ON THE HITCHIN MORPHISM FOR HIGHER-DIMENSIONAL VARIETIES

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Abstract

We explore the structure of the Hitchin morphism for higher-dimensional varieties. We show that the Hitchin morphism factors through a closed subscheme of the Hitchin base, which is in general a nonlinear subspace of lower dimension. We conjecture that the resulting morphism, which we call the spectral data morphism*, is surjective. In the course of the proof, we establish connections between the Hitchin morphism for higher-dimensional varieties, the invariant theory of the commuting schemes, and Weyl's polarization theorem. We use the factorization of the Hitchin morphism to construct the spectral and cameral covers. In the case of general linear groups and algebraic surfaces, we show that spectral surfaces admit canonical finite Cohen–Macaulayfications, which we call the* Cohen–Macaulay spectral surfaces*, and we use them to obtain a description of the generic fibers of the Hitchin morphism similar to the case of curves. Finally, we study the Hitchin morphism for some classes of algebraic surfaces.*

1. Introduction

For a smooth projective curve X over a field k, and a split reductive group G over k of rank n, a G-Higgs bundle over X is a pair (E, θ) consisting of a principal G-bundle E over X and an element $\theta \in \mathrm{H}^0(X, \mathrm{ad}(E) \otimes_{\mathscr{O}_X} \Omega^1_X)$ called a *Higgs field*, where $\mathrm{ad}(E)$ is the adjoint vector bundle associated with E and Ω^1_X is the sheaf of 1-forms of X. In [11], Hitchin constructed a completely integrable system on the moduli space $\mathscr{M}_X$ of G-Higgs bundles over a curve X. This system can be presented as a morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$, where $\mathscr{M}_X$ is the moduli space of Higgs bundles and $\mathscr{A}_X$ is the affine space

$$\mathscr{A}_X = \bigoplus_{i=1}^{n} \mathrm{H}^0(X, \mathrm{S}^{e_i} \Omega^1_X), \tag{1.1}$$

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where $\mathrm{S}^{e_i}\Omega^1_X$ is the e_ith symmetric power of Ω^1_X. The morphism h_X is known as the *Hitchin fibration*. For curves X of genus $g_X > 1$, h_X is surjective and its generic fiber is isomorphic to a disjoint union of Abelian varieties if we discard automorphisms. This work aims at addressing these basic properties of the Hitchin morphism for higher-dimensional algebraic varieties.

Over a higher-dimensional algebraic variety X, a G-Higgs bundle is a G-bundle E over X equipped with a Higgs field

$$\theta \in \mathrm{H}^0\big(X, \mathrm{ad}(E) \otimes_{\mathscr{O}_X} \Omega^1_X\big), \tag{1.2}$$

where $\mathrm{ad}(E)$ is the adjoint vector bundle of E satisfying the integrability condition $\theta \wedge \theta = 0$. With given local coordinates $z_1, \dots, z_d$ in a neighborhood U of $x \in X$ and given a local trivialization of E, we can write $\theta = \sum_{i=1}^d \theta_i \, \mathrm{d}z_i$, where $\theta_i : U \to \mathfrak{g}$ are functions on U with values in the Lie algebra $\mathfrak{g}$ of G. The integrability condition satisfied by the Higgs field is

$$[\theta_i, \theta_j] = 0$$

for all $1 \leq i, \, j \leq d$. Hitchin's construction, generalized to higher-dimensional varieties by Simpson in [23], provides a morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$, where $\mathscr{A}_X$ is the affine space (1.1).

For general higher-dimensional algebraic varieties, the Hitchin morphism is very far from being surjective. We note that $h_X(E, \theta)$ could be defined for any $\theta \in \mathrm{H}^0(X, \mathrm{ad}(E) \otimes_{\mathscr{O}_X} \Omega^1_X)$ independent of whether or not it satisfies the integrability condition $\theta \wedge \theta = 0$. We aim at understanding the equations on $\mathscr{A}_X$ implied by the integrability condition $\theta \wedge \theta = 0$.

Our study of the Hitchin morphism for higher-dimensional varieties follows the method of [18] in the 1-dimensional case, namely, instead of studying the Hitchin morphism for a given variety X, we study certain universal morphisms independent of X. Those morphisms have to do with the construction of G-invariant functions on the scheme $\mathfrak{C}^d_G$ of commuting elements $x_1, \dots, x_d$ in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The reductive group G acts diagonally on $\mathfrak{C}^d_G$ by the adjoint action on $x_1, \dots, x_d$.

Our study of G-invariant functions on $\mathfrak{C}^d_G$ can roughly be divided into two parts. First, we investigate the generalization of the Chevalley restriction theorem to the commuting scheme. Second, we investigate the subring of G-invariant functions on $\mathfrak{C}^d_G$ derived from Weyl's polarization method. Both of these investigations are hindered by notoriously difficult problems in commutative algebra, such as the question of whether the categorical quotient $\mathfrak{C}^d_G /\!\!/ G$ is reduced. We are able to prove the reducedness of $\mathfrak{C}^d_G /\!\!/ G$ in the case $G = \mathrm{GL}_n$ generalizing a theorem of Gan and Ginzburg [9] in the case $d = 2$. Although we cannot prove the reducedness of $\mathfrak{C}^d_G /\!\!/ G$ for general reductive groups, we can work around it and address the problem of description of the image of the Hitchin morphism. Moreover, we will state and hierarchize certain problems which are related to the reducedness of $\mathfrak{C}_G^d /\!\!/ G$, which seem to be worthy of further investigation.

Here is a summary of our results. For a higher-dimensional proper smooth algebraic variety X, the Hitchin morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$, where $\mathscr{M}_X$ is the moduli stack of Higgs bundles on X and $\mathscr{A}_X$ is the affine space defined by the formula (1.1), is not surjective in general. We will define a closed subscheme $\mathscr{B}_X$ of $\mathscr{A}_X$, which is in general a nonlinear subspace of much lower dimension and prove that h_X factors through $\mathscr{B}_X$ (or rather, a thickening of $\mathscr{B}_X$; see Section 5). We conjecture that the resulting morphism $\mathscr{M}_X \to \mathscr{B}_X$, which we call the *spectral data morphism*, is surjective. In the course of the proof, we establish the connections between the Hitchin morphisms for higher-dimensional varieties, the invariant theory of the commuting schemes, and Weyl's polarization theorem in classical invariant theory.

We use the factorization of the Hitchin morphism to construct spectral and cameral covers and establish basic properties of them. In particular, we will see that, unlike the case of curves, the spectral and cameral covers are generally not flat in higher dimension. In the case $G = \mathrm{GL}_n$ and $\dim(X) = 2$, we construct an open subset $\mathscr{B}_X^{\heartsuit}$ of $\mathscr{B}_X$ such that for every $b \in \mathscr{B}_X^{\heartsuit}$, the corresponding spectral surface admits a canonical finite Cohen–Macaulayfication, called the *Cohen–Macaulay spectral surface*, and we use it to obtain a description of the Hitchin fiber $h_X^{-1}(b)$ similar to the case of curves. In particular, we show that $h_X^{-1}(b)$ is nonempty for $b \in \mathscr{B}_X^{\heartsuit}$ and there is a natural action of the Picard stack $\mathscr{P}_b$ of line bundles on the Cohen–Macaulay spectral surface on $h_X^{-1}(b)$. We also construct an open subset $\mathscr{B}_X^{\diamondsuit}$ of $\mathscr{B}_X^{\heartsuit}$ such that for all $b \in \mathscr{B}_X^{\diamondsuit}$ the fiber $h_X^{-1}(b)$ is isomorphic to a disjoint union of Abelian varieties after we discard automorphisms. For some class of algebraic surfaces (including elliptic surfaces), we can prove that $\mathscr{B}_X^{\diamondsuit}$ is an open dense subset of $\mathscr{B}_X^{\heartsuit}$, which is an open dense subset of $\mathscr{B}_X$.

Throughout the present article, we fix an algebraically closed field k of characteristic 0. To remove or weaken the assumption on the characteristic of k, we would have to refine many deep results in invariant theory. We will come back to deal with this task in a future work.

