = 0$. The first possibility is that we assume that $V_\gamma \neq 0$ which implies that:

$$R_0 = 0. \tag{52}$$

Then, Eqs. (48) and (49) result in a simple equation for the singularity location:

$$\frac{\partial a}{\partial t} = -iU_0. \tag{53}$$

Equation (52) means that branch points are zeros of the function R. Equations (52) and (52) can be used for the direct comparison with simulations of the motion of branch points using the numerical analytical continuation into $w \in \mathbb{C}^+$, see Ref. [14].

There is no restriction on the value of γ in the equations in this section, which is a predicament to persistence of branch points of arbitrary types. Nevertheless, the most common type of branch points, observed in our numerical experiments, is $\gamma = \frac{1}{2}$ which is consistent with the results of Refs. [20,23,24,38,39]. Square root singularities have been also intensively studied based on the representation of vortex sheet in Ref. [1–7,11,21,22,31,33,35,47,48].

Particular solution of Eqs. (12)–(14) is Stokes wave which is a nonlinear periodic gravity wave propagating with the constant velocity [36,37]. In the generic situation, when the singularity of Stokes wave is away from the real axis (non-limiting Stokes wave), the only allowed singularity in $\mathbb{C}$ is $\gamma = 1/2$ as was proven in Ref. [38] for the first (physical) sheet of the Riemann surface and in Ref. [27] for the infinite number of other (non-physical) sheets of Riemann surface. References [18,19,28] provided detailed numerical verification of these singularities. The limiting Stokes wave is the special case $\gamma = 1/3$ with $a = iIm(a)$. Also Ref. [39] suggested the possibility in exceptional cases of the existence of $\gamma = 1/n$ singularities with n being any positive integer as well as singularities involving logarithms.

The second possibility to satisfy the compatibility condition (51) is to assume that $V_\gamma = 0$. In that case, either V is the regular function at $w = a$ (while R has a branch point at $w = a$) or one of less singular terms is not zero. We also notice that $(R_0)_t \propto R_0$ [i.e., the case (52) corresponds to the zero initial condition for R_0] as can be obtained from the analysis similar to provided above in this section. A further study of that case is beyond of the scope of this paper.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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