Power Engineering Letters

Reachable Power Flow

Yifan Zhou and Peng Zhang, *Senior Member, IEEE*

***Abstract*—This letter introduces reachable power flow (*ReachFlow*), a formal method providing a provably over-approximated enclosure of the complete set of the power flow solutions that "can be reached" under various uncertainties. The novelty of *ReachFlow* lies in: (1) an ordinary differential equation (ODE) formulation which maps the iterative power flow solving process into a virtual dynamic; (2) a reachability analysis of the virtual ODE model which enclose all possible (infinite) power flow solutions under uncertainties in one calculation. Case studies verify the efficacy of *ReachFlow* as a formal method and, in particular, its capability of handling islanded droop-based microgrids under uncertainties.**

***Index Terms*—Reachable power flow, formal method, reachability, uncertainty, microgrid.**

I. Introduction

POWER flow calculation is the keystone of power system planning and operations studies. Because distributed energy resources (DERs) are increasingly integrated into utility grids, quantifying the impact of uncertainties on power flow is of critical significance to help mitigate the operational risks and ensure grid reliability, security, resilience and beyond.

So far, there is a lack of formal methods for calculating power flow under uncertainties. Simulation-based methods such as Monte Carlo algorithms [1] are not formal as they cannot enumerate all possible scenarios tractably. Analytical probabilistic power flow solutions [2] approximate the probability distributions of power flow states, which may miss certain extreme scenarios due to the lack of information on the tails of events and techniques to calculate extreme value distributions. Another type of analytical approaches, the set-based power flow [3], [4], are reported to suffer from non-formal set computations in the Newton's iterations of the power-mismatch formulation. A robust and tractable formal method for uncertain power flow is therefore in high demand.

This letter bridges the gap by devising Reachable power Flow (*ReachFlow*), a formal method to rigorously enclose all the power flow states that "can be reached" under uncertainties. *ReachFlow* provably overapproximates the complete set of the power flow solutions with various uncertainties. As a reachability- and ordinary differential equation (ODE) simulation- based method, *ReachFlow* adapts to both microgrids and macrogrids with conventional generators or DERs.

Manuscript received November 4, 2019; revised January 26, 2020; accepted February 8, 2020. Date of publication February 14, 2020; date of current version June 22, 2020. This work was supported by the National Science Foundation under Grant ECCS-1831811. Paper no. PESL-00263-2019. *(Corresponding author: Peng Zhang.)*

The authors are with the Department of Electrical and Computer Engineering, Stony Brook University, Stony Brook, NY 11794 USA (e-mail: yifan.zhou.1@stonybrook.edu; p.zhang@stonybrook.edu).

Color versions of one or more of the figures in this article are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TPWRS.2020.2974164

II. ODE-Based Power Flow Model

An ODE-based power flow is devised to tractably quantify the uncertainty propagation in the power flow solution.

A. Static Power Flow Model

Equation (1) presents the general form of static power flow:

$$\begin{cases} P_{sp} = h_p(V, \theta, f) \\ Q_{sp} = h_q(V, \theta, f) \\ 0 = h_f(V, \theta, f) \end{cases} \tag{1}$$

where V and θ respectively denote the vectors of voltage amplitude and angle; f denotes the power system frequency if frequency regulation control is to be modelled (such as islanded microgrids with droop/secondary control); P_{sp} and Q_{sp} denote the active and reactive power injection at specific buses; functions $h_p(\cdot)$, $h_q(\cdot)$, $h_f(\cdot)$ together formulate the power flow model, whose details depend on the specific power system structure, power flow control strategies, etc. Due to the page limit, the detailed power flow model is omitted. Please refer to [7] for an expanded generic power flow model.

As an abstraction, power flow with uncertainty inputs is modeled as nonlinear equations:

$$g(x, u) = 0 \tag{2}$$

where x denotes the state variables, e.g., voltage angles, voltage magnitudes, and frequency; u denotes the uncertainties from DERs and loads.