2. Characteristics of Higgs bundles over curves

Hitchin's construction was revisited in [18] from the point of view of the theory of algebraic stacks. In [18], the Hitchin morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$ was derived from a natural morphism of algebraic stacks

$$h : [\mathfrak{g}/G] \to \mathfrak{g} /\!\!/ G, \tag{2.1}$$

where $[\mathfrak{g}/G]$ and $\mathfrak{g} /\!\!/ G$ are the quotients of the Lie algebra of $\mathfrak{g}$ by the adjoint action of G in the framework of algebraic stacks and geometric invariant theory, respectively.

We recall that for every test scheme S, the groupoid of S-points of $[\mathfrak{g}/G]$ consists of all pairs (E,θ), where E is a principal G-bundle over S and $\theta \in \mathrm{H}^0(S,\mathrm{ad}(E))$ is a global section of the adjoint vector bundle $\mathrm{ad}(E)$ obtained from E by pushing out by the adjoint representation $\mathrm{ad}: G \to \mathrm{GL}(\mathfrak{g})$ of G. The categorical quotient $\mathfrak{g} /\!\!/ G$ is the affine scheme $\mathfrak{g} /\!\!/ G = \mathrm{Spec}(k[\mathfrak{g}]^G)$, where $k[\mathfrak{g}]^G$ is the ring of G-invariant functions on $\mathfrak{g}$. The concept of categorical quotient $\mathfrak{g} /\!\!/ G$ was devised by Mumford in [16] by which he means the initial object in the category of pairs (q,Q), where Q is a k-scheme and $q:\mathfrak{g}\to Q$ is a G-invariant morphism.

We will also use the Chevalley restriction theorem. Let us denote by $\mathfrak{t}$ a Cartan algebra, and by W its Weyl group. Since W-conjugate elements in $\mathfrak{t}$ are G-conjugate as elements of $\mathfrak{g}$, the restriction of a G-invariant function on $\mathfrak{g}$ to $\mathfrak{t}$ is W-invariant and, therefore, defines a homomorphism of algebras $k[\mathfrak{g}]^G \to k[\mathfrak{t}]^W$. The Chevalley restriction theorem asserts that this map is an isomorphism. This is equivalent to stating that the morphism between the categorical quotients

$$\mathfrak{t} /\!\!/ W \to \mathfrak{g} /\!\!/ G \tag{2.2}$$

is an isomorphism.

Let us denote $\mathfrak{c} = \mathfrak{t} /\!\!/ W$. Since W acts on $\mathfrak{t}$ as a reflection group, after another theorem of Chevalley, $\mathfrak{c}$ is also isomorphic to an affine space. The scalar action of $\mathbb{G}_m$ on $\mathfrak{t}$ induces an action of $\mathbb{G}_m$ on $\mathfrak{c}$. In fact, we can choose coordinates $c_1,\dots,c_n$ of the affine space $\mathfrak{c}$ that are homogeneous as polynomial functions of $\mathfrak{t}$, that is,

$$t(c_1,\dots,c_n) = (t^{e_1}c_1,\dots,t^{e_n}c_n). \tag{2.3}$$

The integers $e_1,\dots,e_n$ are independent of the choice of $c_1,\dots,c_n$.

Before proceeding further with the construction of the Hitchin morphism for curves, and as preparation for the higher-dimensional case, let us state an elementary yet useful fact. Let V be a finite-dimensional k-vector space. The space of morphisms $f: V\to \mathbb{A}^1$ satisfying $f(tv)=t^e f(v)$ can be canonically identified with the eth symmetric power $\mathrm{S}^e V^*$ of the dual vector space V^*. This is equivalent to saying that the scalar action of $\mathbb{G}_m$ on V gives rise to the graduation of the algebra of polynomial functions on V, that is, $\mathrm{S}(V^*) = \bigoplus_{e\in\mathbb{Z}_{\geq 0}} \mathrm{S}^e V^*$. Although it may seem completely obvious, this is a useful fact that should not be overlooked. For instance, for $e=1$, this says that any $\mathbb{G}_m$-equivariant polynomial map $f: V\to\mathbb{A}^1$, that is, a polynomial map satisfying $f(tv)=tf(v)$, is automatically linear. For $d=2$, all polynomial maps $f:V\to\mathbb{A}^1$ satisfying $f(tv)=t^2f(v)$ are automatically quadratic and so on.

A Higgs field $\theta\in \mathrm{H}^0(X,\mathrm{ad}(E)\otimes_{\mathcal{O}_X}\Omega^1_X)$ can be seen as an $\mathcal{O}_X$-linear map $\mathcal{T}_X \to \mathrm{ad}(E)$, where $\mathcal{T}_X$ is the $\mathcal{O}_X$-module of local sections of the tangent bundle T_X of X, satisfying the integrability condition (3.1). As the integrability condition is void when X is a smooth algebraic curve, it will be ignored in this section. We note that an $\mathscr{O}_X$-linear map $\mathscr{T}_X \to \mathrm{ad}(E)$ is the same as a $\mathbb{G}_m$-equivariant morphism $\theta : T_X \to [\mathfrak{g}/G]$ lying over the map $X \to \mathbb{B}G$ corresponding to the G-bundle E. Here $\mathbb{B}G$ is the classifying stack of G. By composing with the morphism $[\mathfrak{g}/G] \to \mathfrak{g} /\!/ G$ and the inverse of the isomorphism $\mathfrak{c} \to \mathfrak{g} /\!/ G$, we get a $\mathbb{G}_m$-equivariant morphism $a : T_X \to \mathfrak{c}$. For $i = 1, \dots, n$, by composing with the functions $c_i : \mathfrak{c} \to \mathbb{A}^1$, we obtain $\mathbb{G}_m$-equivariant morphisms $a_i : T_X \to \mathbb{A}^1_{e_i}$, where $\mathbb{A}^1_{e_i}$ is a copy of the affine line on which $\mathbb{G}_m$ acts by the formula $t \cdot x = t^{e_i} x$. We note that the space of all $\mathbb{G}_m$-equivariant functions $a_i : T_X \to \mathbb{A}^1_{e_i}$ is the affine space of global section of the e_ith symmetric power of the cotangent bundle T^*_X of X. Finally, we obtain the Hitchin morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$, where $\mathscr{A}_X$ is the affine space (1.1).

The main result of [11, Section 5] asserts that, under the assumption $g_X \geq 2$, the generic fiber is isomorphic to a union of Abelian varieties if we ignore isotropy groups. For instance, in the case $G = \mathrm{GL}_n$, Hitchin defines for every $a \in \mathscr{A}_X$ a spectral curve $X^\bullet_a$. As a varies, the spectral curves $X^\bullet_a$ form a linear system on the cotangent bundles of X. The assumption on the genus $g_X \geq 2$ implies that the linear system is ample and its generic member is a smooth projective curve. If $X^\bullet_a$ is smooth, then the Hitchin fiber $\mathscr{M}_a = h_X^{-1}(a)$ is isomorphic to the Picard stack $\mathscr{P}\mathrm{ic}(X^\bullet_a)$ which is isomorphic to a disjoint union of Abelian varieties if we ignore automorphisms. For classical groups, Hitchin also constructs certain spectral curves using their standard representations. For a general reductive group, Donagi constructs a cameral cover $\tilde{X}_a$ of X for every $a \in \mathscr{A}_X$ and proves that the Hitchin fiber $\mathscr{M}_a$ is isomorphic to a union of Abelian varieties if the cameral cover $\tilde{X}_a$ is a smooth curve.

