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A conformally invariant gap theorem characterizing $\mathbb{CP}^2$ via the Ricci flow

Sun-Yung A. Chang[1] · Matthew Gursky[2] · Siyi Zhang[1]

Abstract

We extend the sphere theorem of Chang et al. (Publ Math Inst Ht Études Sci 98:105–434, 2003) to give a conformally invariant characterization of $(\mathbb{CP}^2, g_{FS})$. In particular, we introduce a conformal invariant $\beta(M^4, [g]) \geq 0$ defined on conformal four-manifolds satisfying a 'positivity' condition; it follows from Chang et al. (2003) that if $0 \leq \beta(M^4, [g]) < 4$, then M^4 is diffeomorphic to S^4. Our main result of this paper is a 'gap' result showing that if $b_2^+(M^4) > 0$ and $4 \leq \beta(M^4, [g]) < 4(1 + \epsilon)$ for $\epsilon > 0$ small enough, then M^4 is diffeomorphic to $\mathbb{CP}^2$. The Ricci flow is used in a crucial way to pass from the bounds on β to pointwise curvature information.

1 Introduction

In [7], the first two authors with Yang proved a conformally invariant sphere theorem in dimension four. In this paper we extend the results of [7] to give a characterization of complex projective space. To state our results we begin by establishing our notation and conventions.

If (M^4, g) is a smooth, closed Riemannian four-manifold, we denote the Riemannian curvature tensor by Rm (or Rm_g if we need to specify the metric), the Ricci tensor by Ric, and the scalar curvature by R. We also denote the Weyl curvature tensor by W, and the Schouten tensor

✉ Matthew Gursky
  Matthew.J.Gursky.1@nd.edu

  Sun-Yung A. Chang
  chang@math.princeton.edu

  Siyi Zhang
  siyiz@math.princeton.edu

[1] Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

[2] Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA

$$P = \frac{1}{2}\left(Ric - \frac{1}{6}R \cdot g\right). \tag{1.1}$$

We remark that the definition of the Schouten tensor in [7] (denoted by A) differed from the formula in (1.1) by a factor of two; however, in this paper we adopt the more common convention. In terms of the Weyl and Schouten tensors the Riemannian curvature tensor can be decomposed as

$$Rm = W + P \owedge g, \tag{1.2}$$

where $\owedge$ is the Kulkarni–Nomizu product. There are two important consequences of this identity: First, since the Weyl tensor is conformally invariant, it follows that the behavior of the curvature tensor under a conformal change of metric is determined by the transformation of the Schouten tensor. The second consequence is that the splitting induces a splitting of the Euler form, so that the Chern–Gauss–Bonnet formula can be expressed as

$$8\pi^2 \chi(M) = \int \|W\|^2 \, dv + 4 \int \sigma_2(g^{-1}P) \, dv, \tag{1.3}$$

where

- $\|\cdot\|$ denotes the norm of the Weyl tensor, viewed as an endomorphism of $\Omega^2(M)$, the bundle of two-forms. Note that this differs from the norm of Weyl when viewed as a four-tensor, and the two norms are related by

$$\|W\| = \frac{1}{4}|W|^2.$$

- $g^{-1}P$ denotes the $(1,1)$-tensor (interpreted as an endomorphism of the tangent space at each point) obtained by 'raising an index' of the Schouten tensor, and $\sigma_2(g^{-1}P)$ is the second elementary symmetric polynomial applied to its eigenvalues. To simplify notation we will henceforth write $\sigma_2(P)$ in place of $\sigma_2(g^{-1}P)$.

It follows from the conformal invariance of the Weyl tensor that both integrals in (1.3) are conformally invariant. While their sum is a topological invariant, their ratio can be arbitrary. As we now explain, when the scalar curvature is positive the ratio does carry geometric and topological information.

Given a Riemannian manifold (M^n, g) of dimension $n \geq 3$, let $[g]$ denote the equivalence class of metrics pointwise conformal to g, and $Y(M^n, [g])$ denote the Yamabe invariant:

$$Y\left(M^n, [g]\right) = \inf_{\tilde{g} \in [g]} Vol\,(\tilde{g})^{-\frac{n-2}{n}} \int R_{\tilde{g}} \, dv_{\tilde{g}}.$$

We can also express the Yamabe invariant in terms of the first symmetric function of the Schouten tensor: it follows from (1.1) that

$$\sigma_1(P) = \frac{R}{2(n-1)},$$

hence

$$Y\left(M^n, [g]\right) = \inf_{\tilde{g} \in [g]} 2(n-1)Vol(\tilde{g})^{-\frac{n-2}{n}} \int \sigma_1(P_{\tilde{g}}) \, dv_{\tilde{g}}.$$

With this interpretation of the Yamabe invariant, in dimension four we should view the conformal invariant $\int \sigma_2(P) \, dv$ as a kind of "second Yamabe invariant" (see [15,16,37]). We therefore define

$$\mathcal{Y}_1^+\left(M^4\right) = \left\{g \ : \ Y\left(M^4, [g]\right) > 0\right\}, \tag{1.4}$$

and

$$\mathcal{Y}_2^+ \left(M^4 \right) = \left\{ g \in \mathcal{Y}_1^+ (M) \; : \; \int \sigma_2 \left(P_g \right) dv_g > 0 \right\}. \tag{1.5}$$

By a classical result of Lichnerowicz, there are topological obstructions to $\mathcal{Y}_1(M^4)$ being non-empty [33]. There are also topological implications of $\mathcal{Y}_2(M^4)$ being non-empty: by [17], if $\mathcal{Y}_2^+(M^4) \neq \emptyset$ then the first Betti number $b_1(M^4) = 0$. In fact, it follows from [6] that $[g]$ contains a metric $\tilde{g}$ with positive Ricci curvature.

Returning to the Chern–Gauss–Bonnet formula, for metrics $g \in \mathcal{Y}_2^+(M^4)$ we define the conformal invariant

$$\beta \left(M^4, [g] \right) = \frac{\int \| W_g \|^2 \, dV_g}{\int \sigma_2(P_g) \, dv_g} \geq 0. \tag{1.6}$$

We also define smooth invariant

$$\beta \left(M^4 \right) = \inf_{[g]} \beta \left(M^4, [g] \right). \tag{1.7}$$

If $\mathcal{Y}_2^+(M^4) = \emptyset$, we set $\beta(M^4) = -\infty$.

The main results of [7] give a (sharp) range for β that imply the underlying manifold is the sphere:

Theorem 1 *Suppose M^4 is oriented. If $g \in \mathcal{Y}_2^+(M^4)$ with*

$$\beta \left(M^4, [g] \right) < 4, \tag{1.8}$$

then M^4 is diffeomorphic to S^4. In particular, if M^4 satisfies

$$-\infty < \beta(M^4) < 4,$$

then the same conclusion holds.

Furthermore, if M^4 admits a metric with $\beta(M^4, [g]) = 4$, then one of the following must hold:

- *M^4 is diffeomorphic to S^4; or*
- *M^4 is diffeomorphic to $\mathbb{CP}^2$ and $g \in [g_{FS}]$, where g_{FS} denotes the Fubini–Study metric.*

As a corollary we have the following characterization of manifolds for which $\beta(M^4) = 0$:

Corollary 1 *Assume M^4 is oriented. Then $\beta(M^4) = 0$ if and only if M^4 is diffeomorphic to S^4. Furthermore, $\beta(M^4, [g]) = 0$ if and only if $g \in [g_0]$, where g_0 denotes the round metric.*

Remarks 1. For the case of equality, we note that if $\beta(M^4, [g]) = 0$, then the Weyl tensor $W_g \equiv 0$ and it follows that (M^4, g) is locally conformally flat. By our observations above, since $[g]$ admits a metric with positive Ricci curvature, by Kuiper's theorem [28] (M^4, g) is conformally equivalent to (S^4, g_0) or $(\mathbb{RP}^4, g_0)$, where g_0 is the standard metric.

2. There are a number of other sphere-type theorems under integral curvature conditions; see for example [2–5,10,15,29], and the references in [7].

Our first goal in this paper is initiating the study of four-manifolds with

$$\beta \left(M^4, [g] \right) \geq 4.$$

Suppose M^4 is oriented, and let $b_2(M^4)$ denote the second Betti number. Then we can write $b_2 = b_2^+ + b_2^-$, where $b_2^\pm$ denotes the dimension of the space of self-dual/anti-self-dual harmonic two-forms. If $b_2(M^4) \neq 0$, then by changing the orientation if necessary we may assume $b_2^+ > 0$.

Theorem A *Suppose M^4 is oriented and $b_2^+(M^4) > 0$. There is an $\epsilon > 0$ such that if M^4 admits a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$4 \leq \beta\left(M^4, [g]\right) < 4(1 + \epsilon), \tag{1.9}$$

then M^4 is diffeomorphic to $\mathbb{CP}^2$.

This 'gap' theorem immediately gives a characterization of manifolds with $b_2(M^4) \neq 0$ and $\beta(M^4) = 4$:

Theorem B *Suppose M^4 is oriented and $b_2^+(M^4) > 0$. If*

$$\beta\left(M^4\right) = 4, \tag{1.10}$$

then M^4 is diffeomorphic to $\mathbb{CP}^2$. Moreover, $\beta(M^4, [g]) = 4$ if and only if $g \in [g_{FS}]$.

The proof of Theorem A is similar in approach to the proof of Theorem 1 in [7]. The first step is to find a conformal representative satisfying a pointwise curvature condition that encodes the integral assumptions of the theorem. In [7] this involved solving a modified version of the σ_2-Yamabe problem. However, in the proof of Theorem B it is more natural to consider a modified version of the Yamabe problem introduced by the second author in [19]. In particular we show that a metric satisfying the assumptions of Theorem B can be conformally deformed to a metric that is "almost self-dual Einstein" in an L^2-sense, and whose scalar curvature satisfies a condition similar to the condition satisfied by the scalar curvature of a Kähler metric.

