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PK — 12

Exploring the Mathematics of Gravity

Students develop covariational reasoning skills as they explore a simulation that modifies the different factors that determine the force of gravity.

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Starting in middle school, mathematics plays a crucial role in understanding scientific concepts (Roschelle et al. 2007), and a rigorous understanding of the mathematics embedded in different topics in science provides students with the platform to prepare them for future Science, Technology, Engineering and Mathematics (STEM) careers. To help students develop their STEM aptitude, we designed several instructional modules following "a holistic approach that links the disciplines so that the learning becomes connected, focused, meaningful, and relevant to learners" (Smith and Karr-Kidwell 2000, p. 22).

Each module is designed around an earth and environmental science phenomenon, and each uses the power of mathematics and technology as tools to help students think deeply about these phenomena. In this article, we present how our module on gravity (for access to all instructional tasks see https://acmes.online) was used in a sixth-grade mathematics classroom. We discuss how our integrated curriculum design, which focuses on engaging students in covariational reasoning, can be used by mathematics teachers to implement such integrated STEM lessons.

THE SCIENCE OF GRAVITY

To successfully implement integrated curricula, a mathematics teacher must develop a solid understanding of the science topic (Pang and Good 2000). Consequently, before implementing an integrated module, we suggest that mathematics teachers form a peer collaboration with science teachers and work together to identify the connections between the two subjects.

The Next Generation Science Standards for middle school (NGSS Lead States 2013) focus on the role of gravity as a force of attraction that exists between two objects with mass and holds everything together on Earth; they also consider gravity's influence on various phenomena, such as the water cycle and the orbits of planets in our galaxy and beyond. According to Newton's law of gravity, the force of gravity between two bodies is proportional to the product of the masses of the two bodies (m_1 and m_2), and it is inversely proportional to the square of the distance (r) between their centers of mass. Mathematically,

$$F = G\frac{m_1 m_2}{r^2}$$

where G is called the gravitational constant.

The middle school science curriculum introduces students to the formula above, in which the focus is typically on the input of different values of mass and distance to observe their effect on the gravity. Our focus is to engage students in an in-depth, inquiry-based learning of the mathematical relationships that underlie the concept of gravity.

THE MATHEMATICAL RELATIONSHIPS OF GRAVITY

When we decided to explore the mathematics of gravity, our attention was drawn to research on covariational reasoning, which involves coordinating two quantities as the values of those quantities change (Confrey and Smith 1995; Thompson and Carlson 2017). A person reasons covariationally when "she envisions two quantities' values varying and envisions them varying simultaneously" (Thompson and Carlson 2017, p. 425). For instance, when the gravity between two objects (the measure of gravity represented by F in the formula above) increases due to the increase of the mass of one of the two objects, then the two quantities, namely the mass of one object and gravity, are said to covary.

Covariational reasoning can be nonnumeric or numeric. For instance, students reason nonnumerically if they argue that the gravity between two objects increases when the mass of one or both objects increases. This type of reasoning does not involve any calculation. Instead, students focus on the relationship between the two quantities by coordinating the direction of change in one variable (e.g., gravity increases) with

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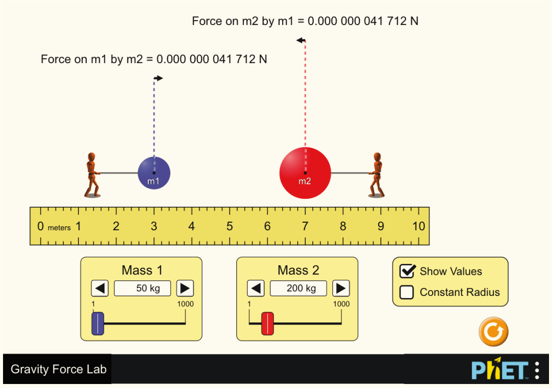
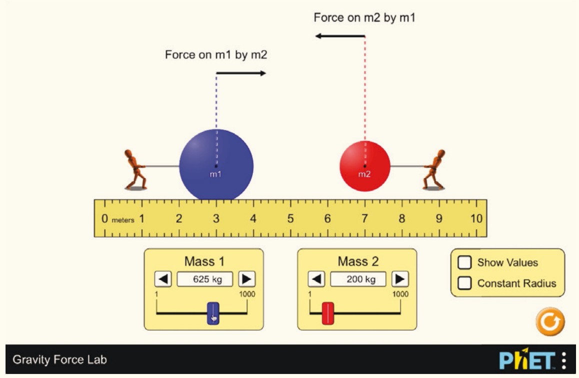
Fig. 1



In the Gravity Force Lab simulation, the size of the circles illustrates the mass of the objects.

