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## Structural dynamic problems in time domain under uncertainty: an interval finite element approach

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**Abstract:** An analysis of the structural dynamic response under uncertainty is presented. Uncertainties in load and material are modelled as intervals exploiting the interval finite element method (IFEM). To reduce overestimation and increase the computational efficiency of the solution, we do not solve the dynamic problem by an explicit step-by-step time integration scheme. Instead, our approach solves for the structural variables in the whole time domain simultaneously by an implicit scheme using discrete Fourier transform and its inverse (DFT and IDFT). Non-trivial initial conditions are handled by modifying the right-hand side of the governing equation. To further reduce overestimation, a new decomposition strategy is applied to the IFEM matrices, and both primary and derived quantities are solved simultaneously. The final solution is obtained using an iterative enclosure method, and in our numerical examples the exact solution is enclosed at minimal computational cost.

**Keywords:** interval finite element method; dynamic response; discrete Fourier transform; matrix decomposition; iterative enclosure method.

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## 1 Introduction

In any physical system, uncertainties are inevitable when dealing with measurement devices and environmental conditions associated with a data acquisition process (Fernández-Martínez et al., 2013). Thus, it is necessary to model and track the propagation of uncertainties in the system and to reliably evaluate the accuracy of predicted system response. Conventional treatment of uncertainties uses probability theory (Lutes and Sarkani, 2004). The probability approach is preferred when measurements are abundant and sufficient to reliably predict the nature of the uncertainties. However, when measurements are scarce non-probabilistic approaches are preferred (Moens and Hanss, 2011; Zhang, 2005) such as Bayesian networks (Igusa et al., 2002; Soize, 2013; Unger and Könke, 2011), fuzzy sets (Adhikari and Khodaparast, 2014; Dehghan et al., 2006; Erdogan and Bakir, 2013; Klir and Wierman, 1999), evidence theory (Bai et al., 2013; Dempster, 1967; Jiang et al., 2013; Shafer, 1968), and intervals (Corliss et al., 2007; Do et al., 2014; Impollonia and Muscolino, 2011; Muhanna et al., 2007).

In this work, we adopt the interval approach modelling uncertainties by way of interval numbers with their respective lower and upper bounds. For the mathematical foundation of interval arithmetic we refer the reader to Alefeld and Herzberger (1984), Kulisch and Miranker (1981), and Moore et al. (2009).

The interval-based approach will be exploited for the analysis of structural dynamic problems in the time domain under uncertainty. In particular, we study the dynamics of elastic structures with uncertain load, geometric and material properties. Uncertain structural parameters are modelled by intervals, and the system response is evaluated by means of the Interval Finite Element Method (IFEM) (Hu and Qiu, 2010; Qiu and Ni,

2010; To, 2012; Xia et al., 2010). From now on, non-italic **bold** letters are used to denote interval variables. The dynamical response of a structure with uncertain parameters is governed by the following interval differential equation

$$\mathbf{K}\mathbf{u} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}, \quad (1)$$

where the interval matrices  $\mathbf{K}$ ,  $\mathbf{D}$ , and  $\mathbf{M}$  are respectively the stiffness, damping, and mass matrix of the structure,  $\mathbf{u}$  is the unknown interval nodal displacement vector,  $\dot{\mathbf{u}}$  and  $\ddot{\mathbf{u}}$  are the corresponding interval nodal velocity and acceleration vector,  $\mathbf{f}$  is the time-varying interval nodal equivalent load. The uncertain initial conditions are expressed in the interval form

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad (2)$$

where  $\mathbf{u}_0$  and  $\mathbf{v}_0$  are the initial nodal displacement and velocity vector, respectively.

In practice, the differential equation (1) is solved at discrete times  $t_k$  uniformly spaced in time. Conventional numerical integration approaches, such as the Newmark- $\beta$  method (De Borst et al., 2012; Dokainish and Subbaraj, 1989; Paz, 1997), solve for equation (1) recursively, viz. the solution at the current time  $t_k$  depends on the solution at the previous time  $t_{k-1}$ . However, a straightforward generalisation of such recursive approaches to intervals yield overestimation due to the interval dependency between successive times. This accumulates at each time step leading to an interval enclosure of the solution that quickly becomes excessively wide after few iterations in time.

To reduce overestimation, we solve for the dynamical equations in the spectral domain (Bae et al., 2014; Yang et al., 2012) using the Discrete Fourier Transform (DFT) (Santamarina and Fratta, 2005). In particular, equation (1) is first transformed into the frequency domain using the DFT. The spectral response is then computed via intervals and transformed back into the time domain by way of the Inverse Discrete Fourier Transform (IDFT). As a result, the solution is as if solved simultaneously at all the time steps.

The paper is structured as follows. First, we present a short background on the formulation of IFEM including the new matrix decomposition strategy followed by the deterministic solver based on the DFT approach. Then the associated interval solver is introduced. The dynamical equation (1) is rewritten in a fixed-point form, and an iterative approach is adopted to obtain a sharp interval enclosure of the exact solution. Finally, the performance of the current method is compared against other available methods by way of several numerical examples.

## 2 Interval finite element formulation

The different formulations of conventional finite element methods in linear elastic systems lead to a linear system equations. When uncertainties are considered in the load and material/geometric properties in the form of intervals the formulation results in the IFEM, which is expressed in an interval linear system of equations whose coefficients are intervals that appear in the system matrices, for example, in the static linear case, the structural equilibrium equation can be described by the following interval linear system

$$\mathbf{K}\mathbf{u} = \mathbf{f}, \quad (3)$$

where  $\mathbf{K}$  is the interval stiffness matrix,  $\mathbf{u}$  is the interval nodal displacement vector, and  $\mathbf{f}$  is the interval nodal equivalent load vector. Then the proposed solver aims to obtain guaranteed interval enclosures for the primary unknown variable  $\mathbf{u}$ . The main challenge is the reduction of overestimation in the interval system solution due to interval dependency. The adopted strategy for overestimation reduction is matrix decomposition. While matrix decomposition is well known in mathematical formulations, however it is new in the context of reducing interval overestimation with the goal of obtaining sharp enclosures in the development of interval finite element methods. A detailed formulation of IFEM based on Element-By-Element and Lagrange multiplier approach is introduced in the previous work of the authors (Xiao, 2015; Rama Rao et al., 2011). Our focus in this work is the system matrices decomposition and its implementations in dynamic problems. To minimise overestimation we propose new matrix decomposition strategies that avoid multiple occurrences of the same interval variables, as a result overestimation due to interval dependency is reduced (Moore et al., 2009). For the sake of clarity and to provide a background for this strategy, we will illustrate the matrix decomposition in the linear static case and later in the paper will be extended to the dynamic case and discussed in details in Section 4. In particular, the interval stiffness matrix  $\mathbf{K}$  and the interval nodal equivalent load  $\mathbf{f}$  are decomposed into

$$\mathbf{K} = A \text{diag}(\Lambda \boldsymbol{\alpha}) A^T, \quad \mathbf{f} = F \boldsymbol{\delta}, \quad (4)$$

where  $A$ ,  $\Lambda$ ,  $F$  are deterministic matrices,  $\boldsymbol{\alpha}$  is the interval stiffness parameter vector that accounts for the geometric and material uncertainties in  $\mathbf{K}$ , and  $\boldsymbol{\delta}$  is the interval load uncertainty vector that accounts for the load uncertainty in  $\mathbf{f}$ .

Note that matrix decomposition in equation (4) is performed at the element level before assembly. First, the element stiffness matrix  $\mathbf{K}_e$  and the element nodal equivalent load vector  $\mathbf{f}_e$  are computed. Their decomposition yields the element matrices  $A_e$ ,  $\Lambda_e$ ,  $F_e$ ,  $\boldsymbol{\alpha}_e$  and  $\boldsymbol{\delta}_e$ . These are further assembled into their global counterparts  $A$ ,  $\Lambda$ ,  $F$ ,  $\boldsymbol{\alpha}$  and  $\boldsymbol{\delta}$ . During the assembly, either the Element-by-Element (EBE) assembly strategy or the conventional strategy can be adopted (Xiao, 2015). In the following subsections, details on the aforementioned decompositions are discussed.

## 2.1 Element matrix decomposition

In this subsection, we present the matrix decomposition strategy applied to the interval element stiffness matrix  $\mathbf{K}_e$  and the element interval nodal equivalent load  $\mathbf{f}_e$ .

### 2.1.1 Decomposition of $\mathbf{K}_e$

According to equation (4), the element stiffness matrix  $\mathbf{K}_e$  is decomposed into

$$\mathbf{K}_e = A_e \text{diag}(\Lambda_e \boldsymbol{\alpha}_e) A_e^T, \quad (5)$$

where  $A_e$  and  $\Lambda_e$  are deterministic matrices, and  $\boldsymbol{\alpha}_e$  accounts for the geometric and material uncertainties in  $\mathbf{K}_e$ . In the following discussion, plane truss, plane frame, and plane stress/strain elements are presented, however the formulation has a general nature and can be implemented for other finite elements.