As in the proof of Theorem 1, the second step involves the Ricci flow. In [7] the weak pinching result of Margerin [34] played a crucial role. To prove Theorem B, we show that the Ricci flow with the conformal representative constructed in the first step as the initial metric, will have uniform bounds on the curvature and the Sobolev constant on a fixed time interval $[0, T_0]$, with $T_0 > 0$ depending on the pinching constant. These estimates together with the convergence theory of Cheeger–Gromov–Taylor [9] imply the family of Ricci flows $g_j(T_0)$ (up to a subsequence) will converge to the Fubini–Study metric.

We remark that the instability of the Fubini–Study metric under the Ricci flow was established by Kroencke [27] (see also [26]). As a consequence, one cannot hope to show that the Ricci flow converges in general to the Fubini–Study metric; this is why we must ultimately appeal to a compactness result in this part of the argument.

In view of Theorems A and B, we make several conjectures. The first is that Theorem B remains valid if we drop the assumption on $b_2^+(M^4)$:

Conjecture 1.1 *If*

$$\beta\left(M^4\right) = 4, \tag{1.11}$$

then M^4 is diffeomorphic to $\pm\mathbb{CP}^2$. Moreover, $\beta(M^4, [g]) = 4$ if and only if $g \in [g_{FS}]$.

It is clear that 4 is a 'special' or 'critical' value of β, at which the topology of the underlying manifold can change. A natural question is the next critical value. As a corollary of [17], we have the following estimate for manifolds with indefinite intersection form, i.e., $b_2^+(M^4), b_2^-(M^4) > 0$.

Theorem C *Suppose M^4 is oriented, $b_2(M^4) \neq 0$, and the intersection form of M^4 is indefinite. If $\mathcal{Y}_2^+(M^4)$ is non-empty, then*

$$\beta\left(M^4\right) \geq 8. \tag{1.12}$$

Moreover, if M^4 admits a metric with $\beta(M^4, [g]) = 8$, then $g \in [g_P]$, where $g_P = g_{S^2} \oplus g_{S^2}$ is the product metric on $S^2 \times S^2$.

Our next conjecture is that we can weaken the condition $\beta(M^4) = 4$, and characterize the possible topological types of manifolds admitting metrics with β between 4 and 8:

Conjecture 1.2 *If M^4 is oriented and admits a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$0 \leq \beta\left(M^4, [g]\right) < 8, \tag{1.13}$$

then M^4 is diffeomorphic to S^4 or $\pm\mathbb{CP}^2$.

2 Preliminaries

In this section, we state and prove some preliminary results, including the proof of Theorem C. Several of the results in this section are based on the following result in [17]:

Theorem 2.1 *Let M^4 be a closed oriented four-manifold with $b_2^+(M^4) > 0$. Then for any metric g with $Y(M^4, [g]) \geq 0$,*

$$\int_M ||W^+||^2 \, dv \geq \frac{4\pi^2}{3} \left(2\chi\left(M^4\right) + 3\tau\left(M^4\right)\right), \tag{2.1}$$

where $\chi(M^4)$ and $\tau(M^4)$ denote the Euler characteristic and signature of M^4, respectively. Furthermore:

1. *Equality is achieved in (2.1) by some metric g with $Y(M^4, [g]) > 0$ if and only if g is conformal to a (positive) Kähler–Einstein metric $g_{KE} = e^{2w}g$.*
2. *Equality is achieved in (2.1) by some metric g with $Y(M, [g]) = 0$ if and only if g is conformal to a Ricci-flat anti-self-dual Kähler–Einstein metric $g_{KE} = e^{2w}g$.*

The following lemma is the first application of this result:

Lemma 2.2 *Let M^4 be a closed, oriented four-manifold admitting a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$\beta\left(M^4, [g]\right) < 8. \tag{2.2}$$

If $b_2^+(M^4) > 0$, then the signature of M^4 satisfies

$$\tau\left(M^4\right) > 0.$$

Proof The signature formula implies

$$\int ||W_g||^2 \, dv_g = \int_M \left||W_g^+\right||^2 \, dv_g + \int \left||W_g^-\right||^2 \, dv_g$$
$$= 2 \int \left||W_g^+\right||^2 \, dv_g - 12\pi^2 \tau(M^4). \tag{2.3}$$

By (2.2),

$$\int \sigma_2(P_g) \, dv_g > \frac{1}{8} \int \|W_g\|^2 \, dv_g.$$

Substituting this into the Chern–Gauss–Bonnet formula, we have

$$8\pi^2\chi(M^4) = \int \left|\left|W_g\right|\right|^2 dv_g + 4\int \sigma_2(P_g)\, dv_g > \frac{3}{2}\int \left|\left|W_g\right|\right|^2 dv_g. \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$8\pi^2\chi(M^4) > 3\int \left|\left|W_g^+\right|\right|^2 dv_g - 18\pi^2\tau(M^4), \tag{2.5}$$

and this inequality can be rewritten as

$$\int \left|\left|W_g^+\right|\right|^2 dv_g < \frac{4}{3}\pi^2(2\chi(M^4) + 3\tau(M^4)) + 2\pi^2\tau(M^4). \tag{2.6}$$

Since $b_2^+(M^4) > 0$ and $Y(M^4, [g]) > 0$, (2.1) in Theorem 2.1 implies

$$\int \left|\left|W_g^+\right|\right|^2 dv_g \geq \frac{4}{3}\pi^2(2\chi(M^4) + 3\tau(M^4)). \tag{2.7}$$

Therefore, combining (2.6) and (2.7), we conclude

$$\tau\left(M^4\right) > 0.$$

□

Remark 2.3 This lemma is sharp in the following sense: suppose (M, g) is isometric to $(S^2 \times S^2, g_{prod})$. In this case, $b_2^+(M^4) = b_2^-(M^4) = 1$, $\tau(M^4) = 0$ and

$$\int \left|\left|W_g\right|\right|^2 dv_g = 8\int \sigma_2(P_g)\, dv_g = \frac{64}{3}\pi^2. \tag{2.8}$$

Corollary 2.4 *Let M^4 be a closed, oriented four-manifold admitting a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$\beta\left(M^4, [g]\right) < 8. \tag{2.9}$$

Then either $b_2(M^4) = 0$, or the intersection form is definite.

Proof Suppose $b_2(M^4) \neq 0$ and $b_2^+(M^4) \cdot b_2^-(M^4) > 0$. Then Lemma 2.2 implies $\tau(M^4) = b_2^+(M^4) - b_2^-(M^4) > 0$. Since $b_2^-(M^4)$ is also non-zero, we can apply Lemma 2.2 to M^4 endowed with the opposite orientation to show that the signature is again positive. This is a contradiction, since changing the orientation changes the sign of the signature. It follows that $b_2^+(M^4) \cdot b_2^-(M^4) = 0$, hence the intersection form is definite.   □

Combining the two previous results with an *a priori* upper bound for the total σ_2-curvature, we can prove the following:

Lemma 2.5 *Let M^4 be a closed, oriented four-manifold admitting a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$\beta\left(M^4, [g]\right) < 8. \tag{2.10}$$

If $b_2(M^4) > 0$, then (after possibly changing the orientation) $b_2^+(M^4) = 1$ and $b_2^-(M^4) = 0$.

Proof By Corollary 2.4 we may choose an orientation for which the intersection form is positive definite, so $b_2^+(M^4) > 0$ and $b_2^-(M^4) = 0$. Also, by Corollary F of [17], if $g \in \mathcal{Y}_2^+(M^4)$, $b_1(M^4) = 0$. Therefore,

$$\chi(M^4) = 2 + b_2^+(M^4),$$
$$\tau(M^4) = b_2^+(M^4) > 0. \tag{2.11}$$

By the Chern–Gauss–Bonnet formula and (2.9),

$$8\pi^2 \chi(M^4) = \int \|W_g\|^2 \, dv_g + 4 \int \sigma_2(P_g) \, dv_g$$
$$< 12 \int \sigma_2(P_g) \, dv_g. \tag{2.12}$$

By Theorem B of [18], we have the bound

$$\int \sigma_2(P_g) \, dv_g \leq 4\pi^2, \tag{2.13}$$

and equality holds if and only if (M^4, g) is conformally equivalent to the round sphere. More generally, since the integral is conformally invariant it is easy to show that

$$\int \sigma_2(P_g) \, dv_g \leq \frac{1}{96} Y(M^4, [g])^2. \tag{2.14}$$

Since $b_2(M^4) > 0$ strict inequality must hold, and substituting this into (2.12) we get

$$8\pi^2 \chi(M^4) < 12 \int \sigma_2(P_g) \, dv_g$$
$$< 48\pi^2, \tag{2.15}$$

hence $\chi(M^4) < 6$. By (2.11), we see that $1 \leq b_2^+(M^4) \leq 3$. It therefore suffices to rule out the possibilities $b_2^+(M^4) = 2$ and $b_2^+(M^4) = 3$.

If $b_2^+(M^4) = 2$ then $\chi(M^4) = 4$, so by the Chern–Gauss–Bonnet formula

$$\int \|W_g\|^2 \, dv_g + 4 \int \sigma_2(P_g) \, dv_g = 32\pi^2.$$

Also, $b_2^+(M^4) = 2$ implies $\tau(M^4) = 2$, so the signature formula gives

$$\int \left\|W_g^+\right\|^2 \, dv_g = \int \left\|W_g^-\right\|^2 \, dv_g + 24\pi^2$$
$$\geq 24\pi^2.$$

It follows that

$$\int \|W_g\|^2 \, dv_g \geq 24\pi^2, \quad \int \sigma_2(P_g) \, dv_g \leq 2\pi^2.$$

Therefore,

$$\int \|W_g\|^2 \, dv_g \geq 12 \int \sigma_2(P_g) \, dv_g,$$

which contradicts (2.10).

