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Hankel transform, Langlands functoriality and functional equation of automorphic L-functions*

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Dedicated to the memory of Prof. Hiroshi Saito, with affection

Abstract This is a survey on recent works of Langlands's work on functoriality conjectures and related works including the works of Braverman and Kazhdan on the functional equation of automorphic L-functions. Efforts have been made to carry out in complete generality the construction of the L-monoid, and certain a kernel which is, we believe, related to the elusive Hankel kernel.

Keywords and phrases: Langlands functoriality, Hankel transform, functional equation

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1. Introduction

The functoriality conjecture of Langlands, as stated in [29], offers a general organizational scheme for all automorphic representations. This conjecture can be roughly stated as follows: if H and G are reductive groups defined over a global field k, $\xi : {}^L H \to {}^L G$ a homomorphism between their Langlands dual groups, then it is possible to attach to each automorphic representation $\pi^H = \bigotimes_v \pi_v^H$ of H a packet of automorphic representations $\pi = \bigotimes_v \pi_v$ of

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G. In the formula $\pi = \bigotimes_v \pi_v$, local components π_v are irreducible admissible representations of $G(k_v)$, k_v being the completion of k at a place $v \in |k|$. The automorphic functoriality is expected to be of local nature in the sense that it should be compatible with the local functoriality principle that consists in a map from the set of packets of irreducible admissible representations of $H(k_v)$ to the set of packets of irreducible admissible representations of $G(k_v)$. Since unramified representations of $H(k_v)$ at an unramified non-archimedean place v are parametrized by certain semi-simple conjugacy classes in ${}^L H$, the local functoriality is required to be compatible with the transfer of conjugacy classes from ${}^L H$ to ${}^L G$.

A crucial part of the functoriality program, known as endoscopy, is now almost completely understood thanks to cumulative efforts of many mathematicians over the last 40 years, see [5]. Its main ingredients include in particular the construction of the general trace formula, mainly due to Arthur [4] and the proof of the transfer conjecture and the fundamental lemma, see [52] and [37]. The endoscopic case is concerned with the case where ${}^L H$ is essentially the centralizer of a semi-simple element in ${}^L G$. In spite of this severe restriction, the endoscopic case is crucial in the general picture for it gives a sense to the concept of packet, both in local and automorphic settings, along with global multiplicity formulae [31].

The endoscopic program also produced the stable trace formula expected to be an essential tool for future works on the conjecture of functoriality. In [32], Langlands proposed a conjectural limiting form of the stable trace formula aiming at isolating the part of the automorphic spectrum of G that comes from a smaller group H by functoriality. These limiting forms of trace formulae are designed to have a spectral expansion involving logarithmic derivatives of automorphic L-functions, as in [32], or a sum of automorphic L-functions themselves, as in [21].

Another possible route to functoriality, proposed by Braverman and Kazhdan in [11], aims first at the functional equations of automorphic L-functions. Historically, the functoriality conjecture springs out of the study of general automorphic L-functions. For every automorphic representation $\pi = \bigotimes_v \pi_v$ of G, and finite dimensional representation $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ of ${}^L G$, Langlands defined the L-function of complex parameter s as an Euler product

$$L(s, \pi, \rho) = \prod_v L_v(s, \pi_v, \rho) \tag{1.1}$$

with local factors

$$L_v(s, \pi, \rho) = \det(1 - \rho(\sigma_v) q_v^{-s})^{-1} \tag{1.2}$$

for unramified local component π_v, where σ_v is the corresponding semi-simple conjugacy class in ${}^L G$. Local factors at archimedean places as well as finitely many ramified places are expected to be defined independently by local means. It is known that the Euler product (1.1) converges absolutely for s in some right half plane and thus converges to a holomorphic function on that domain. It is conjectured that this function can be extended meromorphically to the whole complex plane, and with appropriate local factors at archimedean and ramified places being specified, it should satisfy a functional equation similar to the one of the Riemann ζ-function.

If an automorphic representation π^H of H transfers to an automorphic representation π of G via a homomorphism $\xi : {}^L H \to {}^L G$ of L-groups, then for every finite dimensional representation $\rho : {}^L G \to \mathrm{GL}(V_\rho)$, we must have

$$L(s, \pi^H, \rho \circ \xi) = L(s, \pi, \rho). \tag{1.3}$$

For $G = \mathrm{GL}_n$ and ρ the standard representation of the dual group $\mathrm{GL}_n(\mathbb{C})$, the meromorphic continuation and functional equation of $L(s, \pi, \rho)$ is a theorem of Godement and Jacquet [23]. The functoriality conjecture will allow us to derive the meromorphic continuation and functional equation of general automorphic L-functions from the special case of standard L-functions.

In [11], Braverman and Kazhdan proposed a generalization of Godement–Jacquet's framework to all automorphic L-functions. This framework consists in the construction of some non-standard Schwartz spaces, non-standard Fourier transform and Poisson summation formula. Braverman–Kazhdan's approach to functional equation would bypass the functoriality conjecture, and in fact, may give another route to functoriality if it is combined with suitable forms of the converse theorem.

In this paper, we attempt to give an up-to-date survey on Langlands's functoriality conjecture and the functional equation of automorphic L-functions. This is not an easy task as the topic is as broad as deep. On one hand, there is a depressing feeling that we are nowhere near the end of the journey, but on the other hand, there is a flurry of recent activities with new ideas flying into multiple directions and grounded on as different tools as analytic number theory, representation theory and algebraic geometry. For reviewing many different ideas would entail the risk to make the paper unreadable, thus useless, we will spend most of the time developing the ideas of Langlands and Braverman–Kazhdan and thereafter attempting to connect these to other developments.

Throughout this paper, we espouse the spirit of [29] in treating reductive groups in a complete generality. This is not merely for the sake of generality for itself, or its abstract elegance, the main new construction of the kernel of the Hankel transform relies very much on the fact that all constructions can be carried out canonically in complete generality.

Let us now review the organization of this paper. Materials exposed in Sect. 2 are more or less standard and mainly drawn from Langlands's pioneer paper [29]. We review carefully the root data with Galois action attached to a reductive group defined over an arbitrary base field and Langlands's construction of the L-group. We will also explain how to fix standard invariant measure on tori and reductive groups defined over local fields. These backgrounds are all ingredients in the construction of the Hankel kernel. After this, we review as much of the theory of automorphic forms as is needed to define automorphic L-functions and state the functoriality conjecture. At the end of Sect. 2, we will also briefly review the endoscopic theory. As this section clearly aims at a non-specialist audience, experts are invited to skip it entirely with the exception of Subsect. 2.3 which is not completely standard. Readers may also have to come back occasionally to Subsect. 2.1 to get an acquaintance with the notations we use for root data and Galois action thereon.

In Sect. 3, we review Langlands's proposal "Beyond endoscopy" [32] and further developments in [21] and [33]. Langlands's idea is to construct certain limiting form of the trace formula whose spectral development are weighted by numbers related to the poles of automorphic L-functions. Because the pole of L-functions of an automorphic form is expected to retain information on the functorial source, the smallest group from which the automorphic representation may transfer, we expect this limiting form of the trace formula to be helpful to corner the functoriality principle itself.

In Sect. 4, we review Braverman–Kazhdan's idea from [11] on a possible generalization of the Godement–Jacquet method. In the case of GL_n and the standard representation, the Godement–Jacquet method relies on the Fourier transform operating on the space of Schwartz functions on the space of matrices and the Poisson summation formula. Braverman and Kazhdan suggest for each G and ρ the existence of non-standard Schwartz spaces, Fourier transform and Poisson summation formula and also certain ideas on how to construct them.

In Sect. 5, we discuss the geometry underlying the non-standard Schwartz functions, to be called ρ-functions to emphasize the dependence on ρ. The ρ-functions are expected to be functions with compact support on a certain ρ-monoid that can be constructed canonically out of G and ρ. This construction has been hinted in [11], and emphasized in [38] and [10] in the case G split and ρ irreducible. Here we will construct the ρ-monoid in complete generality in particular without assuming neither G to be a split reductive group nor the representation ρ irreducible. In the toric case, the ρ-monoid is a toric variety. In this case, we emphasize the existence of certain toric stack of which our ρ-toric variety is the coarse space. The Schwartz ρ-functions in the toric case are functions of compact support on the ρ-toric variety that come from smooth functions on the toric stack by integration. We suspect in general the existence of something like a monoid stack without being able to make a precise conjecture. At the end of the section, we review the relation between the singularities of the arc space of the ρ-monoid with a distinguished Schwartz function that we call the basic function, following [10]. The space of ρ-functions has also been investigated by Laffogue [28].

In Sect. 6, we study the kernel of the non-standard Fourier transform mainly in the toric cases. In the toric case, the non-standard Fourier transform are closely related to the classical Hankel transform. The terminology of Hankel transform and Bessel function is inspired by classical harmonic analysis on Euclidean space, and a paper of Sally and Taibleson [44] on special functions of p-adic arguments. We will also recall the construction of Braverman and Kazhdan's construction of the kernel in the case of reductive group over finite fields [12]. We construct a general stably invariant function which gives rise to the correct kernel in the case of tori as well as in the Godement–Jacquet case.

In Sect. 7, we attempt to merge Langlands and Braverman–Kazhdan approach. The common ground here is the space of ρ-functions and its basic function. The trace formula, to be called the ρ-trace formula, envisioned by Langlands, will be an invariant distribution on the space of adelic ρ-functions and its spectral expansion is weighted by values of L-functions. The Poisson summation formula, envisioned by Braverman–Kazhdan can be integrated on the automorphic space and gives rise to a comparison of trace formulas between the ρ-trace formula and its dual. On the spectral side, this comparison is basically the functional equation of L-functions. As usual, it is tempting to approach the comparison from the geometric side. On the geometric side, the Hankel transform on ρ-function descends to a certain orbital Hankel transform on the space of orbital integrals of ρ-functions. A similar transform on the space of orbital integrals was introduced in [21] by Frenkel, Langlands and myself. The orbital Hankel transform and the FLN transform turn out to be completely different in nature: the orbital Hankel transform has a clear group theoretic interpretation; it is difficult to compute, and its Poisson summation formula seems to be very difficult to obtain, the FLN is not known to have a clear group theoretic interpretation, but satisfies in principle a Poisson summation formula, see [1]. The orbital Hankel transform has many similarity with Langlands's local stable transfer factor studied in [33] and Johnstone's thesis. It is noteworthy that the thread of ideas on the local stable transfer factor and the Poisson summation formula has been extensively developed by Sakellaridis in the more general context of spherical varieties and relative functoriality, see [41] and [43].

2. Automorphic L-functions and functoriality

In the epoch making paper [29], Langlands proposed certain general constructions and conjectures on automorphic representations that have shaped the development in this field. Most notably, he defined the L-group ${}^L G$ attached to any reductive group G defined over a local or global field, constructed automorphic L-function $L(s, \pi, \rho)$ of complex variable s attached to an automorphic representation π of G and an algebraic representation ρ of ${}^L G$, formulated the functoriality principle stipulating the transfer of automorphic representations from H to G every time there is a homomorphism ${}^L H \to {}^L G$, and explained how all these concepts may tie together to offer compelling picture of automorphic spectra. The purpose of this section is to briefly review Langlands's constructions and conjectures, mostly in their formal aspects.

2.1. Root data, Langlands dual group and L-group

We recall in this section the construction of the Langlands dual group and his L-groups. In following the surveys [49], [7] as well as the original source [29] in essence, we attempt to frame the discussion in a categorical language. This approach is efficient and necessary for later we want to perform certain constructions for general reductive groups using Galois descent.

First, we discuss the classification of tori by combinatorial data. A split torus over a field k is an algebraic group which is isomorphic to a product finitely many copies of the multiplicative group $\mathbb{G}_m$. For every split torus T over a field k, we will denote by

$$\Lambda(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

the group of cocharacters of T. The functor $T \mapsto \Lambda(T)$ is an equivalence of categories from the category of split tori over k to the category of free abelian groups of finite rank. In particular, it induces an isomorphism on the groups of automorphisms

$$\mathrm{Aut}(T) \longrightarrow \mathrm{Aut}(\Lambda(T))$$

the latter being isomorphic to $\mathrm{GL}_n(\mathbb{Z})$, where n is the rank of $\Lambda(T)$. We also have the inverse functor $\Lambda \mapsto \Lambda \otimes_k \mathbb{G}_m$, where the torus $\Lambda \otimes_k \mathbb{G}_m$ is defined to be the spectrum of the group algebra $k[\Lambda^\vee]$ with $\Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$.

By torus over a field k we mean an affine algebraic group over k whose base change to the separable closure $\bar{k}$ is isomorphic to a split torus. For a torus T over $\bar{k}$, we denote by $\mathrm{Aut}_{\bar{k}}(T)$ the group of $\bar{k}$-automorphisms of T and $\mathrm{Aut}_k(T)$ the group of k-automorphisms of T i.e., k-algebra automorphisms of the algebra $\bar{k}[T]$ preserving also its comultiplication. Such an automorphism preserves necessarily $\bar{k}$ and so induces an element of the Galois group Γ_k of k. We have thus a group homomorphism $\mathrm{Aut}_k(T) \to \Gamma_k$ whose kernel is $\mathrm{Aut}_{\bar{k}}(T)$ i.e., we have the exact sequence:

$$0 \longrightarrow \mathrm{Aut}_{\bar{k}}(T) \longrightarrow \mathrm{Aut}_k(T) \longrightarrow \Gamma_k$$

in which $\mathrm{Aut}_{\bar{k}}(T) = \mathrm{Aut}(\Lambda)$, where $\Lambda = \Lambda(T)$.

We call k-form of a T defined over $\bar{k}$ a torus T_k defined over k such that $T_k \otimes_k \bar{k} = T$. For every k-form T_k, we have a section $\Gamma_k \to \mathrm{Aut}_k(T)$. Since

k-forms exist, the homomorphism $\mathrm{Aut}_k(T) \to \Gamma_k$ is surjective i.e., we have an exact sequence

$$0 \longrightarrow \mathrm{Aut}(\Lambda) \longrightarrow \mathrm{Aut}_k(T) \longrightarrow \Gamma_k \longrightarrow 0.$$

The choice of a k-form of T gives rise to a splitting $\sigma : \Gamma_k \to \mathrm{Aut}_k(T)$, and conversely every splitting $\sigma : \Gamma_k \to \mathrm{Aut}_k(T)$ gives rise to a k-form, namely $T_k = \mathrm{Spec}(\bar{k}[T]^{\sigma(\Gamma_k)})$. We also note that the split k-form $T_k = \Lambda \otimes_k \mathbb{G}_m$ implies an isomorphism

$$\mathrm{Aut}_k(T) \simeq \mathrm{Aut}(\Lambda) \times \Gamma_k.$$

Thanks to the split k-form of T, a splitting $\sigma : \Gamma_k \to \mathrm{Aut}_k(T)$ is equivalent to a continuous homomorphism

$$\sigma_\Lambda : \Gamma_k \to \mathrm{Aut}(\Lambda).$$

To summarize, we have an equivalence of categories between the groupoid of tori over k and the groupoid of pairs (Λ, σ), where Λ is a free abelian group of finite rank and $\sigma : \Gamma_k \to \mathrm{Aut}(\Lambda)$ is a continuous group homomorphism. This equivalence of categories may be regarded as a classification of tori over k by combinatorial data.

In order to classify reductive groups, we will need more elaborate combinatorial structures. To be clear, being hindered of the presence of Galois cohomology, a classification of reductive groups up to isomorphism over an arbitrary base field is not possible but it is possible up to inner equivalence which is rougher.

A root datum $\Psi = (\Lambda, \Phi)$ consists of a free abelian group of finite rank Λ (group of cocharacters), a finite subset (set of roots)

$$\Phi \subset \Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$$

and an injective map $\Phi \to \Lambda$ denoted by $\alpha \mapsto \alpha^\vee$ whose image is denoted by $\Phi^\vee \subset \Lambda$ (set of coroots), satisfying the following properties:

- The equality $\langle \alpha, \alpha^\vee \rangle = 2$ holds for every $\alpha \in \Phi$;
- For every $\alpha \in \Phi$, the reflection $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ in $\Lambda^\vee$ preserves the set of roots Φ, and the reflection $s_\alpha(y) = y - \langle y, \alpha \rangle \alpha^\vee$ in Λ preserves the set of coroots $\Phi^\vee$.

The reflections s_α generate a finite group W, the Weyl group, acting on Λ and $\Lambda^\vee$ in preserving the set of roots $\Phi \subset \Lambda^\vee$ and the set of coroots $\Phi^\vee \subset \Lambda$.