Video 1 Changing Mass to Affect Gravity



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changes in the other variable (mass increases). Students reason numerically if they focus on the values of the quantities and observe a specific relationship between them, such as noticing that when the mass of an object doubles, the value of gravity also doubles.

In middle school, covariational reasoning aligns with multiple Common Core State Standards for Mathematics (CCSSM; NGA Center and CCSSO 2010), especially in the domains of Expressions and Equations (EE), Ratios and Proportions (RP) and Functions (F). For instance, it aligns with standard 6.EE.C.9, which is about using variables to represent two quantities in a real-world problem (in this case, gravity) that change in relationship to one another. Covariational reasoning also aligns with standard 7.RP.A.2, which focuses on recognizing and representing direct proportional relationships, and with standard 8.EE.B.5 on comparing two different proportional relationships represented in different ways (e.g., tables, graphs, and equations). Finally, interpreting covariational relationships in graphs is aligned with content standards focusing on the analysis of the relationship between the dependent and independent variables using graphs and tables (6.EE.C.9), deciding whether two quantities are in a proportional relationship by graphing (7.RP.A.2.A), graphing proportional relationships (8.EE.B.5), and describing qualitatively functional relationships between two quantities in graphs (8.F.B.5).

REASONING MATHEMATICALLY ABOUT GRAVITY

At the beginning of the first lesson of the module, as a way to engage students in a direct physical experience with gravity, the teacher created a vertical number line on the wall and measured the heights of each student's jump above the floor. Then she showed a video on Lunar Olympics from YouTube (view the first 40 seconds of the video at https://www.youtube.com/watch?time_continue=1&v=16D0hmLt-S0) in which an astronaut jumps on the moon. She asked, "Why can an astronaut jump over four feet on the moon with a heavy space suit on when we people can only jump about 20 inches on Earth?" This question triggered students' interest about the topic, and through discussion the teacher examined her students' prior knowledge about gravity.

For our tasks, we used the Gravity Force Lab simulation from PhET (see figure 1), which is a collection of research-based computer simulations for science developed by an interdisciplinary research team at the University of Colorado Boulder. (Simulations provided courtesy of PhEt Interactive Simulations; https://phet.colorado.edu.) The Gravity Force Lab consists of two bodies represented by two circles (blue and red) being pulled by two people. The size of the circles illustrates the mass of the objects; for example, the larger the circle, the bigger the mass of the object. The mass of the objects can be changed using the two Mass sliders in the simulation. The arrows on the top of each object show the value of gravity. The longer the arrows are, the more gravity the two objects exert. The user can also modify the distance between the two objects by dragging them closer to or farther away from each other. The distance can be measured using the on-screen ruler. The simulation also has a Show Values box that the user can toggle to make the gravity values appear or disappear. You may explore the Gravity Force Lab at http://phet.colorado.edu.

At the outset of the session, we asked the students to freely explore the simulation and the user interface. Then we presented them with a series of investigations, which we describe in the following sections, each consisting of a set of tasks focusing on a specific type of covariation reasoning. As table 1 illustrates, students engaged in two types of relationships: (1) the direct proportional relationship between the mass of the two objects and the gravity between them and (2) the inversely proportional relationship between the distance between two objects and gravity. For each type of covariational reasoning, we followed a progression, starting from exploring nonnumeric covariational relationships, then coordinating those nonnumeric relationships, then moving to numeric relationships, and finally interpreting covariational relationships in tables and graphs.