If $b_2^+(M^4) = 3$, we can apply the same argument to conclude

$$\int \|W_g\|^2 \, dv_g \geq 36 \int \sigma_2(P_g) \, dv_g,$$

which also contradicts (2.10). Therefore, $b_2^+(M^4) = 1$. $\qquad\square$

Remark 2.6 The preceding lemma implies that if we take $\epsilon \leq 1$ in Theorem A, then $b_2^+(M^4) > 0$ will show that $b_2^+(M^4) = 1$. Note that the work [13] and [14] of Donaldson and Freedman will imply that the manifold is *homeomorphic* to $\mathbb{CP}^2$ in this case.

The Proof of Theorem C We can now combine Theorem 2.1 and Lemma 2.2 to give the proof of Theorem C. Assuming $b_2^+(M^4) \cdot b_2^-(M^4) > 0$, it follows from Lemma 2.2 that $\beta(M^4, [g]) \geq 8$ for any metric $g \in \mathcal{Y}_2^+(M^4)$. Moreover, if equality holds, then since $b_2^+(M^4) > 0$ we can argue as in the proof of Lemma 2.2 to get equality in (2.6):

$$\int \left\|W_g^+\right\|^2 dv_g = \frac{4}{3}\pi^2 \left(2\chi\left(M^4\right) + 3\tau\left(M^4\right)\right) + 2\pi^2\tau\left(M^4\right). \tag{2.16}$$

By Theorem 2.1 we conclude that $\tau(M^4) \geq 0$. Reversing orientation and applying the same argument (since $b_2^-(M^4) > 0$) we also get $-\tau(M^4) \geq 0$, hence $\tau(M^4) = 0$. Substituting this into (2.16) implies that we have equality in (2.1). Therefore, g is conformal to Kähler–Einstein metric g_{KE}. By Proposition 2 of [12], $\nabla W_{g_{KE}}^+ \equiv 0$.

Applying the same argument with the opposite orientation, we see that g is conformal to a Kähler–Einstein metric g'_{KE}. By Obata's theorem [35], Einstein metrics are unique in their conformal class (except in the case of the sphere, which is ruled out in this case). Therefore, $g_{KE} = g'_{KE}$, and since equality holds in (2.1) with the opposite orientation it follows that $\nabla W_{g_{KE}}^- \equiv 0$. We conclude that g_{KE} is locally symmetric and Einstein; it follows from the classification of such spaces (for example, [24]) that (M^4, g_{KE}) is isometric to $(S^2 \times S^2, g_p)$, and Theorem C follows. $\qquad\square$

2.1 A preliminary lemma

We end this section with a technical lemma that will be used in the proof of Theorem A.

Lemma 2.7 *Let (M^4, g) be a closed, compact oriented Riemannian four-manifold with $b_2^+(M^4) > 0$ and*

$$\beta\left(M^4, [g]\right) = 4\left(1 + \epsilon\right) \tag{2.17}$$

for some $0 \leq \epsilon < 1$. Then

$$\int_M \left\|W^-\right\|^2 dv = \frac{6\epsilon}{2 + \epsilon}\pi^2, \tag{2.18}$$

$$\int \left\|W^+\right\|^2 dv = 12\pi^2 + \int \left\|W^-\right\|^2 dv, \tag{2.19}$$

and

$$Y\left(M^4, [g]\right) \geq \frac{24\pi}{\sqrt{2 + \epsilon}}. \tag{2.20}$$

Proof It follows from Lemma 2.5 that $b_1(M^4) = 0$, $b_2^+(M^4) = 1$, and $b_2^-(M^4) = 0$. Therefore, $\chi(M^4) = 3$ and $\tau(M^4) = 1$. By the Chern–Gauss–Bonnet and signature formulas, we have

$$24\pi^2 = \int ||W||^2\, dv + 4\int \sigma_2(P)dv, \tag{2.21}$$

$$12\pi^2 = \int ||W^+||^2\, dv - \int ||W^-||^2\, dv. \tag{2.22}$$

Since $\beta(M^4, [g]) = 4(1 + \epsilon)$, we have

$$4\int \sigma_2(P)dv = \frac{1}{1+\epsilon}\int ||W||^2\, dv. \tag{2.23}$$

Substituting this into (2.21) we conclude

$$\int ||W||^2\, dv = 24\left(\frac{1+\epsilon}{2+\epsilon}\right)\pi^2. \tag{2.24}$$

By (2.22),

$$12\pi^2 = \int ||W^+||^2\, dv - \int ||W^-||^2\, dv$$

$$= \int ||W||^2\, dv - 2\int ||W^-||^2\, dv$$

$$= 24\left(\frac{1+\epsilon}{2+\epsilon}\right)\pi^2 - 2\int ||W^-||^2\, dv,$$

which implies (2.18). Also, substituting (2.18) into the signature formula (2.22) we get (2.19).

To prove (2.20), we fist observe that (2.23) and (2.24) imply

$$\int \sigma_2(P)dv = \left(\frac{6}{2+\epsilon}\right)\pi^2.$$

Therefore, by (2.25),

$$\left(\frac{6}{2+\epsilon}\right)\pi^2 = \int \sigma_2(P_g)\, dv_g \leq \frac{1}{96}Y(M^4, [g])^2, \tag{2.25}$$

and (2.20) follows. □

3 Modified Yamabe metrics

As mentioned in the Introduction, the proof of Theorems A and B will use the Ricci flow. We will use the fact that our assumptions are conformally invariant and choose an initial metric that satisfies certain key estimates. The metric will be a solution of a modified version of the Yamabe problem introduced in [19], which we now review.

Let (M^4, g) be a Riemannian four-manifold. Define

$$F_g^+ = R_g - 2\sqrt{6}\left|\left|W_g^+\right|\right|, \tag{3.1}$$

and

$$\mathcal{L}_g = -6\Delta_g + R_g - 2\sqrt{6}\left|\left|W^+\right|\right|. \tag{3.2}$$

$\mathcal{L}_g$ is a variant of conformal Laplacian that satisfies the following conformal transformation law:

$$\mathcal{L}_{\widetilde{g}}\phi = u^{-3}\mathcal{L}_g\left(\phi u\right), \tag{3.3}$$

where $\widetilde{g} = u^2 g \in [g]$. In analogy to the Yamabe problem, we define the functional

$$\widehat{Y}_g[u] = \langle u, \mathcal{L}_g u\rangle_{L^2} / \|u\|_{L^4}^2, \tag{3.4}$$

and the associated conformal invariant

$$\widehat{Y}\left(M^4, [g]\right) = \inf_{u \in W^{1,2}(M,g)} \widehat{Y}_g[u]. \tag{3.5}$$

By the conformal transformation law of $\mathcal{L}_g$, the functional $u \to \widehat{Y}_g[u]$ is equivalent to the Riemannian functional

$$\widetilde{g} = u^2 g \to vol(\widetilde{g})^{-\frac{1}{2}} \int F_{\widetilde{g}}^+ \, dv_{\widetilde{g}}. \tag{3.6}$$

The motivation for introducing this invariant is explained in the following result (see Theorem 3.3 and Proposition 3.5 of [19]):

Theorem 3.1   (i) *Suppose M^4 admits a metric g with $F_g^+ \geq 0$ on M^4 and $F_g^+ > 0$ somewhere. Then $b_2^+(M^4) = 0$.*
(ii) *If $b_2^+(M^4) > 0$, then M^4 admits a metric g with $F_g^+ \equiv 0$ if and only if (M^4, g) is a Kähler manifold with non-negative scalar curvature.*
(iii) *If $Y(M^4, [g]) > 0$ and $b_2^+(M^4) > 0$, then $\widehat{Y}(M^4, [g]) \leq 0$ and there is a metric $\widetilde{g} = u^2 g$ such that*

$$F_{\widetilde{g}}^+ = R_{\widetilde{g}} - 2\sqrt{6}\,||W^+||_{\widetilde{g}} \equiv \hat{Y}(M, [g]) \leq 0 \tag{3.7}$$

and

$$\int R_{\widetilde{g}}^2 \, dv_{\widetilde{g}} \leq 24 \int \left||W_{\widetilde{g}}^+\right||^2 dv_{\widetilde{g}}. \tag{3.8}$$

Furthermore, equality is achieved if and only if $F_{\widetilde{g}}^+ \equiv 0$ and $R_{\widetilde{g}} = 2\sqrt{6}||W_{\widetilde{g}}^+|| \equiv const.$

Remark 3.2 Recall that $F_g^+ \equiv 0$ on a Kähler manifold (M^4, g) with $R \geq 0$.

Remark 3.3 In the rest of the paper, we will refer to the metric $\widetilde{g}$ in (iii) of Theorem 3.1, normalized to have unit volume, a *modified Yamabe metric*, and denote it by g_m. To simplify the notation, we write $\hat{Y}(M, [g]) = -\mu_+$. Then g_m satisfies

$$R_{g_m} - 2\sqrt{6}\,||W^+||_{g_m} = -\mu_+ \leq 0 \tag{3.9}$$

and (3.8).

As a preparation for the proof of Theorem A in next section, in the rest of this section we will list some preliminary curvature estimates of the modified Yamabe metric $g_m \in [g]$ with the assumption $b_2^+(M^4) > 0$ and $\beta(M, [g]) = 4(1 + \epsilon)$.