First, we introduce the standard two-node plane truss-bar element. Since the geometry and material properties are affected by uncertainties, we model the cross section area  $\mathbf{A}$  and the Young's modulus  $\mathbf{E}$  as intervals. The corresponding element interval stiffness matrix  $\mathbf{K}_e$  in the local coordinate system is given by

$$\mathbf{K}_e = \begin{Bmatrix} \frac{\mathbf{EA}}{L} & 0 & -\frac{\mathbf{EA}}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\mathbf{EA}}{L} & 0 & \frac{\mathbf{EA}}{L} & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix} \quad (6)$$

where  $L$  is the element length. The above element stiffness matrix is decomposed into the following elementary matrices:

$$\mathbf{A}_e = \{-1 \ 0 \ 1 \ 0\}^T, \quad \Lambda_e = \{1/L\}, \quad \boldsymbol{\alpha}_e = \{\mathbf{EA}\}, \quad (7)$$

where  $\boldsymbol{\alpha}_e$  contains the only interval variables  $\mathbf{EA}$  in the element, and the corresponding deterministic matrices are  $\mathbf{A}_e$  and  $\Lambda_e$ .

For the standard two-node Euler-Bernoulli beam elements, the uncertain parameters; the cross section area  $\mathbf{A}$ , the moment of inertia  $\mathbf{I}$ , and the Young's modulus  $\mathbf{E}$  are modelled as intervals. Considering the axial and bending deformation, the corresponding  $\mathbf{K}_e$  in the local coordinate system is given by

$$\mathbf{K}_e = \begin{Bmatrix} \frac{\mathbf{EA}}{L} & 0 & 0 & -\frac{\mathbf{EA}}{L} & 0 & 0 \\ 0 & \frac{12\mathbf{EI}}{L^3} & \frac{6\mathbf{EI}}{L^2} & 0 & -\frac{12\mathbf{EI}}{L^3} & \frac{6\mathbf{EI}}{L^2} \\ 0 & \frac{6\mathbf{EI}}{L^2} & \frac{4\mathbf{EI}}{L} & 0 & -\frac{6\mathbf{EI}}{L^2} & \frac{2\mathbf{EI}}{L} \\ -\frac{\mathbf{EA}}{L} & 0 & 0 & \frac{\mathbf{EA}}{L} & 0 & 0 \\ 0 & -\frac{12\mathbf{EI}}{L^3} & -\frac{6\mathbf{EI}}{L^2} & 0 & \frac{12\mathbf{EI}}{L^3} & -\frac{6\mathbf{EI}}{L^2} \\ 0 & \frac{6\mathbf{EI}}{L^2} & \frac{2\mathbf{EI}}{L} & 0 & -\frac{6\mathbf{EI}}{L^2} & \frac{4\mathbf{EI}}{L} \end{Bmatrix}, \quad (8)$$

where  $L$  is the element length. Then  $\boldsymbol{\alpha}_e$  contains the element axial stiffness  $\mathbf{EA}$  and the element bending stiffness  $\mathbf{EI}$ . The corresponding  $\mathbf{A}_e$  is a  $6 \times 3$  matrix, whose columns are the eigenvectors of  $\mathbf{K}_e$ , and  $\Lambda_e$  is a  $3 \times 2$  matrix, are

$$\mathbf{A}_e = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & L \\ -1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -1 & L \end{Bmatrix}, \quad \Lambda_e = \begin{Bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \\ 0 & \frac{3}{L^3} \end{Bmatrix}, \quad \boldsymbol{\alpha}_e = \begin{Bmatrix} \mathbf{EA} \\ \mathbf{EI} \end{Bmatrix}. \quad (9)$$

Note that the matrix decomposition strategy that lead to equations (7) and (9) is not unique. Indeed, this approach requires analytic expressions for  $\mathbf{K}_e$ , which are usually not available for the general types of finite elements. A more general approach regardless the finite element type is based on numerical integration. As an example, we consider plane stress/strain elements.

For standard 8-node rectangular isoparametric elements in plane stress/strain problem, the element Young's modulus  $\mathbf{E}$  is modelled as intervals. Then the corresponding  $\mathbf{K}_e$  is given by

$$\mathbf{K}_e = \int_{\Omega} B_e^T(\xi) \mathbf{E}_e(\xi) B_e(\xi) t(\xi) d\Omega, \quad (10)$$

where the integration domain  $\Omega$  is the entire element,  $t$  is the thickness, and  $B_e$  is the strain-displacement matrix. For isotropic material with Poisson's ratio  $\nu$ , the interval constitutive matrix  $\mathbf{E}_e$  for plane stress state is given by

$$\mathbf{E}_e = \frac{\mathbf{E}}{1-\nu^2} \begin{Bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{Bmatrix}. \quad (11)$$

and for plane strain state,

$$\mathbf{E}_e = \frac{\mathbf{E}}{(1-2\nu)(1+\nu)} \begin{Bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{Bmatrix} \quad (12)$$

The double integral in equation (10) can be evaluated numerically by using a  $3 \times 3$  Gaussian integration rule, that is

$$\mathbf{K}_e = \sum_{j=1}^9 B_e^T(\xi_j) \mathbf{E}_e(\xi_j) B_e(\xi_j) w_j J(\xi_j) t(\xi_j), \quad (13)$$

where the coordinates  $\xi_j$  and weights  $w_j$  for all the integration points in the standard domain  $[-1,1] \times [-1,1]$  are given, and  $J$  is the determinant of the Jacobian of the coordinate transformation between the local and global coordinate system. Note that the use of numerical integration has the effect of changing the modulus from a spatial function bounded by interval values to a function described by interval coefficients. Thus the discretisation by the finite element methods results in additional smoothness in the spatial variation of the modulus field.

In equation (13), the only interval term  $\mathbf{E}$  can be factored out, viz.  $\mathbf{K}_e = P_e \mathbf{E}$ , where  $P_e$  is a deterministic matrix. This can be decomposed as  $P_e = A_e \text{diag}(\Lambda_e) A_e^T$ , where  $\Lambda_e$  is the eigenvalue matrix and the columns of  $A_e$  are the corresponding eigenvectors. Note

that  $\Lambda_e$  includes three zero eigenvalues, which correspond to rigid body motion of two translations and one rotation. They are dropped and equation (13) is rewritten as

$$\mathbf{K}_e = A_e \text{diag}(\Lambda_e) A_e^T \boldsymbol{\alpha}_e = A_e \text{diag}(\Lambda_e \boldsymbol{\alpha}_e) A_e^T, \quad (14)$$

where  $\boldsymbol{\alpha}_e = \{\mathbf{E}\}$  is the only interval quantity. One can also use an LDL decomposition of  $P_e$ . As a result,  $A_e$  is a lower triangular matrix, and  $\Lambda_e$  is also different.

In the above decomposition strategy,  $P_e$  is explicitly computed in order to obtain  $A_e$  and  $\Lambda_e$ . In addition, the Young's modulus  $\mathbf{E}$  is assumed to be constant inside the element. Alternatively, the B-matrix approach (Xiao, 2015) can be adopted, in which  $P_e$  is never explicitly computed and the element stiffness parameter vector  $\boldsymbol{\alpha}_e$  contains  $\mathbf{E}_j$  at all the numerical integration points.

Finally, note that the decomposition of diagonal matrix  $\text{diag}(\Lambda_e \boldsymbol{\alpha}_e)$  is novel in the current method, when compared with others in the literature (Gao, 2007; Impollonia, 2006; Neumaier and Pownuk, 2007). The decomposition reduces multiple occurrences of interval terms to the minimum. In later discussions on iterative enclosure method, this decomposition plays an important role.

### 2.1.2 Decomposition of $\mathbf{f}_e$

The element nodal equivalent load vector  $\mathbf{f}_e$  is decomposed into the following form using the  $M - \delta$  method (Mullen and Muhanna, 1999),

$$\mathbf{f}_e = F_e \boldsymbol{\delta}_e. \quad (15)$$

Thus the interval terms in the element load uncertainty vector  $\boldsymbol{\delta}_e$  is completely separated from the deterministic part  $F_e$  of the equivalent load. For an arbitrary element, the nodal equivalent load vector is given by

$$\begin{aligned} \mathbf{f}_e = & \sum_j N^T(\xi_j) \mathbf{f}_0(\xi_j) + \int_{\Omega_1} N^T(\xi) \mathbf{f}_1(\xi) d\Omega_1 \\ & + \int_{\Omega_2} N^T(\xi) \mathbf{f}_2(\xi) d\Omega_2 + \int_{\Omega_3} N^T(\xi) \mathbf{f}_3(\xi) d\Omega_3, \end{aligned} \quad (16)$$

where  $N(\xi)$  is the shape function matrix,  $\xi$  is the local coordinate of the element,  $\xi_j$  are coordinates where concentrated load is applied,  $\mathbf{f}_0(\xi_j)$  are the concentrated loads under consideration,  $\Omega_1, \Omega_2, \Omega_3$  are the integration domains in which line load  $\mathbf{f}_1$ , surface load  $\mathbf{f}_2$ , volume load  $\mathbf{f}_3$  are non-zero.

