Graphs with Tunable Chromatic Numbers for Parallel Coloring *

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Abstract
We consider how to generate graphs of arbitrary size whose chromatic numbers can be chosen (or are well-bounded) for testing graph coloring algorithms on parallel computers. For the distance-1 graph coloring problem, we identify three classes of graphs with this property. The first is the Erdős-Rényi random graph with prescribed expected degree, where the chromatic number is known with high probability. It is also known that the Greedy algorithm colors this graph using at most twice the number of colors as the chromatic number. The second is a random geometric graph embedded in hyperbolic space where the size of the maximum clique provides a tight lower bound on the chromatic number. The third is a deterministic graph described by Mycielski, where the graph is recursively constructed such that its chromatic number is known and increases with graph size, although the size of the maximum clique remains two. For Jacobian estimation, we bound the distance-2 chromatic number of random bipartite graphs by considering its equivalence to distance-1 coloring of an intersection graph. We use a "balls and bins" probabilistic analysis to establish a lower bound and an upper bound on the distance-2 chromatic number. The regimes of graph sizes and probabilities that we consider are chosen to suit the Jacobian estimation problem, where the number of columns and rows are asymptotically nearly equal, and have number of nonzeros linearly related to the number of columns. Computationally we verify the theoretical predictions and show that the graphs are often be colored optimally by the serial and parallel Greedy algorithms.

1 Introduction

We consider the problem of generating graphs of arbitrarily large sizes whose chromatic numbers (or good lower bounds for them) can be chosen within a desired range. Such graphs are helpful for testing parallel graph coloring algorithms. It is well known that many parallel coloring algorithms suffer some loss in the quality of the coloring relative to serial algorithms. Many graphs obtained from "real-life" applications cannot be scaled to arbitrarily large sizes, and hence as parallel computers with ever-increasing numbers of processors become available, we need larger graphs. For many synthetic graphs the number of colors taken by practical coloring algorithms increase with graph size, but if their chromatic numbers are unknown, we cannot tell if this is due to an intrinsic property of the graphs or due to the worsening performance of the coloring algorithm. However, if we can generate test graphs with tunable chromatic numbers and arbitrary sizes, then we can study how parallel coloring algorithms perform as the graphs are scaled with the number of processors.

First we study the distance-1 coloring problem on general graphs. We consider three classes of graphs where the distance-1 chromatic number is either known or can be lower bounded. The first is the well-known Erdős-Rényi graph in the probability regime where the expected mean degree is specified. The second is a random geometric graph embedded in hyperbolic space where the size of a maximum clique is a tight lower bound on the chromatic number, and the asymptotic behavior of this value is known and we can compute a good estimate of it. The third class is the Mycielski graph, which is recursively constructed to have a chromatic number that increases with graph size although the maximum clique size remains two.

Next we consider the distance-2 coloring problem on bipartite graphs. The latter is a partial coloring problem in that only one vertex part is colored. We obtain a lower bound on the partial distance-2 chromatic number of random bipartite graphs by using intersection graphs and "a balls and bins" analysis. We also obtain upper bounds close to the lower bound in parameter regimes of interest to us, and report empirical results that show that the lower bound is reasonably tight. Both the serial and parallel Greedy algorithms color the graphs nearly optimally as well.

Our approach to this problem is by observing the equivalence of the distance-2 chromatic number of bipartite graphs to the distance-1 chromatic number of a column intersection graph of the Jacobian matrix J (equivalently the undirected adjacency graph of the symmetric matrix $J^T J$). Hence we will study the distance-1 chromatic number of random intersection graphs, and show how it differs from the distance-1 chromatic number of random Erdős-Rényi graphs in parameter regimes of interest.

We use asymptotic notation as introduced in [11]. We say $f(n) = O(g(n))$ if there exists a constant $C > 0$ such that $|f(n)| \leqslant C|g(n)|$ as $n \to \infty$. The expression $f = o(g)$ denotes that the ratio of the functions $|f(n)/g(n)|$

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goes to zero as $n \to \infty$; when we wish to specify the size parameter n, we will write $f = o_n(g)$. We write $f(n) = \omega(g(n))$ if this ratio tends to infinity as $n \to \infty$; and $f = \Omega(g(n))$ if there exists a constant $C > 0$ such that $|f(n)| \geqslant C|g(n)|$ as $n \to \infty$. We also write $f \asymp g$ (or, equivalently $f(n) = \Theta(g(n))$) if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, again as $n \to \infty$.

2 The Distance-1 Chromatic Number

2.1 Random Erdős-Rényi Graphs Let $G(n, p)$ denote a random graph on a vertex set V (with $|V| \equiv n$) in which any two vertices are connected with probability $p \in [0, 1]$ independently; this is the well-known Erdős-Rényi model. We choose $p = d/n$ for constant $d > 0$, and then the average degree of a vertex is $(n-1)d/n$. A graph G is k-colorable if it is possible to partition the vertices V into k sets P_1, P_2, …, P_k such that no edge joins any two vertices in each subset P_i. Hence vertices in the subset P_i may be assigned the color i. The chromatic number $\chi(G)$ is the least number k such that G is k-colorable. One of the problems that Erdős and Rényi posed was to obtain the chromatic number of the graph $G(n, d/n)$, and several authors have addressed this question.

The chromatic number of the random graph $G(n, d/n)$ can be specified for "almost all" values of d. Let $d_{k,1} = 2k \log k - \log k - 1 + o_k(1)$, where the last term goes to zero with increasing values of k, and let $d_{k,2} = 2k \log k$. Achlioptas and Naor [1] proved that in the interval $d \in \left(d_{(k-1),1}, d_{(k-1),2}\right)$, the chromatic number of $G(n, d/n)$ is k for large enough values of n with high probability (w.h.p.). In the subsequent interval $d \in \left(d_{(k-1),2}, d_{k,1}\right)$, it is either k or $k+1$. This result has recently been strengthened by Coja-Oghlan and Vilenchik [4] in the following theorem.

Theorem 2.1. *There exists a constant k_0 such that the following statement is true. Let $S_k = (2(k-1)\log(k-1) - \log(k-1) - 0.99, 2k \log k - \log k - 1.38)$, $S = \bigcup_{k \geqslant k_0} S_k$, and $F(d) = k$ for all $d \in S_k$. Then S has asymptotic density 1 and for any $d \in S$, we have*

$$\lim_{n \to \infty} \mathbb{P}\left[\chi(G(n, d/n)) = F(d)\right] = 1.$$

These authors also give a lucid summary of earlier results that led to Theorem 2.1. In this paper, we use this theorem to construct random graphs of any desired size whose distance-1 chromatic numbers are known (w.h.p.). Earlier theoretical analyses show that the Greedy algorithm for coloring (or some modification) colors Erdős-Rényi graphs in several parameter regimes using a number of colors that is at most twice the chromatic number of the graph. It has not been thus far possible to reduce "this vexing factor of two," and there could be reasons why it might not be possible to reduce it [4].