Lemma 3.4 *Let (M^4, g) be a closed, compact oriented Riemannian four-manifold with $b_2^+(M^4) > 0$ and*

$$\beta\left(M^4, [g]\right) = 4\left(1 + \epsilon\right) \tag{3.10}$$

for some $0 < \epsilon < 1$, then we have for the modified Yamabe metric $g_m \in [g]$

$$\int_M \left\| W_{g_m}^- \right\|^2 dv_{g_m} = \frac{6\epsilon}{2+\epsilon} \pi^2, \tag{3.11}$$

$$\int \left\| W_{g_m}^+ \right\|^2 dv_{g_m} = 12\pi^2 + \int \left\| W_{g_m}^- \right\|^2 dv_{g_m}, \tag{3.12}$$

$$Y(M, [g_m]) \geq \frac{24\pi}{\sqrt{2+\epsilon}} \tag{3.13}$$

$$\int |E_{g_m}|^2 dv_{g_m} \leq 6 \int \left\| W_{g_m}^- \right\| dv_{g_m}, \tag{3.14}$$

$$\frac{1}{12} \mu_+ Y \leq 3 \int \left\| W_{g_m}^- \right\|^2 dv_{g_m}, \tag{3.15}$$

$$\frac{1}{24} \int \left(R_{g_m} - \bar{R}_{g_m} \right)^2 dv_{g_m} \leq 3 \int \left\| W_{g_m}^- \right\|^2 dv_{g_m}, \tag{3.16}$$

where $\bar{R}_{g_m} = \int R_{g_m} dv_{g_m}$.

Proof (3.11), (3.12), (3.13) follow from (2.18), (2.19), (2.20) of Lemma 2.7 and confomal invariance. The estimates (2.21) and (2.22) imply

$$\int \left\| W^+ \right\|^2 dv - 4 \int \sigma_2(P_{g_m}) dv_{g_m} = 3 \int \left\| W^- \right\|_{g_m}^2 dv_{g_m}. \tag{3.17}$$

Recall the modified Yamabe metric stasfies $R_{g_m} + \mu_+ = 2\sqrt{6} \|W_{g_m}^+\|$. Squaring both sides of this formula and integrating over M^4, we have

$$\frac{1}{24} \int \left(R_{g_m}^2 + 2\mu_+ R_{g_m} + \mu_+^2 \right) dv_{g_m} = \int \left\| W_{g_m}^+ \right\|^2 dv_{g_m}.$$

With (3.17), we can rewrite this equation in the following way:

$$\frac{1}{2} \int |E_{g_m}|^2 dv + \frac{1}{12} \mu_+ \int R_{g_m} dv_{g_m} + \frac{1}{24} \int \mu_+^2 dv_{g_m} = 3 \int \left\| W_{g_m}^- \right\|^2 dv_{g_m}.$$

Since $\int R_{g_m} dv_{g_m} \geq Y(M^4, [g_m]) > 0$, (3.14) and (3.15) follow from this equation. To see (3.16), we have

$$\frac{1}{24} \int \left(R_{g_m} - \bar{R}_{g_m} \right)^2 dv_{g_m} = \frac{1}{24} \left(\int R_{g_m}^2 dv_{g_m} - \bar{R}_{g_m}^2 \right) \leq \frac{1}{24} \left(\int R_{g_m}^2 dv_{g_m} - Y^2 \right)$$

$$\leq \int \left\| W_{g_m}^+ \right\|^2 dv_{g_m} - 4 \int \sigma_2 \left(P_{g_m} \right) dv_{g_m} = 3 \int \left\| W_{g_m}^- \right\|^2 dv_{g_m}.$$

$\square$

We end this section with a conformally invariant characterization of the Fubini–Study metric:

Lemma 3.5 *Let (M^4, g) be a closed, compact oriented Riemannian four-manifold whose metric g is of positive Yamabe type. In addition, assume $b_1(M^4) = b_2^-(M^4) = 0$ and $b_2^+(M^4) = 1$. Then*

$$\int_M \sigma_2(P) \, dv \leq 12\pi^2$$

and equality holds if and only if (M^4, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$.

Proof By our assumptions we have $\chi(M^4) = 3$ and $\tau(M^4) = 1$. Then the Gauss–Bonnet–Chern and signature formulas read

$$24\pi^2 = \int_M ||W_g||^2 \, dv_g + 4 \int_M \sigma_2(P_g) \, dv_g, \tag{3.18}$$

and

$$12\pi^2 = \int_M \left|\left|W_g^+\right|\right|^2 \, dv_g - \int_M \left|\left|W_g^-\right|\right|^2 \, dv_g. \tag{3.19}$$

Therefore,

$$\int_M \sigma_2(P) dv = 24pi^2 - \int \|W\|^2 \, dv$$

$$\leq 24\pi^2 - \int_M ||W^+||^2 \, dv$$

$$= 12\pi^2 - \int_M ||W^-||^2 \, dv \leq 12\pi^2.$$

If equality holds then $W^- \equiv 0$ and by (3.14) of Lemma 3.4 it immediately follows that $E_{g_m} = 0$. Hence, (M^4, g_m) is self-dual Einstein with positive scalar curvature and $b_2^+(M^4) = 1$. It is easy to check that the equality in (2.1) is achieved, and therefore g_m is conformal to a Kähler–Einstein metric. By Obata's theorem, (M^4, g_m) must be (Kähler–)Einstein with positive scalar curvature and $b_2^+(M^4) = 1$.

Now (M^4, g_m) is a complex surface with a positive Kähler–Einstein metric. For complex surfaces, (see, p. 81 of [1])

$$3c_2(M^4) - c_1(M^4)^2 = \chi(M^4) - 3\tau(M^4) = 0. \tag{3.20}$$

Then the uniformlization of Kähler–Einstein manifolds (e.g., Theorem 2.13 in [38]) implies that the universal cover of (M, g_m) is (up to scaling) isometric to $(\mathbb{CP}^2, g_{FS})$. Hence, (M, g_m) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$ since $\mathbb{CP}^2$ does not have nontrivial smooth quotient space. $\qquad\square$

4 The proofs of Theorems A and B

Suppose M^4 is an oriented four-manifold with $b_2^+(M^4) > 0$ and g is a metric on M^4 with $\beta(M^4, [g]) = 4(1 + \epsilon)$ for $\epsilon > 0$ small. We want to show that if $\epsilon > 0$ is small enough, then M^4 is diffeomorphic to $\mathbb{CP}^2$. By Lemma 3.4, the modified Yamabe metric $g_m \in [g]$ is close to a self-dual Einstein metric in an L^2-sense. Using the Ricci flow, we want to 'smooth' g_m to obtain smallness of W^- and E in a *pointwise* sense. We do this in two stages: first, we show that for a small but uniform time, the Ricci flow applied to g_m gives a metric for which W^- and E are small in an L^p-sense, for some $p > 2$. Next, we appeal to a parabolic Moser iteration estimate of Yang [39] to conclude L^∞-smallness. The Bernstein–Bando–Shi estimates for the Ricci flow then imply bounds for C^∞-norms of the curvature. To complete the proof we apply a contradiction argument using a compactness result of Cheeger–Gromov–Taylor [9].

We begin with some definitions: on (M^4, g), define

$$G_k(g) = |E_g|^k + |R_g - \bar{R}_g|^k + \left|\left|W_g^-\right|\right|^k + |\left(F_g^+\right)_-|^k, \tag{4.1}$$

where $(F_g^+)_- = \min(F_g^+, 0)$. We shall suppress the subscript g when there is no confusion.

Lemma 4.1 *Under the conditions of Theorem A, the modified Yamabe metric $g_m \in [g]$ satisfies $\int G_2(g_m)dv_{g_m} < c(\epsilon)$, where $c(\epsilon) \to 0$ as $\epsilon \to 0$.*

Proof This is a direct consequence of Lemma 3.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Now recall some basic facts about the Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t}g = -2Ric(g) \\ g(0) = g_0 \end{cases} \tag{4.2}$$

The following short time-time existence result of Ricci flow has been established in [21].

Proposition 4.2 *For arbitrary smooth metric g_0, there exists $T = T(g_0)$ such that (4.2) has a unique smooth solution for $t \in [0, T)$.*

Remark 4.3 In general, the time interval $[0, T)$ depends on the initial metric g_0.

Along the Ricci flow, define $G_k(t) = G_k(g(t))$. The following estimate is of fundamental importance for our argument.

Proposition 4.4 *Suppose we have a solution of the Ricci flow whose initial metric satisfies*

$$\int_M G_2(0)dv_{g(0)} = \frac{1}{2}\epsilon_0 \tag{4.3}$$

for some ϵ_0 is sufficiently small. Let

$$T = \inf\left\{t : \int_M G_2(t)\,dv_{g(t)} = \epsilon_0\right\}.$$

Assume in addition for $0 \leq t \leq T$

$$Y(t) = Y\left(M^4, [g(t)]\right) \geq b > 0, \tag{4.4}$$

and

$$0 < \bar{R}(t) \leq a. \tag{4.5}$$

Then for $0 \leq t \leq T$, we have

$$\frac{d}{dt}\int_M G_2(t)\,dv_{g(t)} \leq \tilde{a}\int_M G_2(t)\,dv_{g(t)} - \tilde{b}\left(\int_M G_4(t)dv_{g(t)}\right)^{1/2}, \tag{4.6}$$

where $\tilde{a}$ and $\tilde{b}$ are uniform positive constants independent of ϵ_0. Moreover, there exists T_0, which is independent of ϵ_0 such that $T \geq T_0$, and we may choose $\tilde{a} = \frac{4}{3}a$ and $\tilde{b} = \frac{1}{12}b$.

Proof The proof is based on the evolution of the curvature under the Ricci flow in four dimensions, along with several algebraic inequalities. We begin by summarizing the evolution formulas we will need, most of which can be found in [11]:

Lemma 4.5 *Under (4.2) on Riemannnian four-manifolds,*

$$\frac{\partial}{\partial t}dv = -Rdv, \tag{4.7}$$

$$\frac{\partial}{\partial t}|E|^2 = \Delta|E|^2 - 2|\nabla E|^2 + 4WEE - 4tr\,E^3 + \frac{2}{3}R|E|^2, \tag{4.8}$$

$$\frac{\partial}{\partial t}\left(R^2\right) = \Delta\left(R^2\right) - 2|\nabla R|^2 + 4R|E|^2 + R^3, \tag{4.9}$$

$$\frac{\partial}{\partial t} \left|\left|W^{\pm}\right|\right|^2 = \Delta \left|\left|W^{\pm}\right|\right|^2 - 2 \left|\left|\nabla W^{\pm}\right|\right|^2 + 36 \det W^{\pm} + W^{\pm} EE, \qquad (4.10)$$

$$\frac{\partial}{\partial t} \left|\left|W^{\pm}\right|\right| \leq \Delta \left|\left|W^{\pm}\right|\right| + \sqrt{6} \left|\left|W^{\pm}\right|\right|^2 + \frac{\sqrt{6}}{6} \left|E\right|^2, \qquad (4.11)$$

$$\frac{\partial}{\partial t} \left(\left(F^+\right) -\right) \geq \Delta \left(\left(F^+\right) -\right) - \left(\left(F^+\right) -\right)^2 + 2R \left(F^+\right) - \qquad (4.12)$$

where

$$W^{\pm} EE := W^{\pm}_{ijkl} E_{ik} E_{jl}, \quad F_g^+ := R_g - 2\sqrt{6} \left|\left|W^+\right|\right|, \quad (F^+)_- := \min\{F_g^+, 0\}.$$

Remark 4.6 For the evolution formulas of $W^{\pm}$ we rely on unpublished notes of Knopf [25].