Since we will attempt to generalize the construction of cameral curves for Higgs bundles over higher-dimensional varieties, let us recall their construction in the case of curves. The construction, due to Donagi [7, Section 4.2], derives the cameral covering $\pi_a : \tilde{X}_a \to X$ from the Cartesian diagram

$$\begin{array}{ccc} \tilde{X}_a & \xrightarrow{\tilde{a}} & [\mathfrak{t}/\mathbb{G}_m] \\ \Big\downarrow{\scriptstyle \pi_a} & & \Big\downarrow \\ X & \xrightarrow[a]{} & [\mathfrak{c}/\mathbb{G}_m] \end{array} \tag{2.4}$$

where the morphism $a : X \to [\mathfrak{c}/\mathbb{G}_m]$ at the bottom line comes from the $\mathbb{G}_m$-equivariant morphism $a : T_X \to \mathfrak{c}$. Since the morphism $\pi : \mathfrak{t} \to \mathfrak{c}$ is finite and flat, π_a also has these properties. Away from the discriminant locus $\mathrm{discr}_G \subset \mathfrak{c}$, the morphism $\pi : \mathfrak{t} \to \mathfrak{c}$ is finite, étale, and Galois with Galois group W. In [18], we denote by $\mathscr{A}^\heartsuit_X$ the open subset of $\mathscr{A}_X$ consisting of maps $a : X \to [\mathfrak{c}/\mathbb{G}_m]$ whose image is not contained in $[\mathrm{discr}_G/\mathbb{G}_m]$. By construction, for $a \in \mathscr{A}^\heartsuit_X$, $\tilde{X}_a \to X$ is generically a finite étale Galois morphism with Galois group W. The fibers $\mathscr{M}_a$ are much better understood under the assumption $a \in \mathscr{A}_X^{\heartsuit}$. In particular, there is a natural Picard stack $\mathscr{P}_a$, constructed in [18], acting on $\mathscr{M}_a$ with a dense open orbit.

3. The Higgs stack and the universal spectral data morphism

Let X be a proper smooth variety of dimension d over k. A G-Higgs bundle over X is a G-bundle E over X equipped with an $\mathscr{O}_X$-linear map $\theta : \mathscr{T}_X \to \mathrm{ad}(E)$ from the tangent sheaf $\mathscr{T}_X$ of X to the adjoint vector bundle $\mathrm{ad}(E)$ of E satisfying the integrability condition: for all local sections v_1, v_2 of $\mathscr{T}_X$, we have

$$\big[\theta(v_1), \theta(v_2)\big] = 0. \tag{3.1}$$

Let $\mathfrak{C}_G^d \subset \mathfrak{g}^d$ be the commuting scheme. It is defined as the scheme-theoretic zero fiber of the commutator map

$$\mathfrak{g}^d \to \prod_{i<j} \mathfrak{g}, \qquad (\theta_1, \dots, \theta_d) \to \prod_{i<j} [\theta_i, \theta_j].$$

The k-points of $\mathfrak{C}_G^d$ consist of $(\theta_1, \dots, \theta_d) \in \mathfrak{g}^d(k)$ such that $[\theta_i, \theta_j] = 0$ for $1 \le i, j \le d$. We note that the commuting relations are automatically satisfied in the case $d = 1$. Let V_d denote the dual vector space of k^d equipped with the standard basis $v_1, \dots, v_d$. We will identify $\mathfrak{g}^d$ with the space of all linear maps $\theta : V_d \to \mathfrak{g}$ by attaching to $(\theta_1, \dots, \theta_d) \in \mathfrak{g}^d$ the unique linear map $\theta : V_d \to \mathfrak{g}$ satisfying $\theta(v_i) = \theta_i$. The commuting scheme $\mathfrak{C}_G^d$ can then be identified with the closed subscheme of $\mathfrak{g}^d$ consisting of all k-linear maps $\theta : V_d \to \mathfrak{g}$ such that $[\theta(v), \theta(v')] = 0$ for all $v, v' \in V_d$.

Given this description of $\mathfrak{C}_G^d$, we have an action of $\mathrm{GL}_d \times G$ on $\mathfrak{C}_G^d$ coming from the natural action of GL_d on V_d and the adjoint action of G on $\mathfrak{g}$. We will call the quotient

$$\big[\mathfrak{C}_G^d / (\mathrm{GL}_d \times G)\big], \tag{3.2}$$

in the sense of algebraic stacks, the *Higgs stack*. It attaches to every test scheme S the groupoid of triples $(\mathscr{V}, \mathscr{E}, \theta)$ consisting of a vector bundle $\mathscr{V}$ of rank d over S, a principal G-bundle $\mathscr{E}$ over S, and an $\mathscr{O}_S$-linear map $\theta : \mathscr{V} \to \mathrm{ad}(\mathscr{E})$ satisfying $[\theta(v), \theta(v')] = 0$ for all local sections v, v' of $\mathscr{V}$. A Higgs field on a d-dimensional proper smooth variety X can be represented by a map

$$\theta : X \to \big[\mathfrak{C}_G^d / (\mathrm{GL}_d \times G)\big] \tag{3.3}$$

lying over the map $X \to \mathbb{B}\mathrm{GL}_d$ representing the cotangent bundle T_X^*. Here, we denote by $\mathbb{B}\mathrm{GL}_d$ the classifying stack of GL_d.

The construction of the Hitchin morphism derives from G-invariant functions on $\mathfrak{C}_G^d$. Studying G-invariant functions on $\mathfrak{C}_G^d$ amounts to investigating the morphism

$$[\mathfrak{C}_G^d/G] \to \mathfrak{C}_G^d /\!\!/ G \tag{3.4}$$

between quotients of the commuting scheme $\mathfrak{C}_G^d$ by the diagonal action of G in the sense of algebraic stacks and geometric invariant theory, respectively. By definition, the categorical quotient $\mathfrak{C}_G^d /\!\!/ G$ is the affine scheme whose ring of functions is the k-algebra

$$k[\mathfrak{C}_G^d /\!\!/ G] = k[\mathfrak{C}_G^d]^G$$

of G-invariant functions on $\mathfrak{C}_G^d$.

The commuting scheme $\mathfrak{C}_G^d$ has been studied intensively, especially in the case $d = 2$. It has a nonempty open locus $\mathfrak{C}_G^{d,\mathrm{rss}}$ consisting of commuting linear maps $\theta : V_d \to \mathfrak{g}$ such that the image $\theta(V_d)$ has nonempty intersection with the regular semisimple locus $\mathfrak{g}^{\mathrm{rss}}$ of $\mathfrak{g}$. This open locus is smooth. In the case $d = 2$, Richardson [20] proved that the underlying topological space of $\mathfrak{C}_G^2$ is irreducible, in particular, that the locus $\mathfrak{C}_G^{2,\mathrm{rss}}$ is dense in $\mathfrak{C}_G^2$. Results of Iarrobino [14] on punctual Hilbert schemes on $\mathbb{A}^d$, with $d \geq 3$, imply that irreducibility is no longer true for $d \geq 3$.

There is a long-standing conjecture saying that the commuting scheme $\mathfrak{C}_G^2$ is reduced. The generalization of this conjecture to the case in which $d \geq 3$ seems to be rather doubtful, since we have very little understanding of other components of $\mathfrak{C}_G^d$ other than the component containing $\mathfrak{C}_G^{d,\mathrm{rss}}$.

The categorical quotient $\mathfrak{C}_G^d /\!\!/ G$ behaves better. In [13], Hunziker proved a weak version of the Chevalley restriction theorem for the commuting scheme. If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, then the embedding $\mathfrak{t}^d \to \mathfrak{g}^d$ factors through $\mathfrak{C}_G^d$ since $\mathfrak{t}$ is commutative. Since orbits of the diagonal action of W on $\mathfrak{t}^d$ are contained in orbits of the diagonal action of G on $\mathfrak{C}_G^d$, the restriction of a G-invariant function on $\mathfrak{C}_G^d$ to $\mathfrak{t}^d$ is W-invariant. In other words, we have a morphism

$$\mathfrak{t}^d /\!\!/ W \to \mathfrak{C}_G^d /\!\!/ G. \tag{3.5}$$

Based on Richardson's fundamental result in [20], Hunziker proved that this morphism is a universal homeomorphism, that is, it is a finite morphism inducing a bijection on k-points (see [13, Theorems 6.2, 6.3]).[1] In particular, $\mathfrak{t}^d /\!\!/ W$ is the normalization of the underlying reduced subscheme $(\mathfrak{C}_G^d /\!\!/ G)^{\mathrm{red}}$. Since $\mathfrak{t}^d /\!\!/ W$ is irreducible, the categorical quotient $\mathfrak{C}_G^d /\!\!/ G$ is also irreducible.

[1]In [13, Section 6], Hunziker works with the reduced quotient R^{red} of the ring R of functions on $\mathfrak{C}_G^d$. As we are over $k = \mathbb{C}$, the Reynolds operator implies that there exists an isomorphism between $(R^G)^{\mathrm{red}}$ and $(R^{\mathrm{red}})^G$ for any k-algebra of finite type with G-action (see, e.g., [16, p. 29]). Thus, Hunziker proves that $\mathfrak{t}^d /\!\!/ W \to (\mathfrak{C}_G^d /\!\!/ G)^{\mathrm{red}}$ is a universal homeomorphism. This is equivalent to saying that $\mathfrak{t}^d /\!\!/ W \to \mathfrak{C}_G^d /\!\!/ G$ is a universal homeomorphism.