2.2 Random Hyperbolic Graphs A geometric random graph $G(n, p, r)$ is an undirected graph constructed by randomly placing n nodes in some metric space, usually Euclidean, with a specified probability distribution p, connecting two nodes by a edge if and only if their distance is within a given radius r. A hyperbolic geometric random graph replaces the Euclidean space by the hyperbolic plane. This model was introduced by Papadopoulos, Krioukov, Boguñà, and Vahdat [16] in 2010. von Looz *et al.* [22] have described a hyperbolic geometric random graph generator with time complexity $O(n^{3/2} \log n + m \log n)$. (Here n is the number of vertices and m is the number of edges.)

The chromatic number of hyperbolic random graphs has been studied in [14]. According to their work when the graph size tends to infinity, the ratio of the chromatic number and the clique number would either be 1 or a constant number greater than one, depending on the value of the disk radius. The degree distribution of a hyperbolic random graph follows a power law: $P(k) \sim k^{-\gamma}$, where $P(k)$ is the fraction of nodes with degree k.

The clique number of a graph G, denoted by $\omega(G)$, is the number of vertices in a largest clique of G. It is a lower bound on the chromatic number. The asympotic behavior of the clique number of hyperbolic random graphs has also been studied [7]:

Theorem 2.2. *When the power law exponent $\gamma \in (2, 3)$, then $\omega(G(n)) = \Theta(n^{(3-\gamma)/2})$; when $\gamma \geqslant 3$, $\omega(G(n)) = \Theta(\frac{\log n}{\log \log n})$.*

2.3 Mycielski Graphs The Mycielski graph M_k [15] is an undirected graph defined as follows: Let K_n denote the complete graph on n vertices. $M_1 = K_1$; $M_2 = K_2$; for $k \geqslant 2$, let $V(M_k) = \{v_1, v_2, ..., v_n\}$. The graph M_{k+1} is defined on the vertex set $V(M_{k+1}) = V(M_k) \cup \{w_1, w_2, ..., w_n, z\}$; thus we add a copy W of the vertex set $V(M_k)$, and a distinguished vertex z. The edge set $E(M_{k+1}) = E(M_k) \cup \{(w_i, v_j) : (v_i, v_j) \in E(M_k)\} \cup \{(w_i, z) : 1 \leqslant i \leqslant n\}$. Note that W is an independent set of vertices. These graphs do not contain a triangle, and thus the size of the maximum clique in the graph M_k is two. Hence the clique number is not a good lower bound on the chromatic number here. We list several properties of the Mycielski graphs in what follows: (Many of these are easy to derive, except possibly the generating function for the degree distribution. We are not aware of these results published elsewhere.)

$\omega(M_k) = 2$; $\chi(M_k) = k$; the number of nodes in M_k is $n_k = 3 \times 2^{k-2} - 1$; the number of edges in M_k is

$m_k = 3.5 \times 3^{k-2} - 3 \times 2^{k-2} + 0.5$; and the average degree of M_k is $\overline{\delta_k} = (7 \times 3^{k-2} - 3 \times 2^{k-1} + 1)/(3 \times 2^{k-2} - 1)$.

The degree distribution of the Mycielski graph can be expressed using a polynomial generating function, where the exponent of a term corresponds to the degree, and its coefficient represents the number of vertices of that degree:

$$(2.1) \quad \begin{cases} f_2(x) = 2x, \\ f_{k+1}(x) = f_k(x^2) + x f_k(x) + x^{f_k(1)}, \quad \text{for } k \geqslant 2. \end{cases}$$

The coefficients of the polynomial $f_k(x)$ represent the degree distribution of the graph M_k. A closed form solution for the generating function $f_k(x)$ seems to be difficult to obtain, although we can compute the degree distribution of any M_k using the generating function.

An optimal coloring of M_{k+1} can be obtained from an optimal coloring of M_k as follows. Each newly added vertex at this step $w_i \in W$ can be colored with the color given to vertex $v_i \in V(M_k)$ whose copy it is. This is true since w_i is adjacent to the set of vertices that v_i is adjacent to in the graph M_k, and w_i is not adjacent to v_i. Finally, since the vertex z is adjacent to all vertices in W, it receives the color $k+1$.

Another lower bound on the chromatic number is obtained by taking the ratio of the number of vertices to the size of a maximum independent set in the graph, since the vertices in each color class form an independent set. However, for the Mycielski graph M_{k+1}, the size of a maximum independent set is $|W|$, almost half the vertices in it, and so this lower bound is also weak.

3 The Distance-2 Chromatic Number

Since we have a number of graph classes whose distance-1 chromatic number is known or well-bounded, we describe a transformation of the graph $G = (V, E)$ into a bipartite graph B whose distance-2 chromatic number is also known. The transformation splits each edge in the original graph $(v_i, v_j) \in E$ into two edges (v_i, w_{ij}) and (w_{ij}, v_j), where w_{ij} is a newly added vertex. The w vertices form one vertex part, and the original vertices in V form the other vertex part. After edge splitting, all edges join vertices in V to vertices in W in the graph B. Furthermore, a distance-1 coloring of vertices in G is a distance-2 coloring of vertices in V in the bipartite graph B, and hence these chromatic numbers are equal.

In the following, we obtain lower and upper bounds on the chromatic number of random bipartite graphs, and show how they differ from Erdős-Rényi graphs.

3.1 Binomial Random Intersection Graphs We may view the nonzero structure of a Jacobian matrix by a bipartite graph $H = (V, W, E)$ on two partite sets V and W. We interpret V as a set of n vertices, and W as a set of m features. Associated with every vertex $v \in V$ is a set of features $W(v)$, such that an edge joins v to every feature in $W(v) \subseteq W$. We think of the graph of the Jacobian H as a bipartite vertex-feature inclusion graph as defined above. The (partial) distance-2 coloring problem of the graph of the Jacobian H is equivalent to a distance-1 coloring problem on an intersection graph of H defined on the set of vertices V. A discussion of this well-known equivalence may be found in [9].

An *intersection graph* G is an undirected graph over the set of n vertices V of the vertex-feature inclusion graph H. The edge $(v, v') \in E(G)$ if and only if the sets of features associated with the vertices intersect, that is, $W(v) \cap W(v') \neq \emptyset$. Equivalently, the edge $(v, v') \in E(G)$ in the intersection graph if and only if there exists a length-2 path (using two distinct edges) connecting v and v' in the bipartite graph H. We want to highlight that any intersection graph G is a union of m cliques, each clique corresponding to the vertices associated with each feature w. Furthermore, one can express any arbitrary graph G as an intersection graph with $m = |E(G)|$ features.