Slightly augmenting the Newton-Raphson (NR) iterative process gives the numerical solution to (2), as follows:

$$\begin{cases} x_{n+1} = x_n - (J_g(x_n, u_n))^{-1} g(x_n, u_n) \\ u_{n+1} = u_n \end{cases} \tag{3}$$

where x_n denotes the value of x at the n^{th} iteration; $J_g(x_n) = \partial g/\partial x|_{x=x_n}$ denotes the Jacobian matrix of $g(x)$ at point x_n.

The second subequation of (3) means that the uncertainty input u does not change during a single run of power flow.

B. ODE-Based Power Flow Model

The discrete dynamic in (3) can be viewed as an abstraction of a continuous dynamic as follows:

$$\begin{cases} \dot{x}(t) = -(J_g(x(t), u(t)))^{-1} g(x(t), u(t)) \\ \dot{u}(t) = 0 \end{cases} \tag{4}$$

where t refers to time (i.e., the number of iterations).

The virtual dynamic in (4) is a mathematical equivalent to the NR iterations rather than an actual dynamic process, which is called an ODE-based power flow (ODE-PF) model. Further, the functional form of ODE-PF can be written as:

$$\dot{z}(t) = f(z(t)) \tag{5}$$

where

$$z = [x; u]; \quad f(z) = [f_x(x, u); f_u(x, u)];$$
$$f_x = -(J_g(x, u))^{-1} g(x, u); \quad f_u = 0.$$

An astonishing feature of ODE-PF is that the uncertainty impact naturally propagates in the dynamic process and thus can be explicitly calculated through reachability analysis, making Monte Carlo methods [1] (running power flow repeatedly by trying different u) unnecessary.

III. Reachable Power Flow

A. ReachFlow *Formulation*

Finding a set of all possible power flows is equivalent to solving the reachable power flow (*ReachFlow*) set defined as:

$$\mathcal{R}_{PF} = \left\{ z = \int_0^{\infty} f(z(t)) d\tau \;\middle|\; x(0) \in \mathcal{X}^0, u \in \mathcal{U}^0 \right\} \tag{6}$$

where z and $f(z)$ are defined in (5); $\mathcal{X}^0$ and $\mathcal{U}^0$ are the set of the initial states and uncertainty inputs.

B. Reachability Analysis of ReachFlow

$\mathcal{R}_{PF}$ can be obtained by finding a reachable set [5] of t:

$$\mathcal{R}(t) = \left\{ z(t) = \int_0^{t} f(z(\tau)) d\tau \;\middle|\; z(0) \in \mathcal{Z}^0 \right\} \tag{7}$$

where the initial state $z(0)$ is bounded by set $\mathcal{Z}^0$.

Further, the reachable set during the time interval $[k\Delta t, (k+1)\Delta t]$ ($k \in \mathbb{N}$) can be overapproximated by:

$$\mathcal{R}([k\Delta t, (k+1)\Delta t]) \triangleq \mathcal{R}_k \subseteq \mathcal{R}_k^{lin} + \mathcal{R}_k^{err} \tag{8}$$

Here $\mathcal{R}_k^{err}$ is linearization error due to Lagrange remainder [5]; $\mathcal{R}_k^{lin}$ is the reachable set of the linear abstraction of (5) with the time-point reachable set computed as:

$$\mathcal{R}^{lin}((k+1)\Delta t) = e^{A\Delta t} \mathcal{R}(k\Delta t)$$
$$\subset \left(\sum_{i=0}^{\eta} \frac{(A\Delta t)^i}{(i)!} + \mathcal{E}(\Delta t) \right) \mathcal{R}(k\Delta t)$$