In this paper, by reductive group over a field k, we mean a connected smooth affine group over k which becomes a split reductive group over a finite separable of k. A (T, G)-pair over a base field k is a pair consisting of a torus T defined over k and a reductive group G over a field k of which T is a maximal torus. Over a separably closed field $\bar{k}$, maximal tori in a reductive group are all conjugate.

A root datum $\Psi(T, G)$ can be attached to a (T, G)-pair defined over a separably closed field $\bar{k}$. We will only provide an incomplete account of this construction and refer to [49] for a more detailed treatment. We set $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$ and $\Lambda^\vee = \mathrm{Hom}(T, \mathbb{G}_m)$ to be the groups of cocharacters and characters of T. The set of roots Φ is given by the non-zero eigenvalues of T acting on the Lie algebra of G. The Weyl group W can be identified with the quotient $\mathrm{Nor}_G(T)/T$ of the normalizer $\mathrm{Nor}_G(T)$ of T in G by T. The functor $(T, G) \mapsto \Psi(T, G)$ from the groupoid $\mathrm{TG}_{\bar{k}}$ of (T, G)-pairs over $\bar{k}$ to the groupoid RD of root data

$$\mathrm{TG}_{\bar{k}} \longrightarrow \mathrm{RD} \tag{2.1}$$

induces a bijection on the sets of isomorphism classes. As to automorphisms of objects, we have a surjective homomorphism

$$\mathrm{Aut}_{\bar{k}}(T, G) \longrightarrow \mathrm{Aut}(\Psi(T, G)) \tag{2.2}$$

of kernel $T(\bar{k})/Z(\bar{k})$, where Z is the center of G. Indeed, automorphisms of (T, G) inducing identity on Λ are inner automorphism of G given by the conjugation by an element of $T(\bar{k})$.

Let $\mathrm{rG}_{\bar{k}}$ denote the groupoid of reductive groups over $\bar{k}$. We have a functor $\mathrm{TG}_{\bar{k}} \to \mathrm{rG}_{\bar{k}}$ given by $(T, G) \mapsto G$. Since maximal tori in a reductive groups over $\bar{k}$ are all conjugate, the functor $\mathrm{TG}_{\bar{k}} \to \mathrm{rG}_{\bar{k}}$ induces a bijection of the sets of isomorphism classes of those categories. As a result, we have a canonical bijection between the set of isomorphism classes of reductive groups over $\bar{k}$ and the set of isomorphism classes of root data. However, there is no functor $\mathrm{rG}_{\bar{k}} \to \mathrm{RD}$ from the groupoid of reductive groups to the groupoid of root data for the homomorphism (2.2) does not factor through $\mathrm{Aut}_{\bar{k}}(G)$. We do nevertheless have a functor $\mathrm{rG}_{\bar{k}} \to \mathrm{RD}_{\mathrm{Out}}$, where $\mathrm{RD}_{\mathrm{Out}}$ is the groupoid in which objects are objects of RD and groups of automorphisms of (Λ, Φ) are defined to be

$$\mathrm{Out}(\Lambda, \Phi) := \mathrm{Aut}(\Lambda, \Phi)/W. \tag{2.3}$$

We have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{TG}_{\bar{k}} & \longrightarrow & \mathrm{RD} \\ \downarrow & & \downarrow \\ \mathrm{rG}_{\bar{k}} & \longrightarrow & \mathrm{RD}_{\mathrm{Out}} \end{array} \tag{2.4}$$

that induces on the automorphism groups the commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}_{\bar{k}}(T, G) & \longrightarrow & \mathrm{Aut}(\Psi(T, G)) \\ \downarrow & & \downarrow \\ \mathrm{Aut}_{\bar{k}}(G) & \longrightarrow & \mathrm{Out}(\Psi(T, G)). \end{array} \tag{2.5}$$

The upper arrow in this diagram is surjective and its kernel is the group $T(\bar{k})/Z(\bar{k})$, where Z is the center of G. The lower arrow in this diagram is surjective and its kernel is the group $G(\bar{k})/Z(\bar{k})$ of inner automorphisms of G.

The exact sequence

$$0 \longrightarrow W \longrightarrow \mathrm{Aut}(\Psi) \longrightarrow \mathrm{Out}(\Psi) \longrightarrow 0 \tag{2.6}$$

affords sections given by based root data. To discuss these sections, let us recall the concept of based root data introduced in [49]. Let $\Psi = (\Lambda, \Phi)$ be a root datum and W the associated Weyl group. For every root $\alpha \in \Phi$, consider the hyperplane H_α in the real vector space $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ consisting of $x \in \Lambda_{\mathbb{R}}$ satisfying the equation $\langle \alpha, x \rangle = 0$. The group W acts simply transitively on the set of connected components of $\Lambda_{\mathbb{R}} \backslash \bigcup_{\alpha \in \Phi} H_\alpha$. The closure of each of these connected components will be called a Weyl chamber. A based root datum is a root datum plus a choice of a Weyl chamber $\Lambda_{\mathbb{R}}^+$. For the intersection $\Lambda^+ = \Lambda \cap \Lambda_{\mathbb{R}}^+$ generate $\Lambda_{\mathbb{R}}^+$ as a cone, Λ^+ is determined by and determines the Weyl chamber $\Lambda_{\mathbb{R}}^+$. We will call Λ^+ the set of dominant weights. We will denote by

$$\Psi^+ = (\Lambda, \Phi, \Lambda^+) \tag{2.7}$$

the resulting based root datum.

The group $\mathrm{Aut}(\Psi^+(T, G))$ of automorphisms of the based root system Ψ^+ is a subgroup of $\mathrm{Aut}(\Psi(T, G))$ which has trivial intersection with W and projects isomorphically on $\mathrm{Out}(G)$:

$$\mathrm{Aut}(\Psi^+(T, G)) \simeq \mathrm{Out}(G). \tag{2.8}$$

In other words, the choice of a Weyl chamber $\Lambda_{\mathbb{R}}^+$ provides us with a section in the exact sequence (2.6).

On the level of (T, G)-pairs, the choice of a Weyl chamber corresponds to the choice of a Borel subgroup B of G containing T. We have an exact sequence

$$0 \longrightarrow T(\bar{k})/Z(\bar{k}) \longrightarrow \mathrm{Aut}_{\bar{k}}(T, B, G) \longrightarrow \mathrm{Out}(G) \longrightarrow 0. \tag{2.9}$$

The theory of pinnings provide us with sections for this exact sequence. A pinning of G, also known as épinglage, consists of a triple (T, B, G) as above plus the choice of a non-zero vector x_α in the one-dimensional vector eigenspace $\mathfrak{g}_\alpha$ for each simple root α of B. We know that $T(\bar{k})/Z(\bar{k})$ act simply transitively on the set of vectors x_α and the group of automorphisms of an pinning $(T, B, G, x_\alpha \mid \alpha \in \Delta)$, Δ being the set of simple roots of B, is a subgroup of $\mathrm{Aut}_{\bar{k}}(T, B, G)$ projecting isomorphically on $\mathrm{Out}(G)$:

$$\mathrm{Aut}_{\bar{k}}(T, B, G, x_\alpha \mid \alpha \in \Delta) \simeq \mathrm{Out}(G). \tag{2.10}$$

It follows that

$$\mathrm{Aut}_{\bar{k}}(T, B, G) = T(\bar{k})/Z(\bar{k}) \rtimes \mathrm{Out}(G). \tag{2.11}$$

We derive an isomorphism

$$\mathrm{Aut}_{\bar{k}}(G) = G(\bar{k})/Z(\bar{k}) \rtimes \mathrm{Out}(G). \tag{2.12}$$

We now have a commutative diagram of functors:

$$\begin{array}{ccc} \mathrm{Pin}_{\bar{k}} & \xrightarrow{\simeq} & \mathrm{RD}^{+} \\ \downarrow & & \downarrow \\ \mathrm{TG}_{\bar{k}} & \longrightarrow & \mathrm{RD} \\ \downarrow & & \downarrow \\ \mathrm{rG}_{\bar{k}} & \longrightarrow & \mathrm{RD}_{\mathrm{Out}}, \end{array} \tag{2.13}$$

where $\mathrm{Pin}_{\bar{k}}$ is the groupoid of pinnings and RD^{+} is the groupoid of based root data. All these functors induce bijections on the sets of isomorphism classes. Moreover, the functors $\mathrm{Pin}_{\bar{k}} \to \mathrm{RD}^{+}$ and $\mathrm{RD}^{+} \to \mathrm{RD}_{\mathrm{Out}}$ are equivalences of categories. As a result, we obtain a functor

$$\mathrm{RD}_{\mathrm{Out}} \longrightarrow \mathrm{rG}_{\bar{k}} \tag{2.14}$$

which is a section of the functor $\mathrm{rG}_{\bar{k}} \to \mathrm{RD}_{\mathrm{Out}}$ in the lower line of diagram (2.13). This functor may be seen as the construction of reductive groups over $\bar{k}$ from root data.

We are now ready to describe combinatorial data that can be attached to reductive groups over an arbitrary base field. Let k be an arbitrary base field, $\bar{k}$ its separable closure. Let G be a reductive group over $\bar{k}$, $\mathrm{Aut}_{\bar{k}}(G)$ the group of automorphisms of G over $\bar{k}$ and $\mathrm{Aut}_k(G)$ the group of automorphisms of G over k. As in the case of tori, we have an exact sequence

$$0 \longrightarrow \mathrm{Aut}_{\bar{k}}(G) \longrightarrow \mathrm{Aut}_k(G) \longrightarrow \Gamma_k \longrightarrow 0 \tag{2.15}$$

which is actually split because a split model exists. In other words, we have a splitting of $\mathrm{Aut}_k(G)$ as a semi-direct product

$$\mathrm{Aut}_k(G) = \mathrm{Aut}_{\bar{k}}(G) \rtimes \Gamma_k. \tag{2.16}$$

A k-form of G i.e., a reductive group G_k over k such that $G_k \otimes_k \bar{k} = G$ corresponds to a continuous section

$$\Gamma_k \longrightarrow \mathrm{Aut}_k(G), \quad \gamma \longmapsto (\sigma_G(\gamma), \gamma) \tag{2.17}$$

of the semi-direct product (2.16), where $\sigma_G : \Gamma_k \to \mathrm{Aut}_{\bar{k}}(G)$ is a cocycle i.e., it satisfies the equality

$$\sigma_G(\gamma\gamma') = \sigma_G(\gamma)\gamma(\sigma_G(\gamma'))$$

for all $\gamma, \gamma' \in \Gamma_k$ in which the expression $\gamma(\sigma_G(\gamma'))$ refers to the action of Γ_k on $\mathrm{Aut}_{\bar{k}}(G)$. It follows that the groupoid rG_k of reductive groups over k is equivalent to the groupoid of pairs (G, σ_G) consisting of an object G of $\mathrm{rG}_{\bar{k}}$ and a continuous cocycle $\sigma_G : \Gamma_k \to \mathrm{Aut}_{\bar{k}}(G)$ in the above sense.

By the lower arrow in the diagram (2.13), we have a functor $\mathrm{rG}_k \to \mathrm{RD}_{\mathrm{Out},k}$, where $\mathrm{RD}_{\mathrm{Out},k}$ is the groupoid of pairs $(\Psi, \sigma_{\mathrm{Out}})$, where Ψ is an object of $\mathrm{RD}_{\mathrm{Out}}$ and

$$\sigma_{\mathrm{Out}} : \Gamma_k \longrightarrow \mathrm{Out}(\Psi) \tag{2.18}$$

is a continuous cocycle. We note that as Γ_k acts trivially on the quotient $\mathrm{Out}(\Psi)$ of $\mathrm{Aut}_{\bar{k}}(G)$, the cocycle σ_{Out} is in fact a continuous homomorphism. Two reductive groups over k are said to be inner equivalent if they map to isomorphic objects in $\mathrm{RD}_{\mathrm{Out},k}$. From what precedes we conclude that the functor $\mathrm{rG}_k \to \mathrm{RD}_{\mathrm{Out},k}$ induces a bijection from the set of inner equivalence classes of reductive groups over k to the set of isomorphism classes of $\mathrm{RD}_{\mathrm{Out},k}$ consisting of pairs $(\Psi, \sigma_{\mathrm{Out}})$ as above. By definition, this map is injective. We will prove that it is surjective by constructing a section.

The functor $\mathrm{rG}_k \to \mathrm{RD}_{\mathrm{Out},k}$ affords a section

$$\mathrm{RD}_{\mathrm{Out},k} \longrightarrow \mathrm{rD}_k \tag{2.19}$$

from $\mathrm{RD}_{\mathrm{Out},k}$ to the category of reductive groups defined over k. To construct this section we recall that in the diagram (2.13) the arrows $\mathrm{Pin}_{\bar{k}} \to \mathrm{RD}^+$ and $\mathrm{RD}^+ \to \mathrm{RD}_{\mathrm{Out}}$ are equivalences. By applying the inverses of these functors to (2.18), we obtain an object $\mathrm{pin} \in \mathrm{Pin}_{\bar{k}}$, which consists of a reductive group G defined over $\bar{k}$ endowed with a pinning, and a homomorphism

$$\sigma_{\mathrm{Pin}} : \Gamma_k \longrightarrow \mathrm{Aut}(\mathrm{pin}).$$

Using the isomorphism (2.12), which is Γ_k-equivariant, we obtain a cocycle $\sigma_G : \Gamma_k \to \mathrm{Aut}_{\bar{k}}(G)$ and therefore a reductive group G_k defined over k such that $G_k \otimes_k \bar{k} = G$. An object in rG_k belonging to the essential image of (2.19) is called quasi-split. In each inner class, there exists a unique quasi-split group up to isomorphism.

We are now ready to define the Langlands dual group of a reductive group G defined over $\bar{k}$. We have an involutive functor on the category of root data defined by mapping a root datum $\Psi = (\Lambda, \Phi)$ to the dual root datum $\Psi^\vee = (\Lambda^\vee, \Phi^\vee)$. This involution can be defined compatibly on both categories RD^+ and $\mathrm{RD}_{\mathrm{Out}}$. If G is a reductive group defined over a separably closed field $\bar{k}$, by using the functor $\mathrm{rD} \to \mathrm{RD}_{\mathrm{Out}}$ in the lower line of diagram (2.13), we obtain an object in $\mathrm{RD}_{\mathrm{Out}}$. Then we apply the involution $\Psi \mapsto \Psi^\vee$ in $\mathrm{RD}_{\mathrm{Out}}$. Applying (2.14) for to $\Psi^\vee$, with $\bar{k} = \mathbb{C}$, we obtain a reductive group $G^\vee$ defined over $\mathbb{C}$, the Langlands dual group of G.

We will now define the L-group of a reductive group G_k defined over an arbitrary base field k. The group G_k can be identified with an object of the category rG_k consisting of a reductive group G over $\bar{k}$ and a continuous homomorphism $\sigma_G : \Gamma_k \to \mathrm{Aut}_{\bar{k}}(G)$. Using the lower arrow in the diagram (2.13), we derive an object in $\mathrm{RD}_{\mathrm{Out},k}$ consisting of a root datum Ψ and a continuous homomorphism $\sigma_\Psi : \Gamma_k \to \mathrm{Out}(\Psi)$. We apply the functor $\Psi \mapsto \Psi^\vee$ to get the dual root datum which will also be endowed with a homomorphism $\sigma_{\Psi^\vee} : \Gamma_k \to \mathrm{Out}(\Psi^\vee)$. By applying (2.14) over $\bar{k} = \mathbb{C}$, we get the dual reductive group $G^\vee$ with a homomorphism $\sigma_{\mathrm{Out}} : \Gamma_k \to \mathrm{Out}(G^\vee)$. Langlands's L-group of G is defined to be the semi-direct product

$$^L G_v = G^\vee \rtimes \Gamma_k, \tag{2.20}$$

where Γ_k acts on $G^\vee$ through its group of outer automorphisms acting on $G^\vee$ in fixing a pinning. In many circumstances, it is harmless to replace Γ_k in the semi-direct product by the quotient of Γ_k by a normal closed subgroup acting trivially on $G^\vee$.