In this paper, we consider intersection graphs sampled from appropriate distributions. *We implicitly define a distribution over intersection graphs by defining a distribution over bipartite vertex-feature inclusion graphs.* In the sequel, we shall always represent an intersection graph by G and its associated vertex-feature graph by H. A study of various properties of random intersection graphs is included in [8]. We consider intersection graphs where the number of features $m(n)$ is a suitable function of n, the number of vertices in G. We will find it helpful to consider three related models of random intersection graphs, which we describe below.

We begin with a distribution over vertex-feature inclusion bipartite graphs H with partite sets V and W as follows. We include every edge $(v, w) \in V \times W$ in the bipartite graph H independently with probability $p = p(n)$. The expected number of edges in H is $\mu = nmp$. Let $G(n, m, p)$ be a distribution over intersection graphs with n vertices and m features, referred to as the *binomial random intersection graphs* [12], obtained from the above distribution over vertex-feature inclusion graphs H as defined earlier.

Let $G(n, m, e)$ be the distribution over intersection *multi-graphs* with n vertices and m features, obtained from a distribution over vertex-feature inclusion bipartite graphs H by sampling e pairs of vertices and features (v_i, w_i), where $i \in \{1, \ldots, e\}$, uniformly and independently at random. These feature-vertex pairs *need not* be unique. The edges in the intersection graph G (with self-loops) are defined as follows. We include the edge $(v, v') \in E(G)$ if and only if there exists distinct $i, j \in \{1, \ldots, e\}$ such that $v = v_i$, $v' = v_j$, and $w_i = w_j$. In particular, a self-loop $(v, v) \in E(G)$ indicates that there exists distinct pairs (v_i, w_i) and (v_j, w_j) such that $w_i = w_j$ and $v_i = v_j = v$. We emphasize that e represents the number of edges in the bipartite vertex-feature graph H (and *not* in the intersection graph G).

Let $G^*(n, m, e)$ be a distribution over intersection graphs with n vertices and m features, obtained from a distribution over vertex-feature inclusion bipartite graphs H by sampling e *distinct* edges uniformly at random from the set $V \times W$.

Recall that a valid k*-coloring* of a graph $G = (V, E)$ is a mapping $f : V \to \{1, \ldots, k\}$ such that for any edge $(v, v') \in E(G)$, where $v, v' \in V$ are distinct vertices, we have $f(v) \neq f(v')$. We clarify that self-loops do *not* count as violations of the coloring constraint. We highlight an observation specific to valid colorings $f : V \to \{1, \ldots, k\}$ of an intersection graph G. Note that for any distinct vertices $v, v' \in V$, if $f(v) = f(v')$ then the feature sets $W(v)$ and $W(v')$ are disjoint. Consequently, all vertices $\{v_1, \ldots, v_t\} \subseteq V$ that share a feature w must receive distinct colors.

We prove bounds first for the intersection multigraphs $G(n, m, e)$, then lift them to the simple intersection graphs $G^*(n.m, e)$, and from there to the intersection graphs of interest to us, $G(n, m, p)$. These two additional graph models enable a modular analysis of the problem that circumvents the complexities of employing coupling arguments to account for the subtle correlations in the $G(n, m, p)$ model.

We now discuss the two theorems that provide lower and upper bounds of the chromatic number of intersection graphs, and proofs will be provided later. In the following results, for brevity, we write m and p to represent the functions $m(n)$ and $p(n)$. Also, for expressions e_1 and e_2, we write $\ln e_1 e_2$ to denote $\ln(e_1 e_2)$.

THEOREM 3.1. *Let $m \in \mathbb{N}$ be an increasing function of n representing the number of features. Let $\alpha \in [0, 1)$ and $\varepsilon \in (0, 1)$ be arbitrary constants. For $1/n\,(\ln m)^c \leqslant p = O\left(1/\sqrt{nm}\right)$ (where c is a suitable positive constant) and sufficiently large $n \in \mathbb{N}$, the following bound holds.*

$$\mathbb{P}\left[\chi(G) < k : G \leftarrow G(n, m, p)\right] = o(1), \text{ where}$$

$$k := \frac{\ln m}{\ln \frac{m}{\mu_{1-\varepsilon}} \ln m}\left(1 + \alpha \frac{\ln \ln \frac{m}{\mu_{1-\varepsilon}} \ln m}{\ln \frac{m}{\mu_{1-\varepsilon}} \ln m}\right), \text{ and}$$

$$\mu_{1-\varepsilon} := \lceil (1-\varepsilon) nm \cdot p \rceil.$$

The notation $G \leftarrow G(n, m, p)$ denotes that the graph G is sampled according to the distribution $G(n, m, p)$. The constraint $p \geqslant 1/n(\log m)^c$ ensures that the expected number of edges in the vertex-feature inclusion graph H is $nmp = m/\text{polylog}(m)$, which is necessary to apply Theorem 3.3. The constant $c > 0$ influences the constant in the expression $o(1)$ in the probability expression.

The theorem gives a lower bound on the chromatic number of random intersection graphs drawn from $G(n, m, p)$ for certain ranges of the probability p. It is instructive to take the constants $\alpha \to 1$ and $\varepsilon \to 0$ when interpreting the lower bound on the chromatic number in Theorem 3.1.

Earlier results have obtained equivalence theorems (see, for example, [6, 13, 19]) by reducing the random intersection graph model to a suitable random Erdős-Rényi model in certain ranges of the parameters n, m, and p. We cannot expect such an equivalence theorem in the regime of parameters that are interesting for our problem setting. For example, when $m \asymp n$ and $p \asymp 1/n$, the chromatic number $\chi(G)$ is (roughly) lower bounded by the function $\frac{\ln n}{\ln \ln n}$. In this setting of parameters, note that the expected number of edges in a graph $G \leftarrow G(n, m, p)$ is linear in n. On the other hand, the chromatic number of sparse Erdős-Rényi graphs is a constant when the expected number of edges in the graph linear in the number of vertices [1, 4]. (See Theorem 2.1).

The proof of Theorem 3.1 follows the intuition mentioned below. Consider any feature $w \in W$ and define the set $V(w) := \{v : v \in V, w \in W(v)\}$. Note that the vertices $V(w) \subseteq V$ induce a clique in the intersection graph G. Therefore, the chromatic number $\chi(G)$ is lower bounded by the largest $V(w)$, i.e., $\chi(G) \geqslant \max_{w \in W} |V(w)|$. We determine the largest $V(w)$ by analyzing the maximum load in an appropriate balls and bins experiment where $\mu_{1-\varepsilon} := \lceil (1-\varepsilon)(nm) \cdot p \rceil$ balls are thrown into m bins. Our experiments demonstrate that the number of colors the Greedy coloring algorithm uses tracks this lower bound, increasing with the number of vertices (see Figure 4).

