Statistical properties of a laser beam propagating in a turbulent medium

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We examine statistical properties of a laser beam propagating in a turbulent medium. We prove that the intensity fluctuations at large propagation distances possess a Gaussian probability density function and establish quantitative criteria for realizing the Gaussian statistics depending on the laser propagation distance, laser beam waist, laser frequency, and turbulence strength. We calculate explicitly the laser envelope pair correlation function and corrections to its higher-order correlation functions breaking Gaussianity. We discuss also statistical properties of the brightest spots in the speckle pattern.

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I. INTRODUCTION

Propagation of either the acoustic beam or the electromagnetic beam in a turbulent medium is a classical problem which has been a subject of numerous investigations which especially intensified after the advent of lasers in the 1960s. The shape of the beam wavefront is progressively disturbed with the increase of the propagation distance due to the turbulent fluctuations. These distortions are random due to the chaotic nature of turbulence. Therefore they should be described statistically. Below we refer specifically to a laser beam while assuming that the same theory can be applied for an acoustic beam.

The early advances in beam propagation through a turbulent medium outlined in Refs. [1–4] were focused on lower-order statistical moments of the laser intensity. Among such moments often a scintillation index $\sigma_I^2 \equiv \langle I^2\rangle/\langle I\rangle^2 - 1$ is used. The index σ_I^2 is the measure of the strength of the fluctuations of the irradiance I (the laser beam intensity) at the target plane. Here and below we denote by $\langle\cdots\rangle$ an average over the ensemble of the atmospheric turbulence realizations or, equivalently, over time.

At relatively small propagation distances, where irradiance fluctuation are small ($\sigma_I^2 \ll 1$), the classic perturbative approach well describes modification of the laser beam propagation due to turbulence [1,2]. Statistically averaged beam characteristics at larger distances in strong scintillation regimes ($\sigma_I^2 \gg 1$) were addressed through semiheuristic theory [5]. At such distances, the laser beam disintegrates into speckles with lower-order statistical moments providing only very limited information about structure of the intensity fluctuations. It was found in Ref. [6] that a significant fraction of deviation between the theoretical value of σ_I^2 [5] and simulations is due to rare large fluctuations of laser beam intensity. Such giant fluctuations were also observed in numerical experiments [7,8]. Reference [9] studied the structure of large fluctuations and proposed to use them for the efficient delivery of laser energy over long distances by triggering the pulse laser operations only during the times of such rare fluctuations. It was demonstrated in Ref. [9] that after 7 km propagation of the laser beam with the initial beam waist 1.5 cm in typical atmospheric conditions, about 0.1% atmospheric realizations carry $\gtrsim$28% of initial power in a single giant fluctuation (a single speckle) of the laser intensity.

In this paper we address the statistical properties of the laser intensity fluctuations and identify analytically the spatial profiles of the intense laser speckles. That problem has great long-standing fundamental interest as well as it is important for practical applications of long-distance energy delivery and cleaning of space debris [10]. We consider a propagation of the initially Gaussian beam at large distances corresponding to $\sigma_I^2 \gg 1$. We assume that the typical transverse size of speckles is much smaller than the Gaussian beam width, which is also ensured at large enough distances of propagation through a turbulent medium. The main result of this paper is that we use the ladder sequence of diagrams of stochastic perturbation theory to show that the probability density function (PDF) of laser intensity fluctuations at such distances is well approximated by the Gaussian stochastic process. The properties of such a stochastic process at each propagation distance z are fully determined by the pair correlation function of the laser intensity fluctuation, which has explicit expression in the integral form. The exact equation for the pair correlation function was first found in Ref. [11] (see also Refs. [2,4]), while the statistics of the laser field fluctuations were, to the best of our knowledge, not known before our work (beyond the lower-order statistical moments, which are generally not informative for large fluctuations). The spatial structure of large fluctuations (the bright spots of the laser intensity) of the stochastic Gaussian process has the form of the transverse pair correlation function of the Gaussian process [12–14]. The transverse width scaling of that correlation function for the particular case of laser beam propagation in atmospheric turbulence with the Kolmogorov-Obukhov spectrum [1,2] is $\propto z^{-3/5}$. We also found non-Gaussian corrections to the higher-order correlation functions and established that these corrections decrease with the propagation distance.

The plan of the paper is the following. In Sec. II we introduce basic equations in the physical variables, define the fluctuations of the refractive index (Sec. II A), and discuss the characteristic propagation lengths which control the properties of laser beam fluctuations (Sec. II B). Section III introduces scaled dimensionless variables. Section III A provides a reduction of the refractive index fluctuations to the Gaussian stochastic process with the power-law pair correlation function. Section III B discusses the parameters and different spatial regions in the dimensionless variables. Section IV analyzes the pair, fourth-, and higher-order correlation functions as well as addresses the statistic of the intensity fluctuations. Section V develops the ladder approximation as well as discusses the corrections beyond the ladder approximation identifying the region of the applicability of the Poisson statistics. Section VII summarize the results and discusses their applicability conditions. Appendices provide the details of the ladder approximation, corrections beyond it, and the calculations of the statistics of the large fluctuations (bright spots) of the laser intensity.

II. BASIC EQUATIONS

A propagation of a monochromatic beam with a single polarization through turbulent media is described by the linear Schrödinger equation (LSE) (also called the Leontovich equation) [1,2,15] for the spatiotemporal envelope of the electric field

$$i\frac{\partial}{\partial Z}\Psi+\frac{1}{2k_0}\nabla_\perp^2\Psi+k_0n_1\Psi=0, \tag{1}$$

where all quantities are functions of $\boldsymbol{R}_\perp, Z, t$. Here the beam is aligned along the Z axis, $\boldsymbol{R}_\perp\equiv(X,Y)$ are the transverse coordinates, t is the time, $\boldsymbol{\nabla}_\perp\equiv(\partial_X,\partial_Y)$, $k_0=2\pi n_0/\lambda_0$ is the wave vector in the medium, λ_0 is the wavelength in the vacuum, and $n=n_0+n_1$ is the linear index of refraction with the average value n_0 and the fluctuation contribution n_1, with zero average $\langle n_1\rangle=0$. The beam intensity is expressed as $I=|\Psi|^2$.

We assume that the inverse time of the laser beam propagation is much larger than characteristic rates of the turbulent motions. Then corrections related to the dependence of n_1 on t are negligible, and we can operate in terms of flash realizations of the refractive index n_1. Statistics of such flash realizations can be characterized by simultaneous correlation functions of n_1. Thus t does not explicitly enter Eq. (1) but serves as the parameter distinguishing different atmospheric realizations, so below we omit t in arguments of all functions. The neglect of the temporal derivative (that derivative results from the chromatic dispersion) is justified for laser beams with typical durations longer than a few nanoseconds.

Linear absorbtion (results in exponential decay of the laser intensity with propagation distance) is straightforward to include into Eq. (1). Kerr nonlinearity also can be added to Eq. (1) resulting in the nonlinear Schrödinger equation, which describes the catastrophic self-focusing (collapse) of laser beam for laser powers P above critical power P_c ($P_c\sim$ 3 GW for $\lambda_0=1064$ nm) [15–17] and multiple filamentation for $P\gg P_c$ [18]. At distances well below the nonlinear length, one can consider Kerr nonlinearity as perturbation (see, e.g., Ref. [10]) combining it with the effect of atmospheric turbulence. Such additions are beyond the scope of this paper.

We consider propagation of the beam produced by a laser located at $Z=0$. We assume that at the laser output the beam has a Gaussian shape

$$\Psi_{in}=A\exp\left(-R_\perp^2/w_0^2\right), \tag{2}$$

where w_0 is the initial Gaussian beam waist and A is the initial beam amplitude. Expression (2) should be treated as the initial condition to Eq. (1) posed at $Z=0$.

At small Z one can neglect the term with n_1 in Eq. (1). Then we find the explicit solution of Eq. (1) with the initial condition (2) given by the diffraction-limited Gaussian beam

$$\Psi=\frac{w_0^2}{w_0^2+2iZ/k_0}\exp\left(-\frac{R_\perp^2}{w_0^2+2iZ/k_0}\right). \tag{3}$$

Note that at distances $Z\gg Z_R$, where Z_R is the Rayleigh length

$$Z_R=k_0w_0^2/2, \tag{4}$$

the beam width can be estimated as $Z/(k_0w_0)\gg w_0$, whereas the phase varies on a much smaller transverse spatial scale $\sim\sqrt{Z/k_0}$.

A. Fluctuations of the refractive index

The refractive index variation n_1 is proportional to the density fluctuation of the turbulent medium. Description of principal properties of the atmospheric turbulence can be found in Ref. [19]. Typically, l_0 is in the range of few millimeters or even smaller, while L_0 is ranging from many meters to kilometers. The main contribution to n_1 stems from the turbulent fluctuations with scales of the order of the integral length (the outer scale) of the turbulence L_0. We are, however, interested in atmospheric fluctuations on scales of the order of the laser beam width, which is assumed to be much smaller than L_0. In terms of Eq. (1) such fluctuations on the scale L_0 result in the change of the phase of Ψ which are nearly homogenous and do not affect the laser intensity. Also, the beam width is assumed to be much larger than the Kolmogorov scale (the inner scale of turbulence) l_0. Then l_0 does not significantly affect the PDF of laser beam fluctuations. In this situation the turbulent fluctuations relevant for the problem belong to the inertial scale of turbulence where they possess definite scaling properties [20] and are both homogeneous and isotropic.

Atmospheric fluctuations can be characterized by the structure function of the refractive index fluctuations that is the simultaneous average $\langle[n_1(\boldsymbol{R}_\perp,Z)-n_1(\boldsymbol{0},0)]^2\rangle$. We remind readers that the angular brackets $\langle\cdots\rangle$ mean averaging over realizations of n_1 or, equivalently, averaging over time. In the inertial range of turbulence the Kolmogorov-Obukhov law is valid,

$$\langle[n_1(\boldsymbol{R}_\perp,Z)-n_1(\boldsymbol{0},0)]^2\rangle=C_n^2\rho^{2/3}, \tag{5}$$

where $\rho=\sqrt{R_\perp^2+Z^2}$ and the factor C_n^2 characterizes strength of the turbulence. The expression (5) is correct provided $l_0\ll\rho\ll L_0$.

We now consider a little bit more general case not restricting to the Kolmogorov-Obukhov law (5). It is still natural to

assume homogeneity and isotropy of the turbulence on scales smaller than its integral scale L_0. Then the structure function $\langle [n_1(\boldsymbol{R}_\perp, Z) - n_1(\boldsymbol{0}, 0)]^2 \rangle$ depends solely on ρ, which is the absolute value of the separation between points. We assume the following general power-law coordinate dependence on scales from the inertial interval:

$$\langle [n_1(\boldsymbol{R}_\perp, Z) - n_1(\boldsymbol{0}, 0)]^2 \rangle = C_n^2 \rho^\mu, \tag{6}$$

where $\mu < 1$ is the scaling exponent, characterizing the refractive index fluctuations. The particular case of the Kolmogorov-Obukhov law (5) corresponds to $\mu = 2/3$.