As a corollary of the formulas above we have in four dimensions:

Corollary 4.7 *Under* (4.2),

$$\frac{d}{dt} \int dv = - \int R \, dv, \qquad (4.13)$$

$$\frac{d}{dt} \int |E|^2 \, dv = \int \left(-2|\nabla E|^2 + 4WEE - 4tr E^3 - \frac{1}{3} R |E|^2\right) dv, \qquad (4.14)$$

$$\frac{d}{dt} \int \left(R - \bar{R}\right)^2 dv = \int \left(-2|\nabla R|^2 + 4 \left(R - \bar{R}\right) |E|^2 + \bar{R} \left(R - \bar{R}\right)^2\right) dv, \quad (4.15)$$

$$\frac{d}{dt} \int |\left(F^+\right)_-|^2 dv \leq -2 \int \left(|\nabla(F^+)_-|^2 + \frac{R}{6}|\left(F^+\right)_-|^2\right) dv$$

$$- \int \left(\left(F^+\right)_-\right)^3 dv + \int \left(\frac{4}{3}\bar{R}|\left(F^+\right)_-|^2 + \frac{4}{3}\left(R - \bar{R}\right)|\left(F^+\right)_-|^2\right) dv, \quad (4.16)$$

$$\frac{d}{dt} \int \left|\left|W^-\right|\right|^2 dv \leq - \int \left(2|\nabla \left|\left|W^-\right|\right||^2 + R \left|\left|W^-\right|\right|^2\right) dv$$

$$+ \int \left(2\sqrt{6} \left|\left|W^-\right|\right|^3 + \frac{\sqrt{6}}{3} \left|\left|W^-\right|\right| |E|^2\right) dv, \qquad (4.17)$$

where

$$\bar{R} = \int R \, dv \Big/ \int dv, \quad F_g^+ = R_g - 2\sqrt{6} \left|\left|W^+\right|\right|, \quad (F^+)_- = \min\left\{F_g^+, 0\right\}.$$

For the proof of Proposition 4.4 we will also need some algebraic inequalities. The first appears in ([7], Lemma 4.3), and is based on ([34], Lemma 6):

Lemma 4.8

$$WEE \leq \frac{\sqrt{6}}{3} \left(\left|\left|W^+\right|\right| + \left|\left|W^-\right|\right|\right) |E|^2 \qquad (4.18)$$

Proof Recall the well-known decomposition of Singer-Thorpe:

$$Riem = \begin{pmatrix} W^+ + \frac{R}{12} Id & B \\ B^* & W^- + \frac{R}{12} Id \end{pmatrix} \qquad (4.19)$$

Note the compositions satisfy

$$BB^* : \Lambda_+^2 \to \Lambda_+^2, \quad B^* B : \Lambda_-^2 \to \Lambda_-^2.$$

Fix a point $P \in M^4$, and let $\lambda_1^\pm \leq \lambda_2^\pm \leq \lambda_3^\pm$ denote the eigenvalues of $W^\pm$, where $W^\pm$ are interpreted as endomorphisms of $\Lambda_\pm^2$. Also denote the eigenvalues of $BB^* : \Lambda_+^2 \to \Lambda_+^2$ by $b_1^2 \leq b_2^2 \leq b_3^2$, where $0 \leq b_1 \leq b_2 \leq b_3$. From Lemma 4.3 of [7], we have

$$WEE \leq 4\left(\sum_{i=1}^3 \lambda_i^+ b_i^2 + \sum_{i=1}^3 \lambda_i^- b_i^2\right) \tag{4.20}$$

Recall from Lemma 4.2 of [7] that $|E|^2 = 4\sum_{i=1}^3 b_i^2$. For a trace-free 3×3 matrix A, we have the sharp inequality:

$$|A(X, X)| \leq \frac{\sqrt{6}}{3}\,||A||\,|X|^2. \tag{4.21}$$

Apply (4.21) to $A = diag(\lambda_1^\pm, \lambda_2^\pm, \lambda_3^\pm)$ and $X = (b_1, b_2, b_3)$. We derive

$$4\left(\sum_{i=1}^3 \lambda_i^+ b_i^2 + \sum_{i=1}^3 \lambda_i^- b_i^2\right) \leq \frac{\sqrt{6}}{3}\left(||W^+|| + ||W^-||\right)|E|^2. \tag{4.22}$$

Combining (4.20) and (4.22), we derive the desired inequality. $\qquad\square$

Now we turn to the proof of Proposition 4.4. From the definition of G_k, we have

$$\frac{d}{dt}\int G_2 dv = \frac{d}{dt}\int |E|^2\,dv + \frac{d}{dt}\int |R - \bar{R}|^2 dv$$
$$+ \frac{d}{dt}\int ||W^-||^2\,dv + \frac{d}{dt}\int |(F^+)_-|^2 dv. \tag{4.23}$$

Now estimate each term of the right hand side from formulas in Corollary 4.7.

$$\frac{d}{dt}\int |E|^2\,dv \leq \int\left(-2|\nabla E|^2 - \frac{1}{3}R\,|E|^2 - 4tr\,E^3 + \frac{4\sqrt{6}}{3}(||W^+|| + ||W^-||)\,|E|^2\right)dv$$
$$\leq -\frac{Y}{3}\left(\int |E|^4\,dv\right)^{1/2} + 4\left(\int |E|^2\,dv\right)^{1/2}\left(\int |E|^4\,dv\right)^{1/2}$$
$$+ \frac{4\sqrt{6}}{3}\left(\int ||W^-||^2\,dv\right)^{1/2}\left(\int |E|^4\,dv\right)^{1/2}$$
$$+ \frac{2}{3}\left(\int |(F^+)_-|^2 dv\right)^{1/2}\left(\int |E|^4\,dv\right)^{1/2} + \frac{2}{3}\int (R - \bar{R})\,|E|^2\,dv$$
$$+ \frac{2}{3}\int \bar{R}\,|E|^2\,dv$$
$$\leq -\frac{Y}{6}\left(\int |E|^4\,dv\right)^{1/2} + \frac{2}{3}\bar{R}\int |E|^2\,dv. \tag{4.24}$$

The first inequality follows from Lemma 4.8. The second inequality follows from Cauchy–Schwartz inequality and the conformally invariant Sobolev inequality:

$$Y\left(\int \phi^4 dv\right)^{1/2} \leq \int\left(|\nabla\phi|^2 + \frac{1}{6}R\phi^2\right)dv.$$

The third inequality follows from the smallness assumption of ϵ_0. Next, we esimate

$$\frac{d}{dt}\int (R-\bar{R})^2 dv = \int \left(-2|\nabla(R-\bar{R})|^2 - \frac{1}{3}R(R-\bar{R})^2\right) dv + \frac{1}{3}\int (R-\bar{R})^3 dv$$

$$+\frac{4}{3}\int \bar{R}(R-\bar{R})^2 dv + 4\int (R-\bar{R})|E|^2 dv$$

$$\leq -\frac{Y}{3}\left(\int (R-\bar{R})^4 dv\right)^{1/2} + \frac{1}{3}\left(\int (R-\bar{R})^2 dv\right)^{1/2}\left(\int (R-\bar{R})^4 dv\right)^{1/2}$$

$$+\frac{4}{3}\bar{R}\int (R-\bar{R})^2 dv + 4\left(\int (R-\bar{R})^2 dv\right)^{1/2}\left(\int |E|^4 dv\right)^{1/2}$$

$$\leq -\frac{Y}{6}\left(\int (R-\bar{R})^4 dv\right)^{1/2} + \frac{4}{3}\bar{R}\int (R-\bar{R})^2 dv$$

$$+4\left(\int (R-\bar{R})^2 dv\right)^{1/2}\left(\int |E|^4 dv\right)^{1/2}. \tag{4.25}$$

The first inequality is from Sobolev inequality and Cauchy-Schwartz inequality and the second inequality is from the smallness assumption of ϵ_0.

$$\frac{d}{dt}\int |(F^+)_-|^2 dv \leq -\frac{Y}{3}\left(\int |(F^+)_-|^4 dv\right)^{1/2} + \left(\int |(F^+)_-|^2 dv\right)^{1/2}\left(\int |(F^+)_-|^4 dv\right)^{1/2}$$

$$+\frac{4}{3}\bar{R}\int |(F^+)_-|^2 dv + \frac{4}{3}\left(\int (R-\bar{R})^2 dv\right)^{1/2}\left(\int |(F^+)_-|^4 dv\right)^{1/2}$$

$$\leq -\frac{Y}{6}\left(\int |(F^+)_-|^4 dv\right)^{1/2} + \frac{4}{3}\bar{R}\int |(F^+)_-|^2 dv \tag{4.26}$$

The first inequality is from Sobolev inequality and Cauchy-Schwartz inequality and the second inequality is from the smallness assumption of ϵ_0.

$$\frac{d}{dt}\int ||W^-||^2 dv \leq -\frac{Y}{3}\left(\int ||W^-||^4\right)^{1/2} + \frac{2}{3}\left(\int (R-\bar{R})^2 dv\right)^{1/2}\left(\int ||W^-||^4 dv\right)^{1/2}$$

$$+2\sqrt{6}\left(\int ||W^-||^2 dv\right)^{1/2}\left(\int ||W^-||^4 dv\right)^{1/2}$$

$$+\frac{\sqrt{6}}{3}\left(\int ||W^-||^2 dv\right)^{1/2}\left(\int |E|^4 dv\right)^{1/2}$$

$$\leq -\frac{Y}{6}\left(||W^-||^4\right)^{1/2} + \frac{\sqrt{6}}{3}\left(\int ||W^-||^2 dv\right)^{1/2}\left(\int |E|^4 dv\right)^{1/2} \tag{4.27}$$

The first inequality is from Sobolev inequality and Cauchy-Schwartz inequality and the second inequality is from the smallness assumption of ϵ_0.