We now turn to an upper bound on the chromatic number of random intersection graphs.

THEOREM 3.2. *Let $m \in \mathbb{N}$ represent the number of features, and let $\lambda > 0$ be an arbitrary constant. Suppose p simultaneously satisfies the following constraints.*

$$(1)\ p = o\left(\ln n/m\right),\ (2)\ p = \Omega(1/n), \text{ and } (3)\ p = O\left(1/\sqrt{nm}\right).$$

Then for sufficiently large $n \in \mathbb{N}$, the following bound holds.

$$\mathbb{P}\left[\chi(G) > \lambda \ln n : G \leftarrow G(n, m, p)\right] = o(1).$$

In particular, if $m \asymp n$ and $p \asymp 1/n$, combining the lower and upper bounds, the chromatic number satisfies the simplified bounds $\Theta\left(\frac{\ln n}{\ln\ln n}\right) \leqslant \omega(G) \leqslant \chi(G) \leqslant \Theta(\ln n)$, with probability $1-o(1)$, where $G \leftarrow G(n,m,p)$.

The proof of Theorem 3.2 follows by upper bounding the maximum degree of any vertex in the intersection graph G. Consider any vertex $v \in V$. In the vertex-feature inclusion graph, note that the vertex v contains the features in $W(v)$. The number of edges of the bipartite vertex-feature inclusion graph H belonging to the set $(V \setminus \{v\}) \times W(v)$ is an upper bound on the degree of v in the graph G. We choose $\mu := (nm) \cdot p$ edges at random to define the bipartite graph H, and in this graph we expect to encounter (at most) $\frac{|W(v)|}{|W|} \cdot \mu$ edges in the set $(V \setminus \{v\}) \times W(v)$. So, the expected degree of v in the graph G is at most $|W(v)|\, np$. As in the lower bound case, we upper bound $|W(v)|$ (w.h.p.) by the max-load in an experiment where μ balls are thrown into n bins.

For $m = n$ and $p = d/n$, we explicitly present an interesting consequence of our results. Fix arbitrary positive constants ε, δ close to 0. For brevity, define $T_\beta(n) := \ln\ln n - \ln \beta d$. (The proof of) Theorem 3.1, in fact, shows that (w.h.p.) we have

$$\omega(G(n,m,p)) \geqslant \frac{\ln n}{T_{1-\varepsilon}(n)}\left(1 + (1-\delta)\frac{\ln T_{1-\varepsilon}(n)}{T_{1-\varepsilon}(n)}\right).$$

Furthermore, Theorem 3.2 proves that (w.h.p.) we have

$$\chi(G(n,m,p)) \leqslant \varepsilon \ln n.$$

We conclude that the clique number and the chromatic number of $G \leftarrow G(n,m,p)$ are within a $\Theta(\ln\ln n)$ multiplicative factor of each other (w.h.p.).

Typically the qualitative nature of any graph property of random intersection graphs is sensitive to the density of its edges. If the random intersection graph is too sparse or too dense, then its chromatic number mimics the chromatic number of Erdős-Rényi graphs with an identical edge density. Previous work [3, 20] considers various ranges of the parameters m and p that lead to either too sparse or too dense graphs to be of interest in Jacobian computations. The choice of these parameters that are relevant in Jacobian computations lies at a crucial phase transition for intersection graphs (see, e.g., [2] and the discussion in [20]). The behavior of graph properties of random intersection graphs for parameters lying near this phase transition is relatively poorly understood. Our result approximates the chromatic number of random intersection graphs for parameters in this region up to a $\Theta(\ln\ln n)$ multiplicative factor. Outside this phase transition, the intersection graphs are relatively well-behaved, and [3, 20] prove stronger bounds demonstrating that the chromatic number is within a $(1+o(1))$ factor of the clique number.

3.2 Lower Bounding the Chromatic Number

Similar to [1, 4], we follow the strategy of proving desired results for $G \leftarrow G(n,m,\mu)$ and, then, lifting them to the $G \leftarrow G(n,m,p)$. This strategy circumvents the subtleties in analysis arising due to correlations among the random variables that are being analyzed.

3.2.1 Part 1

First, we shall prove a bound on the quantity related to the graph $G \leftarrow G(n,m,\mu)$, where $\mu = \lceil (nm)\cdot p \rceil$. Recall that the corresponding vertex-feature graph H might have multi-edges, and, hence, the intersection graph G might have self-loops. We will need the following theorem.

THEOREM 3.3. (MAX-LOAD IN BALLS AND BINS [17]) *Let a balls be thrown uniformly and independently at random into b bins. Let M be the random variable that counts the maximum number of balls in any bin.*

For any constant $\alpha \in [0,1)$, if $a \geqslant b/(\ln b)^c$, for some positive constant c, then the following bound holds.

$$\mathbb{P}\left[M < \frac{\ln b}{\ln \frac{b}{a}\ln b}\left(1 + \alpha\frac{\ln\ln\frac{b}{a}\ln b}{\ln\frac{b}{a}\ln b}\right)\right] = o_b(1).$$

For any constant $\alpha > 1$, if $a = o(b\ln b)$, then

$$\mathbb{P}\left[M > \frac{\ln b}{\ln \frac{b}{a}\ln b}\left(1 + \alpha\frac{\ln\ln\frac{b}{a}\ln b}{\ln\frac{b}{a}\ln b}\right)\right] = o_b(1).$$

LEMMA 3.1. *Let $m \in \mathbb{N}$ be an increasing function of n representing the number of features. Let $\alpha \in [0,1)$ be any constant. For $1/n\,(\ln m)^c \leqslant p$ (where c is a suitable positive constant) and sufficiently large $n \in \mathbb{N}$, the following bound holds. For any feature $w \in W$, let $E(w)$ represent the multi-set of edges in the bipartite vertex-feature graph belonging to the set $V \times \{w\}$. Then*

$$\mathbb{P}\left[\max_{w\in W} |E(w)| < k : G \leftarrow G(n,m,\mu)\right] = o(1),$$

where

(3.2)
$$k := \frac{\ln m}{\ln \frac{m}{\mu}\ln m}\left(1 + \alpha\frac{\ln\ln\frac{m}{\mu}\ln m}{\ln\frac{m}{\mu}\ln m}\right) \text{ and } \mu := \lceil nm\cdot p\rceil.$$

Proof: Consider the bipartite vertex-feature inclusion graph H with partite sets V and W. Since there can be multi-edges in H, the multi-set $E(w)$ may be different from the subset $V(w)$, the subset of vertices that have the feature w.