Using Eq. (6), one can represent the pair correlation function of n_1 by the expansion in three-dimensional Fourier harmonics as follows (see, e.g., Refs. [1,2]):

$$\begin{aligned}\langle n_1(\boldsymbol{R}_\perp, Z) n_1(\boldsymbol{0}, 0) \rangle = {} & \frac{C_n^2 \Gamma(\mu+2)}{4\pi^2} \sin\left(\frac{\pi\mu}{2}\right) \\ & \times \int d^2q\, dq_z \exp(i\boldsymbol{q}\cdot\boldsymbol{R}_\perp + iq_z Z) \\ & \times \left(q^2 + q_z^2\right)^{-3/2-\mu/2}, \end{aligned} \tag{7}$$

where $q^2 = q_x^2 + q_y^2$. Here we assumed that the divergence of the integral at small q, q_z is removed by a cutoff at the large scale $\rho \sim L_0$, which can be also taken care by considering the structure function (6) instead of the pair correlation function.

The propagation length Z is assumed to be much larger than the beam width, i.e., one can use the paraxial approximation with the characteristic wave vector q much larger than the characteristic component q_z. Therefore q_z can be neglected in Eq. (7) in comparison with q. Then one obtains by replacing $q^2 + q_z^2 \to q^2$ that

$$\begin{aligned}&\langle n_1(\boldsymbol{R}_\perp, Z) n_1(\boldsymbol{0}, 0) \rangle \\ &= \frac{C_n^2 \Gamma(\mu+2)}{2\pi} \sin\left(\frac{\pi\mu}{2}\right) \delta(Z) \int d^2q \exp(i\boldsymbol{q}\cdot\boldsymbol{R}_\perp) q^{-3-\mu}. \end{aligned} \tag{8}$$

The main contribution to the above integral stems from small wave vectors q (of the order of the inverse integral scale of turbulence L_0) giving an $\boldsymbol{R}_\perp$-independent constant. Extracting also an $\boldsymbol{R}_\perp$-dependent contribution one obtains that

$$\begin{aligned}&\langle n_1(\boldsymbol{R}_\perp, Z) n_1(\boldsymbol{0}, 0) \rangle \\ &= \delta(Z) C_n^2 \left[\text{const} + \frac{\Gamma(\mu+2)\Gamma(-1/2-\mu/2)}{2^{\mu+2}\Gamma(3/2+\mu/2)} \sin\left(\frac{\pi\mu}{2}\right) R_\perp^{\mu+1} \right]. \end{aligned} \tag{9}$$

Note that Eq. (9) can be obtained from Eq. (8) using the regularized version of the integral through the replacement of the power law (6) with the corresponding von Kármán spectrum [1] obtained by the substitution $q^{-3-\mu} \to (q^2 + q_0^2)^{-3/2-\mu/2}$, where $q_0 \sim 1/L_0$.

B. Spatial scales

The basic equation (1) is a linear equation with multiplicative noise. There are some characteristic distances for the laser beam propagation in the random medium which can be extracted by comparison of different terms in Eq. (1) and taking into account expression (9). The first scale is determined by the propagation distance at which the scintillation index σ_I^2 of the initially plane wave, $\Psi|_{Z=0} \equiv \text{const}$, becomes ~ 1. Using the perturbation technique of Refs. [2,21] we obtain that

$$\sigma_I^2 \equiv \sigma_R^2 = p_1 C_n^2 k_0^{3/2-\mu/2} Z^{3/2+\mu/2}. \tag{10}$$

Here

$$p_1 \equiv \sin\left(\frac{1}{2}\pi\mu\right) \cos\left[\frac{\pi(\mu+1)}{4}\right] \Gamma\left(-\frac{\mu}{2} - \frac{3}{2}\right) \Gamma(\mu+2), \tag{11}$$

which is the generalization of Eq. (47.31) of Ref. [2] beyond the particular case $\mu = 2/3$. The quantity σ_R^2 is called the Rytov variance, and for $\mu = 2/3$ it recovers the standard expression

$$\begin{aligned}\sigma_R^2 &= \frac{\sqrt{3}}{4}\sqrt{2-\sqrt{3}}\,\Gamma\left(-\frac{11}{6}\right)\Gamma\left(\frac{8}{3}\right) C_n^2 k_0^{7/6} Z^{11/6} \\ &= 1.22871\ldots C_n^2 k_0^{7/6} Z^{11/6} \end{aligned} \tag{12}$$

(see, e.g., Ref. [5], where 1.22871… was replaced by 1.23 following the approximate numerical value provided in Refs. [2,21]).

Fluctuations of the intensity I are strong at $\sigma_R^2 \gtrsim 1$, so we define the characteristic distance Z_{rytov} from the condition that $\sigma_R^2 = 1$, which gives together with Eq. (10)

$$Z_{\text{rytov}} = \left(p_1 C_n^2 k_0^{3/2-\mu/2}\right)^{-2/(3+\mu)}. \tag{13}$$

We call the distance (13) the *Rytov length*. In our work the Rytov length is assumed to be larger than the Rayleigh length (4),

$$Z_{\text{rytov}} \gtrsim Z_R, \tag{14}$$

which implies a smallness of the initial beam waist w_0 and a relative weakness of the refractive index fluctuations to make sure that the condition (14) is satisfied. We investigate the propagation distances $Z > Z_{\text{rytov}}$ where the Gaussian beam is already disintegrated into speckles.

It follows from Eq. (1) that the envelope Ψ changes with Z due to n_1 as $\Psi \propto \exp\left[ik_0 \int_0^Z dZ' n_1(\boldsymbol{R}_\perp, Z')\right]$. Then we obtain from Eq. (9) that the average square of the phase difference between two points at the same Z but different $\boldsymbol{R}_\perp$ caused by the refractive index fluctuations is

$$\begin{aligned}&k_0^2 \left\langle \left\{ \int_0^Z dZ' [n_1(\boldsymbol{R}_\perp, Z') - n_1(\boldsymbol{0}, Z')] \right\}^2 \right\rangle \\ &= -\frac{k_0^2 \Gamma(\mu+2)\Gamma(-1/2-\mu/2)}{2^{\mu+1}\Gamma(3/2+\mu/2)} \sin\left(\frac{\pi\mu}{2}\right) C_n^2 R_\perp^{\mu+1} Z. \end{aligned}$$

Equating this phase difference to unity, we find the phase correlation length

$$R_{ph} \sim \left(k_0^2 C_n^2 Z\right)^{-1/(\mu+1)}. \tag{15}$$

Thus qualitatively the beam front at a given Z can be considered as multiple cells of the transverse size R_{ph} with independent phases between different cells.

In accordance with the Huygens-Fresnel principle, the wave amplitude can be considered as a superposition of waves emitted by the secondary sources at a wavefront. The source of a transverse size R_0 produces the beam of the transverse length

$\sim\theta Z$ at the propagation distance Z, where $\theta \equiv 1/(k_0 R_0)$ is the corresponding beam divergence. Respectively, the sources of the transverse size $\sim R_{ph}$ located at $Z = Z_1$ produce the beams of the transverse length $\sim Z_1/(k_0 R_{ph})$ at the distance $Z = 2Z_1$. This length becomes larger than the transverse size $2Z_1/(k_0 w_0)$ at $Z = 2Z_1$ of the initial beam if $R_{ph} \gtrsim w_0/2$. This condition is satisfied at the distance $Z > Z_\star$, where $Z_\star$ is estimated from the condition that $w_0 \sim R_{ph}|_{Z=Z_\star}$ and Eq. (15) as

$$Z_\star \sim k_0^{-2} C_n^{-2} w_0^{-\mu-1}. \tag{16}$$

At $Z > Z_\star$ the total transverse beam width $R_{\rm width}$ loses memory of the initial beam waist w_0 and is estimated as

$$R_{\rm width} \sim Z/(k_0 R_{ph}) \sim \left(C_n^2 Z^{\mu+2} k_0^{1-\mu}\right)^{1/(\mu+1)}. \tag{17}$$

Note that the inequality (14) results in $Z_\star \gtrsim Z_{\rm rytov}$.

III. DIMENSIONLESS VARIABLES

Here we introduce dimensionless parameters, which we use below. Namely, the dimensionless coordinates $\boldsymbol{r} = (x, y)$ and z are defined as follows:

$$x = X/w_0,\ y = Y/w_0,\ \boldsymbol{r} = \boldsymbol{R}_\perp/w_0,\ z = Z/(4Z_R). \tag{18}$$

where w_0 is the initial Gaussian beam waist [see Eq. (2)] and Z_R is Rayleigh length (4). Then we obtain from Eq. (1) the following dimensionless stochastic equation:

$$i\frac{\partial}{\partial z}\Psi + \nabla^2\Psi + \xi(\boldsymbol{r}, z)\Psi = 0, \tag{19}$$

where the random factor

$$\xi = 2k_0^2 w_0^2 n_1 \tag{20}$$

determines stochastic properties of the envelope Ψ. The initial condition (2) in the dimensionless units takes the following form:

$$\Psi_{\rm in}(\boldsymbol{r}) = \exp(-r^2), \tag{21}$$

where the initial beam amplitude is set to one without loss of the generality because we consider the linear equation for the wave amplitude.

In the dimensionless units the relation (9) is rewritten as

$$\langle \xi(\boldsymbol{r}_1, z_1)\xi(\boldsymbol{r}_2, z_2)\rangle = \left(\text{const} - D r_{12}^{\mu+1}\right)\delta(z_1 - z_2), \tag{22}$$

where $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ and the factor D is

$$D = -\frac{c_n^2}{2^{\mu+1}} \frac{\Gamma(\mu+2)\Gamma(-1/2-\mu/2)}{\Gamma(3/2+\mu/2)} \sin\left(\frac{\pi\mu}{2}\right). \tag{23}$$

Here we used the dimensionless turbulence strength c_n^2 first introduced in Ref. [9] as

$$c_n^2 \equiv k_0^3 w_0^{\mu+3} C_n^2. \tag{24}$$

For $\mu = 2/3$, Eq. (23) implies that $D/c_n^2 = 2.91438\ldots$.

A. Reduction of the refractive index fluctuations to Gaussian stochastic process with power-law pair correlation function

We introduce the Fourier transform

$$\tilde\xi(\boldsymbol{k}, z) = \int d^2 r\ \exp(-i\boldsymbol{k}\boldsymbol{r})\xi(\boldsymbol{r}, z).$$

Then Eq. (22) implies that

$$\langle \tilde\xi(\boldsymbol{k}, z_1)\tilde\xi(\boldsymbol{q}, z_2)\rangle = \frac{D(2\pi)^2}{p_0 k^{3+\mu}}\delta(z_1 - z_2)\delta(\boldsymbol{k} + \boldsymbol{q}),$$
$$p_0 \equiv \frac{1}{2^{2+\mu}\pi(\mu+1)} \frac{\Gamma(1/2-\mu/2)}{\Gamma(3/2+\mu/2)}. \tag{25}$$

Of course, expression (25) corresponds to Eq. (22).