Investigation 1: Nonnumeric Relationships of Gravity

The goal of investigation 1 is for students to make generalizations about the nonnumeric relationships between the different variables (mass of bodies, distance between

Table 1 Types of Students' Covariational Reasoning

INVESTIGATION 4 Interpreting covariational relationships in graphs	Examining the amount of change of gravity while considering equal changes in the mass or distance	
	Coordinating the nature of the graph (straight line) to the relationship it depicts (increasing mass, increasing distance)	Coordinating the nature of the graph (curve) to the relationship it depicts (increasing distance, decreasing gravity)
INVESTIGATION 3 Numeric covariational relationships	Reasoning about the change in the dependent variable (gravity) as two independent variables (masses of two objects) change	
	Reasoning about the change in the dependent variable (gravity) as one independent variable (mass) changes	Reasoning about the change in the dependent variable (gravity) as one independent variable (distance) changes
INVESTIGATION 2 Coordinating nonnumeric covariational relationships	Coordinating both directly proportional (masses of two objects – gravity) and inversely proportional relationships (distance – gravity) to change the dependent variable (gravity) by changing one independent variable while keeping the other one constant.	
INVESTIGATION 1 Nonnumeric covariational relationships	Reasoning about the change in the dependent variable (gravity) as one independent variable (mass) changes	Reasoning about the change in the dependent variable (gravity) as one independent variable (distance) changes
	DIRECT PROPORTIONAL RELATIONSHIPS	INVERSELY PROPORTIONAL RELATIONSHIPS

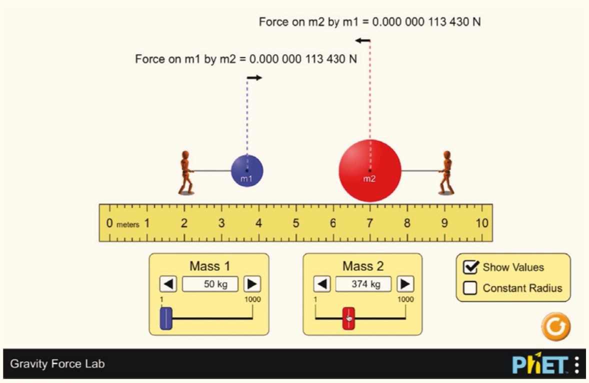
Fig. 2

Refresh and start with the default value of gravity force (0.000 000 041 712 N).

How could you get the force close to F = 0.000001107541 N?

An example of a task in investigation 2

Video 2 Examples of Solutions for Investigation 2



Watch the full video online.

Fig. 3

(a) What do you think will happen to gravity if you double the mass of one of the objects?

(b) Refresh the simulation. The gravity is now 0.000 000 041 712 N. Act it out on the computer. Is your theory correct?

An example of a task in investigation 4

them, and force of gravity). We asked students to uncheck the tool Show Values of gravity, hoping to help them move beyond numbers and numerical computations and to encourage them to focus on the quantities and the relationships among them (Smith and Thompson 2017). Students were asked to explore the simulation and to identify the factors that influence the force of gravity between the two bodies. Next, we included questions to prompt the students to identify how the change (increase or decrease) in one variable influences the gravity between the two objects. For instance, students were asked to play with the mass slider and observe, as they moved the slider of mass 1 from left to right, how that change affected the force between the two bodies (see video 1). As we had hoped, students explored the simulation and identified, for example, that "whenever we decrease the mass, the gravity will decrease." They also recognized that as the distance between two objects increases, the gravity between them decreases.

Investigation 2: Coordination of Nonnumeric Relationships

Following students' reasoning that the gravity between two bodies depends on the mass of the bodies and the distance between them, our next goal was to investigate whether students could coordinate these two relationships. To do that, we asked the students to check the box to now Show Values of gravity, and we presented tasks similar to those shown in figure 2, where they had to manipulate the mass and the distance of the objects to reach a particular value of gravity. The goal was for students to recognize that there is no one correct way to increase or decrease the gravity; instead, gravity can be changed by changing the mass or changing the distance or both (see video 2 for some examples of solutions).

At the end of the investigation, we asked students to share their responses. By hearing other people's solutions, they recognized that the value of gravity can be increased by decreasing the distance and keeping the mass fixed, or by increasing the mass and keeping the distance fixed, or by modifying both through increasing the mass and decreasing the distance.