By rewriting  $\mathbf{f}_j(\xi) = L_j(\xi) \boldsymbol{\delta}_e$  ( $j = 0, 1, 2, 3$ ), the interval element load uncertainty vector  $\boldsymbol{\delta}_e$  can be separated from the deterministic part of  $\mathbf{f}_e$ . Then

$$\begin{aligned} F_e = & \sum_j N^T(\xi_j) L_0(\xi_j) + \int_{\Omega_1} N^T(\xi) L_1(\xi) d\Omega_1 \\ & + \int_{\Omega_2} N^T(\xi) L_2(\xi) d\Omega_2 + \int_{\Omega_3} N^T(\xi) L_3(\xi) d\Omega_3, \end{aligned} \quad (17)$$

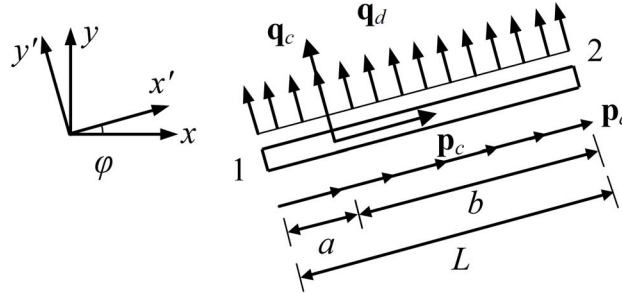
As a first example, consider the Euler-Bernoulli beam element of length  $L$  shown in Figure 1. The element is subject to concentrated loads  $\mathbf{p}_e$  and  $\mathbf{q}_e$  applied at distance  $a$

from node 1, as well as transverse uniform load  $\mathbf{q}_d$  and axial uniform load  $\mathbf{p}_d$  along the element. Then the element nodal equivalent load vector  $\mathbf{f}_e$  and its decomposition are given by

$$\mathbf{f} = \begin{Bmatrix} \frac{b}{L}\mathbf{p}_c + \frac{L}{2}\mathbf{p}_d \\ \frac{b}{L}\mathbf{q}_c + \frac{L}{2}\mathbf{q}_d \\ \frac{L^2}{12}\mathbf{q}_d \\ \frac{a}{L}\mathbf{p}_c + \frac{L}{2}\mathbf{p}_d \\ \frac{a}{L}\mathbf{q}_c + \frac{L}{2}\mathbf{q}_d \\ \frac{L^2}{12}\mathbf{q}_d \end{Bmatrix} = \begin{Bmatrix} \frac{b}{L} & 0 & \frac{L}{2} & 0 \\ 0 & \frac{b}{L} & 0 & \frac{L}{2} \\ 0 & 0 & 0 & \frac{L^2}{12} \\ \frac{b}{L} & 0 & \frac{L}{2} & 0 \\ 0 & \frac{a}{L} & 0 & \frac{L}{2} \\ 0 & 0 & 0 & \frac{L^2}{12} \end{Bmatrix} \begin{Bmatrix} \mathbf{p}_c \\ \mathbf{q}_c \\ \mathbf{p}_d \\ \mathbf{q}_d \end{Bmatrix} = F_e \boldsymbol{\delta}_e, \quad (18)$$

where the load uncertainty vector  $\boldsymbol{\delta}_e$  contains the four intervals in  $\mathbf{f}_e$ , and each column of  $F_e$  corresponds to one of them.

**Figure 1** Distributed and concentrated loads acting on a two-node Euler-Bernoulli beam element

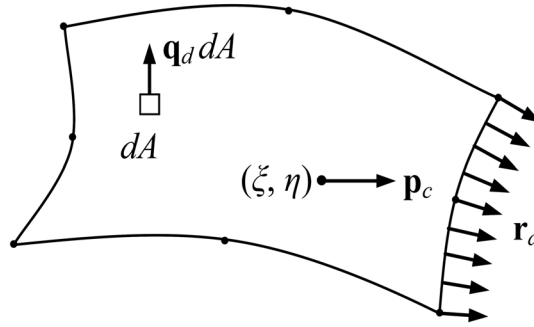


As a second example, consider the 8-node rectangular isoparametric element in Figure 2. The element is subject to a concentrated load  $\mathbf{p}_c$  in the  $x$ -direction at  $(\xi, \eta)$ , a uniform body load  $\mathbf{q}_d$  in the  $y$ -direction, and a uniform line load  $\mathbf{r}_d$  outwards on the right edge. Then  $\boldsymbol{\delta}_e = \{\mathbf{p}_c \mathbf{q}_d \mathbf{r}_d\}^T$  and  $F_e$  is given by

$$F_e = \begin{Bmatrix} N_1(\xi, \eta) & 0 & \int_{x=0}^L N_1 J_{11} dl \\ 0 & \iint_A N_1(\xi, \eta) dA & \int_{x=0}^L N_1 J_{21} dl \\ \vdots & \vdots & \vdots \\ N_8(\xi, \eta) & 0 & \int_{x=0}^L N_8 J_{11} dl \\ 0 & \iint_A N_8(\xi, \eta) dA & \int_{x=0}^L N_8 J_{21} dl \end{Bmatrix}, \quad (19)$$

where  $N_j$  is the shape function with respect to the  $j$ -th node,  $L$  is the length of the right edge, and  $\{J_{11} \ J_{21}\}^T$  is the first column of the Jacobian, accounting for the uniform load  $\mathbf{r}_d$  acting on the right edge. The formulation can be easily extended to cases when the loads are non-uniform or defined in the local coordinate system.

**Figure 2** Concentrated, body and line loads acting on an 8-node rectangular isoparametric element



## 2.2 Element assembly strategies

In this subsection,  $\mathbf{K}_e$ ,  $\mathbf{f}_e$ , and  $B_e$  are assembled into their global counterparts  $\mathbf{K}$ ,  $\mathbf{f}$ , and  $B$ . At the same time, the assembly rules for the decomposed matrices are presented. Two assembly strategies are introduced here: one is the Element-by-Element approach, which assembles the matrices block-by-block; the other is the conventional FEM assembly strategy (Xiao, 2015).

### 2.2.1 Element-by-Element assembly

In the Element-by-Element approach, the structure is modelled by separated elements and common nodes that connect the elements. As a result, the structural nodal displacement vector  $\mathbf{u}$  is a collection of all the element nodal displacement vectors  $\mathbf{u}_e$ , and the nodal displacement vector  $\mathbf{u}_n$  of the common nodes. Then the global stiffness matrix  $\mathbf{K}$  and nodal equivalent load  $\mathbf{f}$  are assembled from their element counterparts

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_e \\ \vdots \\ \mathbf{u}_e \\ \mathbf{u}_n \end{Bmatrix}, \quad \mathbf{K} = \begin{Bmatrix} \mathbf{K}_e & & \\ & \ddots & \\ & & \mathbf{K}_e \\ & & & 0 \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} \mathbf{f}_e \\ \vdots \\ \mathbf{f}_e \\ \mathbf{f}_n \end{Bmatrix}, \quad (20)$$

where  $\mathbf{f}_n$  denotes concentrated forces applied directly on the common nodes. In contrast to the standard FEM assembly,  $\mathbf{K}$  and  $\mathbf{f}$  are assembled block by block from the individual element stiffness  $\mathbf{K}_e$  and individual element load  $\mathbf{f}_e$ , respectively (Rama Rao et al., 2011). Note that  $\mathbf{K}$  is a singular matrix.

To reduce overestimation due to dependency of multiple occurrences of the same variable,  $\mathbf{K}$  and  $\mathbf{f}$  are further decomposed, that is,  $\mathbf{K} = A \text{diag}(\Lambda \boldsymbol{\alpha}) A^T$  and  $\mathbf{f} = F \boldsymbol{\delta}$ . The element vectors  $\boldsymbol{\alpha}_e$  and  $\boldsymbol{\delta}_e$  can be selected components of the global interval vectors  $\boldsymbol{\alpha}$  and  $\boldsymbol{\delta}$ , or they can be interpolated from  $\boldsymbol{\alpha}$  and  $\boldsymbol{\delta}$ . In either way, they can be brought into the following form,

$$\boldsymbol{\alpha}_e = L_\alpha \boldsymbol{\alpha}, \quad \boldsymbol{\delta}_e = L_\delta \boldsymbol{\delta}. \quad (21)$$

The corresponding assembly rule for  $A$ ,  $\Lambda$ , and  $F$  is quite similar to the assembly rule for  $\mathbf{K}$  and  $\mathbf{f}$ . Indeed, they are all assembled block by block,

$$A = \begin{Bmatrix} A_e & & \\ & \ddots & \\ & & A_e \\ 0 & \dots & 0 \end{Bmatrix}, \quad \Lambda = \begin{Bmatrix} \Lambda_e L_\alpha \\ \vdots \\ \Lambda_e L_\alpha \end{Bmatrix}, \quad F = \begin{Bmatrix} F_e L_\delta \\ \vdots \\ F_e L_\delta \\ F_n \end{Bmatrix}. \quad (22)$$

where  $\mathbf{f}_n = F_n \boldsymbol{\delta}$  is the decomposition of  $\mathbf{f}_n$ . In the derivation, it is assumed that each interval component in  $\boldsymbol{\alpha}$  and  $\boldsymbol{\delta}$  varies independently. If two or more components in  $\boldsymbol{\alpha}$  or  $\boldsymbol{\delta}$  represent the same variable, corresponding columns in  $\Lambda$  or  $F$  should be added together. If certain entries in  $\boldsymbol{\delta}$  were equal to zero, the corresponding columns in  $F$  should be deleted. To impose compatibility and equilibrium, Lagrangian multiplier  $\boldsymbol{\lambda}$  is introduced to enforce the constraint  $C\mathbf{u} = 0$ . The energy functional  $\Pi$  of the system is

$$\Pi = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{f} + \boldsymbol{\lambda}^T C \mathbf{u}. \quad (23)$$

Minimising  $\Pi$  with respect to  $\mathbf{u}$  and  $\boldsymbol{\lambda}$  yields the interval governing equations

$$\begin{Bmatrix} \mathbf{K} & C^T \\ C & 0 \end{Bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ 0 \end{Bmatrix}. \quad (24)$$

To reduce overestimation,  $\mathbf{K}$  and  $\mathbf{f}$  are decomposed as in equation (4) Thus

$$\left( \begin{Bmatrix} A \\ 0 \end{Bmatrix} \text{diag}(\Lambda \Delta \boldsymbol{\alpha}) \begin{Bmatrix} A^T & 0 \end{Bmatrix} + \begin{Bmatrix} K_0 & C^T \\ C & 0 \end{Bmatrix} \right) \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \boldsymbol{\delta}, \quad (25)$$

where  $\Delta \boldsymbol{\alpha}$  is the difference between interval vector  $\boldsymbol{\alpha}$  and its reference vector  $\alpha_0$ , viz.