Since ξ is short-correlated in z, it can be regarded to possess Gaussian statistics by the central limit theorem (see, e.g., Ref. [22]). The probability distribution describing fluctuations of ξ corresponding to Eq. (25) can be written as

$$\mathcal{P} \propto \exp\left[-\frac{p_0}{2D}\int dz \int \frac{d^2 q}{(2\pi)^2} q^{3+\mu}|\tilde\xi|^2\right], \tag{26}$$

where the integration over z is taken over the propagation length of the laser beam. The consistency of Eqs. (25) and (26) can be immediately verified by replacing the integrals in Eq. (26) by discrete sums with Gaussian integrals explicitly evaluated which allow one to take the continuous limit back from sum to integrals. The probability density (26) can be a starting point for calculating complicated averages over the ξ statistics.

B. Parameters and regions

In the dimensionless variables (18), using Eqs. (11), (24), and (23), the Rytov length (13) takes the following form:

$$z_{\rm rytov} \equiv \frac{Z_{\rm rytov}}{4Z_R} = \frac{1}{2}\left(p_1 c_n^2\right)^{-\frac{2}{\mu+3}} = p_2 D^{-\frac{2}{\mu+3}}, \tag{27}$$

where

$$p_2 \equiv \frac{1}{2}\Gamma\left[\frac{2^{\mu+3}\pi\cos\left[\frac{1}{4}\pi(\mu+1)\right]}{(\mu+3)^2\Gamma\left(-\frac{\mu}{2}-\frac{3}{2}\right)\cos\left(\frac{\pi\mu}{2}\right)}\right]^{-\frac{2}{\mu+3}}. \tag{28}$$

For $\mu = 2/3$, Eq. (28) reduces to $p_2 = 0.80088\ldots$. The condition (14) together with (13) can be rewritten as $D \lesssim 1$.

In the dimensionless variables (18), the variations of the phase in the transverse direction $\boldsymbol{r}$ due to the noise become of order unity at the scale $r \sim r_{ph}$ where

$$r_{ph} = R_{ph}/w_0 \sim (Dz)^{-1/(\mu+1)}, \tag{29}$$

and R_{ph} is defined by Eq. (17). This quantity can be treated as the phase correlation length of the envelope Ψ in the transverse direction.

Equation (16), with Eqs. (23) and (24) taken into account, transforms in the dimensionless variables (18) into

$$z_\star = Z_\star/(4Z_R) \sim D^{-1}. \tag{30}$$

Note that $z_\star \gtrsim z_{\rm rytov}$ because of the inequality (14). As discussed in Sec. II B, the total transverse beam width $r_{\rm width}$ at the distance $z \gtrsim z_\star$ is determined by the random diffraction, where $r_{\rm width}$ is determined from Eq. (17) as

$$r_{\rm width} = R_{\rm width}/w_0 \sim (Dz)^{1/(\mu+1)} z. \tag{31}$$

The width (31) grows as z increases faster than the pure diffraction case since $\mu < 1$.

The random diffraction leads to a random phase of the field Ψ at $z \gg z_\star$. Therefore Ψ possesses Gaussian statistics at the

scales. Correspondingly, the intensity $I = |\Psi|^2$ has Poissonian statistics. We examine below the accuracy of this statement by calculating corrections to the Poissonian statistics.

IV. CORRELATION FUNCTIONS

Statistical properties of the electric field envelope Ψ can be examined in terms of its correlation functions. The second-order correlation function is the average

$$F_2(\boldsymbol{r}_1, \boldsymbol{r}_2, z) = \langle \Psi(\boldsymbol{r}_1, z)\Psi^\star(\boldsymbol{r}_2, z)\rangle, \tag{32}$$

taken at a given z. We examine also higher-order correlation functions:

$$F_{2n}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{2n}, z) = \langle \Psi(\boldsymbol{r}_1, z)\ldots\Psi(\boldsymbol{r}_n, z)\Psi^\star(\boldsymbol{r}_{n+1}, z)\ldots\Psi^\star(\boldsymbol{r}_{2n}, z)\rangle. \tag{33}$$

The correlation functions (32) and (33) are obviously invariant under a homogeneous phase shift. Therefore they are insensitive to the refraction index fluctuations at the integral scale of turbulence.

The angular brackets in Eqs. (32) and (33) designate averaging over the statistics of ξ. Principally, one should solve Eq. (19) for any realization ξ at a given initial condition, next calculate the product in the angular brackets in Eq. (33), and then average over the realizations with the weight dictated by Eq. (22). Of course, this procedure cannot be performed explicitly. However, Eq. (19) and expression (22) admit derivation of closed equations for correlation functions F_{2n}. First such procedure for the pair correlation function was proposed by Kraichnan [23] and Kazantsev [24] in the contexts of the passive scalar turbulence and turbulent dynamo, correspondingly. It was independently obtained in the optical context in Ref. [11] for both the pair and higher-order correlation functions.

To obtain the equations for F_{2n}, one may start with the relation

$$\Psi(\boldsymbol{r}, z_2) = \mathrm{T}\exp\left[i\int_{z_1}^{z_2} dz\,(\nabla^2 + \xi)\right]\Psi(\boldsymbol{r}, z_1), \tag{34}$$

where T exp means an z-ordered exponent. The relation (34) is a direct consequence of Eq. (19). Equation (34) enables one to relate a product of $\Psi(z_2)$, $\Psi^\star(z_2)$ to the corresponding product of $\Psi(z_1)$, $\Psi^\star(z_1)$. Due to short correlation length of ξ in z direction one can independently average the average of $\Psi(z_1)$, $\Psi^\star(z_1)$ and the exponents. Say,

$$\begin{aligned} F_2(\boldsymbol{r}_1, \boldsymbol{r}_2, z_2) &= \langle \Psi(\boldsymbol{r}_1, z_2)\Psi^\star(\boldsymbol{r}_2, z_2)\rangle \\ &= \left\langle \mathrm{T}\exp\left\{i\int_{z_1}^{z_2} dz\left[\nabla_1^2 + \xi(\boldsymbol{r}_1)\right]\right\}\right. \\ &\quad \left.\times\,\mathrm{T}\exp\left\{-i\int_{z_1}^{z_2} dz\left[\nabla_2^2 + \xi(\boldsymbol{r}_2)\right]\right\}\right\rangle \\ &\quad \times F_2(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1). \end{aligned} \tag{35}$$

Analogous relations can be obtained for other products.

Analyzing closely z_1 and z_2, one can expand the exponents. Since $\langle\xi\rangle = 0$, we should expand terms with ξ up to the second order as follows:

$$\begin{aligned} &\mathrm{T}\exp\left[i\int_{z_1}^{z_2} dz\,(\nabla^2 + \xi)\right] \\ &\quad \approx 1 + i\int_{z_1}^{z_2} dz\,(\nabla^2 + \xi) \\ &\qquad - \int_{z_1}^{z_2} dz\,\xi(\boldsymbol{r}, z)\int_{z_1}^{z} d\zeta\,\xi(\boldsymbol{r}, \zeta). \end{aligned} \tag{36}$$

Substituting the expansion (36) and the analogous expressions for the other exponents, keeping terms up to the second order in ξ and averaging, we relate $F_2(z_2)$ to $F_2(z_1)$. Since $\langle\xi(\boldsymbol{r}, z_1)\xi(\boldsymbol{r}, z_2)\rangle$ is a narrow symmetric function of $z_1 - z_2$, one should take

$$\int^{z_2} dz_1\,\langle\xi(\boldsymbol{r}, z_1)\xi(\boldsymbol{r}, z_2)\rangle = \mathrm{const}/2;$$

see Eq. (22). As a result, we obtain an increment of F_2, proportional to $z_2 - z_1$. Passing from the (small) increment to the differential equation, one finds the equation for the pair correlation function (32)

$$\partial_z F_2 = i\left(\nabla_1^2 - \nabla_2^2\right)F_2 - Dr^{\mu+1}F_2, \tag{37}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. The constant, which appears in Eq. (22), drops from the equation, as it should be. Equation (37) can be rewritten as

$$\partial_z F_2 = 2i\frac{\partial^2}{\partial\boldsymbol{R}\partial\boldsymbol{r}}F_2 - Dr^{\mu+1}F_2, \tag{38}$$

where $\boldsymbol{R} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$.

A. Pair correlation function

A formal solution of Equation (37) can be written in terms of the two-point Green's function $\mathcal{G}$:

$$\begin{aligned} F_2(\boldsymbol{r}_1, \boldsymbol{r}_2, z) &= \int d^2x_1\,d^2x_2\,\mathcal{G}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{x}_1, \boldsymbol{x}_2, z) \\ &\quad \times \Psi_{in}(\boldsymbol{x}_1)\Psi_{in}(\boldsymbol{x}_2), \end{aligned} \tag{39}$$

where $\Psi_{in}(\boldsymbol{x}_1)\Psi_{in}(\boldsymbol{x}_2)$ is the initial value of the pair correlation function; see Eq. (21). The Green's function $\mathcal{G}$ is equal to zero at $z < 0$ and satisfies the equation

$$\begin{aligned} \partial_z\mathcal{G} &= i\left(\nabla_1^2 - \nabla_2^2\right)\mathcal{G} - Dr^{\mu+1}\mathcal{G} \\ &\quad + \delta(z)\delta(\boldsymbol{r}_1 - \boldsymbol{x}_1)\delta(\boldsymbol{r}_2 - \boldsymbol{x}_2). \end{aligned} \tag{40}$$

Note that the Green's function by itself does not know about initial conditions for the envelope Ψ.

To find the Green's function, one can pass to the Fourier transform $\tilde{\mathcal{G}}$ as follows:

$$\mathcal{G}(\boldsymbol{R}, \boldsymbol{r}, \boldsymbol{X}, \boldsymbol{x}, z) = \int \frac{d^2k}{(2\pi)^2}\exp(i\boldsymbol{k}\boldsymbol{R})\tilde{\mathcal{G}}(\boldsymbol{k}, \boldsymbol{r}, \boldsymbol{X}, \boldsymbol{x}, z),$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, $\boldsymbol{R} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$, $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, $\boldsymbol{X} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$. Then Eq. (40) is rewritten as

$$\begin{aligned} \partial_z\tilde{\mathcal{G}} &= -2\boldsymbol{k}\nabla\tilde{\mathcal{G}} - Dr^{\mu+1}\tilde{\mathcal{G}} \\ &= \delta(z)\delta(\boldsymbol{r} - \boldsymbol{x})\exp(-i\boldsymbol{X}\boldsymbol{k}), \end{aligned}$$

where $\nabla = \partial/\partial \boldsymbol{r}$. Solving that equation by characteristics, one finds that

$$\tilde{\mathcal{G}} = \theta(z)\delta(\boldsymbol{r} - 2\boldsymbol{k}z - \boldsymbol{x})\exp(-i\boldsymbol{X}\boldsymbol{k}) \times \exp\left[-D\int_0^z d\zeta\ |\boldsymbol{r} - 2\boldsymbol{k}\zeta|^{\mu+1}\right]. \quad (41)$$

Returning to the real space, one finds

$$\mathcal{G} = \frac{\theta(z)}{16\pi^2 z^2}\exp\left[\frac{i}{2z}(\boldsymbol{r}-\boldsymbol{x})(\boldsymbol{R}-\boldsymbol{X}) - Dz\int_0^1 d\chi\ |\chi\boldsymbol{x} + (1-\chi)\boldsymbol{r}|^{\mu+1}\right]. \quad (42)$$

This quantity is symmetric in permutation of the initial and terminal points, as it should be.