Consider the experiment of throwing μ edges uniformly at random to m features to determine the graph H. By Theorem 3.3, we have the following result.

$$\mathbb{P}\left[\max_{w\in W} |E(w)| < k : G \leftarrow G(n,m,e=\mu)\right] = o_m(1) = o(1),$$

where k and μ are as defined in Equation (3.2). This completes the proof of the lemma. ■

3.2.2 Part 2 Next we prove a result similar to Lemma 3.1, where one samples the intersection graph according to the distribution $G^*(n,m,\mu)$. We emphasize that the total variational distance between the distributions $G(n,m,\mu)$ and $G^*(n,m,\mu)$ may be a constant (due to collisions of edges in the bipartite vertex-feature inclusion graph ensured by the birthday bound), and hence lifting results from one model to the other is not straightforward.

LEMMA 3.2. *Let $m \in \mathbb{N}$ be an increasing function of n representing the number of features. Let $\alpha \in [0,1)$ be any constant. For $1/n(\ln m)^c \leqslant p = O\left(1/\sqrt{nm}\right)$ (where c is a suitable positive constant) and sufficiently large $n \in \mathbb{N}$, the following bound holds.*

$$\mathbb{P}[\chi(G) < k : G \leftarrow G^*(n,m,\mu)] = o(1),$$

where k and μ are as defined in Equation (3.2).

Proof: The intuition of the proof is the following. Consider the vertex-feature inclusion bipartite graph that defines the intersection graph. When one samples an intersection graph from the distribution $G(n,m,\mu)$, then there is no multi-edge in the corresponding bipartite vertex-feature graph with a positive constant probability (equivalently, the intersection graph G does not have a self-loop). Restricted to this set of vertex-feature inclusion bipartite graphs, the distribution $G(n,m,\mu)$ is identical to the distribution $G^*(n,m,\mu)$. Under this restriction, the multi-set $E(w)$ is identical to the set $V(w)$, for any feature $w \in W$. Observe that we have $\chi(G) \geqslant \omega(G) \geqslant \max_{w\in W}|V(w)|$ in intersection graphs. Hence the probability that $\chi(G) < k$ when $G \leftarrow G^*(n,m,\mu)$ is at most a constant times larger than the probability of $\max_{w\in W}|E(w)| < k$ when $G \leftarrow G(n,m,\mu)$.

Let us begin the proof. By Lemma 3.1 we know that

$$\mathbb{P}\left[\max_{w\in W}|E(w)| < k : G \leftarrow G(n,m,\mu)\right] = o(1).$$

We also know that the distribution $G(n,m,\mu)$ conditioned on the event that vertex-feature inclusion bipartite graph H has no multi-edges is identical to the distribution $G^*(n,m,\mu)$. If there are no multi-edges in H then the maximum number of vertices $\in V$ sharing a common feature is identical to $\max_{w\in W}|E(w)|$.

We shall show that the probability of no multi-edge occurring in the vertex-feature inclusion bipartite graph H is at least a positive constant. The probability that all μ edges are distinct is given by the following expression.

$$\begin{aligned}
&\frac{nm(nm-1)\cdots(nm-\mu+1)}{(nm)^\mu} = \prod_{i=0}^{\mu-1}\left(1-\frac{i}{nm}\right)\\
&\geqslant \prod_{i=0}^{\mu-1}\exp(-2i/nm) \quad \because 1-x \geqslant \exp(-2x),\ \text{for } x\in[0,1/2]\\
&= \exp\left(-\frac{\mu(\mu-1)}{nm}\right)\\
&> \exp(-p(nmp+1)) \qquad \because \mu = \lceil nmp\rceil < nmp+1\\
&\geqslant \exp(-p)\exp(-nmp^2).
\end{aligned}$$

The analysis above relies on the fact that $(\mu-1)/nm \leqslant 1/2$. When $nmp^2 = O(1)$, the right-hand side of the expression above is a positive constant. Let $\text{loops}(G) \in \{\texttt{true},\texttt{false}\}$ be a predicate indicating whether the multi-graph G has self-loops or not. We have shown that

$$\mathbb{P}[\neg\text{loops}(G) : G \leftarrow G(n,m,\mu)] = \Theta(1). \tag{3.3}$$

We combine all the ingredients to complete the proof.

$$\begin{aligned}
&\mathbb{P}[\chi(G) < k : G \leftarrow G^*(n,m,\mu)]\\
&\leqslant \mathbb{P}[\omega(G) < k : G \leftarrow G^*(n,m,\mu)]\\
&\qquad \because \chi(G) < k \implies \omega(G) < k\\
&\leqslant \mathbb{P}\left[\max_{w\in W}|V(w)| < k : G \leftarrow G^*(n,m,\mu)\right]\\
&\qquad \because \omega(G) < k \implies \max_{w\in W}|V(w)| < k\\
&= \mathbb{P}\left[\max_{w\in W}|V(w)| < k \,\middle|\, \neg\text{loops}(G) : G \leftarrow G(n,m,\mu)\right]\\
&= \frac{\mathbb{P}[\max_{w\in W}|V(w)| < k, \neg\text{loops}(G) : G \leftarrow G(n,m,\mu)]}{\mathbb{P}[\neg\text{loops}(G) : G \leftarrow G(n,m,\mu)]}\\
&= \frac{\mathbb{P}[\max_{w\in W}|E(w)| < k, \neg\text{loops}(G) : G \leftarrow G(n,m,\mu)]}{\mathbb{P}[\neg\text{loops}(G) : G \leftarrow G(n,m,\mu)]}\\
&\qquad \because \neg\text{loops}(G) \implies \forall w\in W,\ \text{we have } V(w) = E(w)\\
&\leqslant \frac{\mathbb{P}[\max_{w\in W}|E(w)| < k : G \leftarrow G(n,m,\mu)]}{\mathbb{P}[\neg\text{loops}(G) : G \leftarrow G(n,m,\mu)]}\\
&\qquad \because \mathbb{P}[A,B] \leqslant \mathbb{P}[A]\\
&= o(1)/\Theta(1) = o(1),
\end{aligned}$$

using Lemma 3.1 and Equation (3.3). ■

3.2.3 Part 3 We will need the following bound.

CLAIM 1. (CHERNOFF BOUND [11]) *Let S be a binomial distribution with mean μ, and $\delta \in (0,1)$. Then the following bounds hold.*

$$\begin{aligned}
&\mathbb{P}[S < (1-\delta)\mu] < \exp(-\delta^2\mu/2), \quad \textit{and}\\
&\mathbb{P}[S > (1+\delta)\mu] < \exp(-\delta^2\mu/3).
\end{aligned}$$

We emphasize that δ in the Claim may depend on μ.