$\Delta \boldsymbol{\alpha} = \boldsymbol{\alpha} - \alpha_0$ , and  $K_0 = A \text{diag}(\Lambda \alpha_0) A^T$ . Preferably,  $\alpha_0 = \text{mid } \boldsymbol{\alpha}$ .

The Lagrangian multiplier  $\boldsymbol{\lambda}$  denotes negative internal forces between element nodes and common nodes when the constraint is a compatibility condition and  $\boldsymbol{\lambda}$  denotes reactions at the supports when the constraint is an essential boundary condition. Thus internal forces and support reactions are obtained as a by-product as the Lagrangian multiplier  $\boldsymbol{\lambda}$  enforcing the constraint equation  $C\mathbf{u} = 0$ .

### 2.2.2 Conventional assembly

The conventional assembly strategy provides smaller stiffness matrix and is more efficient for large scale problems. In this case, the global nodal displacement vector  $\mathbf{u}$  contains only displacement vector  $\mathbf{u}_n$  of the common nodes. The global stiffness matrix and nodal equivalent load vector are given by

$$\mathbf{K} = \sum_e T_e^T \mathbf{K}_e T_e, \quad \mathbf{f} = \sum_e T_e^T \mathbf{f}_e + \mathbf{f}_n, \quad (26)$$

where  $T_e$  is the transformation matrix between the global and local nodal displacement vector  $\mathbf{u}$  and  $\mathbf{u}_e$ . By inserting  $\mathbf{K}_e = A_e \text{diag}(\Lambda_e \boldsymbol{\alpha}_e) A_e^T$  into equation (26), the decomposition rule for  $\mathbf{K}$  follows as

$$\begin{aligned} \mathbf{K} &= \sum_e T_e^T A_e \text{diag}(\Lambda_e \boldsymbol{\alpha}_e) A_e^T T_e \\ &= \left\{ T_e^T A_e \dots T_e^T A_e \right\} \text{diag} \left\{ \begin{matrix} \Lambda_e \boldsymbol{\alpha}_e \\ \vdots \\ \Lambda_e \boldsymbol{\alpha}_e \end{matrix} \right\} \left\{ \begin{matrix} A_e^T T_e \\ \vdots \\ A_e^T T_e \end{matrix} \right\}. \end{aligned} \quad (27)$$

The assembly rules for  $A$  and  $\Lambda$  are given by

$$A = T_e^T A_e \dots T_e^T A_e, \quad \Lambda = \left\{ \begin{matrix} \Lambda_e L_\alpha \\ \vdots \\ \Lambda_e L_\alpha \end{matrix} \right\}. \quad (28)$$

Similarly, using the decompositions  $\mathbf{f}_e = F_e \boldsymbol{\delta}_e$  and  $\mathbf{f}_n = F_n \boldsymbol{\delta}_n$ , the nodal equivalent load  $\mathbf{f}$  in equation (26) can be written as

$$\mathbf{f} = \sum_e T_e^T F_e \boldsymbol{\delta}_e + F_n \boldsymbol{\delta}_n. \quad (29)$$

The assembly rule for  $F$  is given by

$$F = \sum_e T_e^T F_e L_\delta + F_n. \quad (30)$$

After assembling all matrices, the energy functional in equation (23) is formulated and minimised with respect to  $\mathbf{u}$  and  $\boldsymbol{\lambda}$  which will yield the same form of interval governing equations (24) and (25) but with a different structure of matrices that is consistent with the assembly.

## 3 Deterministic dynamic solver

In this section, the deterministic dynamic solver based on the DFT (Veletsos and Kumar, 1983; Veletsos and Ventura, 1985) is presented. The FEM equations to solve for the dynamic response of a linearly elastic structure are given by

$$K\mathbf{u} + D\dot{\mathbf{u}} + M\ddot{\mathbf{u}} = \mathbf{f}, \quad (31)$$

where  $K$ ,  $D$ , and  $M$  are the stiffness, damping, and mass matrices of the structure, respectively,  $u$  is the nodal displacement vector,  $\dot{u}$  and  $\ddot{u}$  are the first and second derivatives of  $u$  with respect to the time (or, equivalently, nodal velocity and acceleration), and  $f$  is the nodal equivalent load. The initial conditions are given by

$$u(0) = u_0, \quad \dot{u}(0) = v_0. \quad (32)$$

To impose compatibility and equilibrium, Lagrangian multiplier  $\lambda$  is introduced to enforce the constraint  $Cu = 0$ . The energy functional  $\Pi$  of the system is

$$\Pi = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{f} + \lambda^T C \mathbf{u}. \quad (33)$$

Minimising  $\Pi$  with respect to  $\mathbf{u}$  and  $\lambda$  yields the interval governing equations

$$\begin{Bmatrix} \mathbf{K} & C^T \\ C & 0 \end{Bmatrix} \begin{Bmatrix} \mathbf{u} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ 0 \end{Bmatrix}. \quad (34)$$

The dynamics is solved over a given time interval, which is discretised into uniformly spaced times  $t_k$ . The nodal equivalent load at the discrete time  $t_k$  is known, and we want to solve for the nodal displacement vector  $u$  at  $t_k$  and its time derivatives  $\dot{u}$  and  $\ddot{u}$ , viz. velocity and acceleration respectively. That is,  $f(t_k) = f_k$ ,  $u(t_k) = u_k$ ,  $\dot{u}(t_k) = \dot{u}_k$ ,  $\ddot{u}(t_k) = \ddot{u}_k$ . We assume that time steps are uniformly spaced, viz.  $t_k = k\Delta t$ . The sampling interval  $\Delta t$  must be small enough to prevent any potential aliasing (Santamarina and Fratta, 2005). Let  $T$  be the total time length of the signal and  $N$  the number of time steps  $\Delta t$  in the total time length, then  $T = N\Delta t$ .

The governing equation (31) is transformed in the spectral domain via the DFT into

$$(-\omega_j^2 M + i\omega_j D + K) \mathcal{F}_t(u)_j = \mathcal{F}_t(f)_j, \quad (35)$$

or

$$K_{DS} \mathcal{F}_t(u)_j = \mathcal{F}_t(f)_j, \quad (36)$$

where  $K_{DS}$  is the dynamic stiffness,  $i = \sqrt{-1}$  is the imaginary unit,  $\omega_j = j\Delta\omega$  with  $\Delta\omega = 2\pi/T$  being the fundamental frequency,  $\mathcal{F}_t(u)_j$  and  $\mathcal{F}_t(f)_j$  are the Fourier transform of the nodal displacement  $u_k$  and equivalent load  $f_k$ , respectively. Then the nodal displacement vector in the time-domain is obtained by applying the IDFT to  $\mathcal{F}_t(u)_j$ , viz.

$$u_n = \frac{1}{N} \sum_{j=0}^{N-1} \mathcal{F}_t(u)_j e^{-i(2\pi/N)jn} = \frac{1}{N} \sum_{j=0}^{N-1} G_j \mathcal{F}_t(f)_j e^{-i(2\pi/N)jn}, \quad (37)$$

where  $G_j$  is the inverse of the dynamic stiffness matrix in equation (36). To ensure that the final solution  $u_n$  is real, i.e., null imaginary part,  $G_j$  takes the following form,

$$G_j = \begin{cases} (-\omega_j^2 M + i\omega_j D + K)^{-1}, & 0 \leq j < N/2; \\ \text{conjugate of } G_{N-j}, & N/2 \leq j < N. \end{cases} \quad (38)$$

The above approach essentially solves for the stationary response of the structure caused by periodic loads with period  $T$ . The results are identical to the actual dynamic response with trivial initial conditions ( $u_0 = v_0 = 0$ ) when zero-padding is applied. The length of the zero-padding,  $T_p$ , can be estimated from

$$e^{-\zeta\omega T_p} < \tau_{err}, \Rightarrow T_p > \frac{\ln \tau_{err}}{\zeta\omega}, \quad (39)$$

where  $\tau_{err}$  is the error tolerance,  $\omega$  is the lowest natural frequency of the structure, and  $\zeta$  is the corresponding effective damping ratio. Let  $T_0$  be the length of the original signal, then  $T = T_0 + T_p$ .