Integrating expression (42) over $\boldsymbol{R}$ at a given $\boldsymbol{r}$, one obtains that

$$\int d^2R\ \mathcal{G} = \theta(z)\delta(\boldsymbol{r}-\boldsymbol{x})\exp(-Dzr^{\mu+1}). \quad (43)$$

Substituting the expression into Eq. (39), we find that

$$\int d^2R\ F_2 = \frac{\pi}{2}\exp\left(-\frac{r^2}{2} - Dzr^{\mu+1}\right), \quad (44)$$

where expression (21) is substituted. The first term in the exponent dominates at $z \ll z_\star$ and the second term dominates in the opposite limit, $z \gg z_\star$. The dominance of the term $-Dzr^{\mu+1}$ in the second regime $z \gg z_\star$ implies that the transverse correlation length $r_{ph} \sim (Dz)^{-1/(\mu+1)}$ in full agrement with the qualitative analysis of Secs. II B and III B including Eq. (29).

Further we are interested in distances $z \gg z_\star$, where effects of random diffraction are relevant. Then the characteristic values of $\boldsymbol{x}$ and $\boldsymbol{r}$ are determined by the integral in the exponent in Eq. (42). Equating the integral to unity, we find the estimate $|\boldsymbol{x}| \sim |\boldsymbol{r}| \sim r_{ph}$, where r_{ph} is the phase correlation length (29). The correlation length is related to the random diffraction on fluctuations of the refraction index destroying phase correlations. Equating then the first term in the exponent in Eq. (42), we find the characteristic value $R \sim z/r_{ph} = r_{\rm width}$ where $r_{\rm width}$ is determined by Eq. (31). The quantity has the meaning of the beam width, caused by the random diffraction. At distances $z \gg z_\star$ the width is much larger than the pure diffraction width z. In analyzing the pair correlation function in accordance with Eq. (39), one should take into account the initial width of the beam. Just the initial width determines the characteristic value of $\boldsymbol{X}$, and it can be estimated as unity.

Thus in the case $z \gg z_\star$ one obtains from Eq. (39)

$$\langle\Psi(\boldsymbol{R}+\boldsymbol{r}/2)\Psi^\star(\boldsymbol{R}-\boldsymbol{r}/2)\rangle = \int \frac{d^2x\,d^2X}{16\pi^2z^2} \times \exp\left[\frac{i}{2z}(\boldsymbol{r}-\boldsymbol{x})\boldsymbol{R} - Dz\int_0^1 d\chi\ |\chi\boldsymbol{r} + (1-\chi)\boldsymbol{x}|^{\mu+1}\right] \times \Psi_{in}(\boldsymbol{X}+\boldsymbol{x}/2)\Psi_{in}(\boldsymbol{X}-\boldsymbol{x}/2), \quad (45)$$

where we neglected $\boldsymbol{X}$ in the first term in the exponent. If $r \ll r_{ph}$, then the value of x is determined by the second term in the exponent, and then x can be estimated as r_{ph}. If $r_{ph} \ll r \ll 1$, then the value of x is determined by the second term in the exponent as well; however, x can be estimated as r. After integration over $\boldsymbol{x}$ in expression (45), there remains a dependence on $\boldsymbol{r}$ with the characteristic value $r \sim r_{ph}$. Since at $r \ll 1$, $x \ll 1$ as well, we can neglect $\boldsymbol{x}$ in the product $\Psi_{in}(\boldsymbol{X}+\boldsymbol{x}/2)\Psi_{in}(\boldsymbol{X}-\boldsymbol{x}/2)$ in Eq. (45). Then one can integrate over $\boldsymbol{X}$ to obtain

$$\langle\Psi(\boldsymbol{R}+\boldsymbol{r}/2)\Psi^\star(\boldsymbol{R}-\boldsymbol{r}/2)\rangle = \int \frac{d^2x}{32\pi z^2} \times \exp\left[\frac{i}{2z}(\boldsymbol{r}-\boldsymbol{x})\boldsymbol{R} - Dz\int_0^1 d\chi\ |\chi\boldsymbol{r} + x(1-\chi)\boldsymbol{x}|^{\mu+1}\right]. \quad (46)$$

Analyzing expression (46) we conclude that the characteristic value of r is determined by expression (29). Thus, the quantity r_{ph} plays the role of the beam correlation length in the transverse direction as well. We neglected the factor $\exp[-i\boldsymbol{x}\boldsymbol{R}/(2z)]$ in expression (46). For R larger than $z/R_{ph} = D^{1/\mu+1}z^{1/\mu+1+1}$, the exponent is fast oscillating. That leads to diminishing the expression of the pair correlation function in comparison with expression (29). Thus, the quantity (31) is the beam width for $z \gg z_\star$, indeed. The quantity is determined solely by fluctuations.

B. Fourth-order correlation function

The equation for the fourth-order correlation function

$$F_4 = \langle\Psi(\boldsymbol{r}_1,z)\Psi(\boldsymbol{r}_2,z)\Psi^\star(\boldsymbol{r}_3,z)\Psi^\star(\boldsymbol{r}_4,z)\rangle \quad (47)$$

can be derived, similar to Eq. (37), from the representation (34). The corresponding equation is given by

$$\partial_z F_4 = i\left(\nabla_1^2 + \nabla_2^2 - \nabla_3^2 - \nabla_4^2\right)F_4 - D\left[-r_{12}^{\mu+1} + r_{13}^{\mu+1} + r_{14}^{\mu+1} - r_{34}^{\mu+1} + r_{23}^{\mu+1} + r_{24}^{\mu+1}\right]F_4, \quad (48)$$

where $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ and so on.

Generally, the separations between the points in the different spots are of the order of $r_{\rm width}$ (31). Then the real factor in the right-hand side of Eq. (48) is $\sim Dr_{\rm width}^{\mu+1}$. At $z \gg z_\star$ one finds $Dr_{\rm width}^{\mu+1}z \gg 1$. That leads to a strong suppression of the fourth-order correlation function. However, in the geometry where separations between the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_3$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_4$ (or between points $\boldsymbol{r}_1$ and $\boldsymbol{r}_4$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$) are much smaller than $r_{\rm width}$, the factor appears to be much smaller. Therefore the fourth-order correlation function has sharp maxima in the geometries. Further we examine just this case.

Having in mind the geometry where separations r_{13} and r_{24} are much smaller than $r_{\rm width}$, we rewrite Eq. (48) as

$$\frac{\partial F_4}{\partial z} = 2i\left(\frac{\partial^2 F_4}{\partial \boldsymbol{R}_1\partial\boldsymbol{\rho}_1} + \frac{\partial^2 F_4}{\partial \boldsymbol{R}_2\partial\boldsymbol{\rho}_2}\right) - \left[D\left(\rho_1^{\mu+1} + \rho_2^{\mu+1}\right) + U\right]F_4, \quad (49)$$

$$U/D = |\boldsymbol{R} + \boldsymbol{\rho}_1/2 + \boldsymbol{\rho}_2/2|^{\mu+1} - |\boldsymbol{R} + \boldsymbol{\rho}_1/2 - \boldsymbol{\rho}_2/2|^{\mu+1} - |\boldsymbol{R} - \boldsymbol{\rho}_1/2 + \boldsymbol{\rho}_2/2|^{\mu+1} + |\boldsymbol{R} - \boldsymbol{\rho}_1/2 - \boldsymbol{\rho}_2/2|^{\mu+1}, \quad (50)$$

where we introduced

$$\boldsymbol{r}_1 = \boldsymbol{R}_1 + \boldsymbol{\rho}_1/2,\ \boldsymbol{r}_3 = \boldsymbol{R}_1 - \boldsymbol{\rho}_1/2,$$
$$\boldsymbol{r}_2 = \boldsymbol{R}_2 + \boldsymbol{\rho}_2/2,\ \boldsymbol{r}_4 = \boldsymbol{R}_2 - \boldsymbol{\rho}_2/2,$$

and $\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2$. If $R \sim R_{\rm width} \gg \rho_1, \rho_2$. Then the terms in the quantity U (50) cancel each other, and we can neglect U in Eq. (49). Then the operator in the equation becomes a sum of two operators in the equation for the pair correlation function (40). Therefore in the geometry, F_4 is a product of two pair correlation functions, $F_2(\boldsymbol{r}_1, \boldsymbol{r}_3)F_2(\boldsymbol{r}_2, \boldsymbol{r}_4)$.

Let us now estimate accuracy of the approximation. For the purpose we should evaluate corrections to the fourth-order correlation function F_4. In accordance with the equation (49), the first correction is written as

$$\delta F_4 = -\int_0^z d\zeta \int d^2x_1 d^2x_2 d^2x_3 d^2x_4 \times U(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) F_2(\boldsymbol{x}_1, \boldsymbol{x}_3, \zeta) F_2(\boldsymbol{x}_2, \boldsymbol{x}_4, \zeta) \times \mathcal{G}(\boldsymbol{r}_1, \boldsymbol{r}_3, \boldsymbol{x}_1, \boldsymbol{x}_3, z-\zeta)\mathcal{G}(\boldsymbol{r}_2, \boldsymbol{r}_4, \boldsymbol{x}_2, \boldsymbol{x}_4, z-\zeta). \quad (51)$$

In this integral the separations $x_{13} \sim x_{24} \sim r_{ph}$, $x_{14} \approx x_{23} \sim r_{\rm width}$, and the quantity

$$U \sim D x_{14}^{\mu-1} x_{13}^2 \sim D r_{ph}^2 r_{\rm width}^{\mu-1}.$$

Evaluating the Green's functions $\mathcal{G}$ in Eq. (51) as z^{-2} in accordance with expression (42), we find that the correction (51) is evaluated as $\alpha F_2(\boldsymbol{r}_1, \boldsymbol{r}_3)F_2(\boldsymbol{r}_2, \boldsymbol{r}_4)$, where α is the parameter

$$\alpha \sim D z^{-3} r_{ph}^6 r_{\rm width}^{\mu+3}.$$

Substituting here expressions (29) and (31) one obtains

$$\alpha = (z_{\rm rytov}/z)^{(1-\mu)(\mu+3)/(\mu+1)}. \quad (52)$$

The parameter is small provided $z \gg z_{\rm rytov}$. A more accurate calculation is presented in Appendix B.