Investigation 3: Numeric Relationships of Gravity

The goal of investigation 3 was to prompt students to engage in numeric covariation reasoning. Newton's law of gravity ($F = G\, m_1 m_2 / r^2$) indicates that if the mass of a body (m_1) doubles, then the gravity between the bodies doubles. Likewise, if both masses of the bodies double (m_1, m_2), then the gravity becomes four times bigger.

Instead of providing students with a given formula to input different values, we asked students to first create a theory of what they think would happen to the gravity between two bodies if the mass of one of them is doubled and then revise their theories on the basis of their experimentation with the simulation (see figure 3).

In creating their theory for the above task, most students argued, "I think since we are gonna double the mass, I think the gravity will double." Aiming to help them generalize, subsequent tasks asked the students to explore gravity if the mass of one of the objects triples or becomes 100 times bigger. Additionally, we asked them to double the mass of both objects, and they noticed that "when I doubled the one [mass], it became two times bigger; when we doubled both, it became four times bigger." Similarly, the students noticed that when they triple both masses, the gravity becomes nine times bigger. (Video 3 presents one of those discussions.)

Through this exploration, students were able to move from just performing operations on particular numbers to generalization (Blanton and Kaput 2011). They argued, for example, that "when increasing the mass of one object, the gravity increases by how much you increase it by."

Investigation 4: Interpreting Covariation Relationships in Graphs

Graphing plays a crucial role in developing students' conceptual understanding of mathematics; however, students often just focus on the shape of the graph, failing to reason about the covariational relationship between the quantities represented (Moore and Thompson 2015). To help students identify covariational relationships in graphs, the fourth investigation asked them to use the simulation to collect data in a table, plot the ordered pairs on a graph, and use the graph to reason about the relationships.

Teacher: Did you find anything interesting?
Molly: What we found, they are all intervals of 3.
Teacher: They all are in the intervals of 3?
Molly: Yes. 3, 6, 9, 12, 15, 18.
[…]
Teacher: Now if I ask you what will be the force when the mass is 50, how will you do that?
Molly: Then you do 50 times that number.

The excerpt above shows that students recognized that as the mass increases in a uniform way (by 1 kg), the gravity also increases in a uniform way (by 0.000 000 003N). They were also able to use this understanding to generalize that as one quantity increases multiplicatively, the other quantity also increases multiplicatively by the same factor. As a result of the small values in gravitational force, we noticed that some students did not reason about the change in gravity in terms of three billionths but instead in terms of intervals of three (similar to the excerpt above), or three trillionths (see the graph in figure 4), illustrating a difficulty in

Video 3 Doubling and Tripling Masses

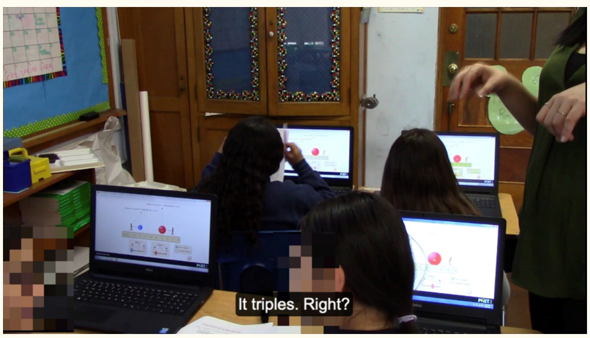


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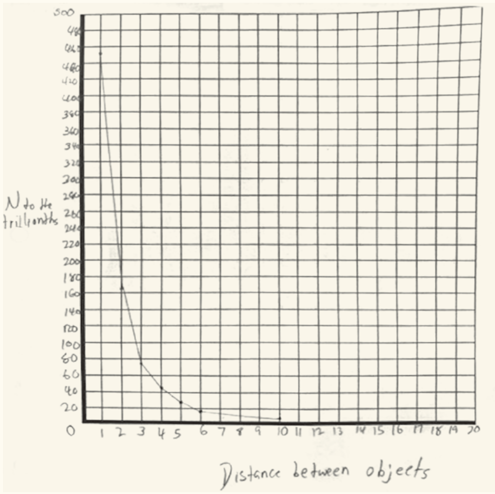
Table 2 Students Gathered Data from the Simulation