Algorithm 1: *ReachFlow* algorithm

Result: $\mathcal{R}_{PF}$
Initialization: $\mathcal{X}^0 = \{x_0\}$ by solving Eq. (2) without uncertainty, $\mathcal{Z}^0 = \mathcal{X}^0 \otimes \mathcal{U}^0$, $n = 0$, $\Delta t = 1$;
Calculate $\mathcal{R}_0$ with $\mathcal{R}(0) = \mathcal{Z}^0$;
while *1* **do**
    $n = n + 1$;
    Update $\mathcal{R}_n$ by Eq. (8) ;
    if *isequal*($\mathcal{R}_n$,$\mathcal{R}_{n-1}$) **then**
        $\mathcal{R}_{PF} = \mathcal{R}_n$;
        break ;
    end
end

where $A = \partial f / \partial z|_{z=z^*}$ is the Jacobian matrix; z^* is the linearization point selected as the power flow solution without uncertainty and hence $f(z^*) = 0$; η is the selected number of Taylor terms for $e^{A\Delta t}$; $\mathcal{E}(\Delta t) = [-\mathbf{1}, \mathbf{1}] \frac{(\|A\|_\infty \Delta t)^{\eta+1}}{(\eta+1)!} \frac{1}{1-\epsilon}$ is the over-approximated remainder of the neglected terms for $e^{A\Delta t}$ beyond η terms.

C. ReachFlow *Algorithm*

The reachability analysis leads to the Algorithm 1.

1) Initialization: $\mathcal{Z}^0$ in (7) is initialized as follows.

a) The state vector x is initialized with $\mathcal{X}^0 = \{x_0\}$ where x_0 is the solution of (2) without uncertainty.

b) The uncertainty vector u is given in a zonotope form:

$$\mathcal{U}^0 = \left\{ u \;\middle|\; u = u_c + \sum_{i=1}^{n_u} \beta_i g^{(i)}, -1 \leq \beta_i \leq 1 \right\} \tag{9}$$

where u_c is the expectation vector of each element in u, $g^{(i)}$ is the maximum deviation of u at specific directions. Zonotope formulates u as the unknown but bounded uncertainties.

c) The overall state variable $z = [x; u]$ is initialized by $\mathcal{Z}^0 = \mathcal{X}^0 \times \mathcal{U}^0$, where $\times$ denotes the Cartesian product operator.

2) Building Jacobian Matrix and Lagrange Remainder: The following are required to perform the reachability analysis in Section III-B.

a) The Jacobian matrix of $f(z)$ is established as:

$$A = \frac{\partial f(z)}{\partial z} = \begin{bmatrix} \partial f_x / \partial x & \partial f_x / \partial u \\ \partial f_u / \partial x & \partial f_u / \partial u \end{bmatrix} \tag{10}$$

where

$$\frac{\partial f_x}{\partial x} = -\frac{\partial}{\partial x}(J_g^{-1} g) = -I - J_g^{-1} \frac{\partial^2 g}{\partial x^2} f_x \tag{11}$$

$$\frac{\partial f_x}{\partial u} = -\frac{\partial}{\partial u}(J_g^{-1} g) = -J_g^{-1} \frac{\partial g}{\partial u} - J_g^{-1} \frac{\partial^2 g}{\partial x \partial u} f_x \tag{12}$$

$$\frac{\partial f_u}{\partial x} = 0, \quad \frac{\partial f_u}{\partial u} = 0 \tag{13}$$

The first order and second order derivative of $g(z)$ can be readily derived from (2).

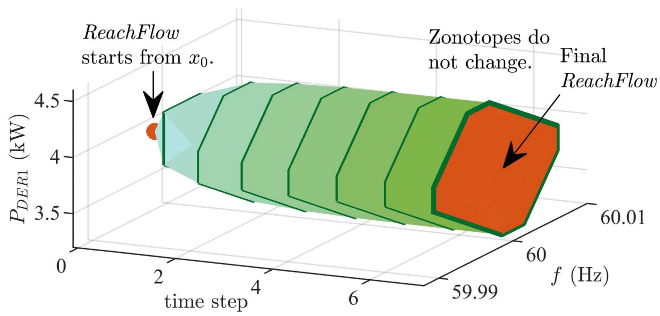


Fig. 1. Reachable set evolution during the "iteration dyanmic".