To analyze the fourth-order correlation function F_4 in the geometry, where all separations between the points are much smaller than $r_{\rm width}$, one should use the Green's function

$$\mathcal{G}_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, z),$$

of Eq. (48), satisfying

$$\partial_z \mathcal{G}_4 - i\big(\nabla_1^2 + \nabla_2^2 - \nabla_3^2 - \nabla_4^2\big)\mathcal{G}_4 + D\big[-r_{12}^{\mu+1} + r_{13}^{\mu+1} + r_{14}^{\mu+1} - r_{34}^{\mu+1} + r_{23}^{\mu+1} + r_{24}^{\mu+1}\big]\mathcal{G}_4 = \delta(z)\delta(\boldsymbol{r}_1 - \boldsymbol{x}_1)\delta(\boldsymbol{r}_2 - \boldsymbol{x}_2)\delta(\boldsymbol{r}_3 - \boldsymbol{x}_3)\delta(\boldsymbol{r}_4 - \boldsymbol{x}_4). \quad (53)$$

The fourth-order correlation function is expressed as

$$F_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4, z) = \int d^2x_1 d^2x_2 d^2x_3 d^2x_4 \times \mathcal{G}_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, z) \times \Psi_{in}(\boldsymbol{x}_1)\Psi_{in}(\boldsymbol{x}_2)\Psi_{in}(\boldsymbol{x}_3)\Psi_{in}(\boldsymbol{x}_4). \quad (54)$$

One can apply the same arguments as for the fourth-order correlation function to the Green's function $\mathcal{G}$. Thus, $\mathcal{G}$ as a function of $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4$ has the sharp maxima in the geometry where separations between the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_3$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_4$ (or between the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_4$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$) are much smaller than $r_{\rm width}$. Therefore in the main approximation

$$\mathcal{G}_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, z) \approx G(\boldsymbol{r}_1, \boldsymbol{r}_3, \boldsymbol{x}_1, \boldsymbol{x}_3, z)\mathcal{G}(\boldsymbol{r}_2, \boldsymbol{r}_4, \boldsymbol{x}_2, \boldsymbol{x}_4, z) + \mathcal{G}(\boldsymbol{r}_1, \boldsymbol{r}_4, \boldsymbol{x}_1, \boldsymbol{x}_4, z)\mathcal{G}(\boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{x}_2, \boldsymbol{x}_3, z). \quad (55)$$

Since the equation for the Green's function $\mathcal{G}_4$ in terms of $\boldsymbol{x}_i$ is the same as in terms of $\boldsymbol{r}_i$ (53), the approximation (55) is correct in terms of $\boldsymbol{x}_i$ as well.

As for any Green's function, one may write

$$\mathcal{G}_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, z) = \int d^2y_1 d^2y_2 d^2y_3 d^2y_4 \times \mathcal{G}_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4, \zeta) \times \mathcal{G}_4(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, z-\zeta). \quad (56)$$

In integration over $\boldsymbol{y}_i$, the main contribution to the integral is produced just by the regions where separations between the points $\boldsymbol{y}_1$ and $\boldsymbol{y}_3$, $\boldsymbol{y}_2$, and $\boldsymbol{y}_4$ (or between the points $\boldsymbol{y}_1$ and $\boldsymbol{y}_4$, $\boldsymbol{y}_2$, and $\boldsymbol{y}_3$) are much smaller than $r_{\rm width}$. Such integration reproduces the approximation (55). We conclude that one can use the approximation (55) for any geometry of the points $\boldsymbol{r}_i, \boldsymbol{x}_i$.

C. Higher-order correlation functions

One can easily generalize the above procedure for correlation functions of arbitrary order. The equation for 2nth-order correlation function is

$$\partial_z F_{2n} = i\big(\nabla_1^2 + \cdots + \nabla_n^2 - \nabla_{n+1}^2 - \cdots - \nabla_{2n}^2\big)F_{2n} + D\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} r_{ij}^{\mu+1} + \sum_{i=n+1}^{2n}\sum_{j=i+1}^{2n} r_{ij}^{\mu+1} - \sum_{i=1}^{n}\sum_{j=n+1}^{2n} r_{ij}^{\mu+1}\right]F_{2n}, \quad (57)$$

where, as above, $r_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j|$. Analogously to Eqs. (39) and (54), a formal solution of Eq. (57) can be written as

$$F_{2n} = \int d^2x_1 \ldots d^2x_{2n} \times \mathcal{G}_{2n}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{2n}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2n}, z) \times \Psi_{in}(\boldsymbol{x}_1)\ldots\Psi_{in}(\boldsymbol{x}_{2n}), \quad (58)$$

where the Green's function $\mathcal{G}_{2n}$ is equal to zero at $z < 0$ and satisfies the equation

$$\partial_z \mathcal{G}_{2n} = i\big(\nabla_1^2 + \cdots + \nabla_n^2 - \nabla_{n+1}^2 - \cdots - \nabla_{2n}^2\big)\mathcal{G}_{2n} + D\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} r_{ij}^{\mu+1} + \sum_{i=n+1}^{2n}\sum_{j=i+1}^{2n} r_{ij}^{\mu+1} - \sum_{i=1}^{n}\sum_{j=n+1}^{2n} r_{ij}^{\mu+1}\right]\mathcal{G}_{2n} + \delta(z)\delta(\boldsymbol{r}_1 - \boldsymbol{x}_1)\ldots\delta(\boldsymbol{r}_{2n} - \boldsymbol{x}_{2n}). \quad (59)$$

Under the condition $z \gg z_\star$, the Green's function $\mathcal{G}_{2n}$ has sharp maxima in the geometry where the points are split into n close pairs $\boldsymbol{r}_i, \boldsymbol{r}_j$ $(i = 1, \ldots, n,\ j = n+1, \ldots, 2n)$, where the separations are smaller or of the order of r_{ph} (29). In the geometry, the "large" differences (of the order of $r_{\rm width}$) in Eq. (59) cancel each other. Then we pass to the differential operator that is a sum of the differential operators of the type appearing in the equation for the pair correlation function (37). Therefore the $2n$th-order correlation function in the geometry is a product of n pair correlation functions. There are $n!$ of such geometries, and in the main approximation the $2n$th-order correlation function can be presented as a sum of $n!$ terms that are products of the pair correlation functions, like expression (56). This is just the case that is the subect of the Wick theorem where Ψ possesses Gaussian statistics. The property is a consequence of random diffraction that makes the phase of Ψ a random variable.

One can estimate corrections to the Gaussian statistics. For this one should evaluate the terms that were discarded in the geometry of close pairs. An analysis, analogous to one produced for the fourth-order correlation function, shows that the approximation is justified by the same small parameter α (52).

D. Statistics of intensity

Let us analyze statistical properties of the intensity $I = |\Psi|^2$, taken at the distance $z \gg z_\star$ inside the diffraction spot $r \ll r_{\rm width}$. One expects that the quantity has Poisson statistics because of the randomness of the phase of Ψ caused by the refractive index fluctuations [25]. We prove the conjecture and give the quantitative criterion determining the applicability region of the statistics.

The average $\langle I^n \rangle$ can be written as

$$\langle I^n \rangle = \int d^2x_1 \cdots d^2x_{2n} \mathcal{G}_{2n}(\boldsymbol{r}, \ldots, \boldsymbol{r}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2n}, z) \times \Psi_{in}(\boldsymbol{x}_1) \cdots \Psi_{in}(\boldsymbol{x}_{2n}), \tag{60}$$

in accordance with Eq. (58). Thus we should establish properties of the Green's function $\mathcal{G}_{2n}$ in the situation where the final points coincide.

We use the following property of any Green's function

$$\mathcal{G}_{2n}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{2n}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2n}, z) = \int d^2y_1 \ldots d^2y_{2n} \times \mathcal{G}_{2n}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{2n}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{2n}, \zeta) \times \mathcal{G}_{2n}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{2n}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2n}, z - \zeta). \tag{61}$$

If we choose $\zeta \sim z$, then the characteristic value of y in the integral can be estimated as $R_{\rm width}$ (31). However, both Green's functions under the integral have have sharp maxima in the geometry where the points are split into n close pairs $\boldsymbol{y}_i, \boldsymbol{y}_j$ $(i = 1, \ldots, n,\ j = n+1, \ldots, 2n)$, provided the separations are smaller or of the order of R_{ph} (29). For the second Green's function, we established the property above. For the first Green's function it follows from the fact that in terms of the variables y, it satisfies the same equation as in terms of r. Thus both Green's functions are represented as sums of the products of the pair of Green's functions. Therefore we arrive at the Gaussian statistics for $\Psi(\boldsymbol{r})$ and, consequently, at the Poisson statistics for I. In other words, the probability distribution function of I is exponential.

Corrections to the Gaussian approximation are controlled by the same small parameter α (52). Now we can estimate the region of applicability of the Poisson approximation. If we analyze $\langle I^n \rangle$, then the relative correction, associated with the neglected terms in the equation for the Green's functions, is estimated as $n^2\alpha$, for large n. Thus the Poisson expression is valid if $n \ll 1/\sqrt{\alpha}$. In other words, the exponential probability distribution is correct one if $I \ll \langle I \rangle / \sqrt{\alpha}$.

V. EFFECTIVE ACTION

Here we propose an alternative language for describing effects associated with fluctuations of the refractive index. Correlation functions of the field Ψ can be examined in the framework of an effective quantum field theory [26–28]. The theory produces a diagrammatic expansion of the type first developed by Wyld in the context of hydrodynamic turbulence [29]. Applications of the technique to the optical problems can be found in Refs. [30–32].

In the framework of the effective quantum theory, the correlation functions of the field Ψ can be found as functional integrals over Ψ, $\Psi^\star$, P, $P^\star$ (where P, $P^\star$ are auxiliary fields) with the weight

$$\exp\left\{-\mathcal{S} + \int d^2r \, [P\Psi^\star_{in} + P^\star \Psi_{in}]\right\}, \tag{62}$$

where the effective action $\mathcal{S}$ is constructed in accordance with Eq. (19). Here Ψ_{in} is the initial condition for the field Ψ posed at $z = 0$. In our setup, the initial condition is determined by Eq. (21).

To find $\mathcal{S}$, we start from the weight

$$\exp\left\{-\mathcal{I} + \int d^2r \, [P\Psi^\star_{in} + P^\star \Psi_{in}]\right\}, \tag{63}$$

where $\mathcal{I}$ forces the equation (19):

$$\mathcal{I} = \int d^2r \, dz \, P^\star (i\partial_z \Psi + \nabla^2 \Psi + \xi \Psi) - \int d^2r \, dz \, P (i\partial_z \Psi^\star - \nabla^2 \Psi^\star - \xi \Psi). \tag{64}$$

The weight (62) is obtained by averaging over the statistics of the refractive index fluctuations in accordance with Eq. (22). The constant term corresponding to homogeneous phase fluctuations cannot contribute to the effective action. Thus we arrive at the additional condition

$$\int d^2r \, (P^\star \Psi + P \Psi^\star) = 0, \tag{65}$$

to be imposed on the field P. Then the constant drops from the consideration.