CONJECTURE 3.1
The morphism (3.5) is an isomorphism.

We note that Conjecture 3.1 is equivalent to asserting that the categorical quotient $\mathfrak{C}_G^d /\!\!/ G$ is reduced and normal. Indeed, since $\mathfrak{t}^d /\!\!/ W$ is obviously reduced and normal, if (3.5) is an isomorphism, then $\mathfrak{C}_G^d /\!\!/ G$ also is reduced and normal. Conversely, if $\mathfrak{C}_G^d /\!\!/ G$ is reduced and normal, then the map (3.5), known to be a normalization, has to be an isomorphism. Note also that Conjecture 3.1 together with (3.4) implies that there is a G-invariant morphism

$$\mathrm{sd} : \mathfrak{C}_G^d \to \mathfrak{t}^d /\!\!/ W \tag{3.6}$$

to be called the *universal spectral data morphism*, making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{t}^d & \longrightarrow & \mathfrak{C}_G^d \\ \downarrow & \swarrow & \downarrow \\ \mathfrak{t}^d /\!\!/ W & \longrightarrow & \mathfrak{C}_G^d /\!\!/ G \end{array} \tag{3.7}$$

As the existence of this morphism would be important to the study of the Hitchin morphism, we state the following conjecture, which is a weaker form of Conjecture 3.1.

CONJECTURE 3.2
There exists a G-invariant morphism $\mathrm{sd} : \mathfrak{C}_G^d \to \mathfrak{t}^d /\!\!/ W$ *making the diagram (3.7) commute.*

We note that Conjecture 3.2 implies that the categorical quotient $\mathfrak{C}_G^d /\!\!/ G$ is reduced. Indeed, the right triangle of (3.7) gives rise to a commutative triangle of rings, which says that the composition of homomorphisms

$$k[\mathfrak{C}_G^d]^G \to k[\mathfrak{t}^d]^W \to k[\mathfrak{C}_G^d]$$

is the inclusion map. It follows that the homomorphism $k[\mathfrak{C}_G^d]^G \to k[\mathfrak{t}^d]^W$ is injective. Since $k[\mathfrak{t}^d]^W$ is an integral domain, $k[\mathfrak{C}_G^d]^G$ is also an integral domain and, in particular, reduced.

In the next section (see Theorem 4.2), we will construct a canonical map $\mathfrak{C}_G^d(k) \to \mathfrak{t}^d /\!\!/ W(k)$ making the diagram (3.7) commute on the level of k-points. For the moment, let us construct this map in the case $G = \mathrm{GL}_n$. A k-point $\theta \in \mathfrak{C}_G^d(k)$ consists of a commuting family of endomorphisms $\theta_1, \ldots, \theta_d$ on the standard n-dimensional k-vector space k^n. It defines a $\mathrm{S}(V_d)$-module structure on k^n where

$v_i \in \mathrm{S}(V_d) = k[v_1, \dots, v_d]$ acts by θ_i. Let F denote the corresponding finite $\mathrm{S}(V_d)$-module. We have a decomposition $F = \bigoplus_{\alpha \in \mathbb{A}^d} F_\alpha$, where F_α is an $\mathrm{S}(V_d)$-module annihilated by some power of the maximal ideal $\mathfrak{m}_\alpha$ corresponding to the point $\alpha \in \mathbb{A}^d(k)$, where $\mathbb{A}^d = \mathrm{Spec}(\mathrm{S}(V_d))$. This decomposition gives rise to a 0-cycle

$$z(\theta) = \sum_{\alpha \in \mathbb{A}^d(k)} \lg(F_\alpha)\alpha$$

of length n in $\mathbb{A}^d$. This construction gives rise to a $G(k)$-invariant map $\mathfrak{C}_G^d(k) \to \mathrm{Chow}_n(\mathbb{A}^d)(k)$, where

$$\mathrm{Chow}_n(\mathbb{A}^d) = \mathrm{Spec}\big(\big(\mathrm{S}(V_d)^{\otimes n}\big)^{\mathfrak{S}_n}\big).$$

As $G = \mathrm{GL}_n$, one can identify $\mathrm{Chow}_n(\mathbb{A}^d)$ with $\mathfrak{t}^d /\!\!/ W$, and we thus obtain the desired map from $\mathfrak{C}_G^d(k)$ to $\mathfrak{t}^d /\!\!/ W(k)$. We shall show that the construction above works in families.

THEOREM 3.3
Conjecture 3.2 holds in the case of GL_n. *In particular, for* $G = \mathrm{GL}_n$, *the categorical quotient* $\mathfrak{C}_G^d /\!\!/ G$ *is reduced.*

Proof
The construction of the universal spectral data morphism $\mathrm{sd} : \mathfrak{C}_G^d \to \mathfrak{t}^d /\!\!/ W = \mathrm{Chow}_n(\mathbb{A}^d)$ in the case $G = \mathrm{GL}_n$ is due to Deligne [6, Section 6.3.1]. For the reader's convenience, we will recall his construction. For any k-algebra R, we will construct a functorial map $\mathfrak{C}_G^d(R) \to \mathrm{Chow}_n(\mathbb{A}^d)(R)$ following Deligne. A collection of d matrices $\alpha_1, \dots, \alpha_d \in \mathfrak{g}(R) = \mathfrak{gl}_n(R)$ gives rise to a k-linear map $\alpha : V_d \to \mathfrak{g}(R)$. If $\alpha_1, \dots, \alpha_d$ commute with each other, then α gives rise to a map

$$\mathrm{S}(\alpha) : \mathrm{S}(V_d) \to \mathfrak{g}(R).$$

By composing with the determinant, we get a map $\det \circ \mathrm{S}(\alpha) : \mathrm{S}(V_d) \to R$ which is a homogeneous algebraic map of degree n on the infinite-dimensional vector space $\mathrm{S}(V_d)$. It must derive from a polynomial linear map

$$z(\alpha) : \big(\mathrm{S}(V_d)^{\otimes n}\big)^{\mathfrak{S}_n} \to R \tag{3.8}$$

characterized by the property that

$$z(\alpha)(f^{\otimes n}) = \det \circ \mathrm{S}(\alpha)(f)$$

for $f \in \mathrm{S}(V_d)$. Since $\det \circ \mathrm{S}(\alpha)$ is multiplicative, $z(\alpha)$ is a homomorphism of k-algebras. In other words, $z(\alpha)$ defines an R-point of $\mathrm{Chow}_n(\mathbb{A}^d)$. This finishes the construction of the map $\mathrm{sd} : \mathfrak{C}_G^d \to \mathfrak{t}^d /\!\!/ W$.

We shall prove that the composition $\mathfrak{C}_G^d \xrightarrow{\mathrm{sd}} \mathfrak{t}^d /\!/ \mathrm{W} \xrightarrow{(3.5)} \mathfrak{C}_G^d /\!/ G$ is the quotient map. Equivalently, the induced map

$$k[\mathfrak{C}_G^d]^G \to k[\mathfrak{t}^d]^{\mathrm{W}} \to k[\mathfrak{C}_G^d]$$

on rings of functions is the natural inclusion map.