b) The Lagrange remainder is required to formulate the linearization error $\mathcal{R}_k^{err}$, which is approximated as follows:

$$\mathcal{L}(z-z^*) = [\mathcal{L}_x(z-z^*); \mathcal{L}_u(z-z^*)]$$

$$\approx \left[-\frac{1}{2}J_g^{-1}((z-z^*)^T\frac{\partial^2 g}{\partial^2 z}(z-z^*)); 0\right]$$

For the virtual dynamic of power flow $f(z)$, the Lagrange remainder can be approximately calculated without compromising the precision of $\mathcal{R}_{PF}$. The reason is that during the NR-based power flow solving process of (3), as long as the iteration converges, any modification to the iterative equation (3) is acceptable and does not influence the final power flow solution.

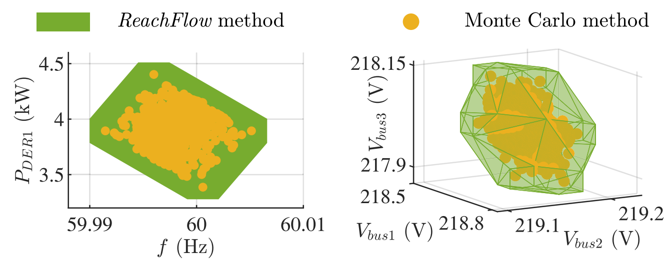
3) Overall Algorithm: *ReachFlow* can be initialized by the crisp power flow results obtained from a conventional power flow calculation. Then, the reachable set of the ODE-PF model is calculated step by step, which reflects the propagation of the uncertainty set $\mathcal{U}^0$ during power flow calculation. The algorithm converges when the reachable set becomes stable. The reachable set at the last time step will be the final *ReachFlow* which is the rigorous enclosure of all possible power flow solutions under $\mathcal{U}^0$. A salient feature of the algorithm is that the reachable set is obtained through an analytical solution, meaning *ReachFlow* is inherently robust and convergent.

IV. Case Study

ReachFlow is verified on a typical microgrid detailed in [6]. The *ReachFlow* algorithm is developed in MATLAB and runs on a 2.50 GHz PC.

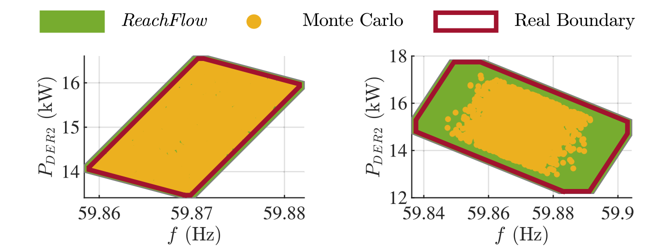
A. Methodology Validity

Figure 1 illustrates the calculation process of *ReachFlow* in a 3-bus microgrid with 3 DERs [6]. f is the system frequency and P_{DER1} is the active power generation of DER1. The uncertainty level of the active power injection from each DER is set as 10%. The whole calculation starts from the initial power flow solution x_0 without uncertainty, as illustrated by the red dot at the 0^{th} iteration. With the iteration going on, the reachable set gradually expands, which reflects the propagation of the uncertainty. At the 7^{th} iteration, the reachable set stabilizes, which leads to the algorithm termination. The algorithm outputs the zonotope at the final step, which gives the exact *ReachFlow*.



(a) *ReachFlow* projected to 2D space (b) *ReachFlow* projected to 3D space

Fig. 2. Comparison between Monte Carlo results and the *ReachFlow* result.



(a) n_u = 2 (b) n_u = 5

Fig. 3. Conservatism comparison for high-dimensional uncertainty input between *ReachFlow* and Monte Carlo results.

Figure 2 compares the *ReachFlow* against the Monte Carlo results. The high-dimensional zonotope obtained by the *ReachFlow* calculation is projected to two-dimensional or three-dimensional spaces as an observational convenience. V_{bus1}, V_{bus2}, V_{bus3} are voltage magnitudes at 3 buses. The green area is the *ReachFlow* while the yellow dots are the power flow solutions obtained with 500 Monte Carlo runs. Because those yellow dots are always contained by the zonotope, this verifies that *ReachFlow* is formal and gives a rigorous enclosure of all possible power flow states under the specific uncertainty.