As a result, we find that the effective action $\mathcal{S}$ is the sum $\mathcal{S} = \mathcal{S}_{(2)} + \mathcal{S}_{\rm int}$, where

$$\mathcal{S}_{(2)} = \int d^2r \, dz \, P^\star (i\partial_z \Psi + \nabla^2 \Psi) - \int d^2r \, dz \, P (i\partial_z \Psi^\star - \nabla^2 \Psi^\star), \tag{66}$$

$$\mathcal{S}_{\rm int} = \frac{1}{2}\int d^2r_1\, d^2r_2\, dz\, (P_1^\star\Psi_1 + P_1\Psi_1^\star) \times (D|\boldsymbol{r}_1 - \boldsymbol{r}_2|^{\mu+1} - \text{const})(P_2^\star\Psi_2 + P_2\Psi_2^\star). \quad (67)$$

Here the first terms describes the free laser beam diffraction, whereas the second term describes the influence of the fluctuations of the refractive index, that is, the random diffraction.

Correlation functions of the fields $\Psi, \Psi^\star, P, P^\star$ can be found in the framework of the perturbation theory. For the purpose one expands $\exp(-\mathcal{S}_{\rm int})$ in $\mathcal{S}_{\rm int}$ and calculates explicitly the resulting expressions that are Gaussian integrals with the weight

$$\exp\left\{-\mathcal{S}_{(2)} + \int d^2r\, [P\Psi_{in} + P^\star\Psi_{in}]\right\},$$

that is, the exponent of the quadratic in the field quantity. The terms of the perturbation series can be represented as Feynman diagrams. The diagrammatic expansion of this type was first used by Wyld in the context of hydrodynamic turbulence [29].

Analytical expressions caused by the diagrams are constructed from the propagators, which are the correlation functions

$$G = \langle\Psi(\boldsymbol{r}, z)P^\star(0,0)\rangle = -\frac{\theta(z)}{4\pi z}\exp\left(i\frac{r^2}{4z}\right), \quad (68)$$

$$G^\star = \langle\Psi^\star(\boldsymbol{r}, z)P(0,0)\rangle = -\frac{\theta(z)}{4\pi z}\exp\left(-i\frac{r^2}{4z}\right). \quad (69)$$

Here angular brackets mean averaging with the weight $\exp[-\mathcal{S}_{(2)}]$ and $\theta(z)$ is the Heaviside step function. Let us stress that there are no corrections, caused by the interaction term (67), to expressions (68) and (69) because of causality.

Ladder approximation

The pair correlation function of the envelope at a given z is written as the integral (39). The Green's function in the relation can be expressed in terms of the correlation function of the introduced fields

$$\mathcal{G} = \langle\Psi(\boldsymbol{r}_1, z)\Psi^\star(\boldsymbol{r}_2, z)P(\boldsymbol{x}_1, 0)P^\star(\boldsymbol{x}_2, 0)\rangle. \quad (70)$$

The expression (70) can be derived if to expand the weight (62) up to the second order in Ψ_{in}.

In zero approximation $\mathcal{G} = G(\boldsymbol{r}_1 - \boldsymbol{r}_4, z)G^\star(\boldsymbol{r}_2 - \boldsymbol{r}_3, z)$. Contributions to $\mathcal{G}$, related to the interaction term (67), can be presented by ladder diagrams; see Fig. 1. Here a line directed to the right represents the average G (68), and a line directed to the left represents the average $G^\star$ (69). The dotted line represents the factor $-Dr^{\mu+1}$, where r is the separation between the points. Summation of the ladder diagrams depicted in Fig. 1 leads to an integral equation for $\mathcal{G}$, analyzed in Appendix A. Solving the integral equation, one obtains expression (42) obtained above by another method. Let us stress that the expression is the exact one in our setup due to the shortness of the refractive index correlations in z.

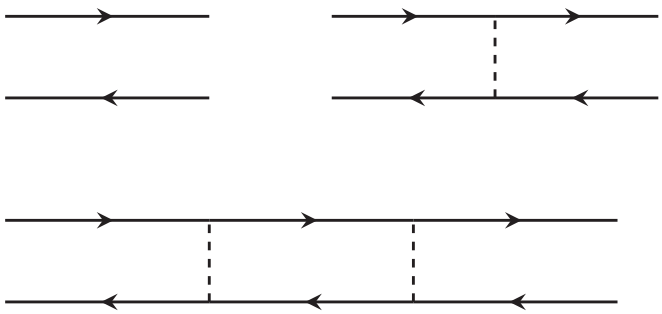

FIG. 1. Ladder sequence of diagrams for the quantity $\mathcal{G}$.

Let us analyze momenta of the intensity I. In the framework of our scheme, the moment $\langle I^n\rangle$ can be written as

$$\langle I^n\rangle = \int d^2x_1\cdots d^2x_{2n}\langle[\Psi(0,z)\Psi^\star(0,z)]^n \times P(\boldsymbol{x}_1,0)\cdots P(\boldsymbol{x}_n,0)P^\star(\boldsymbol{x}_{n+1},0)\cdots P^\star(\boldsymbol{x}_{2n},0)\rangle \times \Psi_{in}(\boldsymbol{x}_1)\cdots\Psi_{in}(\boldsymbol{x}_{2n}). \quad (71)$$

The problem is how to calculate the average in expression (71).

Below we use the ladder approximation, where the average in expression (71) is reduced to a product of factors corresponding to the ladder diagrams. Then one finds

$$\langle I^n\rangle = \int d^2x_1\cdots d^2x_{2n}[\mathcal{G}(0,0,\boldsymbol{x}_1,\boldsymbol{x}_{n+1},z)\cdots \times \mathcal{G}(0,0,\boldsymbol{x}_n,\boldsymbol{x}_{2n},z) + \cdots]\Psi_{in}(\boldsymbol{x}_1)\ldots\Psi_{in}(\boldsymbol{x}_{2n}), \quad (72)$$

where the number of summands is $n!$. Each summand produces $\langle I\rangle^n$, and we find $\langle I^n\rangle = n!\langle I\rangle^n$. In other words, we arrive at the Poisson statistics for I with the probability density $\mathcal{P}(I) = \langle I\rangle^{-1}\exp(-I/\langle I\rangle)$. The validity of the ladder approximation should be checked separately.

Correlation functions of I can be analyzed in the same ladder approximation. If we separate the points in the correlation function $\langle I(\boldsymbol{r}_1, z)I(\boldsymbol{r}_2, z)\cdots\rangle$, a part of the ladders are switched off when the separation becomes larger than R_{ph}. The corresponding analysis is analogous to one made for the pair correlation function $\langle\Psi\Psi^\star\rangle$. Thus cumulants (irreducible parts) of the averages like $\langle I^n(\boldsymbol{r}_1, z)I^n(\boldsymbol{r}_2, z)\rangle$ become parametrically smaller where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ exceeds r_{ph}.

Let us now analyze corrections to the ladder approximation. The first correction is determined by the diagrams including a crossbar connecting two ladders; see Fig. 2. Corrections to the averages like $\langle I^n\rangle$ are analyzed in Appendix B. It is shown there that the small parameter justifying the applicability condition of the ladder approximation is α (52). Now we can estimate the region of applicability of the Poisson approximation. If we analyze $\langle I^n\rangle$, then the relative correction, associated with the crossbar is estimated as $n^2\alpha$, for large n. Thus the Poisson expression is valid if $n \ll 1/\sqrt{\alpha}$. In other words, the exponential probability distribution is correct if $I \ll \langle I\rangle/\sqrt{\alpha}$.

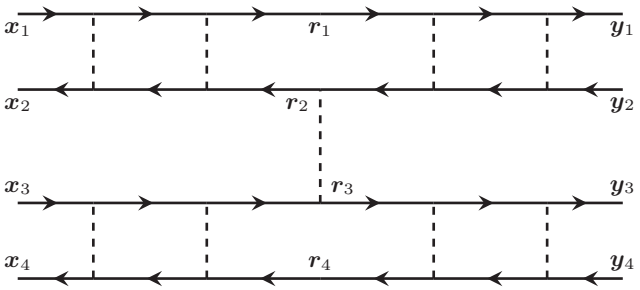

FIG. 2. Crossbar connecting two ladders.

Due to the refraction index fluctuations, the beam cross section is divided into a large number of speckles N. The number of the speckles in the pattern can be estimated as the ratio of the beam cross-sectional area to the square of the correlation length (29). Thus we obtain

$$N \sim D^{4/(\mu+1)} z^{2+4/(\mu+1)}, \quad z \gg z_\star. \tag{73}$$

One can think about statistics of the beam intensity in the brightest speckle in the limit $N \gg 1$. For this purpose, we assume that the intensity statistics in the speckles are independent and are determined by the Poisson statistics. Then the probability distribution $\mathcal{P}(I)$ of the laser intensity I in the brightest speckle is given by Fisher-Tippett-Gnedenko theorem and has the form of so-called double-exponential Gumbel distribution [33]:

$$\mathcal{P}(I) = \frac{N}{\langle I \rangle} \exp\left(-\frac{I}{\langle I \rangle} - N e^{-I/\langle I \rangle}\right). \tag{74}$$

Respectively, the average intensity in the brightest speckle can be estimated as $\langle I \rangle \ln N$. Due to slowness of the logarithmic function this value remains inside the applicability region of Poisson statistics determined by the condition $I \ll \langle I \rangle / \sqrt{\alpha}$. That justifies our conclusions.

VI. STRONG FLUCTUATIONS

Here we consider the case of strong fluctuations where our dimensionless parameter D (23) entering the noise fluctuation function (22) is large, $D \gg 1$. Then we find $z_{\text{rytov}} \gg z_\star$, where $z_{\text{rytov}} \sim D^{-2(\mu+3)}$ (27) is the distance where the refractive index fluctuations destroy the flat wave front, and $z_\star \sim D^{-1}$ (30) is the distance where the initially Gaussian beam is sliced into speckles due to the fluctuations. In the region $z \ll z_\star$ the laser beam is weakly perturbed by the fluctuations. That is why we consider the case $z \gg z_\star$. Then $Dz \gg 1$ and therefore the characteristic size of the speckle is $r_{ph} \sim (Dz)^{-1/(\mu+1)}$ (29), $r_{ph} \ll 1$.

We begin with the pair correlation function that can be written as the convolution (39) of the initial conditions with the Green's function (42). Taking the initial condition (21) and integrating over $\boldsymbol{X}$, one finds

$$\langle \Psi(\boldsymbol{R}+\boldsymbol{r}/2)\Psi^\star(\boldsymbol{R}-\boldsymbol{r}/2)\rangle = \int \frac{d^2x}{32\pi z^2} \times \exp\left[\frac{i}{2z}(\boldsymbol{r}-\boldsymbol{x})\boldsymbol{R} - \frac{1}{2}\frac{(\boldsymbol{r}-\boldsymbol{x})^2}{z^2} - \frac{x^2}{2}\right] \times \exp\left[-Dz\int_0^1 d\chi\, |\chi \boldsymbol{r} + (1-\chi)\boldsymbol{x}|^{\mu+1}\right]. \tag{75}$$

We see that the characteristic value of r is determined by r_{ph}, indeed. The estimate can be obtained by equating to unity the argument of the second exponent in Eq. (75).