Non-trivial initial conditions can be modelled by modifying the equivalent load (Lee et al., 2005; Liu et al., 2015; Mansur et al., 2000). For initial displacement  $u_0$ , it is equivalent to add a constant load  $f_{u0} = Ku_0$ , which exist for the time interval  $T_0 \leq t < T$ . For initial velocity  $v_0$ , it is equivalent to add an impulse load  $f_{v0} = Mv_0 / \Delta t$ , at time  $t = 0$  for a duration of time  $\Delta t$ .

#### 4 Interval dynamic solver

Consider an elastic structure characterised by uncertain parameters of the load, material, and geometry, which are modelled by intervals. The structural system is governed by equations (1) and (2). For simplicity, the Rayleigh damping is adopted. The damping matrix

$$\mathbf{D} = \alpha_d \mathbf{M} + \beta_d \mathbf{K}, \quad (40)$$

where  $\alpha_d$  and  $\beta_d$  are the Rayleigh damping coefficients. To reduce overestimation due to interval dependency, the interval matrix decomposition outlined in section 2 is adopted. Then DFT is used to transform the governing equation into a fixed-point form, which is further solved by a new variant of iterative enclosure method. Details on the current method are presented in the following subsections.

##### 4.1 Interval matrix decomposition

Following the matrix decomposition strategy introduced in section 2, the overestimation reduction due to interval dependency is achieved by avoiding multiple occurrences of the same interval variable in the IFEM formulation. The stiffness matrix  $\mathbf{K}$ , and the mass matrix  $\mathbf{M}$  are decomposed into

$$\mathbf{K} = A \text{diag}(\Lambda \boldsymbol{\alpha}) A^T, \quad \mathbf{M} = A_m \text{diag}(\Lambda_m \boldsymbol{\alpha}_m) A_m^T, \quad (41)$$

where  $A$ ,  $A_m$ ,  $\Lambda$ , and  $\Lambda_m$  are deterministic matrices,  $\boldsymbol{\alpha}$  is the interval stiffness parameter vector that accounts for uncertainties in the stiffness matrix  $\mathbf{K}$ , and  $\boldsymbol{\alpha}_m$  is the interval mass parameter vector that accounts for uncertainties in the mass matrix  $\mathbf{M}$ . Details about the implementation of this decomposition are introduced in the authors' previous work (Xiao, 2015).

By combining the nodal equivalent load vector  $\mathbf{f}_k$  at different time steps  $t_k$ , the interval load matrix  $\mathbf{f}$  is obtained, whose  $k$ -th column is  $\mathbf{f}_k$ . When the structure is subject to external loading and the  $M$ - $\delta$  method is adopted (Muhanna and Mullen 2001),  $\mathbf{f}$  is decomposed into

$$\mathbf{f} = F\delta_t, \quad (42)$$

where  $F$  is a deterministic matrix, and  $\delta_t$  is the time-varying load uncertainty matrix. Usually it is necessary to distinguish the uncertainty in the magnitude of the load and the uncertainty in the time-history of the load. Thus  $\delta_t$  is further decomposed into an interval column vector  $\delta$  and an interval row vector  $\mathbf{d}_t$ , viz.  $\delta_t = \delta \mathbf{d}_t$ , where  $\delta$  models the uncertainties in the load magnitude and  $\mathbf{d}_t$  models the uncertainties in the load time-history. Finally, the nodal equivalent load  $\mathbf{f}$  is decomposed into

$$\mathbf{f} = (F\delta)\mathbf{d}_t. \quad (43)$$

Similarly, when the structure is subject to ground motion,  $\mathbf{f}$  is decomposed into

$$\mathbf{f} = -\mathbf{M}\mathbf{a} = -\mathbf{M}q\delta_t, \quad (44)$$

where  $\delta_t$  denotes the time-varying ground acceleration,  $\mathbf{a}$  represents the resulting nodal acceleration of the structure, and  $q$  relates  $\delta_t$  to  $\mathbf{a}$ , viz.  $\mathbf{a} = q\delta_t$ . By using the same decomposition for  $\delta_t$ , and noting equation (41),

$$\mathbf{f} = -A_m \text{diag}(\Lambda_m \alpha_m) A_m^T q \delta \mathbf{d}_t = A_m (\Lambda_m \alpha_m \circ B_f \delta) \mathbf{d}_t, \quad (45)$$

where  $B_f = -A_m^T q$ , and  $a \circ b$  is the element-by-element Hadamard product of two vectors  $a$  and  $b$ .

When the initial conditions are non-trivial and modelled by intervals, as shown in equation (2), the corresponding nodal equivalent load  $\mathbf{f}$  is given by

$$\mathbf{f} = \mathbf{K}\mathbf{u}_0 d_{u_0} + \mathbf{M}\mathbf{v}_0 d_{v_0}, \quad (46)$$

where  $d_{u_0}$  and  $d_{v_0}$  are two deterministic row vectors. The value of  $d_{u_0}$  is zero for the time interval  $0 \leq t_k < T_0$  and unity for the time interval  $T_0 \leq t_k < T$ , where  $T_0$  and  $T$  are the length of the original and padded signal and  $d_{u_0}$  represents an impulse load which has the value  $1/\Delta t$  at  $t_k = 0$  and zero everywhere else. Noting the decomposition in equation (41),

$$\mathbf{f} = A(\Lambda \alpha \circ A^T \mathbf{u}_0) d_{u_0} + A_m (\Lambda_m \alpha_m \circ A_m^T \mathbf{v}_0) d_{v_0}, \quad (47)$$

which has a similar matrix form as equation (45). Thus the non-trivial initial conditions are treated in the same manner as ground accelerations.

#### 4.2 Interval governing equations

To solve the interval differential equation (1), following the DFT approach outlined in Section 3, the equation is transformed into the frequency domain, viz.

$$(-\omega_j^2 \mathbf{M} + i\omega_j \mathbf{D} + \mathbf{K}) \mathcal{F}_t(\mathbf{u})_j = \mathcal{F}_t(\mathbf{f})_j, \quad (48)$$

where  $\mathcal{F}_t(\mathbf{u})_j$  and  $\mathcal{F}_t(\mathbf{f})_j$  are the Fourier transform of the nodal displacement  $\mathbf{u}_k$  and equivalent load  $\mathbf{f}_k$ , respectively.

To include compatibility requirements and essential boundary conditions in the governing equation, and to ensure that the final solution has zero imaginary part, using equations (33) and (34), equation (48) is brought into the following equivalent form,

$$\begin{Bmatrix} \mathbf{K}_{DS,j} & C^T \\ C & 0 \end{Bmatrix} \begin{Bmatrix} \mathcal{F}_t(\mathbf{u})_j \\ \mathcal{F}_t(\boldsymbol{\lambda})_j \end{Bmatrix} = \begin{Bmatrix} \mathcal{F}_t(\mathbf{f})_j \\ 0 \end{Bmatrix}, \quad (49)$$

where  $\mathbf{K}_{DS,j}$  is the dynamic stiffness matrix corresponding to the  $j$ -th frequency  $\omega_j$ , namely

$$\mathbf{K}_{DS,j} = \begin{cases} -\omega_j^2 \mathbf{M} + i\omega_j \mathbf{D} + \mathbf{K}, & 0 \leq j < N/2; \\ \text{conjugate of } \mathbf{K}_{DS,N-j}, & N/2 \leq j < N, \end{cases} \quad (50)$$

and  $C$  is the constraint matrix that imposes compatibility requirements and essential boundary conditions, and  $\boldsymbol{\lambda}_k$  is the Lagrangian multiplier representing the internal forces and support reactions at  $t_k$ . By adopting the Rayleigh damping and the decomposition of  $\mathbf{K}$  and  $\mathbf{M}$  in equation (41),  $\mathbf{K}_{DS,j}$  can be decomposed into

$$\mathbf{K}_{DS,j} = A_{DS,j} \text{diag}(\Lambda_{DS} \boldsymbol{\alpha}_{DS}) B_{DS}, \quad (51)$$

where  $A_{DS,j}$  is a deterministic matrix depending on the frequency  $\omega_j$ ,

$$A_{DS,j} = \begin{cases} \{(1 + i\beta_d \omega_k) A (-\omega_k^2 + i\alpha_d \omega_k) A_m\}, & 0 \leq j < N/2; \\ \text{conjugate of } A_{DS,N-j}, & N/2 \leq j < N. \end{cases} \quad (52)$$

and  $\Lambda_{DS}$ ,  $B_{DS}$ , and  $\boldsymbol{\alpha}_{DS}$  are time-invariant variables,

$$\Lambda_{DS} = \begin{Bmatrix} \Lambda & 0 \\ 0 & \Lambda_m \end{Bmatrix}, \quad B_{DS} = \begin{Bmatrix} A^T \\ A_m^T \end{Bmatrix}, \quad \boldsymbol{\alpha}_{DS} = \begin{Bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}_m \end{Bmatrix}. \quad (53)$$

Define

$$\Delta \boldsymbol{\alpha}_{DS} = \boldsymbol{\alpha}_{DS} - \boldsymbol{\alpha}_{DS0}, \quad (54)$$

where  $\Delta \boldsymbol{\alpha}_{DS}$  is the difference between  $\boldsymbol{\alpha}_{DS}$  and the reference deterministic vector  $\boldsymbol{\alpha}_{DS0} = \text{mid}(\boldsymbol{\alpha}_{DS})$ , and the deterministic dynamic stiffness matrix as

$$K_{DS,j0} = A_{DS,j} \text{diag}(\Lambda_{DS} \boldsymbol{\alpha}_{DS0}) B_{DS}. \quad (55)$$