With (4.24), (4.25), (4.26), (4.27), it is now easy to see from the smallness assumption of ϵ_0

$$\frac{d}{dt}\int_M G_2 dv \leq \frac{4}{3}a\int_M G_2 dv - \frac{1}{12}b\left(\int G_4 dv\right)^{1/2}. \tag{4.28}$$

Take $\tilde{a} = \frac{4}{3}a$ and $\tilde{b} = \frac{1}{12}b$. Clearly, we have proved the desired inequality. Note that the differential inequality

$$\frac{d}{dt}\int_M G_2 dv \leq \frac{4}{3}a\int_M G_2 dv$$

implies

$$T \geq T_0 = \frac{3 \log 2}{4} a.$$

□

Remark 4.9 The importance of this lemma is that T_0 does *not* depend on ϵ_0, which implies that we may evolve the Ricci flow on a uniform time interval once we derive uniform bounds for curvatures.

With Lemma 4.1, it is easy to see that we may choose ϵ in Theorem A sufficiently small so that (4.3) is satisfied. To apply Propositions 4.4 and 4.13, we also need establish (4.4) and (4.5). We now establish these inequalities and prove the following proposition.

Proposition 4.10 *Suppose the initial metric of Ricci flow is chosen as the modified Yamabe metric for sufficiently small ϵ in Theorem A. Then there exists $\widetilde{T}$ which does not depend on ϵ such that for $0 \leq t \leq \widetilde{T}$ all conditions of Proposition 4.4 are satisfied.*

Proof It is clear from Lemma 4.1 that if we choose sufficiently small ϵ in Theorem A, we can establish (4.3) for arbitrary small ϵ_0. On $0 \leq t \leq T$, (4.3) implies that

$$\int |E|^2 \, dv \leq \frac{1}{2} \epsilon_0, \quad \int ||W^-||^2 \, dv \leq \frac{1}{2} \epsilon_0, \quad \int \left(R - \bar{R} \right)^2 \, dv \leq \frac{1}{2} \epsilon_0. \qquad (4.29)$$

Recall $b_1(M) = 0$, $b_2^-(M) = 0$, and $b_2^+(M) = 1$. From signature and Chern–Gauss–Bonnet formula, we obtain

$$\int ||W^+||^2 \, dv = 12\pi^2 + \int ||W^-||^2 \, dv \leq 12\pi^2 + \frac{1}{2} \epsilon_0 \qquad (4.30)$$

and thereby

$$12\pi^2 \geq 4 \int \sigma_2(P) dv = 24\pi^2 - \int ||W||^2 \, dv \geq 12\pi^2 - \epsilon_0. \qquad (4.31)$$

Now with the same argument in Lemma 2.7, we can derive

$$\frac{1}{96} Y(t)^2 \geq \int \sigma_2 \left(P_{g(t)} \right) dv_{g(t)} \geq 2\pi^2 \qquad (4.32)$$

if we choose sufficiently small ϵ_0. Since the initial metric is of positive Yamabe type and the square of Yamabe constant has a strictly positive lower bound, we have established (4.4).

Note that (4.29) and (4.31) imply that for some $C > 0$

$$\frac{1}{C^2} \leq \int \bar{R}_{g(t)}^2 dv_{g(t)} = \bar{R}_{g(t)}^2 vol\left(M, g(t) \right) \leq C^2, \quad \frac{1}{C^2} \leq \int R_{g(t)}^2 dv_{g(t)} \leq C^2. \, (4.33)$$

To establish (4.5), it now suffices to derive a uniform lower bound for the volume since $\int \bar{R}^2 dv$.

Lemma 4.11 *Under conditions of Proposition 4.10 with $vol(M, g(0)) = 1$, along Ricci flow, there exists constant $T' > 0$ such that for $0 \leq t \leq T' \leq T$*

$$\frac{9}{4} \geq vol\left(M, g(t) \right) \geq \frac{1}{4}. \qquad (4.34)$$

In addition, there exists $T_1 > 0$ which does not depend on ϵ_0 such that $T' > T_1$.

Proof Recall the evolution equation for volume under the Ricci flow:

$$\frac{d}{dt} \int dv = -\int R dv. \tag{4.35}$$

Hence, for $0 \leq t \leq T$

$$\frac{d}{dt} \int dv \geq -\int |R| dv \geq -\left(\int R^2 dv\right)^{1/2} \left(\int dv\right)^{1/2} \geq -C \left(\int dv\right)^{1/2}.$$

Similarly,

$$\frac{d}{dt} \int dv \leq C \left(\int dv\right)^{1/2}.$$

It is then easy to derive

$$|\sqrt{vol(t)} - \sqrt{vol(0)}| \leq Ct.$$

From this inequality, it is easy to choose $T' = \min\{\frac{1}{2C}, T\}$ such that (4.34) is satisfied. It is easy to see such a T_1 exists since $T \geq T_0$, where T_0 does not depend on ϵ_0. $\qquad\square$

With Lemma 4.11, we establish (4.5) on $0 \leq t \leq T' \leq T$. If we choose $\widetilde{T} = T'$, all conditions of Proposition 4.4 are satisfied on $0 \leq t \leq \widetilde{T}$ and clearly $\widetilde{T}$ has a positive universal lower bound. $\qquad\square$

We now derive integral estimates for G_3. For the sake of clearness, we first establish the estiamtes for $\int |E|^3 dv$ and then derive a similar evolution inequality for $\int G_3 dv$ as we did in Lemma 4.4.

Lemma 4.12 *Under the conditions of Proposition 4.4, along the Ricci flow, we have*

$$\int |E_{g(t)}|^3 dv_{g(t)} \leq C\epsilon_0^{3/2} t^{-1},$$

for any $0 < t \leq \widetilde{T}$, where C is a universal constant which does not depend on ϵ_0.

Proof It is clear from (4.24) that

$$\frac{d}{dt} \int |E|^2 dv + C \left(\int |E|^4 dv\right)^{1/2} \leq C \int |E|^2 dv, \tag{4.36}$$

where C is a constant which does not depend on ϵ_0. From (4.8), we can compute

$$\frac{\partial}{\partial t} |E|^p = \frac{p}{2} |E|^{p-2} \frac{\partial}{\partial t} |E|^2$$

$$= \frac{p}{2} |E|^{p-2} \left(\Delta |E|^2 - 2|\nabla E|^2 + 4WEE - 4tr E^3 + \frac{2}{3} R |E|^2\right), \tag{4.37}$$

Note that

$$\Delta |E|^p = \frac{p}{2} |E|^{p-2} \Delta |E|^2 + p(p-2) |E|^{p-2} |\nabla |E||^2.$$

From this identity and (4.37), it is easy to derive for $p \geq 3$

$$\frac{d}{dt} \int |E|^p \, dv = \int \left(\frac{\partial}{\partial t} |E|^p - R |E|^p \right) dv$$

$$\leq \int \left(-p(p-2) |E|^{p-2} |\nabla |E||^2 + 2p |E|^2 \, WEE \right) dv$$

$$+ \int \left(-2p |E|^{p-2} \, tr E^3 + \left(\frac{p}{3} - 1 \right) R |E|^p \right) dv \qquad (4.38)$$

$$= \int \left(-\frac{4(p-2)}{p} |\nabla |E|^{\frac{p}{2}}|^2 - \frac{2(p-2)}{3p} R |E|^p \right) dv$$

$$+ \int \left(2p |E|^2 \, WEE - 2p |E|^{p-2} \, tr E^3 + C_p R |E|^p \right) dv$$

We now estimate the terms in last line of (4.38)

$$\int ||E|^{p-2} \, tr E^3| dv \leq \left(\int |E|^2 \, dv \right)^{1/2} \left(\int |E|^{2p} \, dv \right)^{1/2} \qquad (4.39)$$

$$\int |E|^{p-2} \, WEE dv \leq \frac{\sqrt{6}}{3} \int \left(||W^-|| + ||W^+|| \right) |E|^p \, dv$$

$$\leq \frac{\sqrt{6}}{3} \left(\int ||W^-||^2 \, dv \right)^{1/2} \left(\int |E|^{2p} \, dv \right)^{1/2}$$

$$+ \frac{1}{6} \left(\int ((F^+)_-)^2 dv \right)^{1/2} \left(\int |E|^{2p} \, dv \right)^{1/2} + \frac{1}{6} \int R |E|^p dv \qquad (4.40)$$

$$\int |R| |E|^p dv \leq \int |R - \bar{R}| |E|^p \, dv + \bar{R} \int |E|^p \, dv$$

$$\leq \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int |E|^{2p} \, dv \right)^{1/2} + \bar{R} \int |E|^p \, dv \qquad (4.41)$$

From the smallness assumption of ϵ_0, it is now easy to derive

$$\frac{d}{dt} \int |E|^p \, dv + C_p \left(\int |E|^{2p} \, dv \right)^{1/2} \leq C_p \int |E|^p \, dv. \qquad (4.42)$$

In this proof, we shall only need this formula for $p = 2, 3$. Take two smooth cut-off functions ϕ_1 and ϕ_2 such that $0 \leq \phi_i \leq 1$ on $[0, \widetilde{T}]$ for $i = 1, 2$. Take $\tau < \tau' < \widetilde{T}$. For ϕ_1, we choose $0 \leq \phi_1 \leq 1$ on $[0, \tau]$ and $\phi_1 \equiv 1$ on $[\tau, \widetilde{T}]$. For ϕ_2, we choose $\phi_2 \equiv 0$ on $[0, \tau], 0 \leq \phi_2 \leq 1$ on $[\tau, \tau']$, and $\phi_1 \equiv 1$ on $[\tau', \widetilde{T}]$. Also assume $|\phi_i'|_{L^\infty}$ have appropriate bound. It is now easy to derive

$$\frac{d}{dt} \left(\phi_i \int |E|^p \, dv \right) + C \phi_i \left(\int |E|^{2p} \, dv \right)^{1/2} \leq C \left(\phi_i + |\phi_i'| \right) \int |E|^p \, dv, \quad (4.43)$$

for $i = 1, 2$ and $p = 2, 3$.