Let us consider the interval $z_\star \ll z \ll D^{-1/(\mu+2)}$, then $|\boldsymbol{r}-\boldsymbol{x}| \sim z \ll r_{ph}$. Therefore it is possible to substitute $\boldsymbol{x} \to \boldsymbol{r}$ in the second exponent in Eq. (75) and to neglect the term with x^2 in the first exponent. Then one finds

$$\langle \Psi(\boldsymbol{R}+\boldsymbol{r}/2)\Psi^\star(\boldsymbol{R}-\boldsymbol{r}/2)\rangle = \frac{1}{16}\exp\left(-\frac{1}{8}R^2 - Dz r^{\mu+1}\right). \tag{76}$$

We conclude that the spot width in the region is of order unity. Note that z_{rytov} lies inside the interval $z_\star \ll z \ll D^{-1/(\mu+2)}$. In the case $z \gg D^{-1/(\mu+2)}$ we return to expression (46). Correspondingly, the characteristic speckle size and the size of the spot are determined by expressions (29) and (31).

Now we proceed to the fourth-order correlation function, analyzing it in the interval $z_\star \ll z \ll D^{-1/(\mu+2)}$. One can repeat the same steps as in Sec. IV B to obtain expression (51). Then the integrations over $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ are performed like at deriving expression (76) for the pair correlation function. The integration over $\boldsymbol{x}_1 + \boldsymbol{x}_3$ and over $\boldsymbol{x}_2 + \boldsymbol{x}_4$ can be performed explicitly with account of expression (76) for the pair correlation function. It gives factors of order unity and also leads to exponential contributions forcing $(\boldsymbol{r}_1 - \boldsymbol{r}_3) - (\boldsymbol{x}_1 - \boldsymbol{x}_3) \sim z$ and $(\boldsymbol{r}_2 - \boldsymbol{r}_4) - (\boldsymbol{x}_2 - \boldsymbol{x}_4) \sim z$, like in Eq. (75). Therefore the integration over $\boldsymbol{x}_1 - \boldsymbol{x}_3$ and $\boldsymbol{x}_2 - \boldsymbol{x}_4$ gives the factor $\sim z^4$. Estimating the Green's functions as z^{-2} [see Eq. (42)], the quantity U can be estimated as $D r_{ph}^2$. Thus we find

$$\alpha = (Dz)^{(\mu-1)/(\mu+1)} \ll 1. \tag{77}$$

At $z \sim D^{-1/(\mu+2)}$ expression (77) turns to (52) as it should be.

VII. CONCLUSION

We analyzed statistical properties of the speckle pattern of light intensity in the cross section of the laser beam propagating in the turbulent fluid (atmosphere). The pattern is formed due to refraction index fluctuations caused by the turbulent fluctuations. We demonstrated that there are two characteristic dimensionless propagation length, z_{rytov} and $z_\star$, determined by relations (27) and (30). Our theory is valid under the condition $z_{\text{rytov}} \lesssim z_\star$. We are interested in the region $z \gg z_\star$ where effects of the random diffraction dominate.

If the propagation length z satisfies the condition $z \gg z_\star$, then the beam contains many speckles and its width r_{width} becomes much wider than in the case of the free diffraction propagation. This is due to the refractive index fluctuations.

The beam width is determined by expression (17), that is,

$$r_{\text{width}} \propto z^{1/(\mu+1)+1}.$$

The correlation length of the signal r_{ph}, related to fluctuations of its phase, appears to be much smaller than the beam width and can be estimated in accordance with Eq. (29) by

$$r_{ph} \propto z^{-1/(\mu+1)}.$$

The main result of this paper is that we used the analytical expression for the pair correlation function (46) in the region $z \gg z_\star$ while demonstrating that higher-order correlation functions of the envelope Ψ are split into products of the pair correlation functions. This result is in accordance with the expectation that strong phase fluctuations lead to an effective Gaussianity of the envelope statistics. We analyzed also non-Gaussian corrections to the higher-order correlation functions and established that they are controlled by the z-dependent parameter $\alpha \propto z^{\mu+1-4/(\mu+1)}$; see Eq. (52). As one expects, the parameter diminishes as z grows due to the increasing role of the random diffraction.

We developed also the diagrammatic technique for calculating corrections to the correlations functions of the envelope related to the random diffraction. In the diagrammatic language, the effective Gaussianity of the envelope statistics is explained as the approximation where the so-called ladder diagrams are taken into account. Let us stress that the approximation implies a deep resummation of the diagrams. The diagrammatic technique gives a powerful tool to go beyond the scope of the Gaussian approximation and enables one to calculate analytically non-Gaussian corrections.

The Gaussianity of the envelope statistics results in the Poisson statistics of the beam intensity I. We established the applicability region of the statistics is given by $I \ll \langle I\rangle/\sqrt{\alpha}$, where $\langle I\rangle$ is the average intensity inside the pattern and α is the parameter (52). We examined the statistics of the brightest spot among the large number $N \sim (r_{\text{width}}/r_{ph})^2$ in the speckle pattern. The average value of the intensity inside the brightest spot can be estimated as $\langle I\rangle \ln N$. The quantity lies inside the applicability region of the Poisson approximation.

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APPENDIX A: LADDER REPRESENTATION

Here we demonstrate how to obtain expression (42) for the Green's function (70) using the ladder representation for the object. The quantity $\mathcal{G}$ is depicted by the sum of the ladder sequence of the diagrams depicted in Fig. 1. The ladder representation leads to the following integral equation

$$\mathcal{G}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\boldsymbol{r}_4,z) = G(\boldsymbol{r}_1-\boldsymbol{r}_4,z)G^\star(\boldsymbol{r}_2-\boldsymbol{r}_3,z) - \int d\zeta\, d^2r_5 d^2r_6 D|\boldsymbol{r}_{56}|^{\mu+1} G(\boldsymbol{r}_{15},z-\zeta)G^\star(\boldsymbol{r}_{26},z-\zeta) \times \mathcal{G}(\boldsymbol{r}_5,\boldsymbol{r}_6,\boldsymbol{r}_3,\boldsymbol{r}_4,\zeta). \tag{A1}$$

Using the relation

$$\left(\partial_z - i\nabla_1^2 + i\nabla_2^2\right)[G(\boldsymbol{r}_1,z)G^\star(\boldsymbol{r}_2,z)] = \delta(z)\delta(\boldsymbol{r}_1)\delta(\boldsymbol{r}_2),$$

following from Eqs. (68) and (69), one obtains from Eq. (A1)

$$\left(\partial_z - i\nabla_1^2 + i\nabla_2^2 + Dr_{12}^{\mu+1}\right)\mathcal{G} = \delta(z)\delta(\boldsymbol{r}_1-\boldsymbol{r}_4)\delta(\boldsymbol{r}_2-\boldsymbol{r}_3). \tag{A2}$$

Passing to the variables $\boldsymbol{R} = (\boldsymbol{r}_1+\boldsymbol{r}_2)/2$, $\boldsymbol{r} = \boldsymbol{r}_1-\boldsymbol{r}_2$, we rewrite Eq. (A2) as

$$\left(\partial_z - 2i\frac{\partial^2}{\partial \boldsymbol{r}\partial \boldsymbol{R}} + Dr^{\mu+1}\right)\mathcal{G} = \delta(z)\delta(\boldsymbol{r}-\boldsymbol{x})\delta(\boldsymbol{R}-\boldsymbol{X}), \tag{A3}$$

where $\boldsymbol{X} = (\boldsymbol{r}_3+\boldsymbol{r}_4)/2$, $\boldsymbol{x} = \boldsymbol{r}_4-\boldsymbol{r}_3$. Equation (A3) is equivalent to Eq. (40).

APPENDIX B: CORRECTIONS TO THE LADDER APPROXIMATION

Here we demonstrate how to find corrections to the ladder approximation. For this purpose we consider the first correction to the product of two ladders giving the main contribution to the fourth-order correlation function of Ψ. The correction is determined by the diagrams of the type depicted in Fig. 2, containing the only crossbar connecting two ladders. The sum of the ladder diagrams of the type presented in Fig. 2 gives the first correction to the product $\mathcal{G}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{x}_2,\boldsymbol{x}_1,z)\mathcal{G}(\boldsymbol{y}_3,\boldsymbol{y}_4,\boldsymbol{x}_4,\boldsymbol{x}_3,z)$.

After summation of the ladder sequences, the diagrams depicted in Fig. 2 give the following analytical expression:

$$-D\int d\zeta\, d^2r_1 d^2r_2 d^2r_3 d^2r_4 |\boldsymbol{r}_2-\boldsymbol{r}_3|^{\mu+1} \times \mathcal{G}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{r}_2,\boldsymbol{r}_1,z-\zeta)\mathcal{G}(\boldsymbol{y}_3,\boldsymbol{y}_4,\boldsymbol{r}_4,\boldsymbol{r}_3,z-\zeta) \times \mathcal{G}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{x}_2,\boldsymbol{x}_1,\zeta)\mathcal{G}(\boldsymbol{r}_3,\boldsymbol{r}_4,\boldsymbol{x}_4,\boldsymbol{x}_3,\zeta),$$

where the integral over ζ goes from 0 to z. In derivation of the expression we used the relations

$$\int d^2x\, G(\zeta,\boldsymbol{x})G(z-\zeta,\boldsymbol{r}-\boldsymbol{x}) = -iG(z,\boldsymbol{r}),$$
$$\int d^2x\, G^\star(\zeta,\boldsymbol{x})G^\star(z-\zeta,\boldsymbol{r}-\boldsymbol{x}) = iG^\star(z,\boldsymbol{r}), \tag{B1}$$

for the functions (68) and (69), which can be checked directly.