2.2. *Global fields, local fields and adèles*

We will fix a global field k which may be a number field or the field of rational functions on a curve defined over a finite field. For every absolute value $v : k \to \mathbb{R}_+$ we define k_v to be the completion of k with respect to v. A place of k is an equivalence class of absolute values for the equivalence relation defined by $v \sim v'$ if and only if k_v and $k_{v'}$ are isomorphic as topological fields containing k as a subfield. We will denote by $|k|$ the set of places of k. If k is a number field, the set of places $|k|$ contains only finitely many archimedean places v, those places such that k_v is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. For non-archimedean places $v \in |k|$, the elements $x \in k_v$ with $|x|_v \le 1$ form a complete DVR to be called the ring of integers $\mathcal{O}_v$ of k_v. We will denote by $\mathbb{F}_v$ the residue field of $\mathcal{O}_v$; $\mathbb{F}_v$ is a finite field of cardinality q_v. For every $v \in |k|$, k_v is a locally compact group, and if v is non-archimedean, $\mathcal{O}_v$ is a compact open subgroup of k_v. If k is a function field, every place $v \in |k|$ is non-archimedean and the completion k_v is isomorphic to the field of Laurent formal series with coefficients in the residue field $\mathbb{F}_v$.

We also recall that the ring of adèles $\mathbb{A}_k$ is an inductive limit of infinite products

$$\mathbb{A}_k = \varinjlim_{S} \prod_{v \in S} k_v \times \prod_{v \notin S} \mathcal{O}_v \tag{2.21}$$

over all finite subsets $S \subset |k|$ that contain all the archimedean places. An element of $\mathbb{A}_k$ can be represented as a sequence $(x_v)_{v \in |k|}$ with $x_v \in k_v$ for all v

and $x_v \in \mathcal{O}_v$ for almost all non-archimedean places. The ring of adèles with the inductive limit topology is a locally compact ring.

For every place $v \in |k|$, we denote by $\Gamma_v = \mathrm{Gal}(\bar{k}_v/k_v)$ the Galois group of the local field k_v. If v is non-archimedean, we have an exact sequence

$$0 \longrightarrow I_v \longrightarrow \Gamma_v \longrightarrow \Gamma_{\mathbb{F}_v} \longrightarrow 0, \tag{2.22}$$

where I_v is the inertia group and $\Gamma_{\mathbb{F}_v}$ is the Galois group of the residue field $\mathbb{F}_v$ in which the Frobenius element $\sigma_v(x) = x^{q_v}$ is a progenerator. There is a homomorphism between Galois groups of local and global fields

$$\Gamma_v \longrightarrow \Gamma_k \tag{2.23}$$

which is well defined up to conjugation. For every finite Galois extension k'/k, the induced homomorphism $\Gamma_v \to \mathrm{Gal}(k'/k)$ is trivial on the inertia subgroup I_v if and only if v is unramified with respect to the extension k'/k. This is the case for all but finitely many non-archimedean places. For an unramified place v, the homomorphism (2.23) defines a conjugacy class $[\sigma_v]$ of the image of σ_v in $\mathrm{Gal}(k'/k)$.

The L-group of a reductive group G defined over a global field k is defined in (2.20) to be the semi-direct product

$${}^L G = G^\vee \rtimes \Gamma_k$$

in which Γ_k acts on $G^\vee$ factors through some finite quotient $\mathrm{Gal}(k'/k)$. For every non-archimedean place v of G, the L-group of G over k_v is the semi-direct product ${}^L G_v = G^\vee \rtimes \Gamma_v$ constructed out of the action of Γ_v on $G^\vee$ deriving from the action of Γ_k through (2.23). For almost all non-archimedean places v, the restriction of the homomorphism $\Gamma_k \to \mathrm{Gal}(k'/k)$ to the inertia group I_v is trivial. In this case, we may write ${}^L G = G^\vee \rtimes \langle\sigma_v\rangle$.

2.3. *Canonical invariant measures*

We first recall Weil's construction measures out of top differential forms. Let F be a local field equipped with an invariant measure $\mathrm{d}x$ with respect to addition. If X is a smooth algebraic variety of dimension n over F then every non-zero algebraic top differential form ω over X defines a distribution on $X(F)$ i.e a continuous linear form

$$|\omega| : \mathcal{C}_c^\infty(X(F)) \longrightarrow \mathbb{C}$$

defined as follows. For every point $x \in X(F)$, there exists a compact open neighborhood U of $X(F)$ and a p-adic analytic open embedding $u : U \to F^n$. For any system of coverings of $X(F)$ by compact open subsets U_i with an analytic open embedding $u_i : U_i \to F^n$, by the unit partition theorem, every smooth function with compact support $f \in \mathcal{C}_c^\infty(X(F))$ can be decomposed as a sum $f = f_1 + \cdots + f_n$, where f_i is a smooth function with support contained in U_i. It is thus enough to prove that for every compact open subset U open embeddable in F^n as above, there exists a canonical linear form

$$|\omega|_U : \mathcal{C}^\infty(X(F), U) \longrightarrow \mathbb{C},$$

where $\mathcal{C}_c^\infty(X(F), U)$ is the space of smooth functions on $X(F)$ with support contained in U such that for every compact open subset $V \subset U$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_c^\infty(X(F), V) & \xrightarrow{\iota} & \mathcal{C}_c^\infty(X(F), U) \\ & {\scriptstyle |\omega|_V} \searrow \quad \swarrow {\scriptstyle |\omega|_U} & \\ & \mathbb{C}, & \end{array} \tag{2.24}$$

where ι is the extension by zero.

Let U be a compact open subset of $X(F)$ equipped with an analytic open embedding $u : U \to F^n$. If $x_1, \ldots, x_n$ denote the coordinates of F^n we can write $u_*\omega = \phi\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, where $\phi : F^n \to F$ is an analytic function supported in the compact open subset $u(U)$ of F^n. We define after [53]

$$|\omega|_U(f) = \int_{u(U)} f(u^{-1}(x))|\phi(x)|\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

for every smooth function $f \in \mathcal{C}^\infty(X(F), U)$ with compact support contained in U, and $\mathrm{d}x_1 \cdots \mathrm{d}x_n$ is the product measure on F^n. The chain rule in integration guarantees that the above definition of the linear form $|\omega|_U(f)$ does not depend on the embedding u and therefore must be compatible with the extension by zero as in diagram (2.24). We may also observe that as the definition of the linear form $|\omega|$ applies also to continuous functions with compact support, $|\omega|$ defines not only a distribution but in fact a measure. One may notice that for every analytic function $\xi : X(F) \to F$, the equality

$$|\xi|^{-1}|\xi\omega| = |\omega| \tag{2.25}$$

holds. This can be checked upon the very definition of $|\omega|$.

Let F be a local field equipped with a Haar measure $\mathrm{d}x$. We will define canonical invariant measures on $T_F(F)$ for every torus T_F defined over F. Let T_F be a torus defined over F and $T = T_F \otimes_F \bar{F}$ its base change to a separable closure $\bar{F}$ of F. The group of cocharacters $\Lambda = \Lambda(T)$ is a free $\mathbb{Z}$-module of rank n. As in Subsect. 2.1, the F-form T_F of T is equivalent to a continuous homomorphism $\rho_\Lambda : \Gamma_F \to \mathrm{Aut}(\Lambda)$ from the Galois group Γ_F of

F to the discrete group $\mathrm{Aut}(\Lambda)$ of automorphisms of the lattice Λ. In particular, we recover T_F from ρ_Λ by the formula

$$F[T_F] = (\bar{F} \otimes_F F[\Lambda^\vee])^{\Gamma(\rho_\Lambda)},$$

where $\Gamma(\rho_\Lambda)$ is the graph of ρ_Λ which is a subgroup of $\Gamma_F \times \mathrm{Aut}_\Lambda$ acting on $\bar{F} \otimes F[\Lambda^\vee]$ component wise. We will denote by $\rho_\Lambda(\Gamma_F)$ the image of ρ_Λ which is a finite subgroup of $\mathrm{Aut}(\Lambda)$.

If $\mathfrak{t}$ denotes the Lie algebra of T, then we have $\mathfrak{t} = \Lambda \otimes \bar{F}$. We have $(\bigwedge_{\mathbb{Z}}^n \Lambda) \otimes \bar{F} = \bigwedge^n \mathfrak{t}$. Since $\bigwedge_{\mathbb{Z}}^n \Lambda$ is a free $\mathbb{Z}$-module of rank one, its generator ϵ, which is well defined up to a sign, is then a generator of the one-dimensional $\bar{F}$-vector space $\bigwedge_{\bar{F}}^n \mathfrak{t}$. The dual generator $\epsilon^\vee$ of $\bigwedge_{\mathbb{Z}}^n \Lambda^\vee$ gives rise to a generator of $\bigwedge_{\bar{F}}^n \mathfrak{t}^\vee$ and therefore a non-zero invariant top differential form on T. The automorphism group $\mathrm{Aut}(\Lambda)$ of the lattice Λ acts on $\bigwedge_{\mathbb{Z}}^n \Lambda$ through the determinant character $\det_\Lambda : \mathrm{Aut}(\Lambda) \to \{\pm 1\}$ as well as on the space of invariant top differential form $\bigwedge_{\bar{F}}^n \mathfrak{t}^\vee$ on T. The space of top differential form on the F-model T_F is given by

$$\Big(\bigwedge^n \mathfrak{t} \otimes_F F[\Lambda^\vee]\Big)^{\Gamma(\rho_\Lambda)} = (\bar{F}\epsilon^\vee \otimes_F F[\Lambda^\vee])^{\Gamma(\rho_\Lambda)}. \tag{2.26}$$

We will construct a non-zero element $\xi \in F[\Lambda^\vee]$ satisfying

$$\gamma(\xi) = \det_\Lambda(\gamma)\xi \tag{2.27}$$

for every element γ of the finite subgroup $\rho_\Lambda(\Gamma_F)$ of $\mathrm{Aut}(\Lambda)$. Since the fixed points of each non-trivial element of $\rho_\Lambda(\Gamma_F)$ form a strict linear subspace of Λ, there exists an element $\alpha \in \Lambda^\vee$ which is not fixed by any non-trivial element γ of $\rho_\Lambda(\Gamma_F)$. We set

$$\xi = \sum_{\gamma \in \rho_\Lambda(\Gamma_F)} \det_\Lambda(\gamma)\gamma(\alpha) \in F[\Lambda^\vee].$$

Since the terms $\gamma(\alpha)$ appearing in the above formula are distinct elements of $\Lambda^\vee$ and thus linearly independent in $F[\Lambda^\vee]$, the above linear combination is non-zero. It also obviously satisfies the equation (2.27). If ξ and ξ_1 are two non-zero elements of $F[\Lambda^\vee]$ both satisfying (2.27), we have $\xi_1 = \phi_1\xi$, where ϕ_1 is a rational function on T invariant under the action of $\mathrm{Aut}(\Psi)$. The top differential form $\xi\epsilon^\vee$ on T is now invariant under the action of Γ_F and therefore defines a top differential form on the F-model T_F as in (2.26).

The function ξ is not invariant under $\rho_\Lambda(\Gamma_F)$ but its square is. We define the canonical measure on $T_F(F)$ to be

$$\mathrm{d}\mu = |\xi^2|^{-1/2}|\xi\epsilon^\vee|. \tag{2.28}$$

One can check that this is an invariant measure on $T_F(F)$ which is independent of all choices, including ϵ and ξ. Indeed, ϵ is well defined up to a sign which is canceled by the absolute value. The independence on ξ is guaranteed by the formula (2.25).

We can extend the construction of canonical invariant measures on tori to every reductive group. Let G be a reductive group over F. We denote by C the neutral component of its center and G^{der} its derived group. We know that C is a torus and G^{der} is a semisimple group. We derive an invariant measure on $G(F)$ from the measure on $C(F)$ constructed as above, and the measure on G^{der} constructed out of the Killing form.

2.4. *Unramified local L-factors*

If G is a reductive group defined over a non-archimedean local field F, $G(F)$ is a totally disconnected locally compact group i.e., it has a base of neighborhoods of the identity element consisting of compact open subgroups. A representation π of $G(F)$ on a complex vector space V is said to be smooth if the stabilizer of every vector $v \in V$ is an open subgroup of $G(F)$. It is said to be admissible if for every compact open subgroup K of $G(F)$, the space V^K of K-fixed vectors is finite dimensional. It is known, and it is a deep fact of the theory of smooth representations of reductive groups over non-archimedean local fields, that all irreducible representations of $G(F)$ are admissible, see [6].

By choosing a Haar measure $\mathrm{d}\mu$ on $G(F)$, we can identify $\mathcal{C}_c^\infty(G(F))$ with the space of locally constant measures with compact support in $G(F)$. The choice of Haar measure allows us to define the convolution product and as a result, to equip $\mathcal{C}_c^\infty(G(F))$ with a structure of non-unital associative algebra. For every smooth representation (π, V), V is equipped with a structure of $\mathcal{C}_c^\infty(G(F))$-module by the formula:

$$\pi(f)v = \int_{G(F)} f(g)\pi(g)v \,\mathrm{d}\mu. \tag{2.29}$$

If π is admissible, then for every $f \in \mathcal{C}_c^\infty(G(F))$ the operator $\pi(f)$ has finite dimensional image. It follows that we have an $\mathrm{ad}(G(F))$-invariant linear form

$$\mathrm{tr}_\pi : \mathcal{C}_c^\infty(G(F)) \longrightarrow \mathbb{C}. \tag{2.30}$$

If G is a split reductive group over F, G can be extended as a reductive group scheme over $\mathbb{Z}$ and in particular over the ring of integers $\mathcal{O}_F$ of F. We will also denote by G the smooth reductive group scheme over $\mathcal{O}_F$ whose generic fiber is the group G over F with which we start. The group $K = G(\mathcal{O}_F)$ of $\mathcal{O}_F$-points of G is then a maximal compact open subgroup of $G(F)$. The subalgebra $\mathcal{H}$ of $\mathcal{C}_c^\infty(G(F))$ consisting of $(K \times K)$-invariant functions with compact support in

$G(F)$, is a commutative subalgebra with a unit. An irreducible smooth representation (π, V) of $G(F)$ is said to be unramified if it has a non-zero fixed vector under K. In that case, the space of K-fixed vectors in V is a non-zero simple module over $\mathcal{H}$. Since $\mathcal{H}$ is commutative, simple modules are one dimensional and the set of isomorphism classes of simple modules is in bijection with the set of maximal ideals of $\mathcal{H}$. After Satake [46], there is a canonical isomorphism of algebras:

$$\mathrm{Sat} : \mathcal{H} \longrightarrow \mathbb{C}[G^\vee]^{\mathrm{ad}(G^\vee)} \tag{2.31}$$

from $\mathcal{H}$ to the algebra of algebraic regular functions on $G^\vee$ invariant under the adjoint action. It follows that simple modules over $\mathcal{H}$, and hence unramified representations of $G(F)$, are classified up to isomorphism by semi-simple conjugacy classes in the complex reductive group $G^\vee$. An unramified representation π of $G(F)$ is said to have the Satake parameter α_π in the set of semi-simple conjugacy classes of $G^\vee$ if for every $\phi \in \mathcal{H}$ we have

$$\mathrm{tr}_\pi(\phi) = (\mathrm{Sat}(\phi))(\alpha_\pi). \tag{2.32}$$

This equation defines a canonical bijection between the set of unramified representations of $G(F)$ up to isomorphism and the set of semi-simple conjugacy classes of $G^\vee$.

After Langlands [29], Satake's theory can be generalized to unramified reductive groups G over F. A reductive group G over F is said to be unramified if it admits a reductive model over $\mathcal{O}_F$. For quasi-split groups, the condition of being unramified can be read off the action of the Galois group Γ_F of F on the root datum. Recall that we have an exact sequence

$$0 \longrightarrow I_F \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathbb{F}} \longrightarrow 0, \tag{2.33}$$

where I_F is the inertia group and $\Gamma_{\mathbb{F}}$ is the Galois group of the residue field $\mathbb{F}$ of which the Frobenius element σ_F is a topological generator. Quasi-split groups over F are determined up to isomorphism by a homomorphism $\sigma_{\mathrm{Out}} : \Gamma_F \to \mathrm{Out}(G \otimes_F \bar{F})$ as defined in (2.18). The group G admits a reductive model over $\mathcal{O}_F$ if and only if the restriction of σ_{Out} to the inertia group I_F is trivial. In particular, the action of Γ_F on $G^\vee$ factors through $\Gamma_{\mathbb{F}}$. We may take

$${}^L G = G^\vee \rtimes \langle \sigma_F \rangle. \tag{2.34}$$

We suppose that G has a reductive model over $\mathcal{O}_F$ which we will also denote by G. An irreducible smooth representation $G(F)$ is said to be unramified if it has a non-zero vector under $G(\mathcal{O}_F)$. In this setting, Langlands showed that unramified representations of $G(F)$ are in canonical bijection with the conjugacy classes of $G^\vee$ in the connected component $\sigma_F G^\vee$ of ${}^L G$:

$$\alpha \in \sigma_F G^\vee \subset G^\vee \rtimes \langle \sigma_F \rangle.$$

There is a twisted form of the Satake isomorphism

$$\mathrm{Sat} : \mathcal{H} \longrightarrow \mathbb{C}[\sigma_F G^\vee]^{\mathrm{ad}(G^\vee)} \tag{2.35}$$

from the unramified Hecke algebra of G to the ring of regular functions on the connected component $\sigma_F G^\vee$ of ${}^L G$ which are invariant under the adjoint action of $G^\vee$, see [7, 6.7]. The bijection $\pi \mapsto \alpha_\pi$ between the set of isomorphism classes of unramified representations of $G(F)$ and the set conjugacy classes of $G^\vee$ contained in the connected component $\sigma_F G^\vee$ is characterized by the identity

$$\mathrm{tr}_\pi(\phi) = (\mathrm{Sat}(\phi))(\alpha_\pi). \tag{2.36}$$

More information on this construction may be found in [29], [7] and [13].