Let $X(i) \in \mathfrak{g}(k[\mathfrak{g}^d])$ be the $n \times n$ matrix whose (a,b)-entry is given by the coordinate function for the (a,b)-entry of the ith copy of $\mathfrak{g}$ in $\mathfrak{g}^d$. The embedding $\mathfrak{t}^d \to \mathfrak{g}^d$ gives rise to a map $\mathfrak{g}(k[\mathfrak{g}^d]) \to \mathfrak{g}(k[\mathfrak{t}^d])$, and we define $T(i) \in \mathfrak{g}(k[\mathfrak{t}^d])$ to be the image of $X(i)$ under this map. We use the same notation $X(i) \in \mathfrak{g}(k[\mathfrak{C}_G^d])$ for the image of $X(i)$ under the natural map $\mathfrak{g}(k[\mathfrak{g}^d]) \to \mathfrak{g}(k[\mathfrak{C}_G^d])$. It is known (see, e.g., [19]) that the ring of G-invariant functions $k[\mathfrak{g}^d]^G$ is generated by

$$\mathrm{Tr}\big(X(i_1)\cdots X(i_k)\big),$$

where $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq i_1, \ldots, i_k \leq d$. As the restriction map $k[\mathfrak{g}^d]^G \to k[\mathfrak{C}_G^d]^G$ is surjective and $[X(i), X(j)] = 0 \in \mathfrak{g}(k[\mathfrak{C}_G^d])$, it follows that $k[\mathfrak{C}_G^d]^G$ is generated by the G-invariant functions

$$\mathrm{Tr}\big(X(1)^{a_1}\cdots X(d)^{a_d}\big),$$

where $a_j \in \mathbb{Z}_{\geq 0}$. It is easy to see that the image of $\mathrm{Tr}(X(1)^{a_1}\cdots X(d)^{a_d})$ under the map $k[\mathfrak{C}_G^d]^G \to k[\mathfrak{t}^d]^{\mathrm{W}}$ is equal to

$$\mathrm{Tr}\big(T(1)^{a_1}\cdots T(d)^{a_d}\big).$$

Thus, to prove the desired claim, it suffices to show that

$$z(\alpha)\big(\mathrm{Tr}\big(T(1)^{a_1}\cdots T(d)^{a_d}\big)\big) = \mathrm{Tr}\big(X(1)^{a_1}\cdots X(d)^{a_d}\big), \tag{3.9}$$

where $z(\alpha) = \mathrm{sd}^* : k[\mathfrak{t}^d]^{\mathrm{W}} = (S(V_d)^{\otimes n})^{\mathfrak{S}_n} \to k[\mathfrak{C}_G^d]$ is the map in (3.8) in the universal case: $R = k[\mathfrak{C}_G^d]$ and $\alpha : V_d \to \mathfrak{g}(R)$ corresponds to the identity map $\mathrm{id} \in \mathfrak{C}_G^d(R)$.

Let $v_1, \ldots, v_d$ be the coordinate vectors of V_d. We have

$$S(\alpha)(v_i) = X(i) \in \mathfrak{g}\big(k[\mathfrak{C}_G^d]\big). \tag{3.10}$$

For any $x \in k$, consider the element $f = x - v_1^{a_1}\cdots v_d^{a_d} \in S(V_d) = k[v_1, \ldots, v_d]$. It follows from the definition of $z(\alpha)$ that

$$\begin{aligned} z(\alpha)(f^{\otimes n}) &= \det \circ S(\alpha)(f) = \det\big(x\mathrm{id} - S(\alpha)(v_1^{a_1})\cdots S(\alpha)(v_d^{a_d})\big) \\ &= x^n - \mathrm{Tr}\big(S(\alpha)(v_1^{a_1})\cdots S(\alpha)(v_d^{a_d})\big)x^{n-1} + \cdots . \end{aligned} \tag{3.11}$$

On the other hand, under the canonical identification $(S(V_d)^{\otimes n})^{\mathfrak{S}_n} = k[\mathfrak{t}^d]^{\mathrm{W}}$, the element $f^{\otimes n}$ becomes

$$\det\big(x\mathrm{id} - T(1)^{a_1}\cdots T(d)^{a_d}\big)$$

and it follows that

$$\begin{aligned} z(\alpha)(f^{\otimes n}) &= z(\alpha)\big(\det\big(x\mathrm{id} - T(1)^{a_1}\cdots T(d)^{a_d}\big)\big) \\ &= x^n - z(\alpha)\big(\mathrm{Tr}\big(\big(T(1)^{a_1}\cdots T(d)^{a_d}\big)\big)\big)x^{n-1} + \cdots . \end{aligned} \tag{3.12}$$

Comparing the coefficients of x^{n-1} in (3.11) and (3.12), we obtain

$$z(\alpha)\big(\mathrm{Tr}\big(T(1)\big)^{a_1}\cdots\big(T(d)\big)^{a_d}\big) = \mathrm{Tr}\big(S(\alpha)(v_1)^{a_1}\cdots S(\alpha)(v_d)^{a_d}\big), \tag{3.13}$$

which implies that

$$\begin{aligned} z(\alpha)\big(\mathrm{Tr}\big(T(1)\big)^{a_1}\cdots\big(T(d)\big)^{a_d}\big) &\overset{(3.13)}{=} \mathrm{Tr}\big(S(\alpha)(v_1)^{a_1}\cdots S(\alpha)(v_d)^{a_d}\big) \\ &\overset{(3.10)}{=} \mathrm{Tr}\big(X(1)^{a_1}\cdots X(d)^{a_d}\big). \end{aligned}$$

Equation (3.9) follows. This completes the proof of Theorem 3.3. ☐

Although we do not know the validity of Conjectures 3.1 and 3.2 in general, we know they are true on the level of topological spaces. This will allow us to work around and predict the image of the Hitchin morphism.

Remark 3.1
In [9], Gan and Ginzburg proved the reducedness of $\mathfrak{C}_G^d /\!\!/ G$ in the case $G = \mathrm{GL}_n$, $d = 2$, by a different method.

4. Weyl's polarization and the universal Hitchin morphism

Weyl's polarization is a method used to construct G-invariant functions on the space $\mathfrak{g}^d$ of d arbitrary elements $\theta_1, \dots, \theta_d \in \mathfrak{g}$. The idea is as follows. Given a G-invariant function c on $\mathfrak{g}$ and $x_1, \dots, x_d \in k$, the map

$$(\theta_1, \dots, \theta_d) \mapsto c(x_1\theta_1 + \cdots + x_d\theta_d)$$

defines a G-invariant function on $\mathfrak{g}^d$. Although those G-invariant functions on $\mathfrak{g}^d$ in general may not generate $k[\mathfrak{g}^d]^G$ (see, e.g., [15]), as we shall see, they are close to forming a set of generators of the ring $k[\mathfrak{C}_G^d]^G$ of G-invariant functions on the commuting scheme, and they do in the case $G = \mathrm{GL}_n$.

We will formalize the construction above as follows. For every affine variety Y equipped with an action of $\mathbb{G}_m$, the functor on the category of k-algebras which associates with each k-algebra R the set of $\mathbb{G}_m$-equivariant maps $V_d \otimes_k R \to Y$ is representable by an affine scheme, denoted by $Y_{\mathbb{G}_m}^{V_d}$. For instance, if Y is the affine line $\mathbb{A}^1 = \mathrm{Spec}(k[x])$ equipped with an action of $\mathbb{G}_m$ given by $t \cdot x = t^e x$, then $Y_{\mathbb{G}_m}^{V_d}$ is the eth symmetric tensor of $\mathbb{A}^d = \mathrm{Spec}(\mathrm{S}(V_d))$. For $Y = \mathfrak{g}$, the space $\mathfrak{g}_{\mathbb{G}_m}^{V_d}$ can be identified with $\mathfrak{g}^d$. Let us also consider the case $Y = \mathfrak{c}$, where $\mathfrak{c} = \mathfrak{g} /\!\!/ G$. Since $\mathfrak{c}$ is isomorphic to an n-dimensional affine space with homogeneous coordinates $c_1, \ldots, c_n$ of degree $e_1, \ldots, e_n$, the space $A = \mathfrak{c}_{\mathbb{G}_m}^{V_d}$ is isomorphic to

$$A \simeq \prod_{i=1}^{n} \mathrm{S}^{e_i} \mathbb{A}^d. \tag{4.1}$$

The isomorphism depends on the choice of homogeneous coordinates $c_1, \ldots, c_n$.