Set $p = 2$ and $i = 1$. Integrate (4.43) over $[0, t]$ for some $t > \tau'$:

$$\int |E|^2 \, dv + C \int_\tau^t \left(\int |E|^4 \, dv \right)^{1/2} ds \leq C \left(1 + \frac{1}{\tau} \right) \int_0^t \left(\int |E|^2 \, dv \right) ds \quad (4.44)$$

Set $p = 3$ and $i = 2$. Integrate (4.43) over $[0, t]$ for some $t > \tau'$:

$$\int |E|^3 \, dv + C \int_{\tau'}^t \left(\int |E|^6 \, dv \right)^{1/2} ds \leq C \left(1 + \frac{1}{\tau' - \tau} \right) \int_\tau^t \left(\int |E|^3 \, dv \right) ds$$

(4.45)

Now we have

$$\int_\tau^t \left(\int |E|^3 \, dv \right) ds \leq \int_\tau^t \left(\int |E|^2 \, dv \right)^{1/2} \left(\int |E|^4 \, dv \right)^{1/2} ds$$

$$\leq C \epsilon_0^{1/2} \int_\tau^t \left(\int |E|^4 \right)^{1/2} ds$$

$$\leq C \epsilon_0^{1/2} \left(1 + \frac{1}{\tau} \right) \int_0^t \left(\int |E|^2 \, dv \right) ds$$

where second line follow from taking $\epsilon = \frac{1}{2} \epsilon_0$ in Lemma 5.5 and third line follows from (4.44). It then follows

$$\int |E|^3 \, dv \leq C \left(1 + \frac{1}{\tau' - \tau} \right) \int_\tau^t \left(\int |E|^3 \, dv \right) ds$$

$$\leq C \epsilon_0^{1/2} \left(1 + \frac{1}{\tau} \right) \left(1 + \frac{1}{\tau' - \tau} \right) \int_0^t \left(\int |E|^2 \, dv \right) ds$$

$$\leq C \epsilon_0^{3/2} \left(1 + \frac{1}{\tau} \right) \left(1 + \frac{1}{\tau' - \tau} \right) t$$

Take $\tau = \frac{1}{4} t$ and $\tau' = \frac{1}{2} t$ and we get desired estimate. In particular, if we choose $t \in [\frac{1}{4} \widetilde{T}, \widetilde{T}]$, we have

$$\sup_{\widetilde{T}/4 \leq t \leq \widetilde{T}} \int |E|^3 \, dv \leq C \epsilon_0^{3/2}.$$

(4.46)

$\square$

Now we prove an evolution inequality for $\int G_3 dv$ similar as (4.6).

Proposition 4.13 *Under the same conditions of Proposition 4.4, for $\frac{1}{4} \widetilde{T} \leq t \leq \widetilde{T}$, we have*

$$\frac{d}{dt} \int_M G_3(t) \, dv_{g(t)} \leq \widetilde{a}' \int_M G_3(t) \, dv_{g(t)} - \widetilde{b}' \left(\int_M G_6(t) dv_{g(t)} \right)^{1/2},$$

(4.47)

where $\widetilde{a}'$ and $\widetilde{b}'$ are uniform positive constants independent of ϵ_0 and we may choose $\widetilde{a}' = c_1 a$ and $\widetilde{b}' = c_2 b$, where c_1 and c_2 are universal positive constants.

Proof The proof is similar to that of (4.6). From the definition of G_k, we have

$$\frac{d}{dt} \int G_3 dv = \frac{d}{dt} \int |E|^3 \, dv + \frac{d}{dt} \int |R - \bar{R}|^3 dv$$

$$+ \frac{d}{dt} \int ||W^-||^3 \, dv + \frac{d}{dt} \int |(F^+)_-|^3 dv.$$

(4.48)

Note that we shrink the time interval to $[\frac{1}{4} \widetilde{T}, \widetilde{T}]$, so from the previous lemma, we have known that $\int |E|^3$ is bounded by $c(\epsilon_0)$, where $c(\epsilon_0) \to 0$ as $\epsilon_0 \to 0$.

We now estimate the right hand side of (4.48) term by term

$$\frac{d}{dt} \int |E|^3 \, dv = \int \left(\frac{\partial}{\partial t} |E|^3 - R |E|^3 \right) dv$$

$$= \int \left(\frac{3}{2} |E| \left(\Delta |E|^2 - 2|\nabla E|^2 + 4WEE - 4tr E^3 \right) \right) dv$$

$$\leq -c_2 Y \left(\int |E|^6 \, dv \right)^{1/2} + c_1 \bar{R} \int |E|^3 \, dv, \qquad (4.49)$$

where the inequality is established similarly to (4.24).

$$\frac{d}{dt} \int ||W^-||^3 \, dv = \int \left(\frac{\partial}{\partial t} ||W^-||^3 - R ||W^-||^3 \right) dv$$

$$\leq \int \left(3 ||W^-||^2 \Delta ||W^-|| + 3\sqrt{6} ||W^-||^3 + \frac{\sqrt{6}}{2} ||W^-||^2 |E|^2 - R ||W^-||^3 \right) dv$$

$$(4.50)$$

Recall convexity inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad (4.51)$$

for $a, b \geq 0$ and $1/p + 1/q = 1$. Take $p = 3/2$, $q = 3$, $a = \left(\int ||W^-||^6 \, dv \right)^{1/3}$ and $b = \left(\int |E|^3 \, dv \right)^{2/3}$. We have

$$\int ||W^-||^2 |E|^2 \, dv \leq \left(\int ||W^-||^6 \, dv \right)^{1/3} \left(\int |E|^3 \, dv \right)^{2/3}$$

$$\leq \frac{2K}{3} \left(\int ||W^-||^6 \, dv \right)^{1/2} + \frac{1}{3K} \left(\int |E|^3 \, dv \right)^2 \qquad (4.52)$$

We may take K to be a small multiple of the Yamabe constant and absorb the first term of (4.52) by the Sobolev inequality. The second term of (4.52) is bounded by a constant multiple of $\int |E|^3 \, dv$ from (4.46) and the smallness assumption of ϵ_0. It is then easy to derive from Sobolev inequality and Cauchy–Schwartz inequality

$$\frac{d}{dt} \int ||W^-||^3 \, dv \leq -c_2 Y \left(\int ||W^-||^6 \, dv \right)^{1/2} + c_1 \bar{R} \int ||W^-||^3 \, dv + C \int |E|^3 \, dv$$

$$(4.53)$$

$$\frac{d}{dt} \int |(F^+)_-|^3 dv = -\frac{d}{dt} \int \left((F^+)_- \right)^3 dv$$

$$= \int \left(-3 \left((F^+)_- \right)^2 \frac{\partial}{\partial t} \left((F^+)_- \right) - R |(F^+)_-|^3 \right) dv \qquad (4.54)$$

$$\leq -c_2 Y \left(\int |(F^+)_-|^6 dv \right)^{1/2} + c_1 \bar{R} \int |(F^+)_-|^3 dv$$

The inequality follows from evolution inequality (4.12), Sobolev inequality and the following trick:

$$\int R |(F^+)_-|^3 dv \leq \int (R - \bar{R}) |(F^+)_-|^3 dv + \bar{R} \int |(F^+)_-|^3 dv$$

$$\leq \left(\int (R - \bar{R})^2 \right)^{1/2} \left(\int |(F^+)_-|^6 dv \right)^{1/2} + C \int |(F^+)_-|^3 dv$$

(4.55)

Note the first term of the last line can be absorbed by Sobolev inequality with the smallness asssumption of ϵ_0.

$$\frac{d}{dt} \int |R - \bar{R}|^3 dv = \int \left(\frac{\partial}{\partial t} |R - \bar{R}|^3 - R|R - \bar{R}|^3 \right) dv$$

$$= \int \left(\frac{3}{2} |R - \bar{R}| \frac{\partial}{\partial t} (R - \bar{R})^2 - R|R - \bar{R}|^3 \right) dv \qquad (4.56)$$

$$\frac{\partial}{\partial t} (R - \bar{R})^2 = \frac{\partial}{\partial t} R^2 - 2 \frac{\partial}{\partial t} (R\bar{R}) + \frac{\partial}{\partial t} \bar{R}^2$$

$$= \Delta(R^2) - 2|\nabla R|^2 + 4R|E|^2 + R^3 - 2R \frac{\partial}{\partial t} \bar{R} - 2\bar{R} \frac{\partial}{\partial t} R + 2\bar{R} \frac{\partial}{\partial t} \bar{R}$$

$$= \Delta(R - \bar{R})^2 - 2|\nabla(R - \bar{R})|^2 + 4(R - \bar{R})|E|^2$$

$$+ R^2(R - \bar{R}) - 2(R - \bar{R}) \frac{\partial}{\partial t} \bar{R} \qquad (4.57)$$