Using the identity

$$\begin{aligned} A_{DS,j} \text{diag}(\Lambda_{DS} \Delta \alpha_{DS}) B_{DS} \mathcal{F}_t(\mathbf{u})_j &= A_{DS,j} (\Lambda_{DS} \Delta \alpha_{DS} \circ B_{DS} \mathcal{F}_t(\mathbf{u})_j) \\ &= A_{DS,j} \text{diag}(B_{DS} \mathcal{F}_t(\mathbf{u})_j) \Lambda_{DS} \Delta \alpha_{DS}, \end{aligned} \quad (56)$$

and given the external load as

$$\mathcal{F}_t(\mathbf{f})_j = \mathcal{F}_t(F \delta \mathbf{d}_t)_j = F \delta \mathcal{F}_t(\mathbf{d}_t)_j, \quad (57)$$

then equation (49) takes the following equivalent decomposed form

$$\begin{Bmatrix} K_{DS,j0} & C^T \\ C & 0 \end{Bmatrix} \begin{Bmatrix} \mathcal{F}_t(\mathbf{u})_j \\ \mathcal{F}_t(\lambda)_j \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \delta \mathcal{F}_t(\mathbf{d}_t)_j - \begin{Bmatrix} A_{DS,j} \\ 0 \end{Bmatrix} \text{diag}(B_{DS} \mathcal{F}_t(\mathbf{u})_j) \Lambda_{DS} \Delta \alpha_{DS}, \quad (58)$$

However, when the structure is subject to ground motion, according to equation (45), the load vector takes the form

$$\begin{aligned} \mathcal{F}_t(\mathbf{f})_j &= A_f (\Lambda_m \alpha_m \circ B_f \delta) \mathcal{F}_t(\mathbf{d}_t)_j \\ &= (A_f (\Lambda_m \alpha_{m0} \circ B_f \delta) + A_f (\Lambda_m \Delta \alpha_m \circ B_f \delta)) \mathcal{F}_t(\mathbf{d}_t)_j \\ &= F_0 \delta \mathcal{F}_t(\mathbf{d}_t)_j + A_f \text{diag}(B_f \delta \mathcal{F}_t(\mathbf{d}_t)_j) \Lambda_m \Delta \alpha_m, \end{aligned} \quad (59)$$

where  $\Delta \alpha_m$  is the difference between  $\alpha_m$  and the reference vector  $\alpha_{m0}$ , viz.  $\Delta \alpha_m = \alpha_m - \alpha_{m0}$ , and  $F_0 = A_f \text{diag}(\Lambda_m \alpha_{m0}) B_f$ . Then the generalised equivalent load in equation (49) is decomposed into

$$\begin{Bmatrix} \mathcal{F}_t(\mathbf{f})_j \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \delta \mathcal{F}_t(\mathbf{d}_t)_j + \begin{Bmatrix} A_f \\ 0 \end{Bmatrix} \text{diag}(B_f \delta \mathcal{F}_t(\mathbf{d}_t)_j) \Lambda_m \Delta \alpha_m, \quad (60)$$

and the equivalent decomposed form of equation (49) becomes

$$\begin{aligned} \begin{Bmatrix} K_{DS,j0} & C^T \\ C & 0 \end{Bmatrix} \begin{Bmatrix} \mathcal{F}_t(\mathbf{u})_j \\ \mathcal{F}_t(\lambda)_j \end{Bmatrix} &= \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \delta \mathcal{F}_t(\mathbf{d}_t)_j \\ &\quad - \begin{Bmatrix} A_{DS} & A_f \\ 0 & 0 \end{Bmatrix} \text{diag} \left( \begin{Bmatrix} B_{DS} \mathcal{F}_t(\mathbf{u})_j \\ -B_f \delta \mathcal{F}_t(\mathbf{d}_t)_j \end{Bmatrix} \right) \begin{Bmatrix} \Lambda_{DS} \\ 0 & \Lambda_m \end{Bmatrix} \Delta \alpha_{DS}. \end{aligned} \quad (61)$$

Due to the similarities between the decomposition of the equivalent load in equations (45) and (47), the above formulation can be extended to cases when the initial conditions are non-trivial.

### 4.3 Iterative enclosure method

To solve the interval linear system of equations (58) and (61), they are recast into the following form

$$K_{g,j} \mathcal{F}_t(\mathbf{u}_g)_j = F_g \delta \mathcal{F}_t(\mathbf{d}_t)_j - A_{g,j} \text{diag}(\mathcal{F}_t(\mathbf{v}_g)_j) \Lambda_g \Delta \alpha_{DS}, \quad (62)$$

where  $K_{g,j}$ ,  $F_g$ ,  $A_{g,j}$ ,  $\Lambda_g$  are given deterministic matrices,  $\mathbf{u}_g$  is the unknown interval vector,  $\delta$ ,  $\mathbf{d}_t$ , and  $\Delta \alpha_{DS}$  are given interval vectors, and  $\mathbf{v}_g$  linearly depend on  $\mathbf{u}_g$ , viz.

$\mathbf{v}_g = \mathbf{v}_0 + B_g \mathbf{u}_g$ . Here subscripts  $j$  denotes variables associated with the  $j$ -th frequency  $\omega_j$ . Note that matrices  $K_{g,j}$  and  $A_{g,j}$  are functions of the frequency  $\omega_j$ . In the most general case,  $\mathbf{u}_g$  includes  $\mathbf{u}$  and  $\lambda$ , and the auxiliary variable  $\mathbf{v}_g$  includes  $B_{DS} \mathbf{u}, -B_f \delta d_t$ , and  $A_s^T \mathbf{u}$ .

Now introduce  $G_j = K_{g,j}^{-1}$ . Multiplying both sides of equation (62) by  $G_j$  yields

$$\mathcal{F}_t(\mathbf{u}_g)_j = (G_j F_g) \delta \mathcal{F}_t(\mathbf{d}_t)_j - (G_j A_{g,j}) \text{diag}(\mathcal{F}_t(\mathbf{v}_g)_j) \Lambda_g \Delta \alpha_{eff}, \quad (63)$$

then

$$\mathbf{u}_{g,k} = (\mathcal{F}_t^{-1}(G_j F_g) * \mathbf{d}_t)_k \delta - (\mathcal{F}_t^{-1}(G_j A_{g,j}) * \text{diag}(\mathbf{v}_g))_k \Lambda_g \Delta \alpha_{DS}, \quad (64)$$

where  $\mathcal{F}_t^{-1}$  is the IDFT and  $(a * b)_k$  denotes the convolution between two discrete signals  $a_k$  and  $b_k$ . equation (64) can be recast into the following summation form,

$$\mathbf{u}_{g,k} = \left( \sum_{l=0}^{N-1} \mathcal{F}_t^{-1}(G_j F_g)_{k-l} \mathbf{d}_{t,l} \right) \delta - \left( \sum_{l=0}^{N-1} \mathcal{F}_t^{-1}(G_j A_{g,j})_{k-l} \text{diag}(\mathbf{v}_{g,l}) \right) \Lambda_g \Delta \alpha_{DS}. \quad (65)$$

Then a fixed-point form for  $\mathbf{v}_{g,k}$  is obtained as

$$\mathbf{v}_{g,k} = \mathbf{v}_{0,k} + B_g (\mathcal{F}_t^{-1}(G_j F_g) * \mathbf{d}_t)_k \delta - B_g (\mathcal{F}_t^{-1}(G_j A_{g,j}) * \text{diag}(\mathbf{v}_g))_k \Lambda_g \Delta \alpha_{DS}. \quad (66)$$

A guaranteed outer enclosure for  $\mathbf{v}_{g,k}$  is obtained by iteratively using equation (66), starting from the trivial initial guess  $\mathbf{v}_{g,k}^1 = \mathbf{v}_{0,k} + (\mathcal{F}_t^{-1}(G_j F_g) * \mathbf{d}_t)_k \delta$ . The iteration stops when no improvement in  $\mathbf{v}_{g,k}^j$  is observed for two consecutive iterations, and the converged solution is denoted as  $\mathbf{v}_{g,k}^n$ . Then the outer solution  $\mathbf{u}_{g,k}^{out}$  is obtained by substituting  $\mathbf{v}_{g,k}$  in equation (64) with the converged solution  $\mathbf{v}_{g,k}^n$ .

The convolution between a deterministic signal and an interval signal is computed multiple times, as shown in equations (64) and (66). To increase the computational efficiency and reduce overestimation in the final solution, the FFT-based fast interval convolution algorithm, proposed by Liu and Kreinovich (2010), is adopted. During the iteration in equation (66), only the radius of  $\mathbf{v}_g$  is updated, and that will require about 10 iterations for different types and sizes of problems. All other vectors and matrices do not change after the first iteration.