Since the morphism $\mathfrak{g} \to \mathfrak{c}$ is G-invariant and $\mathbb{G}_m$-equivariant, it induces a G-invariant morphism

$$\mathrm{pol} : \mathfrak{g}^d \to A \simeq \prod_{i=1}^{n} \mathrm{S}^{e_i} \mathbb{A}^d \tag{4.2}$$

which embodies Weyl's polarization method for the diagonal action of G on $\mathfrak{g}^d$. For example, in the case in which $G = \mathrm{GL}_n$, given d arbitrary matrices $\theta = (\theta_1, \ldots, \theta_d) \in (\mathfrak{gl}_n)^d$, the trace of the ith power of $x_1\theta_1 + \cdots + x_d\theta_d$ is an ith symmetric form in the variables $x_1, \ldots, x_d$ and thus defines a point $\mathrm{pol}_i(\theta)$ in $\mathrm{S}^i \mathbb{A}^d$, and we have $\mathrm{pol}(\theta) = (\mathrm{pol}_1(\theta), \ldots, \mathrm{pol}_n(\theta))$. Instead of using trace of powers of an endomorphism, we may also use the homogeneous coordinates of $\mathfrak{c}$ given by the ith coefficient of the characteristic polynomial of an endomorphism for $1 \le i \le n$. The latter invariant function is used by Simpson [23] to define the Hitchin morphism for GL_n for higher-dimensional varieties. We have seen that the choice of coordinates of $\mathfrak{c}$ is unimportant as it just gives rise to different isomorphisms (4.1).

By restricting (4.2) to the commuting scheme $\mathfrak{C}_G^d$, we obtain a G-invariant morphism

$$h : \mathfrak{C}_G^d \to A \tag{4.3}$$

to be called the *universal Hitchin morphism*. To study the structure of the Hitchin morphism, and in particular the image thereof, we need to understand the image of the map (4.3) and its relation to the Chevalley restriction morphism (3.5). For that purpose, we will also need to use Weyl's polarization construction for the diagonal action of W on $\mathfrak{t}^d$. The morphism $\mathfrak{t} \to \mathfrak{c} = \mathfrak{t} /\!\!/ W$ is W-invariant and $\mathbb{G}_m$-equivariant. As a result, we have a W-invariant morphism

$$\mathrm{pol}_W : \mathfrak{t}^d /\!\!/ W \to A. \tag{4.4}$$

We recall the following result.

THEOREM 4.1 ([15, Theorem 2.15])
The morphism pol_W *of (4.4) is finite and induces an injective map on* k*-points. In other words, there exists a unique reduced closed subscheme* B *of* A *such that* pol_W *factors through a morphism*

$$b : \mathfrak{t}^d /\!\!/ W \to B, \tag{4.5}$$

which is a universal homeomorphism and normalization. For $G = \mathrm{GL}_n$*,* pol_W *is a closed embedding and* b *is an isomorphism.*

Remark 4.1
In the case $G = \mathrm{GL}_n$, the preceding theorem is the first fundamental theorem for symmetric groups, which is a classical theorem of Weyl [25, Chapter II.A.3]. According to Hunziker [13], pol_W is a closed embedding for groups of types B and C. According to Wallach [24], pol_W fails to be a closed embedding for groups of type D.

Example 4.2
Let us describe the closed subscheme B of A in the case $G = \mathrm{SL}_2$ and $d = 2$. In this case, the Cartan algebra can be identified with $\mathfrak{t} \simeq \mathrm{Spec}(k[t])$. The Weyl group $W = \mathfrak{S}_2$ acts on $\mathfrak{t}$ by $w(t) = -t$, where w is the nontrivial element of W. The categorical quotient $\mathfrak{c} = \mathrm{Spec}(k[u])$ with $u = t^2$ and the morphism $\mathfrak{g} \to \mathfrak{c}$ is given by $u = \det(g)$. Since the exponent $e = 2$, we have $A = \mathrm{S}^2\mathbb{A}^2$, which is a 3-dimensional vector space. The map $\mathfrak{t}^2 = \mathbb{A}^2 \to A = \mathrm{S}^2(\mathbb{A}^2)$ is given by $v \mapsto v^2$. In coordinates, this is the map $\mathbb{A}^2 \to \mathbb{A}^3$ given by $(x, y) \mapsto (x^2, 2xy, y^2)$. Thus, B is the closed subscheme of $A = \mathbb{A}^3$ defined by the equation $b^2 - 4ac = 0$, which can be identified with the categorical quotient of $\mathbb{A}^2$ by the action of $\mathfrak{S}_2$ given by $(x, y) \mapsto (-x, -y)$.

We have the following factorization of the universal Hitchin morphism $h : \mathfrak{C}_G^d \to A$.

THEOREM 4.2
There exists a closed subscheme B' *of* A*, which is a thickening of the closed subscheme* B *of* A*, as in Theorem 4.1, such that the universal Hitchin morphism* $h : \mathfrak{C}_G^d \to A$ *in (4.3) factors through a morphism*

$$\mathrm{sd}' : \mathfrak{C}_G^d \to B'. \tag{4.6}$$

In particular, there is a canonical $G(k)$-equivariant morphism $\mathfrak{C}_G^d(k) \to \mathfrak{t}^d /\!\!/ \mathrm{W}(k)$. For $G = \mathrm{GL}_n$, we have $B' = B$ and (4.6) is equal to the universal spectral data morphism $\mathrm{sd} : \mathfrak{C}_G^d \mapsto \mathfrak{t}^d /\!\!/ \mathrm{W} \simeq B$ *constructed in Theorem 3.3.*

Proof
By [13, Theorem 6.3], the Chevalley restriction map $\mathfrak{t}^d /\!\!/ \mathrm{W} \to \mathfrak{C}_G^d /\!\!/ G$ is a homeomorphism. Therefore, the diagram (3.7) implies that the G-invariant morphism $h : \mathfrak{C}_G^d \to A$ factors through a thickening B' of the closed subscheme B of A. The first claim follows. The second claim follows from Theorem 3.3. ☐

One may ask whether Theorem 4.2 holds for $B' = B$ for general G. This would follow from Conjecture 3.2.

5. The spectral data morphism, postulated image of the Hitchin morphism, and cameral covers

Let X be a proper smooth algebraic variety over k of dimension d. A Higgs bundle over X is represented by a map $\theta : X \to [\mathfrak{C}_G^d/(G \times \mathrm{GL}_d)]$ lying over the map $\tau_X^* : X \to \mathbb{B}\mathrm{GL}_d$ given by its cotangent bundle T_X^*. By composing it with the map $[h] : [\mathfrak{C}_G^d/(G \times \mathrm{GL}_d)] \to [A/\mathrm{GL}_d]$ derived from (4.3), we obtain the Hitchin morphism

$$h_X : \mathscr{M}_X \to \mathscr{A}_X,$$

where $\mathscr{A}_X$ is the space of maps $X \to [A/\mathrm{GL}_d]$ lying over τ_X^*. By choosing a system of homogeneous coordinates $c_1, \dots, c_n$ of $\mathfrak{c}$ of degrees $e_1, \dots, e_n$, we can identify $\mathscr{A}_X$ with the vector space $\bigoplus_{i=1}^n \mathrm{H}^0(X, \mathrm{S}^{e_i} \Omega_X^1)$.

Let $\mathscr{B}_X$ denote the space of maps $X \to [B/\mathrm{GL}_d]$, where B is the closed subscheme of A defined in Theorem 4.1, lying over τ_X^*. It is clear that $\mathscr{B}_X$ is a closed subscheme of $\mathscr{A}_X$. We call it the *postulated image* of the Hitchin morphism h_X. By replacing B by its thickening B' as in Theorem 4.2, we have a thickening $\mathscr{B}'_X$ of $\mathscr{B}_X$. The schemes $\mathscr{B}_X$ and $\mathscr{B}'_X$ have the same underlying topological space.

PROPOSITION 5.1
Let X be a proper smooth algebraic variety of dimension d over an algebraically closed field k of characteristic 0, and let $\mathscr{M}_X$ be the moduli stack of Higgs bundles over X. Then the Hitchin morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$ factors through a map

$$\mathrm{sd}'_X : \mathscr{M}_X \to \mathscr{B}'_X$$

to be called the spectral data morphism. *In particular, the image of every geometric point $\theta \in \mathscr{M}_X(k)$ under the Hitchin morphism belongs to $\mathscr{B}_X(k)$.*

Proof
By Theorem 4.2, for any S-point $\theta : S \times X \to [\mathfrak{C}^d_G/(G \times \mathrm{GL}_d)]$ in $\mathscr{M}_X(S)$ where S is a k-scheme, its image $h_X(\theta) : S \times X \to [A/\mathrm{GL}_d]$ factors through $b' : S \times X \to [B'/\mathrm{GL}_d]$. This gives the desired factorization $\mathrm{sd}'_X : \mathscr{M}_X \to \mathscr{B}'_X$ of the Hitchin morphism. Assume that $\theta \in \mathscr{M}(k)$. Since X is reduced, its image $b' : X \to [B'/\mathrm{GL}_d]$ factors through a morphism $b : X \to [B/\mathrm{GL}_d]$; that is, we have $h_X(\theta) \in \mathscr{B}_X(k)$. The proposition follows. ☐

CONJECTURE 5.2
For every $b \in \mathscr{B}_X(k)$, the fiber $h_X^{-1}(b)$ is nonempty.