For every algebraic representation $\rho : {}^L G \to \mathrm{GL}(V_\rho)$, we define the local L-factor of an unramified representation π to be

$$L(s, \pi, \rho) = \det(1 - \rho(\alpha_\pi) q^{-s})^{-1}, \tag{2.37}$$

where q is the cardinality of the residue field of F, and s is a complex variable. Using the Newton identity

$$\det(1 - At)^{-1} = 1 + \mathrm{tr}(A)t + \mathrm{tr}(\mathrm{sym}^2(A))t^2 + \cdots \tag{2.38}$$

valid for every matrix A, the formal variable t being q^{-s}, we obtain the development of $L(s, \pi, \rho)$ as formal series:

$$L(s, \pi, \rho) = \sum_{d=0}^{\infty} \mathrm{tr}((\mathrm{sym}^d \rho)(\alpha_\pi)) q^{-ds}. \tag{2.39}$$

An unramified representation π of $G(F)$ is said to be tempered if its parameter α_π is a compact element of ${}^L G$. In that case, all eigenvalues of the matrix $(\mathrm{sym}^d \rho)(\alpha_\pi)$ have absolute value 1, and therefore

$$|\mathrm{tr}((\mathrm{sym}^d \rho)(\alpha_\pi))| \le \dim(\mathrm{sym}^d \rho) < (d+1)^r, \tag{2.40}$$

where $r = \dim(\rho)$. We infer that the formal series (2.39) converges absolutely and uniformly on the domain $\Re(s) > 0$ of the complex plane. In general, the formal series (2.39) converges absolutely and uniformly on some domain $\Re(s) > \sigma$, where σ can be explicitly specified in function of absolute values of eigenvalues of the matrix $\rho(\alpha_\pi)$. More information can be found in [29] and [7, 39] and [13, 16].

The local L-factor is conjectured to exist for all irreducible admissible representations of $G(F)$. There is a conjectural formula based on the local Langlands correspondence. The local L-factor should also exist at the archimedean places as well. This is known in some cases: for instance if $G = \mathrm{GL}_n$, and ρ is the standard representation of the dual group $G^\vee = \mathrm{GL}_n(\mathbb{C})$ thanks to Godement and Jacquet [23], and Tamagawa [50].

2.5. *Automorphic L-functions*

Let G be a reductive group defined over a number field k, ${}^LG = G^\vee \rtimes \Gamma_k$ its L-group defined by a continuous homomorphism $\sigma_{\mathrm{Out}} : \Gamma_k \to \mathrm{Out}(G)$. By continuity, σ_{Out} factors through a finite quotient i.e., $\mathrm{Gal}(k'/k)$ for some finite Galois extension of k. For every non-archimedean place, we have an induced homomorphism $\sigma_{v,\mathrm{Out}} : \Gamma_v \to \mathrm{Out}(G)$ well defined up to conjugacy. We know that $G \otimes_k k_v$ is unramified if and only if the restriction of $\sigma_{v,\mathrm{Out}}$ to the inertia subgroup I_v of Γ_v is trivial. This is true for all non-archimedean place $v \in |k|$ with finitely many exceptions.

Automorphic representations of G are irreducible representations π of $G(\mathbb{A}_k)$, occurring in the space of automorphic functions on $[G] = G(k)\backslash G(\mathbb{A}_k)$ as defined in the paper by Borel and Jacquet [8], followed by a supplement by Langlands [30]. An automorphic representation π can be decomposed into tensor product

$$\pi = \bigotimes_{v \in |k|} \pi_v, \tag{2.41}$$

where π_v are irreducible admissible representations of $G(F)$. There exists a finite set $S_\pi \subset |k|$ containing all archimedean places such that for all $v \in |k| - S_\pi$, $G \otimes_k k_v$ is an unramified group over k_v and π_v is an unramified representation of $G(k_v)$.

Following Langlands [29], for every representation $\rho : {}^LG \to \mathrm{GL}(V_\rho)$, we define the incomplete L-function as the Euler product

$$L^{S_\pi}(s, \pi, \rho) = \prod_{v \in |k| - S_\pi} L_v(s, \pi_v, \rho), \tag{2.42}$$

where $L_v(s, \pi_v, \rho)$ is defined by (2.37). An automorphic representation is said to be of the Ramanujan type if for all $v \in |k| - S_\pi$, $\alpha(\pi_v)$ is a compact element of ${}^LG$. In this case, with the development of local L-factor as formal series (2.39) and the estimate (2.40), one can prove that $L^{S_\pi}(s, \pi, \rho)$ converges absolutely and uniformly to a holomorphic function on the domain $\Re(s) > 1$. It is known that in general the Euler product (2.42) converges absolutely and uniformly on a domain of type $\Re(s) \geq \sigma$ for some real number σ, see [29] and [7, 52], which only depends on G.

In [29], Langlands asked no less than seven questions of which the first is whether is it possible to define local L and ϵ factors at the ramified places so that the complete L-function

$$L(s, \pi, \rho) = \prod_{v \in |k|} L_v(s, \pi_v, \rho) \tag{2.43}$$

has a meromorphic continuation in the entire complex plane with finitely many poles which satisfies the functional equation

$$L(s,\pi,\rho)=\epsilon(s,\pi,\rho)L(1-s,\pi,\rho^\vee), \tag{2.44}$$

where $\rho^\vee$ is the dual representation of ρ, and

$$\epsilon(s,\pi,\rho)=\prod_v \epsilon_v(s,\pi,\rho). \tag{2.45}$$

We call this the standard properties of automorphic L-functions.

Again, this conjecture is known in some cases. For instance for $G = \mathrm{GL}_n$, ρ the standard representation of $G^\vee = \mathrm{GL}_n(\mathbb{C})$, this is a theorem of Godement and Jacquet [23], [26, 83–84] and Tamagawa [50]. In other words, standard L-functions satisfy standard properties. There are at least two other methods of proving particular instances of standard properties of automorphic L-functions namely the Langlands–Shahidi and Rankin–Selberg methods which are out of the scope of this survey.

2.6. *Functoriality principle*

Question 5 of [29, 19] is nowadays known as the functoriality principle. This conjecture stipulates that for reductive groups H and G defined over a global field k, given with a homomorphism of complex groups ${}^LH \to {}^LG$, for every automorphic representation π_H of H there exists an automorphic representation π of G such that for every representation $\rho : {}^LG \to \mathrm{GL}(V_\rho)$, there are equalities of local L and ϵ factors

$$L_v(s,\pi_{H,v},\rho|_{{}^LH})=L_v(s,\pi_v,\rho) \ \text{ and } \ \epsilon_v(s,\pi_{H,v},\rho|_{{}^LH})=\epsilon_v(s,\pi_v,\rho). \tag{2.46}$$

In other words, there should be a transfer of automorphic representations from H to G depending "functorially" on a homomorphism of complex groups ${}^LH \to {}^LG$. It is quite clear from this formulation that there should be a similar transfer from smooth representations of $H(k_v)$ to smooth representations of $G(k_v)$ satisfying a local-global compatibility. This is the content of question 4 of [29, 19].

An immediate consequence of the functoriality principle is that every automorphic L-function is equal to some standard L-function of Godement–Jacquet. In particular, all automorphic L-functions have meromorphic continuations satisfying the functional equation (2.44) as standard L-functions do. Thus the standard properties of automorphic L-functions follow from the principle of functoriality.

In the opposite direction, at least when $G = \mathrm{GL}_n$, the functoriality principle follows from the standard properties of automorphic L-functions. This is the content of different formulations of the converse theorem [17].

2.7. L-packets and endoscopy

In the case of $G = \mathrm{GL}_n$, L-functions determine representations both in local and global settings. The transfer of representations of H to representations of $G = \mathrm{GL}_n$ in both local and global setting, according to the functoriality principle, should be a mapping. It is not necessarily so for a general reductive group G. Locally, there are usually a finite number of representations of $G(F)$ having the same local L-factors. These representations are considered as elements of an L-packet. Globally, there are inequivalent irreducible representations of $G(\mathbb{A}_k)$ whose local components belong to the same L-packet, occurring in the space of automorphic forms with possibly different multiplicities. The functorial transfer from representations of H to representations of G, both in local and global settings, are then at best correspondences, or has to be reformulated on the level of L-packets.

This phenomenon, which seems to be an impediment at first view, turns out to be a chance. In [31], Langlands formulates a program aiming at explicating the L-packet phenomenon as a particular case of the functoriality principle in which $H^\vee$ is the neutral component of the centralizer of a semi-simple element of ${}^L G$. The groups H arising in this way are called the endoscopic groups of G. The functoriality principle, in the endoscopic setting, is to be established by the way of comparison of trace formulae. It took more than thirty years to see the endoscopy to come essentially to completion.

As opposed to other partial approaches to functoriality, L-functions and its standard properties do not play a major role in endoscopy theory. The main method in the theory of endoscopy is the comparison of trace formulae. In some sense, the comparison of trace formulae may be seen as a comparison of two distributions, and the transfer of automorphic representations may be seen as the resulting comparison between the spectral supports of those distributions. Both distributions are certain linear combinations of the trace formulae. The comparison is to be proved by comparing their geometric sides, which derives from an equality between certain linear combination of orbital integrals on H and on G. The problem of comparisons of the orbital integrals are to be divided into similar looking but different problems in local harmonic analysis, known as the transfer and the fundamental lemma. A summary of this topic may be found in [36].

Endoscopy theory and its twisted variants culminate in the results exposed in the recent book of Arthur [5]. Twisted endoscopy theory has found a great number of applications in number theory, for instance via the theory of base change and the calculation of cohomology of Shimura varieties. As it is advocated in [21], in addition to the precise description of local L-packets and global multiplicity formula, endoscopy theory produced the stable trace formula which is expected be one of the most powerful tools for constructing functoriality.

3. Langlands's first strategy for beyond endoscopy

In [32], Langlands proposed to construct certain limits of trace formula as means to prove the functoriality principle. Admitting the reciprocity principle, automorphic representations π of G should be classified by homomorphisms

$$\sigma_\pi : \Gamma_F^L \longrightarrow {}^L G$$

from the hypothetical Langlands–Galois group Γ_F^L to ${}^L G$. Without having properly defined Γ_F^L, the Zariski closure of $\sigma_\pi(\Gamma_F^L)$ in ${}^L G$ should be well defined and will be denoted by $H(\pi)$. As an ersatz to functoriality, one may attempt to sort out automorphic representations π of G according to subgroups $H(\pi)$ of ${}^L G$ classified up to conjugation.

For every representation $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ we will denote by

$$m_\pi(\rho) = m_{H(\pi)}(\rho) = \dim(V_\rho^{\rho(H(\pi))})$$

the dimension of the subspace of V_ρ consisting of fixed vectors under the action of $H(\pi)$. The numerical data $\rho \mapsto m_\pi(\rho)$ will tell a great deal about $H(\pi)$ up to conjugation. For instance, for $G = \mathrm{GL}_n$ the numerical data $m_\pi(\rho)$, as ρ varies, determine $H(\pi)$ up to inner automorphism of $\mathrm{GL}_n(\mathbb{C})$ according to a result of Larsen and Pink [34]. It is not necessarily so for other reductive groups but we may restrict ourselves to $G = \mathrm{GL}_n$ for the time being. One may therefore attempt to solve the problem of sorting out automorphic representations π of G according to subgroups $H(\pi)$ of ${}^L G$ by finding formulae that have access to the numbers $m_\pi(\rho)$. Langlands envisions a new breed of trace formula consisting of a stably invariant linear form

$$\ell^\rho : \mathcal{C}_c^\infty(G) \longrightarrow \mathbb{C}$$

whose spectral expansion is of the form

$$\ell^\rho(f) = \sum_\pi m_\pi(\rho) \mathrm{tr}_\pi(f) + \cdots . \tag{3.1}$$

Conjecturally at least, we have access to numbers $m_\pi(\rho)$ by means of the L-function $L(s, \pi, \rho)$. Indeed, one expects that $m_\pi(\rho)$ is equal to the order of the pole of $L(s, \pi, \rho)$ at $s = 1$. In other words,

$$m_\pi(\rho) = -\mathrm{res}_{s=1} \mathrm{dlog}\, L(s, \pi, \rho),$$

where $\mathrm{dlog}\,(f) = f'/f$. For representations π of Ramanujan type, the Euler product $L^{S_\pi}(s, \pi, \rho)$ converges absolutely and uniformly on the domain $\Re(s) > 1$ and we may expect an asymptotic formula as $X \to \infty$

$$\sum_{q_v^d < X} \log(q_v) \mathrm{tr}(\rho(\alpha_{\pi_v})) \mathrm{tr}((\mathrm{sym}^{d-1} \rho)(\alpha_{\pi_v})) = m_\pi(\rho) X + o(X) \tag{3.2}$$

the sum ranging over all places $v \in |k| - S_\pi$ and $d \in \mathbb{N}$ such that $q_v^d < X$, $\alpha_{\pi_v} \in {}^L G/\sim$ being the Langlands parameter of unramified components π_v. Although the terms corresponding to exponents $d \geq 2$ may look unpleasant, they should not contribute to the asymptotic, and therefore we may expect

$$\sum_{v \in |k| - S_\pi, q_v < X} \log(q_v) \mathrm{tr}(\rho(\alpha_{\pi_v})) = m_\pi(\rho) X + o(X) \tag{3.3}$$

for all automorphic representations of the Ramanujan type. Langlands proposes to study the asymptotic of

$$\sum_{v \in |k| - S, q_v < X} \log(q_v) \mathrm{tr}_{[G]}(\phi_v^\rho \otimes f_S) \tag{3.4}$$

as $X \to \infty$, for every fixed finite set of places $S \subset |k|$ containing all the archimedean places, ϕ_v^ρ being the Hecke operator corresponding to ρ and $f_S \in \bigotimes_{v \in S} \mathcal{C}_c^\infty(G(k_v))$.

This compelling perspective faces serious difficulties:

1. There are discrete automorphic representations not of Ramanujan type. The corresponding L-functions may have poles, say at some real number $m > 1$, in which case the leading term of (3.3) should be X^m instead of X. Therefore one should first remove the contribution of representations not of Ramanujan type.
2. Available analytic techniques to deal with a sum over primes as (3.4) are extremely limited. Even if explicit information on orbital integrals of ϕ_v^ρ are available, which is the case only if $G = \mathrm{GL}_2$ and ρ is the standard representation, fundamentally new ideas will be needed to find an asymptotic formula (3.4).

As suggested by Sarnak in [45], there may be some advantages to work with the Kuznetsov trace formula which discards representations not of Ramanujan type. He also suggests that it may be better to construct a trace formula whose spectral expansion involves a sum of automorphic L-functions instead of their logarithmic derivatives, and whose geometric expansion involves a sum over integers instead of a sum over primes. Sarnak's suggestion has been implemented in Venkatesh's PhD thesis [51] treating the case $G = \mathrm{GL}_2$ and ρ the symmetric square representation.

A general form of the trace formula with sum of L-functions appearing in the spectral expansion has been initiated in [21]. Instead of (3.4), [21] aims a spectral expansion of the form

$$L^\rho(f) = \sum_\pi \mathrm{res}_{s=1} L(s, \pi, \rho) \mathrm{tr}_\pi(f) + \cdots . \tag{3.5}$$

For the geometric expansion of L^ρ will be a sum over integers instead of a sum over primes as (3.4), there is more hope of finding estimate of the geometric expansion and derive some information on the spectral expansion. In [21], we propose to use a Poisson summation formula for this endeavor. This strategy has been implemented with success by Altug [1].

The formal structure of (3.5) can be greatly clarified by the introduction of certain Schwartz spaces whose existence is conjectured by Braverman and Kazhdan [11], we will review their conjectures before coming back to the trace formula of [21] in Sect. 7.