## 5 Numerical examples

The current IFEM algorithm is implemented using the interval MATLAB toolbox INTLAB (Rump, 1990). Interval enclosures of the structural responses of the following sample problems are calculated: (i) a four-storey rigid frame and (ii) a simply supported truss. The performance of the current method is compared against other available methods in the literature: (i) the endpoint combination method (EC) and (ii) the Monte Carlo (MC) simulation. The result shows that the current method is applicable to the transient analysis of structural dynamic problems with uncertain parameters. Guaranteed

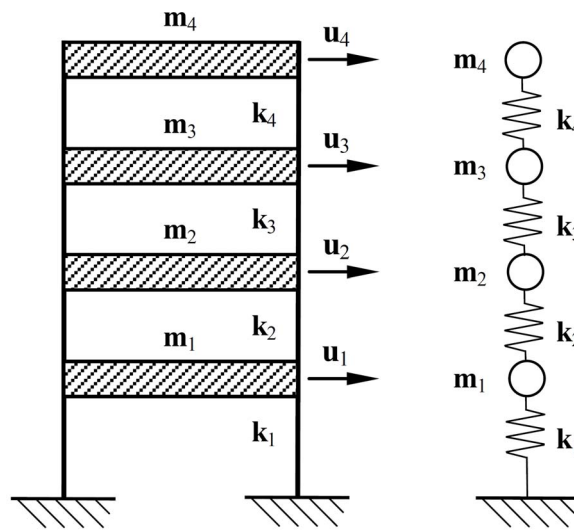
interval enclosures of the exact structural responses in the time domain are obtained with small overestimations. In addition, the computational time is negligible when compared with other competing methods.

If deterministic solution is considered as a reference, the run time of interval calculations currently takes 10–15 times the deterministic ones (if interval operations are implemented in the machine hardware, CPU, there will not be any penalty in comparison with floating point calculations), where EC takes  $2^n$  times with  $n$  interval parameters, and MC depends on the used number of simulations, 10,000 in the frame example and 100,000 in the truss example.

### 5.1 Four-storey rigid frame

The first example is a four-storey frame shown in Figure 3. The floors of the frame are assumed to be rigid enough to model the structure as an equivalent spring-mass system (shown in the right-hand side of Figure 3). The mass  $\mathbf{m}_j$  and the inter-storey shear stiffness  $\mathbf{k}_j$  of each floor ( $j = 1, \dots, 4$ ) are modelled by independent interval variables, and given in Table 1.

**Figure 3** A four-storey rigid frame and the equivalent spring-mass system



**Table 1** Interval mass and stiffness for the five-storey rigid frame of Figure 3, including 1% uncertainties in mass, and 5% uncertainties in stiffness

| Floor | Mass (kg)      |                    |                    | Stiffness (kN/m) |                    |                    |
|-------|----------------|--------------------|--------------------|------------------|--------------------|--------------------|
|       | $\mathbf{m}_j$ | mid $\mathbf{m}_j$ | rad $\mathbf{m}_j$ | $\mathbf{k}_j$   | mid $\mathbf{k}_j$ | rad $\mathbf{k}_j$ |
| 1     | [5.416, 5.470] | 5.443              | 0.027              | [1.180, 1.240]   | 1.210              | 0.030              |
| 2     | [5.416, 5.470] | 5.443              | 0.027              | [1.677, 1.763]   | 1.720              | 0.043              |
| 3     | [5.416, 5.470] | 5.443              | 0.027              | [1.862, 1.958]   | 1.910              | 0.048              |
| 4     | [5.416, 5.470] | 5.443              | 0.027              | [1.775, 1.865]   | 1.820              | 0.045              |

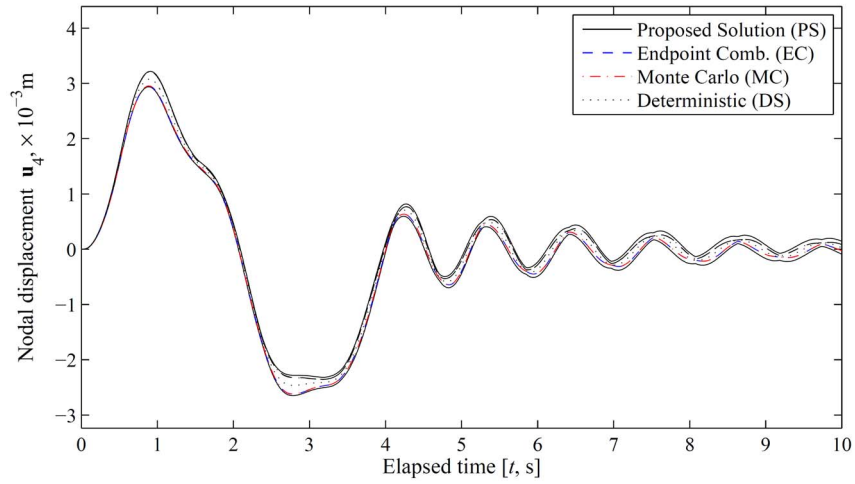
Consider the structural response of the frame under a concentrated impact force acting on the top floor. The force has a duration of 4 s, and its variation during that time is deterministic, viz.

$$\mathbf{f}(t) = \begin{cases} \mathbf{P} \sin(\pi t / 2), & 0 \leq t \leq 4 \text{ s}; \\ 0, & t > 4 \text{ s}, \end{cases} \quad (67)$$

where  $\mathbf{P} = [0.99, 1.01]$  kN (2% uncertainty in the magnitude of the load). The damping matrix  $\mathbf{D} = 0.5\mathbf{M} + 5 \times 10^{-3}\mathbf{K}$ . The sampling rate is 100 Hz, so the sampling interval  $\Delta t = 0.01$  s.

Figure 4 compares the lower and upper bounds of  $\mathbf{u}_4$  for the first 10 s, obtained from the current Proposed Solution (PS, black solid lines), Monte Carlo predictions (MC, red dashed-dotted lines) from an ensemble of 10,000 simulations, the reference solution obtained from endpoint combination (EC, blue dash lines), and the deterministic solution (DS, black dotted line). Note that PS always contains the reference solution EC, and MC is always contained by EC. In addition, the overestimation level of the current method slightly increases as the time increases. The MC solution is obtained using the DFT approach, which is indistinguishable from the solution obtained from a recursive Newmark- $\beta$  method.

**Figure 4** Lower and upper bounds of the nodal displacement  $\mathbf{u}_4$  for the four-storey frame of Figure 3 under a sinusoidal force. MC from an ensemble of 10,000 simulations. Material uncertainty is 1% for mass, and 5% for stiffness. Load uncertainty is 2% for the magnitude



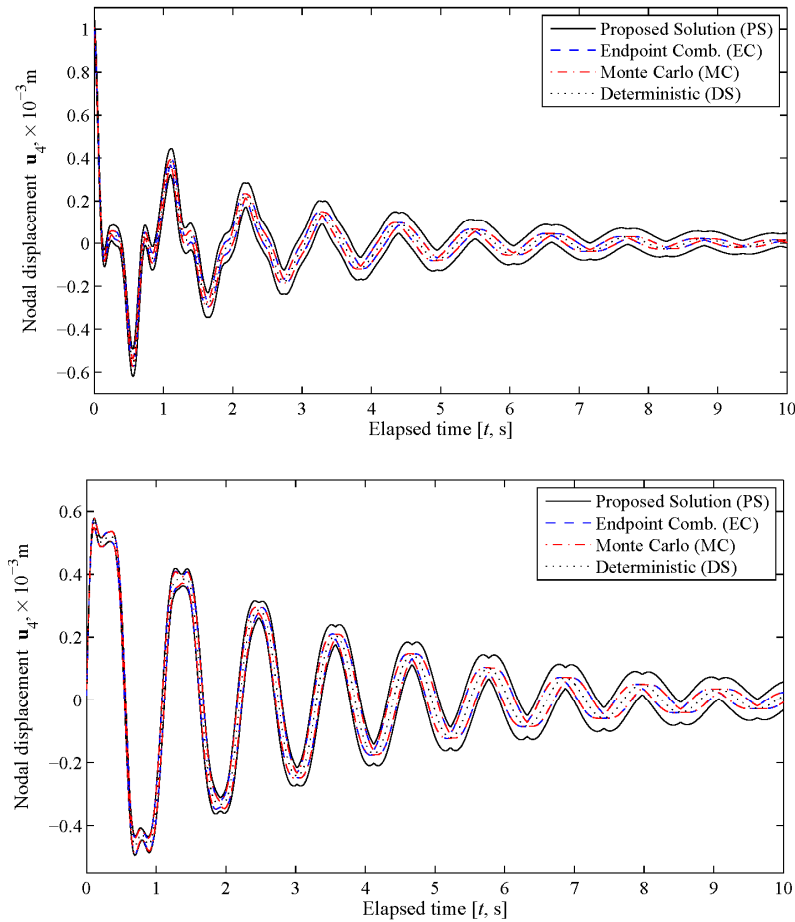
Then the concentrated force  $\mathbf{f}(t)$  is removed, and the structure is subject to non-trivial initial conditions. Figure 5 shows the nodal displacement  $\mathbf{u}_4$  at the top floor for the first

10 s with non-trivial initial nodal displacement  $\mathbf{u}_0$  (top) and nodal velocity  $\mathbf{v}_0$  (bottom), respectively. Here 2% uncertainty is considered for  $\mathbf{u}_0$  and  $\mathbf{v}_0$ , viz.

$$\begin{aligned}\mathbf{u}_0 &= \{0 \ 0 \ 0 \ 0 \ [0.99, 1.01]\}^T \times 10^{-3} \text{ m}; \\ \mathbf{v}_0 &= \{0 \ 0 \ 0 \ 0 \ [0.99, 1.01]\}^T \times 10^{-2} \text{ m/s}.\end{aligned}\quad (68)$$

Figure 5 shows that the high frequency components dissipate quickly. After about 3 s, the response of the structure is dominated by the lowest frequency vibration. Observe that the performance of the current method is the same as in the previous case. The obtained interval solution guarantees to enclose the reference solution (endpoint combination, EC), and the overestimation level increases slightly as the time increases. Thus non-trivial initial conditions are handled successfully.

