Fast and Efficient Evaluation of Integrals Arising in Iso-geometric Analysis

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***Abstract*—Integral equation based analysis of scattering (both acoustic and electromagnetic) is well known and has been studied for several decades. Analysis typically proceeds along the following lines: representation of the geometry using a collection of triangles, representation of physics using piecewise constant basis function defined on each triangle, and then solving the resulting discrete system. In the past few decades, this area has seen several improvements in algorithms that reduce the computational complexity of analysis. But, as we delve into higher order isogeometric analysis (IGA), these algorithms are bogged down by the cost of integration. In this paper, we seek to address this challenge. Our candidate for modeling geometry and physics are subdivision basis sets. The order of these basis sets is sufficiently high to make the challenge apparent. We will present a methodology to ameliorate the cost for both acoustic and electromagnetic integral equations and demonstrate its efficacy.**

I. INTRODUCTION

Surface integral equation (SIE) methods for computing fields scattered from piecewise homogeneous scatterers have been well studied and understood [1]. Their applications has grown in a number of areas thanks to advances in algorithms that significantly reduce the computational complexity. Over the past couple decades, there has been advances in both development of higher order basis sets and representation of geometry enabling more accurate solutions to SIE with fewer degrees of freedom. All these advancements have been paired with fast methods resulting in reliable solutions at a computational complexity that scales as $\mathcal{O}(N_s \log N_s)$, where N_s is the number of degrees of freedom. However, the constant of proportionality hides the impeding cost of integration that arises in higher order representations.

The cost of any integral equation method comprises of (a) matrix evaluation and (b) evaluation of matrix-vector product that scales as $\mathcal{O}(N_s^2 p^2)$ and $\mathcal{O}(N_s^2)$, respectively; p is the order of the quadrature rule. Under certain assumptions, the cost reduction for both these operations using fast multipole method is $\mathcal{O}(N_s p^2)$ and $\mathcal{O}(N_s \log N_s)$. These assumptions are carryovers from low order analysis wherein the domain of support of each basis is small. Unfortunately, these assumptions do not hold for higher geometry representation whose *objective* is to better represent the geometry using fewer, but more significant, sub-domains [2]. Therefore, higher order analysis demands more integration points, which leads to p quickly dominating the overall cost.

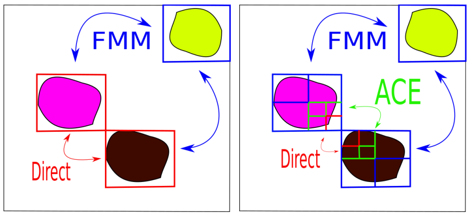
Whereas [2] developed methods to address these problems in generalized method of moments, we take this a step further. While our overall goal is the same, we implement a subdivision-based IGA; IGA enables the same basis function (subdivision) to represent both the geometry and physics. The resulting geometry is C^2 almost everywhere, and as a result p needs to be high. To address these problems, we invoke wideband multilevel fast multipole acceleration (MLFMA) to ameliorate costs such that the overall cost for both operations scales as $\mathcal{O}(N_p p)$ and $\mathcal{O}(N_p \log N_p)$, where $N_p = \nu N_s$, where the constant $0 < \nu < 0.3$. The rest of the paper, elaborates upon this procedure and presents resuls for acoustics. Resuts for electromagnetics will be presented at the conference.

II. FORMULATION AND DISCRETIZATION

Consider a scatter that occupies a region $\Omega \in \mathbb{R}^3$, whose surface $\partial\Omega$ is sufficiently smooth such that one can define a unique normal $\hat{n}(\mathbf{r})$. Assume an incident plane wave, in the case of electromagnetics, on a perfect electrically conducting body and hard body in the case of acoustics. The induced current/pressure may be obtained from a surface integral equation (SIE) that is formulated using Green's theorems, imposing boundary condition on $\partial\Omega$ and infinity [1]. Let $\mathbf{J}(\mathbf{r})$ denote an equivalent electric current on $\partial\Omega$ and $\phi(\mathbf{r})$ denote the velocity potential. Then one can prescribe either a combined field integral equation or a Burton-Miller equation to obtain a solution to the current/potential that is unique at all frequencies; these are prescribed in general using

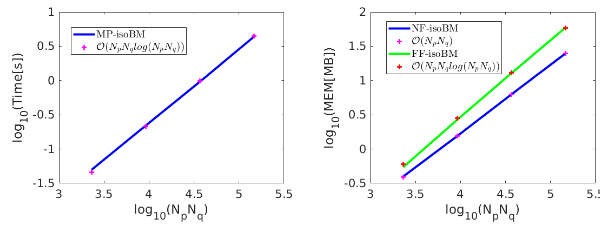
$$\mathcal{L}\left\{\xi(\mathbf{r})\right\} = \zeta^i(\mathbf{r}) \tag{1}$$