4. Braverman–Kazhdan conjectures

Braverman and Kazhdan propose in [11] a conjectural generalization of Godement–Jacquet's construction of standard automorphic L-functions. Before stating the Braverman–Kazhdan conjectures, we will briefly recall Godement–Jacquet's construction. We will call the case $G = \mathrm{GL}_n$ and ρ the standard representation of $G^\vee$ the standard case, and qualify all other cases as non-standard.

4.1. Standard case

Let G be the group GL_n and M^{std} be vector space of $(n \times n)$-matrices containing G as a Zariski open subset. Let F be a non-archimedean local field. The Schwartz space $\mathcal{S}^{\mathrm{std}}(G(F))$ in this case consists of all locally constant functions with compact support on $M^{\mathrm{std}}(F)$. The restriction of $\phi \in \mathcal{S}^{\mathrm{std}}(G(F))$ to $G(F)$ is a locally constant function on $G(F)$ but no longer of compact support. For ϕ is completely determined by its restriction to $G(F)$, $\mathcal{S}^{\mathrm{std}}(G(F))$ may be regarded as the space of locally constant functions on $G(F)$ that can be extended to a locally constant function with compact support in $M^{\mathrm{std}}(F)$.

The Schwartz space determines the standard L-factor $L(s, \pi, \mathrm{std})$ in terms of Godement–Jacquet's zeta integrals. These integrals, depending on a function $\phi \in \mathcal{S}^{\mathrm{std}}(G(F))$, a matrix coefficient f of π and a complex variable s

$$Z(\phi, f, s) = \int_G \phi(g) f(g) |\det(g)|^s \,\mathrm{d}g \tag{4.1}$$

is convergent for $\Re(s) > s_G$ for some positive real number s_G depending only on G. For a given test function ϕ and a matrix coefficient f, $Z(\phi, f, s)$ can be extended to a rational function $Q(q^{-s})$ of q^{-s}, q being the cardinality of the residue field of F. The standard local L-factor is then

$$L(s, \pi, \mathrm{std}) = P(q^{-s-\frac{n-1}{2}})^{-1}, \tag{4.2}$$

where $P(t)$ is a polynomial such that $P(t)^{-1}$ generates the fractional ideal of rational functions $Q(t)$ as above, with $P(t)$ being normalized such that $P(0) = 1$.

Disregarding the convergence issue that can be circumvented by an appropriate application of the principle of analytic continuation, the above zeta integral can be expressed as a convolution product

$$Z(\phi, f, s) = (\phi \star \check{f}|\det|^{-s})(\delta_G) = (\check{\phi} \star f|\det|^{s})(\delta_G), \tag{4.3}$$

where $\check{f}(g) = f(g^{-1})$, $\check{\phi}(g) = \phi(g^{-1})$ and δ_G is the neutral element of G.

In this construction the characteristic function $\mathbb{L}^{\mathrm{std}}$ of the compact set $M^{\mathrm{std}}(\mathcal{O})$ of integral matrices plays a prominent role for it singles out unramified representations and produces unramified L-factors:

$$\mathrm{tr}_\pi(\mathbb{L}^{\mathrm{std}}) = \begin{cases} L\Big(-\dfrac{n-1}{2}, \pi, \mathrm{std}\Big) & \text{if } \pi \text{ is unramified,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

As $M^{\mathrm{std}}(F)$ is an F-vector space, the Schwartz space $\mathcal{S}^{\mathrm{std}}(G(F))$ is equipped with a Fourier transform, once we chose a non-trivial additive character of $\psi : F \to \mathbb{C}^\times$:

$$\hat{\phi}(x) = \int_{M^{\mathrm{std}}(F)} \phi(y)\psi(\mathrm{tr}(xy))\, \mathrm{d}^+ y, \tag{4.5}$$

where $\mathrm{d}^+ y$ is the additive invariant measure on $M^{\mathrm{std}}(F)$ which can be normalized by the formula:

$$\mathrm{d}^+ g = |\det(g)|^n\, \mathrm{d}g. \tag{4.6}$$

The Fourier transform can be written as the multiplicative convolution

$$\phi \longmapsto \hat{\phi} = \mathcal{F}^{\mathrm{std}}(\phi) = J^{\mathrm{std}}|\det|^n\, \mathrm{d}g \star \check{\phi}, \tag{4.7}$$

where the convolution kernel J^{std} is the conjugation invariant function

$$J^{\mathrm{std}}(g) = \psi(\mathrm{tr}(g)). \tag{4.8}$$

We observe that $J^{\mathrm{std}}\, \mathrm{d}g$ is an invariant measure on $G(F)$ which is essentially of compact support in $M^{\mathrm{std}}(F)$. Indeed, for every compact open subgroup K of $G(F)$, e_K the invariant measure on K on volume one, it is not hard to check that $e_K \star J^{\mathrm{std}}$ is of compact support in $M^{\mathrm{std}}(F)$. In a suitable sense, $J^{\mathrm{std}}\, \mathrm{d}g$ defines an element of a generalized Bernstein center. For every irreducible representation π of $G(F)$, for every matrix coefficient f of π, the convolution product $J^{\mathrm{std}} \star f|\det|^s$ converges for $\Re(s)$ large enough and there exists a constant $\gamma^{\mathrm{std}}(s, \pi)$ such that

$$J^{\mathrm{std}}\, \mathrm{d}g \star f|\det|^s = \gamma^{\mathrm{std}}(s, \pi) f|\det|^s. \tag{4.9}$$

After analytic continuation, by setting $s = 0$, we have $\gamma^{\mathrm{std}}(\pi) = \gamma^{\mathrm{std}}(0,\pi)$ and

$$J^{\mathrm{std}}\,\mathrm{d}g \star f = \gamma^{\mathrm{std}}(\pi) f, \tag{4.10}$$

where $\pi \mapsto \gamma^{\mathrm{std}}(\pi)$ is a rational function on the Bernstein variety of G. Provisionally, for generalized Bernstein center we mean the total ring of rational functions on the Bernstein variety. In this sense, $J^{\mathrm{std}}\,\mathrm{d}g$ defines an element of the generalized Bernstein center.

One can derive from (4.10) the functional equation for zeta integrals (4.1):

$$Z(\hat{\phi}, \check{f}, s) = \gamma^{\mathrm{std}}(s,\pi) Z(\phi, f, n-s). \tag{4.11}$$

In manipulating formally convolution integrals, we have

$$\begin{aligned} Z(\hat{\phi}, \check{f}, n-s) &= (\hat{\phi} \star f|\det|^{-n+s})(\delta_G) \\ &= (J^{\mathrm{std}}|\det|^{n} \star \check{\phi} \star f|\det|^{-n+s})(\delta_G) \\ &= (\check{\phi} \star J^{\mathrm{std}} \star f|\det|^{s})(\delta_G) \\ &= \gamma^{\mathrm{std}}(s,\pi) Z(\phi, f, s). \end{aligned}$$

Using the Poisson summation formula on the vector space M^{std}, Godement and Jacquet derive the functional equation of global L-functions by establishing "morally" the product formula for γ-factors

$$\prod_{v} \gamma^{\mathrm{std}}(\pi) = 1. \tag{4.12}$$

We must however remind ourselves that it does not seem possible to make sense of this formula as the infinite product in its left hand side never converges.

4.2. *Nonstandard case*

Taking Godement–Jacquet's construction of standard L-functions as a paradigm, Braverman and Kazhdan made a series of conjectures aiming at establishing the meromorphic continuation and functional equation of general automorphic L-functions. Before recalling their conjectures, we will put ourselves in a slightly more favorable situation. We will assume that G is given with a homomorphism to $\nu : G \to \mathbb{G}_m$ similar to the determinant in the case of GL_n:

$$0 \longrightarrow G' \longrightarrow G \longrightarrow \mathbb{G}_m \longrightarrow 0. \tag{4.13}$$

Dually we have a homomorphism $\nu^\vee : \mathbb{C}^\times \to {}^L G$. We will only consider irreducible representation $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ such that $\rho \circ \nu$ is the scalar multiplication of $\mathbb{C}^\times$ on V_ρ. In fact, we can always put ourselves in this setting by adding an extra $\mathbb{G}_m$ factor to G. The advantage of this setting is that we can get rid of the complex parameter s in the L-function by putting

$$L(\pi,\rho) = L\Big(-\frac{n_\rho - 1}{2}, \pi, \rho\Big) \quad \text{with } n_\rho = \langle 2\eta_G, \lambda_\rho\rangle + 1, \tag{4.14}$$

where $2\eta_G$ is the sum of positive roots of G and λ_ρ is the highest weight of ρ. We recover the whole L-function from these L-values by the formula

$$L(s,\pi,\rho) = L(\pi \otimes |\det|^{s+\frac{n_\rho-1}{2}}, \rho).$$

We will reformulate Braverman–Kazhdan's conjectures as follows. More precisions will be added in Sects. 5 and 6.

1. There exists a Schwartz space $\mathcal{S}^\rho(G(F))$ of ρ-functions on $G(F)$, locally constant on $G(F)$ and characterized by asymptotic conditions on a certain boundary such that the zeta integral (4.1) defines L-factors of irreducible representations of $G(F)$.
2. The Schwartz space of ρ-functions contains a distinguished vector, the ρ-basic function:
$$\mathbb{L}^\rho \in \mathcal{S}^\rho(G(F))$$
satisfying
$$\mathrm{tr}_\pi(\mathbb{L}^\rho) = \begin{cases} L(\pi,\rho) & \text{if } \pi \text{ is unramified,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$
3. With the help of the basic function, we can construct the global Schwartz space of adelic ρ-functions associated with a reductive group G defined over a global field k
$$\mathcal{S}^\rho(G(\mathbb{A}_k)) = \varinjlim_{S\subset|k|} \bigotimes_{v\in S} \mathcal{S}^\rho(G(k_v)) \tag{4.16}$$
over all finite sets of places of F containing all the archimedean places, the transition map in the inductive system being the tensor product with the basic functions:
$$\bigotimes_{v\in S} \phi_v \longmapsto \bigotimes_{v\in S'-S} \mathbb{L}_v^\rho \otimes \bigotimes_{v\in S} \phi_v \tag{4.17}$$
for all finite subsets $S \subset S'$ of $|k|$.
4. There exists an involutive Hankel transform $\mathcal{F}^\rho$ of $\mathcal{S}^\rho(G(F))$ to itself of the form $\phi \mapsto J^\rho|\nu|^{n_\rho} \star \check{\phi}$, where J^ρ, the ρ-Bessel function, is a locally integrable invariant function on $G^{\mathrm{srss}}(F)$, the set of (strongly regular semi-simple) elements of $G(F)$ whose centralizer is a torus. We expect that the equality $J^\rho(\mathbb{L}^\rho) = \mathbb{L}^\rho$ holds for all non-archimedean places v, where v is

unramified and the conductor of ψ is $\mathcal{O}_v$. We expect the γ-factor $\gamma^\rho(\pi)$ to be defined by the formula

$$J^\rho \star f = \gamma^\rho(\pi) f \tag{4.18}$$

for all matrix coefficients f of π.

5. Local Hankel transforms on $\mathcal{S}^\rho(G(k_v))$ induce the adelic Hankel transform on the global Schwartz space $\mathcal{S}^\rho(G(\mathbb{A}_k))$. We expect that the Poisson summation formula

$$\sum_{\gamma \in G(k)} \phi(\gamma) = \sum_{\gamma \in G(k)} \mathcal{F}^\rho(\phi)(\gamma) \tag{4.19}$$

holds for all $\phi \in \mathcal{S}^\rho(G(\mathbb{A}_k))$ subject to certain local conditions. (Note that in the standard case the Poisson sum is a sum over all rational matrices and in order to eliminate non-invertible matrices on both sides, we need to impose some local conditions on test functions.)

5. Geometry underlying the ρ-functions

In order to add more precision to the conjectures of Braverman and Kazhdan, we will need a space M^ρ, containing G as an open subset, that plays the role of the space of matrices in the standard case. For instance, the asymptotic conditions defining the ρ-Schwartz functions should refer to the boundary of G in the monoid M^ρ. A good candidate for M^ρ turns out to be a reductive monoid, that can be constructed canonically from G and ρ. This construction is an application of the general theory of reductive monoids due to Putcha and Renner of which the survey [40] is a good reference.

We will also relax the condition on the central character of ρ and no longer request ρ to be irreducible. Under some mild conditions on the central characters of ρ, we will define a monoid M^ρ of which G is the group of units, generalizing the space of matrices in the standard case. Following [11], we expect that the space $\mathcal{S}^\rho(G(F))$ of ρ-functions is defined by local conditions on $M^\rho(F)$ with respect to its totally disconnected topology. In other words, there should exist a sheaf $\tilde{\mathcal{S}}^\rho$ on the totally disconnected topological space $M^\rho(F)$ of which $\mathcal{S}^\rho(G(F))$ is the space of sections with compact support:

$$\mathcal{S}^\rho(G(F)) = \Gamma_c(M^\rho(F), \tilde{\mathcal{S}}^\rho). \tag{5.1}$$

For every $x \in M^\rho(F)$, the stalk $\tilde{\mathcal{S}}^\rho_x$ of $\tilde{\mathcal{S}}^\rho$ at x would define the asymptotic condition for a smooth function on $G(F)$ to be a ρ-function. The existence of $\tilde{\mathcal{S}}^\rho$ is only known in the case $G = \mathrm{GL}_n$ and ρ is the standard representation and in the toric case. In the toric case, it is necessary to relax the condition ρ being irreducible to produce interesting examples.

5.1. Renner's construction of monoids

A reductive monoid is a normal affine algebraic variety equipped with a monoid structure such that the unit group, which is the open subset consisting of invertible elements, is a reductive group. We will recall some basic elements of the construction of reductive monoids out of certain combinatorial data, following [40].

Let G be a reductive group defined over a separably closed field $\bar{k}$. Let M be a reductive monoid of unit group G. Let T be a maximal torus of G. If we denote by M_T the normalization of the closure of T in M, then M_T is a normal affine toric variety of torus T which is W-equivariant. This toric variety is thus completely determined by a W-equivariant strongly convex rational polyhedral cone $\sigma_M \subset \Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$, where $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$. For our purpose, the main outcome of the theory of reductive monoids is that the reductive monoid M determines and is uniquely determined by the W-equivariant strongly convex rational polyhedral cone $\sigma_M \subset \Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$, see [40, Theorems 5.2, 5.4].

We first observe that there is no reductive monoid with semisimple unit group. If G is semisimple, there is no W-equivariant strictly convex polyhedral cone $\sigma \subset \Lambda_{\mathbb{R}}$ other than zero. Indeed, if $\lambda \in \Lambda_{\mathbb{R}}$ is a non-zero vector, the average of the W-orbit of λ, being a W-invariant vector, must be 0 as $\Lambda_{\mathbb{R}}^W = 0$ for G is semisimple. It follows that 0 belongs to the interior of convex span of $W\lambda$ with respect to the topology of the linear span of $W\lambda$. It also follows that every W-equivariant cone containing the ray passing through λ contains a line, in other words, is not strictly convex.

The simplest example of W-equivariant strictly convex cone is related to the setting fixed for the Subsect. 4.2. We recall that G is equipped with a homomorphism $\nu : G \to \mathbb{G}_m$ such that dual homomorphism $\nu^\vee : \mathbb{C}^\times \to G^\vee$ induces the scalar action of $\mathbb{C}^\times$ on the vector space V_ρ. If $\Omega(\rho)$ denotes the convex span of weights of ρ, then for every $\omega \in \Omega(\rho)$ we have $\langle \nu, \omega \rangle = 1$. It follows that the cone $\xi(\rho)$ generated by $\Omega(\rho)$ is contained in the open half-space consisting of vectors $x \in \Lambda_{\mathbb{R}}$ satisfying $\langle \nu, x \rangle > 0$ and therefore is strictly convex.

This construction can be generalized as follows. Let G be a split reductive group and ρ a representation $\rho : G^\vee \to \mathrm{GL}(V_\rho)$ which is not necessarily irreducible. We will denote by $\Omega(\rho)$ the convex set in $\Lambda_{\mathbb{R}}$ generated by the weights of ρ. We will denote by $\xi(\rho)$ the cone in $\Lambda_{\mathbb{R}}$ generated by rays through elements of $\Omega(\rho)$. Whether $\xi(\rho)$ is a strictly convex cone or not depends only on how the center of $G^\vee$ acts on the vector space V_ρ.