Now we are ready to prove Theorem 3.1. The overview of the proof is as follows. Let $\varepsilon \in (0,1)$ be an arbitrary constant. Using the multiplicative form of the Chernoff-bound, we shall prove that the total number of edges in H corresponding to $G \leftarrow G(n,m,p)$ is $\geqslant \lceil(1-\varepsilon)nmp\rceil$ with probability $1-o(1)$. Using Lemma 3.2, and the fact that "not k colorable" is a monotonically increasing property with respect to the number of edges, we shall show that the sample $G \leftarrow G^*(n,m,e')$, where $e' \geqslant \lceil(1-\varepsilon)nmp\rceil$, is k-colorable with probability $o(1)$. These two results shall yield the theorem.

Let us begin the proof. Recall that the expected number of edges in the bipartite vertex-feature graph H when we sample $G \leftarrow G(n,m,p)$ is $\mu = nmp$. From the multiplicative form of the Chernoff-bound (Claim 1),

$$\begin{aligned}&\mathbb{P}\left[|E(H)| < (1-\varepsilon/2)\mu : G \leftarrow G(n,m,p)\right]\\ &< \exp\left(-(\varepsilon/2)^2\mu/2\right)\\ &= \exp(-\varepsilon^2 nmp/8) = o(1),\\ &\qquad \because nmp \geqslant m/\mathrm{polylog}(m) = \omega(1).\end{aligned}$$

We conclude that with probability $1-o_m(1)$ the bipartite vertex-feature graph H corresponding to the graph $G \leftarrow G(n,m,p)$ has $\geqslant (1-\varepsilon/2)\mu$ edges. Now, we shall show that $(1-\varepsilon/2)\mu \geqslant \lceil(1-\varepsilon)nmp\rceil$, for large enough m. Towards this objective it suffices to prove that the following difference is positive.

$$\begin{aligned}&(1-\varepsilon/2)\mu - \lceil(1-\varepsilon)nmp\rceil\\ &> (1-\varepsilon/2)nmp - (1-\varepsilon)nmp - 1\\ &= (\varepsilon/2)nmp - 1.\end{aligned}$$

Since the right-hand side expression is > 0 for large enough m, for such cases we conclude that

$$\mathbb{P}\left[|E(H)| < \lceil(1-\varepsilon)nmp\rceil : G \leftarrow G(n,m,p)\right] = o(1). \tag{3.4}$$

Note that the distribution $G(n,m,p)$ conditioned on the number of edges in the bipartite vertex-feature graph H being e' is identical to the distribution $G^*(n,m,e')$. Hence for any $e' \geqslant \mu_{1-\varepsilon} := \lceil(1-\varepsilon)nmp\rceil$ and

$$k := \frac{\ln m}{\ln \frac{m}{\mu_{1-\varepsilon}} \ln m}\left(1 + \alpha\frac{\ln\ln \frac{m}{\mu_{1-\varepsilon}} \ln m}{\ln \frac{m}{\mu_{1-\varepsilon}} \ln m}\right),$$

we have the following result (using Lemma 3.2 and the fact that the property of a graph H being "not k-colorable" is a monotonically increasing property of the number of edges).

$$\begin{aligned}&\mathbb{P}\left[\chi(G) < k \mid |E(H)| = e' : G \leftarrow G(n,m,p)\right]\\ &= \mathbb{P}\left[\chi(G) < k : G \leftarrow G^*(n,m,e')\right] = o(1).\end{aligned} \tag{3.5}$$

Based on these two observations, we can prove our main theorem. The following manipulation holds for a large enough n.

$$\begin{aligned}&\mathbb{P}\left[\chi(G) < k : G \leftarrow G(n,m,p)\right]\\ =&\mathbb{P}\left[\chi(G) < k, |E(H)| < \mu_{1-\varepsilon} : G \leftarrow G(n,m,p)\right]\\ &\quad + \mathbb{P}\left[\chi(G) < k, |E(H)| \geqslant \mu_{1-\varepsilon} : G \leftarrow G(n,m,p)\right]\\ \leqslant&\mathbb{P}\left[|E(H)| < \mu_{1-\varepsilon} : G \leftarrow G(n,m,p)\right]\\ &\quad + \mathbb{P}\left[\chi(G) < k, |E(H)| \geqslant \mu_{1-\varepsilon} : G \leftarrow G(n,m,p)\right]\\ &\quad \because \mathbb{P}[A,B] \leqslant \mathbb{P}[A]\\ =&o(1) + \mathbb{P}\left[\chi(G) < k, |E(H)| \geqslant \mu_{1-\varepsilon} : G \leftarrow G(n,m,p)\right],\\ &\quad \text{using Equation (3.4)}\\ =&o(1) + \sum_{e' \geqslant \mu_{1-\varepsilon}} \mathbb{P}\left[\chi(G) < k, |E(H)| = e' : G \leftarrow G(n,m,p)\right]\\ =&o(1) + \sum_{e' \geqslant \mu_{1-\varepsilon}} \mathbb{P}\left[\chi(G) < k \mid |E(H)| = e' : G \leftarrow G(n,m,p)\right]\\ &\qquad\qquad \cdot \mathbb{P}\left[|E(H)| = e' : G \leftarrow G(n,m,p)\right]\\ =&o(1) + \sum_{e' \geqslant \mu_{1-\varepsilon}} o(1) \cdot \mathbb{P}\left[|E(H)| = e' : G \leftarrow G(n,m,p),\right]\\ &\quad \text{using Equation (3.5)}\\ =&o(1) + o(1) \cdot \mathbb{P}\left[|E(H)| \geqslant \mu_{1-\varepsilon} : G \leftarrow G(n,m,p)\right]\\ <&o(1) + o(1) \cdot 1 = o(1).\end{aligned}$$

This completes the proof.

3.3 Upper Bounding the Chromatic Number

The proof proceeds in three steps similar to the lower bound. Due to space limitations, we offer a proof sketch.

3.3.1 Part 1

We shall prove a rough upper bound on the maximum degree of a vertex in $G \leftarrow G(n,m,e=\mu)$ that holds (w.h.p.). We will need the following upper bound on large deviations in a binomial distribution.