Example 5.1
Consider the case in which X is a d-dimensional Abelian variety. By choosing an isomorphism between the Lie algebra of X and the d-dimensional vector space V_d, we will have an isomorphism $\mathscr{A}_X = A$ and $\mathscr{B}_X = B$ which is a strict subset of A for $d \geq 2$. We can also prove that the spectral data map $\mathscr{M}_X(k) \to \mathscr{B}_X(k)$ is surjective by restricting ourselves to the subset of $\mathscr{M}_X(k)$ consisting of Higgs bundles (E, θ), where E is the trivial G-bundle.

One can think of $\mathscr{B}_X(k)$ as the subset of $\mathscr{A}_X(k)$ consisting of points $b \in \mathscr{A}_X(k)$ for which one can construct a cameral covering. For any scheme Y with an action of GL_d, we can form the twist $Y_{T^*_X}$ of Y by the GL_d-torsor given by T^*_X. Then a point $b \in \mathscr{B}_X(k)$ gives rise to a map $b : X \to B_{T^*_X}$ and, since the map $(\mathfrak{t}^d /\!\!/ W)_{T^*_X} \to B_{T^*_X}$ induced from $\mathfrak{t}^d /\!\!/ W \to B$ is the normalization and X is normal, the map b lifts to a map $X \to (\mathfrak{t}^d /\!\!/ W)_{T^*_X}$. We define $\tilde{X}_b$ to be the fiber product

$$\begin{array}{ccc} \tilde{X}_b & \longrightarrow & (\mathfrak{t}^d)_{T^*_X} \\ \downarrow & & \downarrow \\ X & \longrightarrow & (\mathfrak{t}^d /\!\!/ W)_{T^*_X} \end{array}$$

The projection $\tilde{X}_b \to X$, which is a finite surjective morphism, is called the cameral covering associated with b.

Let B° denote the open dense locus of B, where the morphism $\mathfrak{t}^d \to B$ is a finite étale Galois morphism with Galois group W. This is a GL_d-equivariant open subset of B.

Definition 5.3
We define $\mathscr{B}_X^{\heartsuit}(k)$ to be the open locus of $\mathscr{B}_X(k)$ consisting of maps $b : X \to [B/\mathrm{GL}_d]$ whose image has nonempty intersection with $[B^\circ/\mathrm{GL}_d]$.

For every $b \in \mathscr{B}_X^{\heartsuit}(k)$, the cameral covering $\tilde{X}_b \to X$ is generically a finite étale Galois morphism with Galois group W. We will prove Conjecture 5.2 in the case $G = \mathrm{GL}_n$ and $d = 2$ for all $b \in \mathscr{B}_X^{\heartsuit}(k)$. In the 1-dimensional case, and for $G = \mathrm{GL}_n$, it is well known that spectral curves are more convenient than cameral curves for the purpose of constructing Higgs bundles. Cameral and spectral covers are generally not flat in higher dimension, but in the case of dimension 2, there is a canonical way to make them flat.

From now on, we will assume that $G = \mathrm{GL}_n$.

6. Spectral covers

Let us first review the construction of the universal spectral cover for $d = 1$. For the group GL_n, $\mathfrak{t} = \mathbb{A}^n = \mathrm{Spec}(k[x_1, \dots, x_n])$ is the space of diagonal matrices with entries $x_1, \dots, x_n$. The Weyl group W is the symmetric group $\mathfrak{S}_n$ acting on $\mathbb{A}^n$ by permutation of coordinates $x_1, \dots, x_n$. By the fundamental theorem of symmetric polynomials, the categorical quotient $\mathfrak{c} = \mathbb{A}^n /\!\!/ \mathfrak{S}_n$ is the affine space of coordinates

$$
\begin{aligned}
c_1 &= x_1 + \cdots + x_n,\\
c_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n,\\
&\cdots\\
c_n &= x_1 \cdots x_n.
\end{aligned}
$$

The universal spectral cover is a finite flat covering $\mathfrak{c}^\bullet \to \mathfrak{c}$ of degree n. To construct it we consider the action of the subgroup $\mathfrak{S}_{n-1}$ of $\mathfrak{S}_n$ on $\mathbb{A}^n$ permuting the coordinates $(x_1, \dots, x_{n-1})$ and leaving x_n fixed. The categorical quotient $\mathfrak{c}^\bullet = \mathbb{A}^n /\!\!/ \mathfrak{S}_{n-1}$ is the affine space of coordinates $(c'_1, \dots, c'_{n-1}, x_n)$ with

$$
c'_1 = x_1 + \cdots + x_{n-1}, \qquad \dots, \qquad c'_{n-1} = x_1 \dots x_{n-1}.
$$

The induced morphism $p : \mathfrak{c}^\bullet \to \mathfrak{c}$ is a finite flat morphism of degree n. One can represent the finite morphism $\mathfrak{c}^\bullet \to \mathfrak{c}$ in terms of equations by considering the morphism $\iota : \mathfrak{c}^\bullet \to \mathfrak{c} \times \mathbb{A}^1$ given by $(c'_1, \dots, c'_{n-1}, x_n) \mapsto (c_1, \dots, c_n, t)$ with

$$
t = x_n, \qquad c_1 = c'_1 + x_n, \qquad c_2 = c'_2 + c'_1 x_n, \qquad \dots, \qquad c_n = c'_{n-1} x_n. \tag{6.1}
$$

This is a closed embedding that identifies $\mathfrak{c}^\bullet$ with the closed subscheme of $\mathfrak{c} \times \mathbb{A}^1$ defined by the equation $t^n - c_1 t^{n-1} + \cdots + (-1)^n c_n = 0$.

We will now generalize this construction to the case $d \geq 2$. For $G = \mathrm{GL}_n$, we have $\mathfrak{t}^d = (\mathbb{A}^d)^n$. The categorical quotient $\mathfrak{t}^d /\!\!/ W$ can be identified with the Chow scheme $\mathrm{Chow}_n(\mathbb{A}^d) = (\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_n$ classifying 0-dimensional cycles of length n of $\mathbb{A}^d$. We will represent a point of $\mathrm{Chow}_n(\mathbb{A}^d)$ as an unordered collection of n points of $\mathbb{A}^d$

$$[x_1, \dots, x_n] \in \mathrm{Chow}_n(\mathbb{A}^d). \tag{6.2}$$

By Theorem 4.1, the morphism

$$\mathrm{pol}_{\mathrm{W}} : \mathrm{Chow}_n(\mathbb{A}^d) \to \prod_{i=1}^{n} \mathrm{S}^i \mathbb{A}^d, \qquad [x_1, \dots, x_n] \to (c_1, \dots, c_n), \tag{6.3}$$

where $c_i \in \mathrm{S}^i \mathbb{A}^d$ is the ith elementary symmetric polynomial of variables $x_1, \dots, x_n \in \mathbb{A}^d$, is a closed embedding. We will construct the universal spectral covering of $\mathrm{Chow}_n(\mathbb{A}^d)$ as follows. Consider the morphism

$$\chi_{\mathbb{A}^d} : \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d \to \mathrm{S}^n \mathbb{A}^d \tag{6.4}$$

given by

$$\chi_{\mathbb{A}^d}\big([x_1, \dots, x_n], x\big) = (x - x_1) \cdots (x - x_n) = x^n - c_1 x^{n-1} + \cdots + (-1)^n c_n. \tag{6.5}$$

We define the closed subscheme $\mathrm{Cayley}_n(\mathbb{A}^d)$ to be

$$\mathrm{Cayley}_n(\mathbb{A}^d) = \chi_{\mathbb{A}^d}^{-1}\big(\{0\}\big), \tag{6.6}$$

the fiber over $0 \in \mathrm{S}^n \mathbb{A}^d$.