Proposition 5.1. *Let G_k be a reductive group over a field k, which becomes split over a finite separable extension of k; $G = G_k \otimes_k \bar{k}$. Let $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ be a finite dimensional representation whose central characters are non-zero and generate a strictly convex cone. More precisely, if C denotes the connected center of $G^\vee$ and if we decompose the restriction of ρ to C as a sum* *of characters*

$$\rho|_C = \chi_1 \oplus \cdots \oplus \chi_r,$$

$\chi_1, \ldots, \chi_r \in \Lambda_C = \mathrm{Hom}(C, \mathbb{C}^\times)$ *possibly appearing with multiplicity, then we assume that* $\chi_1, \ldots, \chi_r$ *are non-zero and generate a strictly convex cone in* $\Lambda_C \otimes \mathbb{R}$*. Then there exists a unique reductive monoid* M_k^ρ *containing* G_k *as the open subset of invertible elements such that for every maximal torus* T *of* G*, the normalization of its closure* M_T^ρ *is the toric variety of torus* T *corresponding to the strictly convex cone generated by the set of weights* $\Omega(\rho)$ *of representation* ρ*.*

Proof. If the central weights $\chi_1, \ldots, \chi_r$ are non-zero and generate a strictly convex cone in $\Lambda_{C,\mathbb{R}}$, then there exists a central cocharacter $\nu : \mathbb{C}^\times \to C$ such that $\langle \nu, \chi_i \rangle > 0$ for all $i = 1, \ldots, r$. It follows that for every $\lambda \in \Omega(\rho)$, we have $\langle \nu, \lambda \rangle > 0$ and therefore the cone generated by $\Omega(\rho)$ is strictly convex.

By [40, Theorems 5.2, 5.4], there exists a unique reductive monoid M^ρ containing G as the open subset of invertible elements such that for every maximal torus T of G, the normalization of its closure M_T^ρ is the toric variety of torus T corresponding to the strictly convex cone $\xi(\rho)$ generated by the set of weights $\Omega(\rho)$ of representation ρ.

In order to justify the descent to the base field k, we need to recall the construction of the monoid. Let T be a maximal torus of G and M_T^ρ the normal affine toric variety of torus T corresponding to the strictly convex cone $\xi(\rho)$ generated by the set of weights $\Omega(\rho)$ of the representation ρ. By construction, the ring of regular functions on M_T^ρ is $\bar{k}[\xi(\rho)^\vee]$, where $\xi(\rho)^\vee \subset \Lambda^\vee$ is the sub-monoid consisting of $\alpha \in \Lambda^\vee$ such that the restriction of α to $\Omega(\rho)$ takes non-negative values. The assumption that the cone $\xi(\rho)$ generated by $\Omega(\rho)$ is strictly convex guarantees that $\xi(\rho)^\vee$ generates $\Lambda^\vee$ as an abelian group and therefore T acts on the variety M_T^ρ that contains T as an open subset. The fact that $\Omega(\rho)$ is stable under the action of W implies that the action of W on T extends to an action of W on M_T^ρ.

Let $\alpha_1, \ldots, \alpha_r \in \Lambda^\vee$ such that their orbits under W generate $\xi(\rho)^\vee$. By replacing α_i by a W-conjugate if necessary, we may assume that the α_i lie in the positive Weyl chamber $\Lambda^{\vee,+}$. Let $\omega_{\alpha_i} : G \to \mathrm{GL}(V_{\alpha_i})$ the Weyl module of G of highest weight α_i and $\omega : G \to \mathrm{GL}(V)$ the representation in the direct sum $V = \bigoplus_i V_{\alpha_i}$. The monoid M^ρ, which is defined as the normalization of the closure of $\rho(G)$ in the space of matrices $\mathrm{End}(V)$, is independent of the choice of the generators $\alpha_1, \ldots, \alpha_r \in \Lambda^{\vee,+}$.

The reductive group G defined over k gives rise to a continuous homomorphism $\sigma_{\mathrm{Out}} : \Gamma_k \to \mathrm{Out}(G_{\bar{k}})$. We denote by Out_σ the image of σ_{Out} in $\mathrm{Out}(G_{\bar{k}})$. Since Γ_k is compact and $\mathrm{Out}(G_{\bar{k}})$ is discrete, Out_σ is a finite subgroup of $\mathrm{Out}(G_{\bar{k}})$. Since the representation $\rho : G^\vee \to \mathrm{GL}(V_\rho)$ extends to ${}^L G$, the set $\Omega(\rho)$ is stable under the action of Out_σ, and so are the cones $\xi(\rho) \subset \Lambda_R$

and $\xi(\rho)^\vee \subset \Lambda^\vee_{\mathbb{R}}$. We can choose a set $\alpha_1, \ldots, \alpha_r \in \Lambda^{\vee,+}$ stable under Out_σ such that the W-conjugates of $\alpha_1, \ldots, \alpha_r$ generate $\xi(\rho)^\vee$. The homomorphism $\omega : G \to \mathrm{GL}(V)$ is then Out_σ-equivariant. It follows that the closure of the image of ω and its normalization is acted on by $G(\bar{k})/Z(\bar{k}) \rtimes \mathrm{Out}_\sigma$ extending the action of this group on $G(\bar{k})$.

A k-form G_k of G corresponds to a section of the exact sequence

$$0 \longrightarrow \mathrm{Aut}_{\bar{k}}(G) \longrightarrow \mathrm{Aut}_k(G) \longrightarrow \Gamma_k \longrightarrow 0.$$

The induced cocycle $\sigma_G : \Gamma_k \to \mathrm{Aut}_{\bar{k}}(G_{\bar{k}}) = G(\bar{k})/Z(\bar{k}) \rtimes \mathrm{Out}(G_{\bar{k}})$ has image contained in $G(\bar{k})/Z(\bar{k}) \rtimes \mathrm{Out}_\sigma$. It follows that the σ_G-action of Γ_k on $G_{\bar{k}}$ extends to M^ρ, and therefore we obtain a k-form M^ρ_k of M^ρ given by

$$M^\rho_k = \mathrm{Spec}(\bar{k}[M^\rho_{\bar{k}}]^{\sigma_G(\Gamma_k)}).$$

We have thus defined a monoid M^ρ_k of G_k which is a k-form of the monoid M^ρ of G. □

After taking some pain to construct the monoid in full generality for future reference, let us come back to the familiar case where G is split and ρ is an irreducible representation of $G^\vee$. In this case, the connected center C of $G^\vee$ acts on V_ρ via a character $\chi : C \to \mathbb{G}_m$. The assumption of Proposition 5.1 is then simply that the character χ is non-trivial, and in particular, C itself is a non-trivial torus. This encompasses the automorphic L-functions setting we fixed in the Subsect. 4.2: G is a split reductive group given with a homomorphism $\nu : G \to \mathbb{G}_m$, the dual central homomorphism $\nu : \mathbb{C}^\times \to G^\vee$ induces the scalar multiplication of $\mathbb{C}^\times$ on V_ρ. The monoids constructed in the particular setting of Subsect. 4.2 are to be called L-monoids.

The construction of the L-monoid M^ρ is motivated by the combinatorics of the basic function $\mathbb{L}^\rho$. According to the expansion of the unramified L-function $L(s,\pi,\rho)$ as a formal series (2.39), the basic function $\mathbb{L}^\rho$ has an expansion as an infinite sum

$$\mathbb{L}^\rho = \sum_{d=0}^{\infty} \mathbb{L}^\rho_d, \tag{5.2}$$

where $\mathbb{L}^\rho_d \in \mathcal{H}(G(F))$ is the element of the Hecke algebra whose Satake transform is the invariant regular function on $G^\vee$ given by $g \mapsto \mathrm{tr}(g, \mathrm{sym}^d(\rho))$. Under the assumption of Subsect. 4.2, the infinite sum (5.2) is locally finite and therefore it makes sense as a function on $G(F)$, invariant under $G(\mathcal{O}) \times G(\mathcal{O})$ whose support is not compact in general. In fact, the spherical function $\mathbb{L}^\rho_d$ has support contained in

$$\mathrm{Supp}(\mathbb{L}^\rho_d) \subset \bigsqcup_{\mu \in \Lambda^+ \cap \Omega(\mathrm{sym}^d(\rho))} K\mu K, \tag{5.3}$$

where $\Omega(\mathrm{sym}^d(\rho))$ is the set of weights of $\mathrm{sym}^d(\rho)$ and therefore

$$\mathrm{Supp}(\mathbb{L}^\rho) \subset \bigsqcup_{\mu\in\Lambda^+\cap\xi(\rho)} K\mu K = G(F)\cap M^\rho(\mathcal{O}). \tag{5.4}$$

The relation (5.4) was the main motivation behind the introduction of the monoid M^ρ for the study of L-functions in [38]. The same monoid also appears earlier in works of Braverman–Kazhdan [11] and Lafforgue [27]. More recently, Wen-wei Li and Shahidi have studied this monoid in the setting of Piatetski–Shapiro and Rallis's doubling method, see [35], [47] and [48].

We expect that the supports of ρ-functions on $G(F)$ have compact closure in $M^\rho(F)$. Moreover those functions, beside being locally constant on $G(F)$ must be characterized by asymptotic properties on the boundary $(M^\rho - G)(F)$.

5.2. *Toric varieties and toric stacks*

Let T be a split torus over a field k, and $\rho : T^\vee \to \mathrm{GL}(V_\rho)$ a finite dimensional representation of $T^\vee$ satisfying the assumption of Proposition 5.1. Suppose that ρ decomposes as a direct sum of characters

$$V_\rho = \mathbb{C}_{\mu_1} \oplus \cdots \oplus \mathbb{C}_{\mu_r},$$

where $\mu_1, \dots, \mu_r \in \Lambda$ are characters of $T^\vee$ which are not necessarily distinct. Assuming that the cone $\xi(\rho) \in \Lambda_\mathbb{R}$ generated by $\mu_1, \dots, \mu_r$ is strictly convex, we have the normal affine toric variety M_T^ρ of torus T characterized by the property that a homomorphism $\lambda : \mathbb{G}_m \to T$ extends to a morphism $\mathbb{A}^1 \to M_T^\rho$ if and only if $\lambda \in \Lambda \cap \xi(\rho)$. The ring of coordinates $k[M_T^\rho]$ of M_T^ρ is the k-algebra $k[M_T^\rho] = k[\xi(\rho)^\vee]$ associated to the subsemigroup $\xi(\rho)^\vee$ of the group $\Lambda^\vee$ consisting of $\alpha \in \Lambda^\vee$ such that $\langle\lambda, \alpha\rangle \geq 0$ for all $\lambda \in \xi(\rho)$. The assumption $\xi(\rho)$ being strictly convex guarantees that $\xi(\rho)^\vee$ generates $\Lambda^\vee$ as abelian group, or in other words, T is an open subset of M_T^ρ.

The cocharacters $\mu_1, \dots, \mu_r$ induce a homomorphism of tori $\rho_T : \mathbb{G}_m^r \to T$ given by

$$\rho_T(x_1, \dots, x_r) = \mu_1(x_1) \cdots \mu_r(x_r). \tag{5.5}$$

Let U denote the kernel of ρ_T. For each μ_i extends to a homomorphism of monoids $\mu_i : \mathbb{A}^1 \to M_T^\rho$, the formula (5.5) gives rise to a homomorphism of monoids $\rho_{M_T} : \mathbb{A}^r \to M_T^\rho$ such that we have a Cartesian diagram

$$\begin{array}{ccc} \mathbb{G}_m^r & \longrightarrow & \mathbb{A}^r \\ {\scriptstyle\rho_T}\downarrow & & \downarrow{\scriptstyle\rho_{M_T}} \\ T & \longrightarrow & M_T^\rho. \end{array} \tag{5.6}$$

We consider the quotient stack of $\mathbb{A}^r$ by the action of U

$$\mathcal{M}_T^\rho = \mathbb{A}^r/U \tag{5.7}$$

of which M_T^ρ is the coarse space. As opposed to M_T^ρ, the quotient stack $\mathcal{M}_T^\rho$ depends on the cocharacters $\mu_1, \ldots, \mu_r \in \Lambda$ i.e., on the representation $\rho : T^\vee \to \mathrm{GL}(V_\rho)$ and not merely on the cone $\xi(\rho)$ generated by $\mu_1, \ldots, \mu_r$. As we will see, the space of ρ-functions depends on the finer geometry of $\mathcal{M}_T^\rho$. Before coming to that point, let us show that this construction can be generalized to an arbitrary torus.

Let T be a torus over an arbitrary base field k. If Λ denotes the group of cocharacters $\mathbb{G}_m \to T$ defined over a separable closure of k, then the Galois group Γ_k of k acts on Λ through a finite quotient. The Langlands dual group is then ${}^L T = T^\vee \rtimes \Gamma_k$, where $T^\vee = \mathrm{Hom}(\Lambda, \mathbb{C}^\times)$. Let $\rho : {}^L T \to \mathrm{GL}(V_\rho)$ be an r-dimensional algebraic representation of ${}^L T$ satisfying the assumption of Proposition 5.1. The restriction of ρ to $T^\vee$ is a direct sum of characters, possibly with multiplicity

$$\lambda|_{T^\vee} = \lambda_1^{r_1} \oplus \cdots \oplus \lambda_m^{r_m}, \tag{5.8}$$

where $\lambda_1, \ldots, \lambda_m \in \Lambda$ are distinct characters of $T^\vee$ and multiplicities $r_1, \ldots, r_m$ adding up to r. The weights $\lambda_1, \ldots, \lambda_m$ given with multiplicities $r_1, \ldots, r_m$ determine a finite subset

$$R_\rho = \{(\lambda_1, 1), \ldots, (\lambda_1, r_1), \ldots, (\lambda_m, 1), \ldots, (\lambda_m, r_m)\} \tag{5.9}$$

of $\Lambda \times \mathbb{N}$ of cardinality $r_1 + \cdots + r_m = r$. For the representation $\lambda|_{T^\vee}$ extends to $T^\vee \rtimes \Gamma_k$, this subset is invariant under the action of Γ_k on $\Lambda \times \mathbb{N}$.

Over $\bar{k}$, we have a homomorphism $\rho_T : \mathbb{G}_m^{R_\rho} \to T_{\bar{k}}$ given by the formula (5.5). As this homomorphism is equivariant with respect to the action of Γ_k, it can be descended to a homomorphism between tori over k

$$\rho_T : D^\rho \to T, \tag{5.10}$$

where D^ρ is the unique torus defined over k satisfying $D^\rho \otimes_k \bar{k} = \mathbb{G}_m^{R_\rho}$ and such that the action of Γ_k on the scalar restriction $\mathbb{G}_m^{R_\rho}$ coincides with the one derived from the action of Γ_k on R_ρ. We will denote by U the kernel of ρ_T. Let A^ρ denote the affine space over k satisfying $A^\rho \otimes_k \bar{k} = \mathbb{A}^{R_\rho}$ and such that the action of Γ_k on the scalar restriction $\mathbb{A}^{R_\rho}$ coincides with the action of Γ_k on the set of indices R_ρ. By construction $A^\rho(k)$ is a product of finite separable extensions of k. The homomorphism of tori can be extended to a morphism of monoids $\rho_{M_T} : A^\rho \to M_T^\rho$ such that we have a Cartesian diagram

$$\begin{array}{ccc} D^\rho & \longrightarrow & A^\rho \\ {\scriptstyle \rho_T}\downarrow & & \downarrow{\scriptstyle \rho_{M_T}} \\ T & \longrightarrow & M_T^\rho. \end{array} \tag{5.11}$$

We will denote by $\mathcal{M}_T^\rho = A^\rho/U$ the stack quotient of A^ρ by the action of U of which M_T^ρ is the coarse space i.e., the quotient in the sense of invariant theory. Now we will explain how the geometry of $\mathcal{M}_T^\rho$ gives rise to the space of ρ-functions when the base field k is a non-archimedean local field.

The simplest example one may have in mind is the case where $T = \mathbb{G}_m$ and $\rho = \mathrm{std} \oplus \mathrm{std}$ is the 2-dimensional representation of $\mathbb{C}^\times$ acting on $\mathbb{C}^2$ as scalar multiplication. In this case, the toric variety M_T^ρ is the affine line $\mathbb{A}^1$ and the homomorphism $\rho_T : \mathbb{G}_m^2 \to \mathbb{G}_m$ is the multiplication map $\rho_T(x, y) = xy$ which can be extended to $\mathbb{A}^2 \to \mathbb{A}^1$. The stack $\mathcal{M}_T^\rho$ is $\mathbb{A}^2/U$, where U is isomorphic to $\mathbb{G}_m$ and acts on $\mathbb{A}^2$ by the hyperbolic action $t(x, y) = (tx, t^{-1}y)$. Now let the base field k be a non-archimedean local field F with ring of integers $\mathcal{O}_F$. The basic function $\mathbb{L}^\rho$ defined by (4.15) can be identified with the push-forward of the characteristic function of $\mathbb{A}^2(\mathcal{O}_F) - \{0\}$ to $\mathbb{A}^1(\mathcal{O}_F) - \{0\}$ with multiplicative Haar measure normalized as in Subsect. 2.3. By a straightforward calculation, we see that this function is the function supported in $\mathcal{O}_F$ and given by the formula $z \mapsto \mathrm{val}(z) + 1$. In this case, every ρ-function is a linear combination of the basic function $\mathbb{L}^\rho$ and smooth compactly supported functions on F.