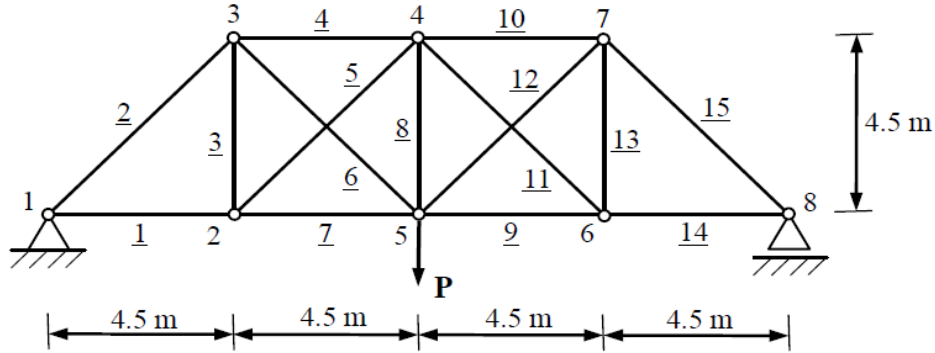
**Figure 5** Lower and upper bounds of the nodal displacement  $\mathbf{u}_4$  for the four-storey frame of Figure 3 under non-trivial initial conditions: (top) non-trivial initial displacement  $\mathbf{u}_0$ , (bottom) non-trivial initial velocity  $\mathbf{v}_0$ . MC from an ensemble of 10,000 simulations. Material uncertainty is 1% for mass, and 5% for stiffness. Uncertainty in the initial condition is 2%



### 5.2 Simply supported truss

The second example is a simply supported symmetric truss composed of 15 bars, as shown in Figure 6. The joints are labelled from 1 to 8, and the bars are labelled from 1 to 15. Time-varying concentrated load  $\mathbf{P}$  acts at joint 5. Bars 1 to 3, 13 to 15 have the same cross section area  $A = 1.0 \times 10^{-3} \text{ m}^2$ , and all other bars, viz. bars 4 to 12, have smaller cross section area  $A = 6.0 \times 10^{-4} \text{ m}^2$ . All the bars are made of steel. They have the interval mass density  $\rho$  with midpoint value  $\rho = 7.8 \times 10^3 \text{ kg/m}^3$ , and the interval Young's modulus  $\mathbf{E}$  with midpoint value  $E = 200 \text{ GPa}$ .

**Figure 6** A simply supported symmetric truss subject to concentrated force



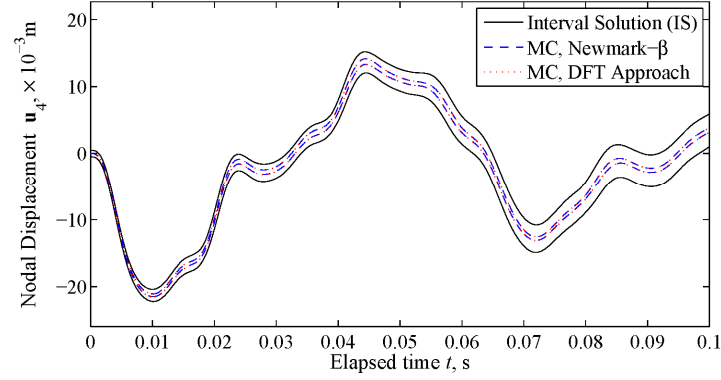
Fifteen bar elements are used to model the truss in Figure 6. Element mass density  $\rho$  and Young's modulus  $\mathbf{E}$  are assumed independent, and they are modelled by 30 interval variables. The midpoint of the load  $\mathbf{P}$  is a sinusoid with a frequency of 50 Hz and an amplitude of 200 kN, viz.

$$P = 200 \sin(100\pi t) \text{ kN.} \quad (69)$$

The damping matrix  $\mathbf{D} = 20\mathbf{M} + 3 \times 10^{-5} \mathbf{K}$ . The sampling rate is 10 kHz, so  $\Delta t = 1 \times 10^{-4} \text{ s}$ .

Then vertical displacement  $\mathbf{v}_5$  at joint 5 is selected for comparison among the various methods mentioned previously. Consider 10% uncertainty for the load magnitude and 1% for the load time-history, as well as Young's modulus and mass density of each bar. Figure 7 plots the lower and upper bounds of  $\mathbf{v}_5$  for the first 0.1 s obtained from the proposed current solution (PS, solid lines) and the Monte Carlo predictions (MC, dashed lines from the Newmark- $\beta$  approach, and dash-dotted lines from the DFT approach) from an ensemble of 100,000 simulations. Observe that the current method obtains guaranteed enclosures of the MC prediction.

**Figure 7** Lower and upper bounds of the nodal displacement  $\mathbf{v}_5$  at joint 5 for the truss of Figure 6 under external loads. MC predictions (Newmark- $\beta$  and DFT) from an ensemble of 100,000 simulations. Parameter uncertainties are 10% for load magnitude, 1% for load history, Young's modulus, and mass density



**Figure 8** Lower and upper bounds of the nodal displacement  $\mathbf{v}_5$  at joint 5 of the truss of Figure 6 with: (top) 2% uncertainty only in load history; (bottom) 20% uncertainty in load magnitude, 1% uncertainties in Young's modulus and mass density. MC predictions (Newmark- $\beta$  and DFT from an ensemble of 100,000 simulations)

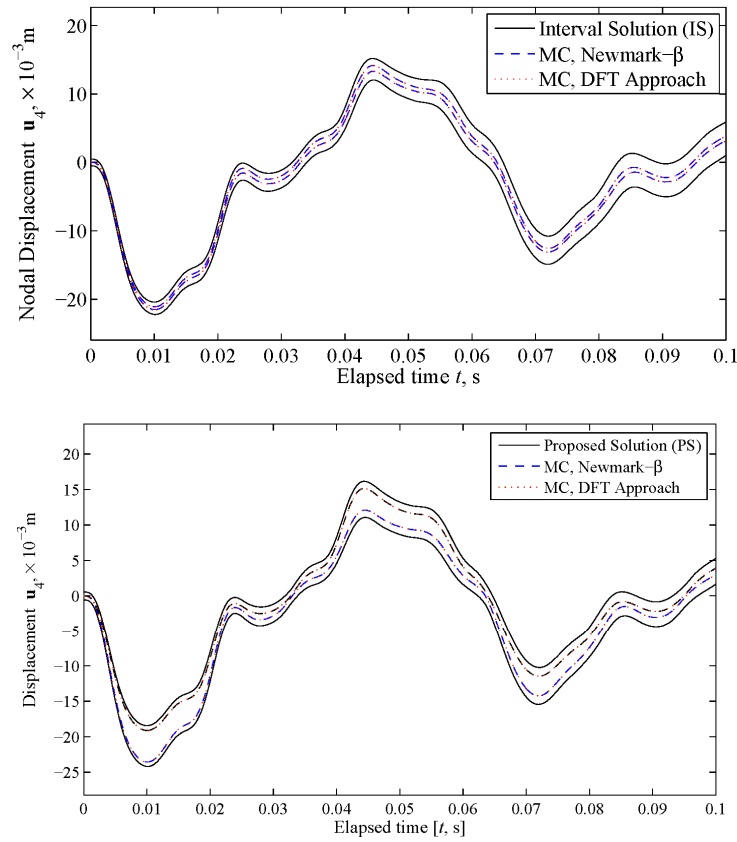


Figure 7 shows that the uncertainties in the structural responses increase significantly over time. This behaviour is due to the fact that the load history uncertainties are modelled by independently varied intervals at different time steps. In the current example, this means  $0.1 \text{ s} \times 10 \text{ kHz} = 1000$  independent interval variables. As a result, the overall uncertainty level is much higher than 1%. This also explains the growing differences between PS and MC predictions over time. Figure 8 considers (top) 2% uncertainties in load time-history, (bottom) 20% uncertainties in load magnitude, and 1% Young's modulus, and mass density. Observe that in the bottom subplot, the uncertainties now do not increase over time, and the difference between PS and MC is much smaller than the top subplot. So it is indeed the increased number of interval variables that caused the increased uncertainty and the difference between PS and MC.

## 6 Conclusion

We present a new spectral-based interval finite element formulation for the time-domain dynamic analysis of elastic structures with uncertain load, geometric, and material properties. Ground motion and non-trivial initial conditions are successfully handled, resulting in a method that is both efficient and widely applicable.

Uncertain parameters of the structure are modelled as intervals. The interval enclosures guarantee to enclose the exact solution set with small overestimation, even for large uncertainty levels. Numerical examples show that the proposed method gives guaranteed sharp bounds on the dynamic structural responses in comparison to other methods that give over-optimistic predictions on lower and upper bounds.

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