CLAIM 2. (LARGE DEVIATION BOUND [5]) *Let S be a binomial distribution with n trials and expectation $\leqslant \mu$. For any $A > \mu$, we have*

$$\mathbb{P}[S \geqslant A] \leqslant \left(\frac{e\mu}{A}\right)^A \cdot \exp(-\mu).$$

LEMMA 3.3. *Let $m \in \mathbb{N}$ represent the number of features, and $\lambda > 0$ be an arbitrary constant. For $p = o(\ln n/m)$, $p = \Omega(1/n)$, and sufficiently large $n \in \mathbb{N}$, the following bound holds. For $v \in V$, let $D(v)$ denote the multi-set $E(H) \cap (V \setminus \{v\} \times W(v))$ and $\mu := \lfloor nm \cdot p\rfloor$, then*

$$\mathbb{P}\left[\max_{v\in V} |D(v)| > \lambda \ln n : G \leftarrow G(n,m,\mu)\right] = o(1).$$

Proof: Consider any vertex $v \in V$. Recall that $W(v)$ is the set of features associated with the vertex v. The degree of v in the intersection graph G is upper bounded by the size of the multi-set $D(v)$. Note that the expected size of $D(v)$ is at most

$$\frac{|W(v)|\, n}{mn} \cdot \mu.$$

Now, we need to upper bound the typical value of $|W(v)|$. Towards this objective, consider the experiment where μ edges are thrown into n bins. By Theorem 3.3, with probability $1 - o(1)$ we have $|W(v)| \leqslant k$, for all $v \in V$, where $\alpha > 1$ is an arbitrary constant and

$$k := \frac{\ln n}{\ln \frac{n}{\mu} \ln n}\left(1 + \alpha \frac{\ln\ln \frac{n}{\mu} \ln n}{\ln \frac{n}{\mu} \ln n}\right).$$

That is, with probability $1 - o(1)$, we have $\mathbb{E}[|D(v)|] \leqslant \mu' := k\mu/m$, for all $v \in V$. By Claim 2 and the fact that $\mu' = \omega(1)$ and $\mu' = o(\ln n)$, we conclude that (for large enough n)

$$\mathbb{P}[|D(v)| \geqslant \lambda \ln n] \leqslant \left(\frac{e\mu'}{\lambda \ln n}\right)^{\lambda \ln n} \exp(-\mu') \leqslant \frac{1}{n} \cdot o(1).$$

By union bound, we conclude that

$$\mathbb{P}[\exists v \in V \text{ s.t. } |D(v)| \geqslant \lambda \ln n] = o(1),$$

whence the lemma. ■

3.3.2 Part 2 Next, just as in the lower bound, we lift the result to $G \leftarrow G^*(n, m, \mu)$.

LEMMA 3.4. *Let $m \in \mathbb{N}$ represent the number of features. Let $\lambda > 0$ be an arbitrary constant and $\mu := \lfloor nmp \rfloor$. For $p = o(\ln n/m)$, $p = \Omega(1/n)$, $p = O\left(1/\sqrt{nm}\right)$, and sufficiently large $n \in \mathbb{N}$, the following bound holds.*

$$\mathbb{P}[\chi(G) > \lambda \ln n : G \leftarrow G^*(n, m, \mu)] = o(1).$$

3.3.3 Part 3 Finally, we can prove Theorem 3.2. The outline of the argument is as follows. With high probability, the number of edges in H when we sample $G \leftarrow G(n, m, p)$ is $\leqslant \mu_{1+\varepsilon} := \lfloor (1+\varepsilon)\, nmp \rfloor$. Observe that "max-degree being $< k$" is a monotonically decreasing property with respect to the number of edges. Then the proof of the upper bound follows similar to the proof of Part 3 of the lower bound.

4 Experiments and Results

In this section, we show the results of greedy coloring algorithms applied to the different classes of graphs in our study. For the Random Erdős-Rényi and Hyperbolic graphs, we generate graphs with vertex sets ranging in size from 1 to $2^{20} \approx 10^6$. We generate 1000 instances of graphs for each vertex size, and report the average number of edges in the graphs, and the average number of colors taken by the Greedy algorithm using a specific vertex ordering. The vertex orderings we have chosen to evaluate include: (1) Largest Degree First (LF), (2) Dynamic Largest Degree First (DY), (3) Smallest Degree Last (SL), (4) Incidence Degree (ID), and (5) Random (Rd) [10].

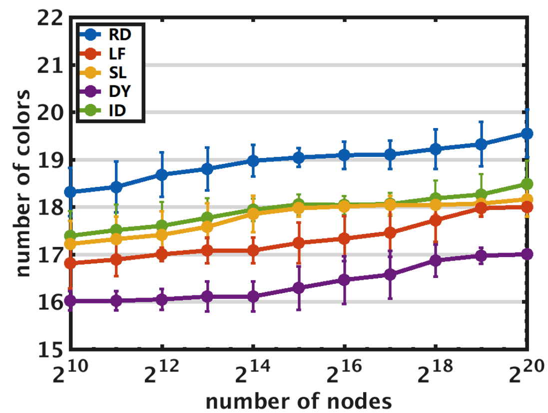

Figure 1: Number of colors taken by the Greedy algorithm with different orderings on Random Erdős-Rényi graphs whose chromatic number is 11.

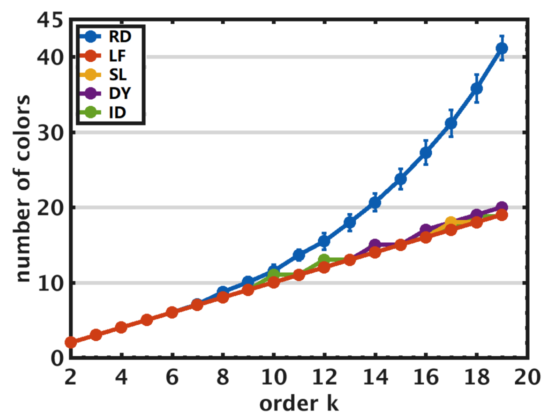

Figure 2: Number of colors taken by the Greedy algorithm with different orderings on Mycielski graphs.

4.1 Random Erdős-Rényi graphs We plot the number of colors taken by the Greedy algorithm with different orderings on Random Erdős-Rényi graphs whose chromatic number is 11 in Figure 1. Recall that the results reported are the arithmetic mean of 1000 instances of each graph size; the variability in the number of colors is shown as a whisker-plot with the minimum and the maximum number of colors as range. For both values of the expected chromatic number, we find that the DY ordering uses the fewest colors, followed by LF, and then SL. The random ordering performs the worst. While none of these orderings yield an optimal number of colors, all of them require fewer colors than twice the expected chromatic number, as proven theoretically [4]. Furthermore, as the number of vertices increases, the number of colors increases quite slowly. This is a satisfying result since the expected chromatic number is fixed even as the number of vertices increases.

4.2 Random Hyperbolic graphs We generated these graphs using Networkit [21]. We repeat the experiments for two values of the average vertex degree, 20 and 50, and two values of the power law exponent, 2.5 and 5. Lower bounds on the clique numbers were computed using a program of Rossi et al. [18].