Let us now consider a more general case. Let the base field k be a non-archimedean local field F. We assume that $\rho_T : D^\rho \to T$ is a surjective homomorphism of tori whose kernel is a split torus over F. Under this assumption, we have $\mathrm{H}^1(F, U) = 0$ and the ρ_T induces a surjective homomorphism $D^\rho(F) \to T(F)$ on F-points. In this case, ρ-functions on $T(F)$ are obtained by integration along the fibers from smooth functions with compact support in $A^\rho(F)$ or in other words

$$\mathcal{S}^\rho(T(F)) = \mathcal{S}(A^\rho(F))_{U^\rho(F)}, \tag{5.12}$$

where $\mathcal{S}(A^\rho(F))_{U^\rho(F)}$ is the space of $U(F)$-coinvariants in the space of compactly supported smooth functions on the F-vector space $A^\rho(F)$. This is the space of ρ-functions defined by L. Lafforgue in [28, II.1].

Without the assumption $\mathrm{H}^1(U, F) = 0$, for the map $D^\rho(F) \to T(F)$ may not be surjective, the above definition of ρ-functions has to be corrected. We expect the space of ρ-functions in the toric case to be Sakellaridis's space of Schwartz functions on F-points of the smooth stack A^ρ/U defined in [42].

The case of tori suggests that in addition to the monoid M^ρ, one may want to construct an algebraic stack $\mathcal{M}^\rho$ of which M^ρ is the coarse space. As opposed to M^ρ which only depends on the cone $\xi(\rho)$ generated by the set $\Omega(\rho)$ of weights of ρ, one expects an algebraic stack $\mathcal{M}^\rho$ retains enough information to reconstruct ρ. Over a finite field, the category of ℓ-adic sheaves over $\mathcal{M}^\rho$ should be equipped with an involutive Fourier transform, similar to the Fourier transform on the space of matrices in the standard case. This question has been addressed by Laumon and Lettelier whose work on progress sheds some lights on the nature of $\mathcal{M}^\rho$.

5.3. Perverse sheaves on arc spaces

When the representation $\rho : G^\vee \to \mathrm{GL}(V_\rho)$ is irreducible, singularities of the associated monoid M^ρ must bear relation with the asymptotic behavior of ρ-functions. In particular, as it is shown in [10], the basic function $\mathbb{L}^\rho$ is the Frobenius trace function on the intersection complex of the arc space of $\mathbb{L}^\rho$. To state this result with some precision, we need some background on arc spaces.

We will assume that F is a local field of positive characteristic i.e., $F = k((t))$ and $\mathcal{O} = k[[t]]$, k being a finite field of characteristic p. Let M be an affine algebraic variety over k. For every $d \in \mathbb{N}$, we define the jet space $\mathcal{L}_d M$ of order d to be the affine variety representing the functor $R \mapsto M(R[[t]]/t^{d+1})$ from the category of k-algebras to the category of sets. The arc space $\mathcal{L}M$ is defined to be the projective limit of $\mathcal{L}_d M$

$$\mathcal{L}M = \varprojlim_{d \in \mathbb{N}} \mathcal{L}_d M. \tag{5.13}$$

We have $\mathcal{L}M(R) = M(R[[t]])$ and in particular $\mathcal{L}M(k) = M(\mathcal{O})$.

For the arc space $\mathcal{L}M$ is infinite dimensional when $\dim(M) > 0$, it is not clear how to make sense of the intersection complex of $\mathcal{L}M$. In [24] and [19], Grinberg, Kazhdan and Drinfeld proved that the formal completion of $\mathcal{L}M$ at a non-degenerate arc is equivalent to the formal completion of a finite-dimensional variety up to formally smooth equivalence. More precisely, a k-point $x \in \mathcal{L}M(k)$ is said to be non-degenerate if as a morphism $x : \mathrm{Spec}(\mathcal{O}) \to M$, it sends the generic point $\mathrm{Spec}(F)$ of $\mathrm{Spec}(\mathcal{O})$ in the smooth locus of M. The theorem of Grinberg–Kazhdan–Drinfeld says that for every non-degenerate point $x \in \mathcal{L}M(k)$ of the arc space, there exists a finite dimensional k-variety Y and a k-point $y \in Y(k)$ such that there exists an isomorphism

$$\mathcal{L}M_x \simeq Y_y \hat{\times} \mathbb{D}^\infty, \tag{5.14}$$

where $\mathcal{L}M_x$ is the formal completion of the arc space $\mathcal{L}M$ at x, Y_y the formal completion of Y at y and $\mathbb{D}^\infty$ the infinite power of the formal disc.

This result is not enough to construct the intersection complex on $\mathcal{L}M$ for it only gives a finite dimensional description of the formal completion instead of henselization. Nevertheless, if k is a finite field, we can define the trace of the Frobenius on the stalk of the sought for intersection complex on the arc space by means of its finite dimensional formal model. In [10, Proposition 1.2] we proved that this definition is well grounded in the sense that the function we construct is independent of the finite dimensional formal model we choose. As a result, we have a canonical function IC on the subset of $\mathcal{L}M(k)$ consisting of non-degenerate arcs. In [10], we prove:

Theorem 5.2. *If G a split reductive group and ρ is an irreducible representation of $G^\vee$ as in Subsect. 4.2, then up to a normalization constant, the function* IC *on $\mathcal{L}M^\rho(k)$ is the basic function $\mathbb{L}^\rho$.*

We believe that not only the basic function but many relevant ρ-functions may be related to other perverse sheaves on arc spaces. In [9], complemented by [39], Bouthier and Kazhdan outlined a theory of perverse sheaves on arc spaces. A definitive theory is yet to be established.

6. Kernel of the Hankel transform

In this section, we will construct a general stable class function which coincide with the correct kernel of the Hankel transform in the toric case as well as the standard case. This construction is similar to a construction of Braverman and Kazhdan for reductive groups over finite fields, which we will also recall.

6.1. ρ-Bessel functions on tori

Let F be a non-archimedean local field and $\psi : F \to \mathbb{C}^\times$ a non-trivial additive character. Let T be a torus defined over a non-archimedean local field F and $\rho : {}^L T \to \mathrm{GL}(V_\rho)$ a finite-dimensional representation of its L-group satisfying the assumption of Proposition 5.1. We will define a canonical smooth function

$$J_T^\rho : T(F) \longrightarrow \mathbb{C} \tag{6.1}$$

to be called the ρ-Bessel function of T.

As in Subsect. 5.2, the representation ρ of the L-group gives rise to a canonical induced torus D^ρ defined over F endowed with a homomorphism $\rho_T : D^\rho \to T$. This homomorphism of tori extends to a monoid homomorphism $A^\rho \to M_T^\rho$. We recall that after base change to the separable closure $\bar{F}$ of F, we have $D^\rho \otimes_F \bar{F} = \mathbb{G}_m^{R_\rho}$ and $A^\rho \otimes_F \bar{F} = (\mathbb{A}^1)^{R_\rho}$, where R_ρ is the finite set defined in (5.9) which is equipped with a canonical action of the Galois group Γ_F of F.

The morphism $h : (\mathbb{A}^1)^{R_\rho} \to \mathbb{G}_a$ defined by

$$(x_\alpha)_{\alpha\in R_\rho} \longmapsto \sum_{\alpha\in R_\rho} x_\alpha$$

is clearly invariant under the action of Γ_F and therefore descends to a morphism

$$h : A^\rho \longrightarrow \mathbb{A}^1 \tag{6.2}$$

defined over F. We denote by $h_\psi : A^\rho(F) \to \mathbb{C}^\times$ the function defined by $x \mapsto \psi(h(x))$.

We define the function $J_T^\rho : T(F) \to \mathbb{C}$ by integrating the restriction of the function h_ψ to $D^\rho(F)$ along the fibers of $\rho_T : D^\rho(F) \to T(F)$. In other words, for every $t \in T(F)$, we aim at defining the value of J_T^ρ at t by the formula

$$J_T^\rho(t) = \int_{\rho_T^{-1}(t)} h_\psi(x)\,\mathrm{d}x. \tag{6.3}$$

To give a sense to this integral we need to show how to normalize the measure dx on the fiber $\rho_T^{-1}(t)$ and how to regularize the integral as the function h_ψ is not of compact support.

First, if x does not belong to the image of $D^\rho(F) \to T(F)$ we declare that $J_T^\rho(x) = 0$. If x belongs to the image of $D^\rho(F) \to T(F)$, then by choosing a F-point of $\rho_T^{-1}(t)$, we get an isomorphism $\rho_T^{-1}(t)(F) \simeq U(F)$, where U is the kernel of $\rho_T : D^\rho \to T$. The canonical invariant measure on tori defined in Subsect. 2.3 gives rise to a measure on $\rho_T^{-1}(t)(F)$ which is independent of the choice of the base F-point on the principal homogenous space $\rho_T^{-1}(t)$. We have thus a canonical measure dx on $\rho_T^{-1}(t)(F)$ as long as t belongs to the image of $D^\rho(F) \to T(F)$.

Proposition 6.1. *Let T be a torus defined over a non-archimedean local field F and $\rho : {}^LT \to \mathrm{GL}(V_\rho)$ a finite-dimensional representation of its L-group satisfying the assumption of Proposition 5.1. Let $\rho_T : D^\rho \to T$ be the associated homomorphism of tori and U its kernel. For every compact open subgroup K_U of $U(F)$, we denote by e_{K_U} the distribution with compact support on $U(F)$ defined by the invariant measure on the compact subgroup K_U with total mass one. Then the function*

$$h_\psi \star e_{K_U} : D^\rho(F) \longrightarrow \mathbb{C}$$

is a smooth function of proper support relatively to $\rho_T : D^\rho(F) \to T(F)$. The integral

$$J_T^\rho(t) = \int_{\rho_T^{-1}(t)} (h_\psi \star e_{K_U})(x)\,\mathrm{d}x \tag{6.4}$$

defines a smooth function on $T(F)$ which is independent of the choice of K_U.

We regularize of the integral (6.3) by setting (6.4) as the definition of the ρ-Bessel function $J_T^\rho : T(F) \to \mathbb{C}$.

In the case $G = \mathbb{G}_m$ and $\rho = \mathrm{std} \oplus \mathrm{std}$, the induced torus D^ρ is $\mathbb{G}_m^2$ which is endowed with the homomorphism $\rho_T : D^\rho \to T$ given by $(x, y) \mapsto xy$. In this case, the integral (6.10) defines the p-adic Kloosterman integral. In this case, the function J_T^ρ was introduced and calculated by Sally and Taibleson in [44, Section 4].

6.2. ρ-Bessel functions on reductive groups

Let G be a reductive group over a non-archimedean local field F and $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ a representation of its L-group satisfying the assumption of Proposition 5.1. We will define a canonical ρ-Bessel function $J_G^\rho : G^{\mathrm{srss}}(F) \to \mathbb{C}$, where G^{srss} is the strongly regular semi-simple open subset of G consisting of elements $x \in G$ such that the centralizer G_x is a maximal torus. It will be the unique stably invariant function satisfying

$$J_G^\rho(x) = J_T^\rho(x), \tag{6.5}$$

where J_T^ρ is the ρ-Bessel function of the torus $T = G_x$ centralizer of x.

There is a priori a difficulty to make sense of the right hand side of (6.5) for the representation $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ of ${}^L G$ does not induce a representation of ${}^L T$ in general. To circumvent this difficulty we observe that the construction of the ρ-Bessel function presented in Subsect. 6.1 does not require a representation ${}^L T$ but merely a finite set R_ρ of cardinality $r = \dim(V_\rho)$ endowed with an action of Γ_F.

To clarify the matters, let us recall the construction of the L-groups of T and G and compare them. As in Subsect. 2.1, we can associate to a pair (T, G) consisting of a reductive group G and a maximal torus T both defined over F, a root datum $\Psi(T, G) = (\Lambda, \Phi)$ object of RT endowed with a continuous action of Γ_F

$$\sigma_\Psi : \Gamma_F \longrightarrow \mathrm{Aut}(\Lambda, \Phi) = W \rtimes \mathrm{Out}(\Lambda, \Phi).$$

Going one step down to the category $\mathrm{RD}_{\mathrm{Out}}$ in the fundamental diagram (2.13) we obtain a homomorphism

$$\sigma_{\Psi_{\mathrm{Out}}} : \Gamma_F \longrightarrow \mathrm{Out}(\Lambda, \Phi)$$

which depends only on G and not on the maximal torus T. By using an inverse of the equivalence $\mathrm{RD}^+ \to \mathrm{RD}_{\mathrm{Out}}$ in the diagram (2.13), we obtain an object $(\Lambda, \Phi, \Lambda^+)$ of $\mathrm{RD}_{\mathrm{Out}}$ equipped with a homomorphism

$$\sigma_{\Psi^+} : \Gamma_F \longrightarrow \mathrm{Aut}(\Lambda, \Phi, \Lambda^+).$$

By exchanging roots and coroots, we get a homomorphism

$$\sigma_{\Psi^+} : \Gamma_F \longrightarrow \mathrm{Aut}(\Lambda^\vee, \Phi^\vee, \Lambda^{\vee,+}).$$

An inverse of the equivalence $\mathrm{Pin}_{\mathbb{C}} \to \mathrm{RD}^+$ in the diagram (2.13) gives rise to a reductive group $G^\vee$ over $\mathbb{C}$ endowed with a pinning. The pinning encompasses many data including a maximal torus $T^\vee$ with $\mathrm{Hom}(T^\vee, \mathbb{C}^\vee) = \Lambda$.

We form ${}^L G = G^\vee \rtimes \Gamma_F$ with Γ_F acting on $G^\vee$ in preserving the pinning by means of σ_{Ψ^+}. On the other hand, we form ${}^L T = T^\vee \rtimes \Gamma_F$ by means of σ_Ψ. As the two actions of Γ_F on $T^\vee$ through σ_Ψ and σ_{Ψ^+} are different, there is a priori no given homomorphism ${}^L T \to {}^L G$. Nevertheless, the restriction of $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ to $T^\vee$ defines a finite subset $R_\rho \subset \Lambda \times \mathbb{N}$

$$R_\rho = \{(\lambda, d) \in \Lambda \times \mathbb{N} \mid 1 \le d \le r_\rho(\lambda)\}, \tag{6.6}$$

where $r_\rho(\lambda)$ is the multiplicity of the weight $\lambda \in \Lambda$ in ρ. This set is stable under the action of the Weyl group. If we denote by Out_σ the finite image of $\sigma_{\Psi_{\mathrm{Out}}}$ in $\mathrm{Out}(\Lambda, \Phi)$, then R_ρ is also stable under the action of Out_σ. It follows that R_ρ is stable under the action of $W \rtimes \mathrm{Out}_\sigma$. Since its image is contained in $W \rtimes \mathrm{Out}_\sigma$, the homomorphism σ_Ψ induces an action of Γ_F on the finite subset R_ρ of $\Lambda \times \mathbb{N}$ compatible with its action on Λ.

We are now in position to apply the construction in Subsect. 6.1 and make sense of the right hand side of formula (6.5). This formula gives rise to a function $x \mapsto J_G^\rho(x)$ defined for all elements $x \in G(F)$ whose centralizer is a maximal torus.

Proposition 6.2. *Let G be a reductive group over a non-archimedean local field F and $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ a representation of its L-group satisfying the assumption of Proposition 5.1. Then the function $J_G^\rho : G^{\mathrm{srss}}(F) \to \mathbb{C}$ defined by (6.5) is stably invariant.*

Proof. It is equivalent to prove that the function J_G^ρ is well-defined on the adjoint quotient. To discuss question of rationality, it is convenient to switch to notations of Subsect. 2.1. We denote by G_F the reductive group defined over F and use the letter G to denote $G = G_F \otimes_F \bar{F}$. We denote by T a maximal torus of G and $W = \mathrm{Nor}_G(T)$ the associated Weyl group. We have the adjoint quotient morphism

$$c : G \longrightarrow C, \tag{6.7}$$

where $C = T/\!/W$ is the coarse quotient defined by $\bar{F}[C] = \bar{F}[T]^W$. We note that the morphism c is invariant under the inner automorphisms of G. There is an open subset C^{srss} of C consisting of all $a \in C$ such that for all $x \in G$ mapping to a, the centralizer G_x is a maximal torus. The fiber of c over $a \in C^{\mathrm{srss}}$ is a homogenous space for the inner action of G. The Weyl group W acts freely on the preimage T^{srss} of C^{srss} in T.