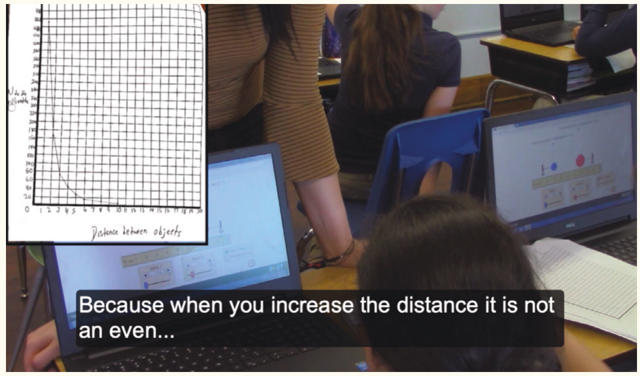
Mass of one object in kg	Gravity Force in N
1	0.000 000 003
2	0.000000006
3	0,000000069
4	0,000 000 012
5	0,000 000015
6	0.000000018
10	0,000000030
100	0,000000303

Fig. 4



Graph that students constructed by gathering data from the simulation

Video 4 Distance-Gravity Discussion



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naming the correct place value. This is particularly common when working with raw data of values with multiple zeros after the decimal point. These experiences can be learning opportunities for initiating a discussion around naming place values correctly while also recognizing the need for unit conversions. For example, table 2 can be modified to include a third column that asks for a conversion from Newtons (N) to Nanonewtons (nN) so that students have easier values to work with, especially in graphing.

By experimenting with graphing, students were able to reason covariationally, arguing that the gravity increases as mass increases because the curve representing the two quantities "went in, like, an upward, like, diagonal line. So, like, when the mass increases, you can see like the points going upward." This shows that students were able to reason about the direction of change of gravity while considering the changes in mass.

Next, we asked the students to plot the relationship between distance and gravity (see figure 4). By graphing both relationships, mass – gravity and distance – gravity, students noticed the differences between the two graphs (straight line and a curve.) In video 4, Molly and Kim explain the reasons they think the distance-gravity relationship is not depicted as a straight line.

As video 4 shows, Kim focused on the amount of change of gravitational force for each interval of distance and identified that as the distance between two objects increased, the gravitational force decreased unevenly. When we asked her to reflect on the uneven decrease of gravitational force, she focused on the interval size of gravitational force and stated that the intervals became smaller with a uniform increase in distance. Although students did not specifically mention the rate of change of gravity (students of that grade do not have a formal instruction on rate of change), we anticipate that by considering the nonuniform length of the gravitational force intervals for uniform increments of the distance, they were able to coordinate the rate of change of one quantity with respect to the other, which in turn establishes their understanding of nonlinear relationship.

SUGGESTIONS FOR IMPLEMENTATION

Both mathematics and science teaching and learning focus on discovering patterns and expressing relationships (Pang and Good 2000), and covariational reasoning can be the bridge that connects these two disciplines. The module on gravity was one of the integrated modules we developed focusing on covariational reasoning, but the types of covariational reasoning presented in table 1 can also be used to describe other science phenomena, such as the water cycle (e.g., the higher the land temperature, the higher the evaporation) and the greenhouse effect (e.g., as the global temperature is increasing by 0.5 degrees, the height of the future sea level is increasing by 4 feet).

In addition, a teacher can include explicit questions to focus on specific mathematical ideas they are learning in class, such as focusing on the concept of a ratio and ratio relationships between two quantities (e.g., CCSSM standard 6.RP.A.1), examining equivalent ratios in tables (e.g., 7.RP.A.2.A), representing proportional relationships using equations (e.g., 6.EE.C.9; 7.RP.A.2.C), or even constructing functions to model linear relationships (e.g., 8.F.B.4).

This article shows how powerful covariational reasoning can be for integrating science and mathematics and also for connecting multiple mathematical ideas of middle school, such as ratios and proportions, expressions and equations, graphing, and functions. As Dugger (2010) argues, these types of experiences help students develop a better understanding of the integrated world they live in, rather than having a fragmented knowledge about it.

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