These results are plotted in Figure 3 for both values of the average degree and both values of γ. Whisker plots are shown as before to indicate the range of colors used for 1000 graphs of a specific size. Note that as the graph size n increases while keeping the average degree $\overline{d}$ fixed, the clique number and the number of colors increase proportionately with the $\log n/\log\log n$ function (corresponding to $\gamma = 5$) and $n^{(3-\gamma)/2}$ function ($\gamma = 2.5$). This is expected by Theorem 2.2. The SL ordering obtains number of colors equal to the computed lower bound on the clique number in every instance. The LF, DY, ID orderings all obtain about the same number of colors on the Random Hyperbolic graph, but the natural and random orderings perform worse.

4.3 Mycielski graphs We generated the graphs for $k = 2, 3, \cdots, 19$, and colored these graphs with the Greedy algorithm in ColPack [10], again using different orderings. For the random orderings, we repeated the experiment 100 times for each k. The results in Figure 2 show that the LF, DY, SL, and ID orderings obtain nearly optimal coloring for all the values of k from 2 to 19. The random ordering again performs worse, with the number of colors increasing non-linearly as n increases.

4.4 Distance-2 coloring of Random Bipartite graphs We generated random bipartite graphs with n vertices and $m = n$ features, with edge present with probability $d/\sqrt{(mn)}$, where d is a parameter to be varied. We computed the maximum number of vertices that a feature is associated with, and call this the experimental maximum load (EML). We have plotted the theoretical lower bound we have obtained on this quantity, counting vertices with their multiplicity, the lower bound from Theorem 3.1 (LBT3.1), and also EML for two values of d. We compare these against the number of colors taken by a Greedy distance-2 coloring algorithm, both a serial algorithm and a parallel algorithm running on 20 threads. The theoretical lower bound and the EML are reasonably close for larger values of n (our results hold asymptotically). EML is also a lower bound on the distance-2 chromatic number. Since the Greedy algorithm requires number of colors equal to the EML for larger values of n, this number is equal to the chromatic number. Thus these graphs are colored optimally by the Greedy algorithm. We will report on parallel coloring results for the other classes of graphs elsewhere, but for some of these classes (e.g., Mycielski graphs) the number of colors increases significantly with parallelism.

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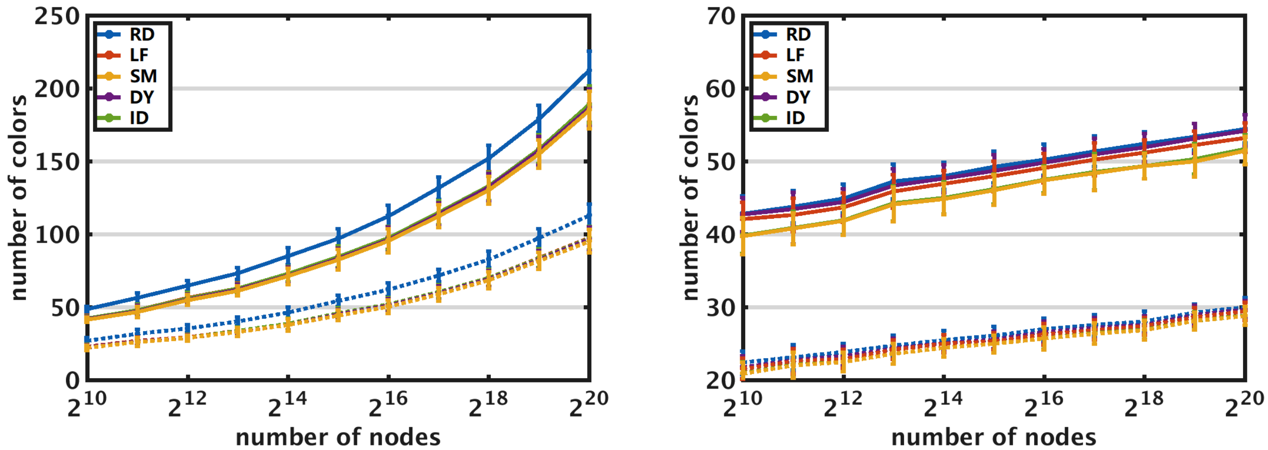
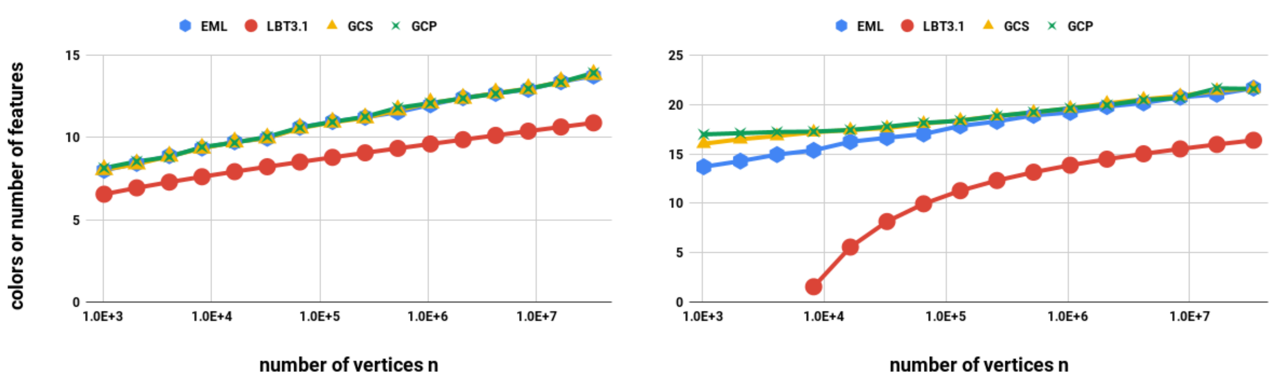


(a) power law exponent $\gamma = 2.5$.

(b) power law exponent $\gamma = 5$.

Figure 3: Number of colors taken by the Greedy algorithm with different orderings on Random Hyperbolic graphs. The solid lines represent results for graphs with average degree 50, while the dashed lines represent results for graphs with average degree 20.



(a) $d = 2$.

(b) $d = 5$.

Figure 4: Greedy partial distance-2 coloring algorithms on Random bipartite graphs. The theoretical lower bound on the maximum load (LBT3.1) is plotted in red dots; the experimental maximum load (EML) is plotted in blue hexagons; and the number of colors taken by the serial Greedy algorithm (GCS) is plotted in yellow triangles, while the result for the parallel Greedy algorithm (GCP) on 20 threads is plotted in green X.

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