We recall the exact sequence

$$0 \longrightarrow G(\bar{F})/Z(\bar{F}) \longrightarrow \mathrm{Aut}_{\bar{F}}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 0, \tag{6.8}$$

where $\mathrm{Aut}_{\bar{F}}(G)$ is the group of $\bar{F}$-automorphisms of G, $G(\bar{F})/Z(\bar{F})$ is the subgroup on inner automorphisms and $\mathrm{Out}(G)$ is the group of outer automorphisms which is determined by the associated root datum. We note that the morphism c is equivariant with respect to the action of $\mathrm{Aut}_{\bar{F}}(G)$ on G and $\mathrm{Out}(G)$ on C.

As discussed in Subsect. 2.1, a k-form G_F of G corresponds to a homomorphism $\sigma : \Gamma_F \to \mathrm{Aut}_{\bar{F}}(G)$. We will write $G_F = G^\sigma$. The homomorphism

$\sigma : \Gamma_F \to \mathrm{Aut}_{\bar{F}}(G)$ induces a homomorphism $\sigma_{\mathrm{Out}} : \Gamma_F \to \mathrm{Out}(G)$. Since $\mathrm{Aut}_{\bar{k}}(G)$ and $\mathrm{Out}(G)$ act on G and C compatibly, we obtain a σ-twist of morphism c

$$c^\sigma : G^\sigma \longrightarrow C^\sigma$$

which is invariant under inner automorphisms of G^σ. For every $x \in G^{\mathrm{srss},\sigma}(F)$, the stable conjugacy class of x is the set $c^{\sigma,-1}(c^\sigma(x))(F)$ of F-points on the fiber of c^σ passing through x. We will define a function $C^{\mathrm{srss},\sigma}(F) \to \mathbb{C}$ whose pullback to $G^{\mathrm{srss},\sigma}(F)$ is J_G^ρ defined by the formula (6.5). This implies in particular that J_G^ρ is stably invariant.

The representation $\rho : {}^L G \to \mathrm{GL}(V_\rho)$ gives rise to a finite subset $R_\rho \subset \Lambda \times \mathbb{N}$ defined in (6.6) with a canonical homomorphism $\rho_T : \mathbb{G}_m^{R_\rho} \to T$. As the Weyl group W acts canonically on R_ρ, we have an induced morphism on the coarse quotients

$$\rho_T^W : \mathbb{G}_m^{R_\rho} // W \longrightarrow T // W = C.$$

Twisting by σ, we get a morphism

$$\rho_T^{W,\sigma} : (\mathbb{G}_m^{R_\rho} // W)^\sigma \longrightarrow C^\sigma.$$

The formula (6.3) can be used to define a function on $C^{\mathrm{srss},\sigma}(F)$ whose pullback to $G^{\mathrm{srss},\sigma}(F)$ is the function J_G^ρ defined by (6.5). The function J_G^ρ is therefore stably invariant. □

Proposition 6.3. *Extending the function J_G^ρ from $G^{\mathrm{srss}}(F)$ to $G(F)$ by zero, we obtain a locally integrable function on $G(F)$. For every compact open subgroup K of $G(F)$, the measure $J_{G(F)}^\rho \star e_K$ has compact support in the monoid $M^\rho(F)$.*

Proof. It is enough to prove that J_G^ρ is locally bounded. Let $x \in G(F)$ and and $a \in C(F)$ its image. There are only finitely many tori T_α up to conjugacy whose image in $C(F)$ contains a. Since $J_{T_\alpha}^\rho$ are smooth function on $T_\alpha(F)$, in particular locally bounded, the function J_G^ρ is locally bounded. Similarly, the compactness of the support in the monoid can also be reduced to the torus case. □

The Bessel function J_G^ρ defined as above coincide with the kernel of the Hankel transform in the toric case, as well as in the standard case. Lafforgue pointed out that even for the case of symmetric power of GL_2, one needs to modify further J_G^ρ to obtain the correct Hankel transform. In recent works of Shahidi [48] as well as Dihua Jiang progress of Jiang and his students, the correct Hankel transform has been explicitly computed. It is important to understand the precise relationship between the naive kernel J_G^ρ constructed as above in general as the correct kernel that has been computed in many cases, but still lacks a general description.

6.3. ρ-Bessel sheaves for reductive groups over finite fields

Our construction of the kernel of the Hankel transform is inspired by the construction of Braverman and Kazhdan in the case of reductive group over finite fields in [12]. For simplicity, we first assume that G is split over a finite field k and that the restriction of $\rho : G^\vee \to \mathrm{GL}(V_\rho)$ to the maximal torus $T^\vee$ of $G^\vee$ is multiplicity free. In other words

$$\rho|_{T^\vee} = \lambda_1 \oplus \cdots \oplus \lambda_r, \tag{6.9}$$

where $\lambda_1, \ldots, \lambda_r$ are distinct characters of $T^\vee$. The multiplicity-free assumption implies that there exists a canonical homomorphism $\rho_W : W \to \mathfrak{S}_r$, where W is the Weyl group of G such that the decomposition (6.9) is W-equivariant. The dual homomorphism of tori $\rho_T : \mathbb{G}_m^r \to T$ is thus W-equivariant. We define the ℓ-adic ρ-Bessel sheaf

$$\mathcal{J}_T^\rho = \check{\rho}_{T,!}\mathcal{J}^{\mathrm{std}}, \tag{6.10}$$

where $\mathcal{J}^{\mathrm{std}} = (x_1 + \cdots + x_r)^* \mathcal{L}_\psi$, where $x_1, \ldots, x_n$ are the standard coordinates of $\mathbb{G}_m^r$ and $\mathcal{L}_\psi$ is the Artin–Schreier sheaf on $\mathbb{G}_a$ attached to a non-trivial additive character $\psi : k \to \mathbb{Q}_\ell^\times$. In [16], we proved that the assumption (4.13) guarantees that $\mathcal{J}_T^\rho$ is, up to a degree shift, a local system [16]. This is a generalization of Deligne's theorem on hyper-Kloosterman sums in [18].

Let T^{rss} be the open subset of T where W acts freely. With help from the canonical action of W on $\mathcal{J}^\rho$, we can descend $\mathcal{J}_T^\rho|_{T^{\mathrm{rss}}}$ to a local system $\tilde{\mathcal{J}}_T^\rho$ on T^{rss}/W. If $j : G^{\mathrm{rss}} \hookrightarrow G$ denotes the subset of G of regular semisimple elements, we have a natural G-invariant map $p : G^{\mathrm{rss}} \to T^{\mathrm{rss}}/W$. Following Braverman and Kazhdan, we set

$$\mathcal{J}^\rho = j_{!*} p^* \tilde{\mathcal{J}}_T^\rho. \tag{6.11}$$

This construction is very similar to the construction of character sheaves. The main difference is that we start on the torus with the local system $\mathcal{J}_T^\rho$ of Kloosterman type with large monodromy instead of Kummer local systems as in the construction of character sheaves.

Without the multiplicity free assumption, we can still define a canonical action of W on $\mathcal{J}_T^\rho$, on the ground of the Hasse–Davenport relation for Kloosterman sums, as follows. The decomposition in weight spaces of V_ρ defines a Levi subgroup M of $\mathrm{GL}(V_\rho)$. If W_M denotes the Weyl group of M, there exists a canonical extension W' of W by W_M:

$$0 \longrightarrow W_M \longrightarrow W' \longrightarrow W \longrightarrow 0. \tag{6.12}$$

By construction, the finite group W' is equipped with canonical homomorphisms $W' \to \mathfrak{S}_r$ and $W' \to W$ such that the homomorphism of tori $\mathbb{G}_m^r \to T$

is W'-equivariant. It follows that $\mathcal{J}_T^\rho = \check{\rho}_{T,!}\mathcal{J}^{\mathrm{std}}$ is W'-equivariant i.e., for every $w' \in W'$, we have a canonical morphism $\alpha(w') : w'^* \mathcal{J}_T^\rho \to \mathcal{J}_T^\rho$, satisfying the usual cocycle equation. The Hasse–Davenport relation implies that the induced action of W_M on $\mathcal{J}_T^\rho$ is given by the sign character

$$\mathrm{sgn}_r : W_M \longrightarrow \mathfrak{S}_r \longrightarrow \{\pm 1\}. \tag{6.13}$$

We construct a new action of W' on $\mathcal{J}_T^\rho$ by setting

$$\alpha'(w') = \alpha(w')\mathrm{sgn}_r(w'). \tag{6.14}$$

The restriction of α' to W_M is then trivial, and therefore it induces an action of W on $\mathcal{J}_T^\rho$. In order to be compatible with the standard case, we need to correct this action by the sign character of W and set

$$\alpha''(w) = \alpha'(w)\mathrm{sgn}_W(w), \tag{6.15}$$

where $\mathrm{sgn}_W : W \to \{\pm 1\}$ is the sign character of W. This construction provides us with an action of W on $\mathcal{J}_T^\rho$. This action of W being granted, we can continue the construction of the γ-Bessel sheaf $\mathcal{J}^\rho$ as in the multiplicity-free case (6.11).

As the construction of $\mathcal{J}^\rho$ is purely geometric, it is not a priori obvious that it acts on matrix coefficients with expected γ-factors compatible with Deligne–Lusztig's induction. This is implied by the Conjecture 9.19 of Braverman and Kazhdan [11]. This conjecture may be seen as the geometric reason for why the kernel $\mathcal{J}^\rho$ behaves correctly with respect to Deligne–Lusztig induction. In [12], Braverman and Kazhdan proved this conjecture for groups of semi-simple rank 1. In [16], S. Cheng and I proved this conjecture for GL_n and arbitrary ρ. T.H. Chen proved this conjecture for arbitrary G in the setting of D-modules in [14].

7. ρ-trace formula and Poisson sums

This section aims to describe global problems related to the trace formula and the Poisson summation formulas. As the current state of my knowledge is limited and piecemeal, the discussion will have to be imprecise. We aim only to draw an impressionistic picture connecting relevant works.

7.1. ρ-trace formula for beyond endoscopy

The spaces of ρ-functions $\mathcal{S}^\rho(G(F_v))$ along with the basic function $\mathbb{L}_v^\rho \in \mathcal{S}^\rho(G(F_v))$ provide the natural theoretical framework for the beyond endoscopic trace formula (3.5). Assume that we have defined local Schwartz space $\mathcal{S}^\rho(G(F))$ for all places v, both archimedean and non-archimedean. For non-archimedean place, we have also defined the basic function $\mathbb{L}_v^\rho \in \mathcal{S}^\rho(G(F))$. We then define the global ρ-Schwartz space as the inductive limit

$$\mathcal{S}^\rho(G(\mathbb{A})) = \varinjlim_{S} \bigotimes_{v\in S} \mathcal{S}^\rho(G(F)) \tag{7.1}$$

over all finite sets of places of F containing all the archimedean places. The inductive system is formed with the basic functions.

Leaving aside very serious analytic difficulties, the stable ρ-trace formula must consists in a stably invariant linear form $S^\rho : \mathcal{S}^\rho(G(\mathbb{A})) \to \mathbb{C}$ endowed with a geometric expansion

$$S^\rho(\phi) = \sum_\gamma \mathrm{SO}_\gamma(\phi) + \text{ hyperbolic terms} \tag{7.2}$$

and a spectral expansion

$$S^\rho(\phi) = \sum_\pi m_\pi L^{\rho,S}(\pi) \prod_{v\in S} \chi^{\mathrm{st}}_{\pi_v}(\phi_v) + \text{ continuous terms.} \tag{7.3}$$

In the spectral expansion, the traces $\prod_{v\in S} \chi^{\mathrm{st}}_{\pi_v}(\phi_v)$ are weighted by values of partial L-functions $L^{\rho,S}(\pi)$ at complex parameter s specified as in (4.14). Other values of s can be obtained by twisting. In fact, one need to move s in some half-plane $\Re(s) \gg 0$ to assure absolute convergence. It is necessary to shift s at least further right than the pole of the L-function of the trivial representation π. Before attaining essentially interesting arithmetic information concealed in the poles at $s = 1$ of L functions of tempered representations, one should have an analytic control on the removal of non-tempered representations whose L-functions have poles to the right of $s = 1$.

For this purpose, one of the proposals of [21] is to treat the geometric side (7.2) as a Poisson sum. Regular semisimple stable conjugacy classes in G over a field F may be identified with F-points of the invariant quotient $T//W$, which is an affine space under the assumption G semisimple and simply connected. In [21], we call that affine space the Steinberg–Hitchin base. It is thus theoretically possible to analyze the analytic behavior of (7.2) by using the Poisson summation over the Steinberg–Hitchin base. This completely new approach to the trace formula gives rise to a formidable technical difficulties of which we have not yet grasped the essence. This approach has been pursued in Altug's in his PhD thesis [1], [2], [3], in the case $G = \mathrm{GL}_2$ and ρ being either the standard representation or the symmetric square representation. Among significant works in this direction, on should mention those of Getz, Herman and Sakellaridis [25], [22] and [41], [43].

One difficulty that would immediately arise when we look at the general case is that we do not have a good understanding the Fourier transform on the Steinberg–Hitchin base from the representation theoretic standpoint.

7.2. Trace formula and the functional equation

The trace formula (7.3) may also provide a new road to the functional equation of L-functions. By taking average over π, the functional equation of L-function would boil down to a comparison between $S^\rho(\phi)$ and $S^\rho(\mathcal{F}^\rho(\phi))$, where $\mathcal{F}^\rho(\phi)$ is the Hankel transform of an adelic ρ-function $\phi \in \mathcal{S}^\rho(G(\mathbb{A}))$. We want to establish this comparison by the way of the geometric sides. One should observe that the comparison between $S^\rho(\phi)$ and $S^\rho(\mathcal{F}^\rho(\phi))$ can be derived from the conjectural Poisson summation formula (4.19) by integration over the automorphic quotient $G(F)\backslash G(\mathbb{A}_k)$.

Comparison between the geometric sides (7.2) boils down to compare $\sum_\gamma \mathrm{SO}_\gamma(\phi) + \cdots$ and $\sum_\gamma \mathrm{SO}_\gamma(\mathcal{F}^\rho(\phi)) + \cdots$. Stable orbital integrals of ϕ induces a function $\theta_\phi : (T//W)(\mathbb{A}_k) \to \mathbb{C}$ such that $\mathrm{SO}_\gamma(\phi) = \theta_\phi(a)$, where $a \in (T//W)(\mathbb{A}_k)$ is the characteristic polynomial of γ. On the space $\Theta^\rho((T//W)(\mathbb{A}_k))$ of all stable orbital integral of ρ-function, we have the induced Hankel transform H^ρ_Θ. The comparison between $\sum_\gamma \mathrm{SO}_\gamma(\phi) + \cdots$ and $\sum_\gamma \mathrm{SO}_\gamma(\mathcal{F}^\rho(\phi)) + \cdots$ boils down to a Poisson type summation formula for the orbital Hankel transform H^ρ_Θ. In the case of $G = \mathrm{GL}_n$ and $\rho = \mathrm{std}$, H^ρ_Θ coincide with the Harish-Chandra transform. The above mentioned Poisson summation type formula has been carried out by S. Cheng in his thesis [15].

In all other cases, there are important obstacles to this approach. For instance, we have very little understanding of the stable orbital Hankel transform H^ρ_Θ. In the case $G = \mathrm{GL}_2$ and $\rho = \mathrm{std}$, the paper [20] provides explicit formula for the kernel of H^ρ_Θ. Staring at the hyperbolic part in the Everling formula i.e., the part of the kernel concerning hyperbolic orbital integrals, one observe that the kernel is essentially the same as the kernel of the Hankel transform for the split torus. In fact, the diagonal part in the Everling formula has the same shape as the Hankel transform for different tori of GL_2. We also recall that the Poisson summation formula for the Hankel transform has been established by Lafforgue in [28]. One may hope to use the Poisson summation formula for tori as an approximation of the Poisson summation formula for the stable orbital Hankel transform on $T//W$.

Finally, we note that Sakellaridis has found explicit formulas for the transfer operator, very similar to the above stable orbital Hankel transform, in many examples in the framework of relative functoriality. In the cases where explicit formula for the transfer operator can be explicitly computed, he derived the nonlinear Poisson summation formula from the usual Poisson summation formula.

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