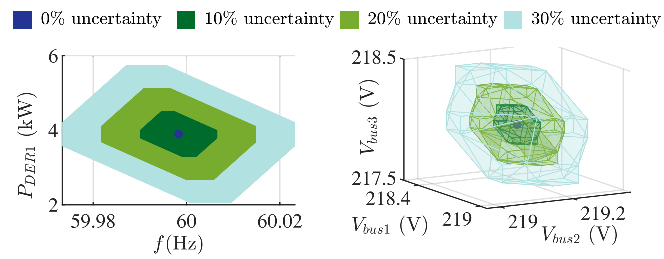
Moreover, as a formal and analytical method, *ReachFlow* exhibits superior performance with high-dimensional uncertainty inputs. Figure 3 compares the performances between *ReachFlow* and 3,000 Monte Carlo runs on a 33-bus microgrid with 5 DERs [7]. Let n_u denote the number of non-dispachable DERs with uncertain power injections into the microgrid. The red-line box represents the true boundary of power flow states under uncertainties by traversing the space of u. Figure 3(a) shows that with a small number of uncertainty inputs (i.e., $n_u = 2$), the results from *ReachFlow* and Monte Carlo are almost the same, which again verifies the efficacy of *ReachFlow*. Figure 3(b), however, shows that Monte Carlo simulations miss most of the extreme power flow scenarios, meaning it could lead to overly optimistic and hazardous decisions. In contrast, *ReachFlow* always provides a rigorous and conservative estimation of all power flow scenarios, as shown in Figure 3(b).

Further, Table I presents the computational performance of *ReachFlow* for microgrids of different scales. The algorithm performs well for complicated microgrids in terms of computational efficiency and convergence.

Figure 4 investigates the impact of the uncertainty level. *ReachFlow* expands with the increase of the uncertainty from DERs, which reflects that larger uncertainty leads to larger

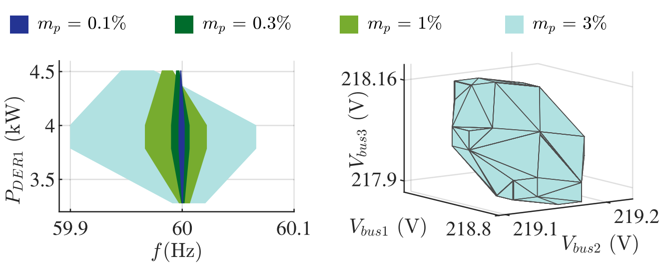
TABLE I
COMPUTATIONAL PERFORMANCE OF *REACHFLOW*

Testing system	3-bus microgrid with 3 DERs [9]	modified 33-bus microgrid with 5 DERs [10]
Dimension of x [1]	52	252
Dimension of u	3	5
Computing time (s)	1.5269	16.6727
Iterations	7	8



(a) *ReachFlow* projected to 2D space (b) *ReachFlow* projected to 3D space

Fig. 4. Impact of the uncertainty level on *ReachFlow*.



(a) *ReachFlow* projected to 2D space (b) *ReachFlow* projected to 3D space

Fig. 5. Impact of the active power gain of droop control on *ReachFlow*.

deviation of the power flow states. When there is no uncertainty, *ReachFlow* shrinks to a point, which is identical to the deterministic power flow result. This again verifies the correctness of the *ReachFlow* method. Simulation results in Figure 4 demonstrate the efficacy of *ReachFlow* as a powerful tool for power system operators to analytically assess the uncertainty impact on power flow states.