for both electromagnetics and acoustics. Specifically, for electromagnetics, $\mathcal{L} = \mathcal{L}_{\text{CFIE}}$, $\xi(\mathbf{r}) = \mathbf{J}(\mathbf{r})$, and $\zeta^i(\mathbf{r}) = \mathbf{f}^i(\mathbf{r})$. For acoustics, $\mathcal{L} = \mathcal{L}_{\text{BM}}$, $\xi(\mathbf{r}) = \phi(\mathbf{r})$, and $\zeta^i(\mathbf{r}) = f^i(\mathbf{r})$; the explicit details of both can be obtained from [3], [4]. Discretization of this system proceeds as follows: one discretizes Ω using a collection of patches such that $\Omega = \bigcup_l^{N_p} \Omega_l$, where Ω_n denotes a patch. Then each patch, Ω_l, supports a unique, locally indexed, set of basis functions $\psi_n(\mathbf{r})$, $n \in \{1, \cdots, N_v\}$. In other words, each basis function, ψ_n, is defined over a unique collection of patches $\Gamma_n = \{\Omega_\alpha, \cdots, \Omega_l, \cdots, \Omega_\beta\}$. The basis functions for $\mathbf{J}(\mathbf{r})$ can be constructed in terms of $\psi_n(\mathbf{r})$ as $\mathbf{J}_{n,1}(\mathbf{r}) = \nabla_s \psi_n(\mathbf{r})$, and $\mathbf{J}_{n,2} = \hat{n}(\mathbf{r}) \times \nabla_s \psi_n(\mathbf{r})$ [3]. In what follows, the basis functions for both acoustics and electromagnetics are represented using φ_n, and is to be understood in the appropriate context. In the Galerkin framework, one obtains a linear system of the form $\mathcal{Z}\mathbf{I} = \mathcal{V}$, where $\mathbf{I}$ is a column of



(a) (b)

Fig. 1. Comparison of interactions between FMM and wideband MLFMA. (a) FMM: Limit on traditional leaf box size results in large patches computed directly. (b) Wideband MLFMA: large patches may be split between nearfield and farfield maintaining favorable scaling.



(a) (b)

Fig. 2. Comparison of interactions between FMM and wideband MLFMA on a series of boxes of size $1.151\lambda - 9.212\lambda$. (a) Scaling of matrix vector product timings (b) Scaling in nearfield and farfield memory.

unknowns coefficients of basis sets, $V_m \in \mathcal{V} = \langle \varphi_m(\mathbf{r}), \zeta^i(\mathbf{r}) \rangle$ and $Z_{mn} \in \mathcal{Z} = \langle \varphi_m(\mathbf{r}), \mathcal{L}\left\{\varphi_n(\mathbf{r})\right\} \rangle$.

III. Evaluation of Inner products

The crux of this paper is overcome the dominating cost of evaluating a matrix element Z_{mn}. We can illustrate the cost by considering a pair of fourth order basis function interacting; assume that the order of integration for evaluating the inner products scale as $p \propto 16$ per triangle, this gives an effective scaling of $p^2 \propto 256N_\Delta^2$ where N_Δ is the number of patches associated with the basis function m and n. The approach we espouse to handle this higher order integration rule is to construct an adaptive quadrature scheme that can be accelerated via wideband MLFMA. The scheme beings by assuming that each triangle in the domain Γ_n of the nth basis function is partitioned recursively into 4^l subpatches where $l \in \mathbb{N}$. The domain of each of these partitions is now denoted by Γ_n^γ for $\gamma = 1, \cdots, 4^l N_\Delta$. It follows that any matrix element can be computed in terms of its partial contributions such that

$$Z_{mn} = \sum_{\alpha,\beta} \int_{\Gamma_m^\alpha} \varphi_m(\mathbf{r}) \mathcal{L}^\beta \left[\varphi_n(\mathbf{r}), \Gamma_n^\beta\right] dS = \sum_{\alpha,\beta} Z_{mn}^{\alpha\beta} \quad (2)$$

where $\mathcal{L}^\beta$ denotes the evaluation of the operator over the domain Γ_n^β, and the summations is over the number of source and observation subpatches. We note the following:

- The size of each sub-patch is typically sub-wavelength; often less than a tenth of a wavelength.
- If the interaction is a self interaction, i.e., $m = n$, and $\Gamma_m^\alpha = \Gamma_n^\beta$, one needs an self-integration method tailored to the order of the singularity.

Once the patches have been partitioned, we use a tree based algorithm to reduce the cost of evaluating equation (2). The algorithm makes use of the Accelerated Cartesian Expansions (ACE) at leaf levels due its robustness at small length scales [5]. This allows seamlessly transitions to MLFMA at higher levels as illustrated in Fig. 1. The use of ACE permits leaf sizes to be arbitrarily small, permitting population of the tree using sub-triangles. Note, one needs a mixed potential formulation for electromagnetic problems as elucidated in [2].

IV. Results

In section we show studies of the algorithm's scaling for both complexity and storage in terms of total number of number of quadrature points per patch or N_pN_q; we note that all experiments are conducted on a series of boxes of size $1.151\lambda - 9.212\lambda$. In Fig. 2(a), we study a single matrix-vector product, where significant contribution stems from the farfield, allowing the scaling to behave as $\mathcal{O}(N_pN_qlog(N_pN_q))$. This results is as expected due to our system being higher order, thus demanding a high order integration rule. Scaling in storage for the near and far fields are shown in Fig. 2(b). As expected, the nearfield memory scales linearly with respect to N_pN_q. The farfield memory, for our given experiment, scales as $\mathcal{O}(N_pN_qlog(N_pN_q))$ which is mainly attributed to the precomputation of the aggregation/disaggregation and translation operator.

V. Conclusion

In this paper, we have developed a technique to ameliorate costs associated with evaluation of integrals in higher order analysis for both scalar and vector integral equations. The adaptive quadrature together with wideband MLFMA makes this possible. We will demonstrate applications of this technique to a variety of challenging problems at the meeting.

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