Recall the evolution equation of $\bar{R}$ under Ricci flow:

$$\frac{\partial}{\partial t} \bar{R} = \frac{1}{vol} \int \left(2|E|^2 - \frac{1}{2} R^2 \right) dv + \bar{R}^2. \qquad (4.58)$$

Plugging (4.57) and (4.58) into (4.56), we can derive

$$\frac{d}{dt} \int |R - \bar{R}|^3 dv \leq \int \left(\frac{3}{2} |R - \bar{R}| \Delta(R - \bar{R})^2 - R|R - \bar{R}|^3 \right) dv$$

$$+ 6 \int (R - \bar{R})^2 |E|^2 dv + \frac{3}{2} \int |R^2 - \bar{R}^2|(R - \bar{R})^2 dv \qquad (4.59)$$

$$+ \frac{3}{2} \left| \bar{R}^2 - \frac{\int R^2 dv}{vol} \right| \int (R - \bar{R})^2 dv + \frac{6}{vol} \int |E|^2 dv \int (R - \bar{R})^2 dv.$$

Now we estimate the terms in second and third line of (4.59):

$$\int (R - \bar{R})^2 |E|^2 dv \leq \left(\int |R - \bar{R}|^6 dv \right)^{1/3} \left(\int |E|^3 dv \right)^{2/3}$$

$$\leq \frac{2K}{3} \left(\int |R - \bar{R}|^6 dv \right)^{1/2} + \frac{1}{3K} \left(\int |E|^3 dv \right)^2 \qquad (4.60)$$

We may take K to be a small multiple of the Yamabe constant and absorb the first term of (4.60) by the Sobolev inequality. The second term of (4.60) is bounded by a constant multiple of $\int |E|^3 dv$ from (4.46) and the smallness assumption of ϵ_0.

$$\int |E|^2 dv \int (R - \bar{R})^2 dv \leq C \left(\int |E|^3 dv \right)^{2/3} \left(\int (R - \bar{R})^6 dv \right)^{1/3}$$

$$\leq CK \left(\int (R - \bar{R})^6 dv \right)^{1/2} + \frac{C}{K} \left(\int |E|^3 dv \right)^2 \qquad (4.61)$$

The C in first line just depends on volume. For the second line, we may take K to be a small multiple of the Yamabe constant and absorb the first term of (4.61) by the Sobolev inequality.

The second term of (4.61) is bounded by a constant multiple of $\int |E|^3 \, dv$ from (4.46) and the smallness assumption of ϵ_0.

$$
\begin{aligned}
\int \left| R^2 - \bar{R}^2 \right| (R - \bar{R})^2 dv &= \int \left| R + \bar{R} \right| \left| R - \bar{R} \right|^3 dv \\
&\leq \bar{R} \int (R - \bar{R})^3 dv + \int |R| \left| R - \bar{R} \right|^3 dv \\
&\leq 2\bar{R} \int (R - \bar{R})^3 dv + \int \left| R - \bar{R} \right|^4 dv \\
&\leq 2\bar{R} \int (R - \bar{R})^3 dv + \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int \left| R - \bar{R} \right|^6 dv \right)^{1/2}
\end{aligned}
\tag{4.62}
$$

The last term can be absorbed by Sobolev inequality from the smallness assumption of ϵ_0.

$$
\begin{aligned}
\left| \bar{R}^2 - \frac{\int R^2 dv}{vol} \right| \int (R - \bar{R})^2 dv &= \frac{1}{vol} \left(\int (R - \bar{R})^2 dv \right)^2 \\
&\leq C \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int (R - \bar{R})^6 dv \right)^{1/2}
\end{aligned}
\tag{4.63}
$$

This term can be absorbed by Sobolev inequality from the smallness assumption of ϵ_0.

Combining all these estimates for $|R - \bar{R}|$, we derive

$$
\frac{d}{dt} \int |R - \bar{R}|^3 dv \leq -c_2 Y \left(\int |R - \bar{R}|^6 dv \right)^{1/2} + c_1 \bar{R} \int |R - \bar{R}|^3 dv + C \int |E|^3 dv
\tag{4.64}
$$

Now we combine (4.49), (4.50), (4.54), (4.64) to derive 4.47. □

Lemma 4.14 *With the modified Yamabe metric chosen as initial metric, under the Ricci flow, we have*

$$
\sup_{\widetilde{T}/2 \leq t \leq \widetilde{T}} \int G_3(t) dv_{g(t)} \leq C \epsilon_0^{3/2}.
$$

Proof The proof is fundamentally the same as that of Lemma 4.12. The only difference is to replace $|E|^k$ by $G_k(t)$ since we have evolution equations of same type as is shown in Proposition 4.4 and Proposition (4.13). □

To derive the L^∞-boundedness, we shall apply the following result established by Yang in [39].

Lemma 4.15 *Assume that with respect to the metric $g = g(t)$, $0 \leq t \leq T$, the following Sobolev inequality holds:*

$$
\left(\int |\varphi|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq C_S \left[\int |\nabla \varphi|^2 dv + \int \varphi^2 dv \right], \varphi \in W^{1,2}(M^n).
\tag{4.65}
$$

Also, let $b \geq 0$ on $M^n \times [0, T]$ satisfy

$$
\frac{\partial}{\partial t} dv \leq b dv.
\tag{4.66}
$$

Let $q > n$, and suppose $u \geq 0$ is a function on $M^n \times [0, T]$ satisfying

$$\frac{\partial u}{\partial t} \leq \Delta u + bu, \tag{4.67}$$

and that

$$\sup_{0 \leq t \leq T} |b|_{L^{q/2}} \leq \beta. \tag{4.68}$$

Given $p_0 > 1$, there exists a constant $C = C(n, q, p_0, C_S, \beta)$ such that for $0 \leq t \leq T$,

$$|u(t, \cdot)|_\infty \leq C e^{Ct} t^{-\frac{n}{2p_0}} |u(0, \cdot)|_{p_0}. \tag{4.69}$$

Moreover, given $p \geq p_0 > 1$, the following inequality holds for $0 \leq t \leq T$:

$$\frac{d}{dt} \int u^p dv + \int \left| \nabla \left(u^{p/2} \right) \right|^2 dv \leq C p^{\frac{2n}{q-n}} \int u^p dv \tag{4.70}$$

where $C = C(n, q, p_0, C_S)$.

Lemma 4.16 *With the modified Yamabe metric chosen as the initial metric, we have*

$$\sup_{3\widetilde{T}/4 \leq t \leq \widetilde{T}} \{|E| + |R - \bar{R}| + ||W^-|| + |F_-^+|\} \leq C\epsilon_0.$$

Proof Apply Lemma 4.15 to $u = G_2$, $q = 6 > n = 4$ and $p_0 = 3/2$ on $t \in [\widetilde{T}/2, \widetilde{T}]$. Condition (4.68) is satisfied by Lemma 4.14. Hence, we can prove the desired estimate. □

Now recall the Bernstein–Bando–Shi estimate (see for example Chapter 7 of [11]).

Lemma 4.17 *Let $(M^4, g(t))$ be a solution to the Ricci flow. For every $m \in \mathbb{N}$, there exists a constant C_m depending only on m such that if*

$$\sup_{x \in M} |Rm(x, t)|_{g(t)} \leq K, \quad t \in \left[0, \frac{1}{K}\right],$$

then

$$\sup_{x \in M} |\nabla^m Rm(x, t)|_{g(t)} \leq \frac{C_m K}{t^{m/2}}, \quad t \in \left[0, \frac{1}{K}\right],$$

Now we are at the position to prove Theorem A.

Proof of Theorem A We argue by contradiction. Suppose there is a sequence of manifolds (M_j, g_j) satisfying $\beta(M_j, [g_j]) < 4(1 + \epsilon_j)$ with $\epsilon_j \to 0$ and each of them is *not* diffeomorphic to standard $\mathbb{CP}^2$. For each conformal class $[g_j]$, we choose the modified Yamabe metric $(g_j)_G$ as initial metric and evolve the metric along Ricci flow. Then Lemma 4.16 and Lemma 4.17 will imply that there is a time $\widetilde{T}$ such that the curvatures of $g_j(\widetilde{T})$ are uniformly bounded in C^∞-norm and the Sobolev constants are also uniformly bounded. The convergence theory [9] established by Cheeger et al. then shows that there is a subsequence of $\{(M_j, g_j(\widetilde{T}))\}$ which converges smoothly to a manifold (M_∞, g_∞). As $\epsilon_j \to 0$, we obtain that (M_∞, g_∞) satisfies

$$\int_{M_\infty} ||W||^2 dv_\infty = 4 \int_{M_\infty} \sigma_2 dv_\infty$$

Note that we also have $b_1(M_\infty) = 0$, $b_2^+(M_\infty) = 1$ and $b_2^-(M_\infty) = 0$. Hence, by Chern–Gauss–Bonnet and signature formula, we can easily derive that (M_∞, g_∞) is self-dual

Einstein. The same argument in Lemma 3.5 will show that (M_∞, g_∞) is conformal equivalent to $(\mathbb{CP}^2, g_{FS})$. Since the convergence is smooth, we thereby obtain that (M_j, g_j) must be diffeomorphic to $\mathbb{CP}^2$ with standard differentiable structure when j is sufficiently large. This is clearly a contradiction to our assumption. Hence, we have proved the theorem. □

Proof of Theorem B To prove Theorem B, suppose M^4 is oriented with $b_2^+(M^4) > 0$. If $\beta(M^4) = 4$, then by definition we can find a metric g with

$$\beta(M^4, [g]) < 4(1 + \epsilon/2),$$

where $\epsilon > 0$ is from Theorem A. From Theorem A we conclude that M^4 is diffeomorphic to $\mathbb{CP}^2$. In addition, if g is a metric on $\mathbb{CP}^2$ for which $\beta(M^4, [g]) = 4$, then taking $\epsilon = 0$ in Lemma 2.7 we see that g is self-dual. It follows, for example, from [36] that $(M^4, [g])$ is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$. □

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