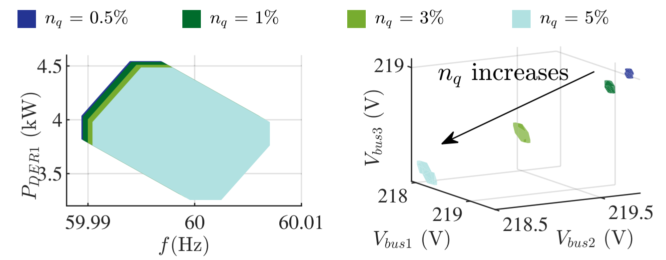
B. Impact Analysis of Droop Control Gains

In the test case [6], the following droop control is adopted:

$$\Delta\omega = m_p \Delta P, \quad \Delta V = n_q \Delta Q \tag{14}$$

where ω, V, P, Q are respectively the frequency, voltage, active power and reactive power of DER; m_p and n_q are respectively the active and reactive power droop gain, reflecting the adjustment of the DER power output in response to the deviation of the microgrid frequency and voltages.

Figure 5 shows the impact of the P-f droop coefficient m_p on the microgrid *ReachFlow*. With the increasing of m_p, the zonotope obtained by *ReachFlow* expands, indicating higher uncertainty of the power flow results. Specifically, with the deviations of the active power generation from DERs, larger



(a) *ReachFlow* projected to 2D space (b) *ReachFlow* projected to 3D space

Fig. 6. Impact of the reactive power gain of droop control on *ReachFlow*.

m_p leads to amplified impact on the system frequency while the bus voltage remains nearly unchanged.

Figure 6 demonstrates the impact of the Q-V droop coefficient n_q. With larger n_q, voltage magnitude at each bus decreases while the system frequency is slightly influenced, since the increase of n_p leads to a larger negative deviation of the bus voltage. Monte Carlo simulations also show that larger droop gain leads to larger power flow uncertainty, which is similar to Figure 5. Further, Figure 5 and Figure 6 indicate that *ReachFlow* can assist controller parameter tuning by providing a formal verification of the steady-state performance of the controllers.

V. CONCLUSION

This letter presents a reachable power flow (*ReachFlow*) method to rigorously enclose the complete set of all the possible power flow solutions under uncertainties. Test results show the efficacy and efficiency of *ReachFlow*. *ReachFlow* is a guaranteed overapproximation approach which can provably verify the power system power flow states under uncertainties.

As a robust algorithm free of divergence, *ReachFlow* is promising to significantly expedite the uncertainty analysis in power system operation and planning. A future direction is to develop a distributed *ReachFlow* which serves as a more scalable formal verification tool for the steady-state analysis of large networked microgrids or macrogrids.

REFERENCES

[1] R. Y. Rubinstein and D. P. Kroese, *Simulation and the Monte Carlo Method*. Hoboken, NJ, USA: Wiley, 2016.
[2] Z. Ren, W. Li, R. Billinton, and W. Yan, “Probabilistic power flow analysis based on the stochastic response surface method,” *IEEE Trans. Power Syst.*, vol. 31, no. 3, pp. 2307–2315, May 2016.
[3] X. Jiang, Y. C. Chen, and A. D. Domínguez-García, “A set-theoretic framework to assess the impact of variable generation on the power flow,” *IEEE Trans. Power Syst.*, vol. 28, no. 2, pp. 855–867, May 2013.
[4] Z. Wang and F. L. Alvarado, “Interval arithmetic in power flow analysis,” *IEEE Trans. Power Syst.*, vol. 7, no. 3, pp. 1341–1349, Aug. 1992.
[5] M. Althoff, O. Stursberg, and M. Buss, “Reachability analysis of nonlinear systems with uncertain parameters using conservative linearization,” in *Proc. 47th IEEE Conf. Decis. Control*, 2008, pp. 4042–4048.
[6] N. Pogaku, M. Prodanovic, and T. C. Green, “Modeling, analysis and testing of autonomous operation of an inverter-based microgrid,” *IEEE Trans. Power Electron.*, vol. 22, no. 2, pp. 613–625, Mar. 2007.
[7] G. Díaz, J. Gómez-Aleixandre, and J. Coto, “Direct backward/forward sweep algorithm for solving load power flows in AC droop-regulated microgrids,” *IEEE Trans. Smart Grid*, vol. 7, no. 5, pp. 2208–2217, Sep. 2016.