Taking into account expression (67) for the interaction and relations (B1), one finds ultimately the following correction Θ

to the product $\mathcal{G}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{x}_2,\boldsymbol{x}_1,z)\mathcal{G}(\boldsymbol{y}_3,\boldsymbol{y}_4,\boldsymbol{x}_4,\boldsymbol{x}_3,z)$:

$$\Theta = -D\int d\zeta\, d^2r_1 d^2r_2 d^2r_3 d^2r_4 (|\boldsymbol{r}_2-\boldsymbol{r}_3|^{\mu+1} + |\boldsymbol{r}_1-\boldsymbol{r}_4|^{\mu+1} - |\boldsymbol{r}_1-\boldsymbol{r}_3|^{\mu+1} - |\boldsymbol{r}_2-\boldsymbol{r}_4|^{\mu+1}) \times \mathcal{G}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{r}_2,\boldsymbol{r}_1,z-\zeta)\mathcal{G}(\boldsymbol{y}_3,\boldsymbol{y}_4,\boldsymbol{r}_4,\boldsymbol{r}_3,z-\zeta) \times \mathcal{G}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{x}_2,\boldsymbol{x}_1,\zeta)\mathcal{G}(\boldsymbol{r}_3,\boldsymbol{r}_4,\boldsymbol{x}_4,\boldsymbol{x}_3,\zeta), \tag{B2}$$

Let us consider the case $\boldsymbol{x}_i = \boldsymbol{y}_i = 0$. We introduce the quantities

$$\boldsymbol{r}_1 = \boldsymbol{X} + \boldsymbol{R}/2 + \boldsymbol{y}/2,\ \boldsymbol{r}_2 = \boldsymbol{X} + \boldsymbol{R}/2 - \boldsymbol{y}/2,$$
$$\boldsymbol{r}_3 = \boldsymbol{X} - \boldsymbol{R}/2 + \boldsymbol{s}/2,\ \boldsymbol{r}_4 = \boldsymbol{X} - \boldsymbol{R}/2 - \boldsymbol{s}/2, \tag{B3}$$

where $\boldsymbol{s}$ and $\boldsymbol{y}$ measure the ladder thickness whereas $\boldsymbol{R}$ is the separation between the ladders. Taking expression (42) into account, we conclude that integration over $\boldsymbol{X}$ in Eq. (B2) produces a δ function fixing $\boldsymbol{y} = -\boldsymbol{s}$. After integrating over $\boldsymbol{s}$, one arrives at expression

$$\Theta = \frac{D}{2^{14}\pi^6}\int \frac{d\zeta\, d^2R\, d^2y}{z^2\zeta^2(z-\zeta)^2} \times (|\boldsymbol{R}+\boldsymbol{y}|^{\mu+1} + |\boldsymbol{R}-\boldsymbol{y}|^{\mu+1} - 2R^{\mu+1}) \times \exp\left[\frac{iz}{2\zeta(z-\zeta)}\boldsymbol{y}\boldsymbol{R} - \frac{2Dz}{2+\mu}y^{\mu+1}\right]. \tag{B4}$$

One can take the integral over $\boldsymbol{R}$ in expression (B4) explicitly to obtain that

$$\Theta = \frac{2^{2c}cD}{2^8\pi^4 z^{3-c}}\frac{\Gamma(2+\mu/2)}{\Gamma(1-c/2)}\int_0^1 d\chi\,[\chi(1-\chi)]^{\mu+1} \times \int_0^\infty \frac{dy}{y^{2+\mu}}\exp\left(-\frac{2Dzy^{\mu+1}}{2+\mu}\right)\sin^2\left[\frac{y^2}{4z\chi(1-\chi)}\right]. \tag{B5}$$

One can worry about a singular contribution related to small $\chi(1-\chi)$. At $y^2 \lesssim z\chi(1-\chi)$ the integral (B5) converges. Thus, at small $\chi(1-\chi)$ the integral over y produces a singular contribution $\propto [\chi(1-\chi)]^{-(\mu+1)/2}$, suppressed by the factor $[\chi(1-\chi)]^{\mu+1}$ in Eq. (B5). Thus we arrive at the natural estimates $\zeta \sim z$, $y \sim (Dz)^{-1/\mu+1}$, $R \sim D^{1/(\mu+1)}z^{1+1/(\mu+1)} \gg y$. Therefore the integral (B5) is estimated as

$$\Theta \sim \frac{D}{z^{5-c}}y^{3-\mu} \sim \frac{1}{z^4}\alpha. \tag{B6}$$

The factor α (52) characterizes the small corrections to the ladder diagrams.

Let us examine the pair correlation function $\langle II\rangle$ at distances much larger than the correlation length R_{ph} (29). The main contribution to the irreducible part of the correlation function is determined by the same diagram depicted in Fig. 2. If $z \ll z_\star$, then $\boldsymbol{x}_i$ are of order of a and can be neglected. Then $\boldsymbol{y} = -\boldsymbol{s} \sim (Dz)^{-1/(\mu+1)}$, and we arrive at the same smallness (52). If $z \gg z_\star$ then $x_{12} \sim x_{34} \sim (Dz)^{-1/(\mu+1)}$ and $y \sim s \sim (Dz)^{-1/(\mu+1)}$ as well. And we return to the same smallness (52). The correlation length of the correction to the pair correlation function $\langle II\rangle$ is r_{width} (31). An analogous analysis can be done for the other correlation functions of I.

APPENDIX C: BRIGHT SPECKLES

Here we consider bright speckles, that are characterized by the inequality $I \gg \langle|\Psi(\boldsymbol{y})|^2\rangle$. Here $\boldsymbol{y}$ is the center of the speckle, and I is the beam intensity at the point. We are interested in the shape of such a bright speckle. To solve the problem, one can use the saddle-point approach. However, we assume that we are still inside the Gaussian approximation.

Let us analyze the probability density P, that at some point $\boldsymbol{y}$ the beam intensity is I. In the Gaussian approximation the probability density can be written as

$$P(I,\boldsymbol{y}) = \int D\Psi\, D\Psi^\star\, \mathcal{N} e^{-\mathcal{S}}\delta[\Psi^\star(\boldsymbol{y})\Psi(\boldsymbol{y}) - I], \tag{C1}$$

where $\mathcal{N}$ is the normalization factor and $\mathcal{S}$ is the effective action. It is written as

$$\mathcal{S} = \int d^2r\, \Psi^\star\hat{K}\Psi, \tag{C2}$$

where $\hat{K}$ is the operator, related to the pair correlation function:

$$\hat{K}F_2(\boldsymbol{r}_1,\boldsymbol{r}_2) = \delta(\boldsymbol{r}_1-\boldsymbol{r}_2). \tag{C3}$$

Obviously, $\int dI\, P(I,\boldsymbol{y}) = 1$.

We rewrite expression (C1) as

$$P(I,\boldsymbol{y}) = \int D\Psi\, D\Psi^\star\, \mathcal{N}\int\frac{d\lambda}{2\pi i} \times \exp\left[-\int d^2r\, \Psi^\star\hat{K}\Psi + \lambda\Psi^\star(\boldsymbol{y})\Psi(\boldsymbol{y}) - \lambda I\right], \tag{C4}$$

where integration over λ goes along the imaginary axis. If I is high, then the integration in expression (C4) can be performed in the saddle-point approximation. The saddle-point equation for Ψ is

$$\hat{K}\Psi = \lambda\Psi(\boldsymbol{y})\delta(\boldsymbol{r}-\boldsymbol{y}). \tag{C5}$$

Taking into account Eq. (C3), one finds from Eq. (C5)

$$\Psi(\boldsymbol{r}) = \lambda\Psi(\boldsymbol{y})F_2(\boldsymbol{r},\boldsymbol{y}). \tag{C6}$$

Therefore

$$\lambda F_2(\boldsymbol{y},\boldsymbol{y}) = 1. \tag{C7}$$

Multiplying relation (C6) by $\Psi^\star(\boldsymbol{y})$, one obtains

$$\Psi(\boldsymbol{r})\Psi^\star(\boldsymbol{y}) = \lambda I F_2(\boldsymbol{r},\boldsymbol{y}), \tag{C8}$$

where the self-consistency condition $\Psi^\star(\boldsymbol{y})\Psi(\boldsymbol{y}) = I$ is used. As we see from Eq. (C8), the saddle-point profile is determined by the pair correlation function, which recovers the result of Refs. [12–14].

[1] V. I. Tatarskii, *Wave Propagation in a Turbulent Medium*, McGraw-Hill Series in Electrical Engineering (McGraw-Hill, New York, 1961).

[2] V. I. Tatarskii, *The Effects of the Turbulence Atmosphere on Wave Propagation* (Israel Program for Scientific Translations, Jerusalem, 1971).

[3] J. W. Goodman, in *Laser Speckle and Related Phenomena*, edited by J. C. Dainty (Springer-Verlag, Berlin, 1975), pp. 9–75.

[4] *Laser Beam Propagation in the Atmosphere*, edited by J. W. Strohbehn (Springer, New York, 1978).

[5] L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media*, PressMonograph Series Vol. PM53 (SPIE, Bellingham, Washington, 1998).

[6] S. L. Lachinova and M. A. Vorontsov, J. Opt. **18**, 025608 (2016).

[7] M. A. Vorontsov, G. W. Carhart, V. S. R. Gudimetla, T. Weyrauch, E. Stevenson, S. L. La chinova, L. A. Beresnev, J. Liu, K. Rehder, and J. F. Riker, in Proc. 2010 AMOS Conf. (2010).

[8] M. A. Vorontsov, V. R. Gudimetla, G. W. Carhart, T. Weyrauch, S. L. Lachinova, E. Polnau, J. R. L. A. Beresnev, J. Liu, and J. F. Riker, in Proc. 2011 AMOS Conf. (2011).

[9] P. M. Lushnikov and N. Vladimirova, Pis'ma Zh. Eksp. Teor. Fiz. **108**, 611 (2018) [JETP Lett. **108**, 571 (2018)].

[10] I. A. Vaseva, M. P. Fedoruk, A. M. Rubenchik, and S. K. Turitsyn, Sci. Rep. **6**, 30697 (2016).

[11] V. I. Tatarskii, Sov. Phys JETP **29**, 1133 (1969).

[12] G. Lindgren, Ark. Mat. **10**, 195 (1972).

[13] H. A. Rose and D. DuBois, Phys. Fluids B **5**, 3337 (1993).

[14] J. Garnier, Phys. Plasmas **6**, 1601 (1999).

[15] S. N. Vlasov, V. A. Petrishchev, and V. I. Talanov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **14**, 1353 (1971).

[16] V. E. Zakharov, Sov. Phys. JETP **35**, 908 (1972).

[17] P. M. Lushnikov, S. A. Dyachenko, and N. Vladimirova, Phys. Rev. A **88**, 013845 (2013).

[18] P. M. Lushnikov and N. Vladimirova, Opt. Lett. **35**, 1965 (2010).

[19] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics, Mechanics of Turbulence*, Vols. I & II (MIT Press, Cambridge, MA, 1971 and 1979).

[20] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).

[21] M. S. Rytov, Y. A. Kravtsov, and V. I. Tatarskii, *Principles of Statistical Radiophysics 4: Wave Propagation through Random Media* (Springer-Verlag, Berlin, 1989).

[22] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1957).

[23] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).

[24] A. P. Kazantsev, Sov. Phys. JETP **26**, 1031 (1968).

[25] A. Weinrib and B. I. Halperin, Phys. Rev. B **26**, 1362 (1982).

[26] P. E. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 483 (1973).

[27] H. K. Janssen, Z. Phys. B **23**, 377 (1976).

[28] C. de Dominicis, J. Phys. Colloq. **1**, 247 (1976).

[29] H. W. Wyld, Ann. Phys. **14**, 143 (1961).

[30] I. V. Kolokolov, V. V. Lebedev, E. V. Podivilov, and S. S. Vergeles, Zh. Eksp. Teor. Fiz. **146**, 1295 (2014) [JETP **119**, 1134 (2014)].

[31] D. V. Churkin, I. V. Kolokolov, E. V. Podivilov, I. D. Vatnik, M. A. Nikulin, S. S. Vergeles, I. S. Terekhov, V. V. Lebedev, G. Falkovich, S. A. Babin, and S. K. Turitsyn, Nat. Commun. **6**, 6214 (2015).

[32] L. L. Ogorodnikov and S. S. Vergeles, Opt. Lett., **43**, 651 (2018).

[33] S. Kotz and S. Nadarajah, *Extreme Value Distributions: Theory and Applications* (World Scientific, Singapore, 2000).