PROPOSITION 6.1

(1) *The projection* $p : \mathrm{Cayley}_n(\mathbb{A}^d) \to \mathrm{Chow}_n(\mathbb{A}^d)$ *is a finite morphism which is étale over the open subset* $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$ *of* $\mathrm{Chow}_n(X)$ *consisting of multiplicity-free* 0*-cycles.*

(2) *For every point* $a = [x_1^{n_1}, \dots, x_m^{n_m}] \in \mathrm{Chow}_n(\mathbb{A}^d)$*, where* $x_1, \dots, x_m$ *are distinct points of* $\mathbb{A}^d$*, and* $n_1, \dots, n_m$ *are positive integers such that* $n_1 + \cdots + n_m = n$*, the fiber of* $p : \mathrm{Cayley}_n(\mathbb{A}^d) \to \mathrm{Chow}_n(\mathbb{A}^d)$ *over* a *is the finite subscheme of* $\mathbb{A}^d$

$$\mathrm{Cayley}_n(a) = \bigsqcup_{i=1}^{m} \mathrm{Spec}(\mathscr{O}_{\mathbb{A}^d, x_i} / \mathfrak{m}_{x_i}^{n_i}), \tag{6.7}$$

where $\mathscr{O}_{\mathbb{A}^d, x_i}$ *is the local ring of* $\mathbb{A}^d$ *at* x_i*, and* $\mathfrak{m}_{x_i}$ *its maximal ideal. In particular, as soon as* $d \geq 2$ *and* $n \geq 2$*, then the cover* $\mathrm{Cayley}_n(\mathbb{A}^d) \to \mathrm{Chow}_n(\mathbb{A}^d)$ *is not flat.*

(3) *Let F be a finite $\mathscr{O}_{\mathbb{A}^d}$-module of length n, and let $a \in \mathrm{Chow}_n(\mathbb{A}^d)$ be its spectral datum. Then F is supported by the finite subscheme $\mathrm{Cayley}_n(a)$ of $\mathbb{A}^d$. (This is a generalization of the Cayley–Hamilton theorem.)*

Proof

We will first describe a set of the generators of the ideal defining the closed subscheme $\mathrm{Cayley}_n(\mathbb{A}^d)$ of $\mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$. Let V_d be the space of linear forms on $\mathbb{A}^d$. Every $v : \mathbb{A}^d \to \mathbb{A}^1$ in V_d induces a map on Chow varieties $[v] : \mathrm{Chow}_n(\mathbb{A}^d) \to \mathrm{Chow}_n(\mathbb{A}^1)$ mapping $a = [x_1, \dots, x_n] \in \mathrm{Chow}_n(\mathbb{A}^d)$ to

$$v(a) = \big[v(x_1), \dots, v(x_n)\big] \in \mathrm{Chow}_n(\mathbb{A}^1).$$

As the diagram

$$\begin{array}{ccc} \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d & \xrightarrow{\chi_{\mathbb{A}^d}} & \mathrm{S}^n \mathbb{A}^d \\ {\scriptstyle [v]\times v}\downarrow & & \downarrow{\scriptstyle \mathrm{S}^n(v)} \\ \mathrm{Chow}_n(\mathbb{A}^1) \times \mathbb{A}^1 & \xrightarrow[\chi_{\mathbb{A}^1}]{} & \mathrm{S}^n \mathbb{A}^1 = \mathbb{A}^1 \end{array} \tag{6.8}$$

is commutative, the function $f_v = \chi_{\mathbb{A}^1} \circ ([v] \times v) : \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d \to \mathbb{A}^1$ vanishes on $\mathrm{Cayley}_n(\mathbb{A}^d)$. Explicitly, for every $a = [x_1, \dots, x_n] \in \mathrm{Chow}_n(\mathbb{A}^d)$, we have

$$f_v(a, x) = \big(v(x) - v(x_1)\big) \cdots \big(v(x) - v(x_n)\big). \tag{6.9}$$

Moreover, since $\mathrm{S}^n(v)$ generates the ideal defining 0 in $\mathrm{S}^n \mathbb{A}^d$ as v varies in V_d, the functions f_v generate the ideal defining $\mathrm{Cayley}_n(\mathbb{A}^d)$ inside $\mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$. This provides a convenient set of generators of this ideal, albeit infinite and even innumerable as k may be.

(1) Let $v_1, \dots, v_d$ be the standard basis of V_d whose symmetric algebra $\mathrm{S}(V_d)$ is the ring of functions of $\mathbb{A}^d$. The functions $f_{v_1}, \dots, f_{v_d}$ cut out a closed subscheme Z of $\mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$ which is finite flat of degree n^d over $\mathrm{Chow}_n(\mathbb{A}^d)$. Since $\mathrm{Cayley}_n(\mathbb{A}^d)$ is a closed subscheme of Z, it is also finite over $\mathrm{Chow}_n(\mathbb{A}^d)$. This proves the first assertion of the proposition.

(2) We will prove that for $a = [x_1^{n_1}, \dots, x_m^{n_m}] \in \mathrm{Chow}_n(\mathbb{A}^d)$, where $x_1, \dots, x_m$ are distinct points of $\mathbb{A}^d$, and $n_1, \dots, n_m$ are positive integers such that $n_1 + \cdots + n_m = n$, $\mathrm{Cayley}_n(a)$ is the closed subscheme of $\mathbb{A}^d$ defined by the ideal $\mathfrak{m}_{x_1}^{n_1} \cdots \mathfrak{m}_{x_m}^{n_m}$ of $\mathrm{S}(V_d)$, where $\mathfrak{m}_{x_i}$ is the maximal ideal corresponding to the point $x_i \in \mathbb{A}^d$.

Let us denote by I_a the ideal of $\mathrm{S}(V_d)$ defining the finite subscheme $\mathrm{Cayley}_n(a)$ in $\mathbb{A}^d$. We first prove that $I = I_{x_1} \cdots I_{x_n}$, where A/I_{x_i} is supported by some finite thickening of the point x_i. For this we only need to prove that for every $x \notin \{x_1, \dots, x_m\}$, there exists a function $f \in I_a$ such that $f \notin \mathfrak{m}_x$. We recall that the ideal I_a is generated by the functions $f_v(a) : \mathbb{A}^d \to \mathbb{A}^1$ as v varies in V_d. Choose a linear form $v \in V_d$ on $\mathbb{A}^d$ such that $v(x) \neq v(x_i)$ for all $i \in \{1, \dots, m\}$. Then we have $f_v(a)(x) \neq 0$ by (6.9).

As $x_1, \dots, x_m$ play equivalent roles, we can focus our attention on x_1. It only remains to prove that the images of the functions $f_v(a)$ in the localization $\mathrm{S}(V_d)_{x_1}$ of $\mathrm{S}(V_d)$ at x_1, as v varies in V_d, generate the ideal $\mathfrak{m}_{x_1}^{n_1}$. From (6.9), we already know that $f_v(a) \in \mathfrak{m}_{x_1}^{n_1}$ for every $v \in V_d$. By Nakayama's lemma, we only need to prove that the images of $f_v(a)$ in $\mathfrak{m}_{x_1}^{n_1}/\mathfrak{m}_{x_1}^{n_1+1}$ generate this vector space as v varies in V_d. We observe that for $v \in V_d$ such that $v(x_1) \neq v(x_i)$ for $i \in \{2, \dots, m\}$, the factors $v(v) - v(x_2), \dots, v(v) - v(v_m)$ are all invertible at x_1, so it is enough to prove that for $v \in V_d$ satisfying the open condition $v(x_1) \neq v(x_i)$ for $i \in \{2, \dots, m\}$, the functions $(v(v) - v(x_1))^{n_1}$ generate $\mathfrak{m}_{x_1}^{n_1}/\mathfrak{m}_{x_1}^{n_1+1}$. Here we again use the fact that the image of the nth power map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ spans $\mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ and this conclusion does not change even after we remove from $\mathfrak{m}_x/\mathfrak{m}_x^2$ a closed subset of smaller dimension.

(3) By the Chinese remainder theorem, we are easily reduced to proving that if F is a finite $\mathrm{S}(V_d)$-module of length n, supported by a finite thickening of $x \in \mathbb{A}^d$, then F is annihilated by $\mathfrak{m}_x^n$. Since F is supported by a finite thickening of $x \in \mathbb{A}^d$ it has the structure of an $\mathrm{S}(V_d)_x$-module, where $\mathrm{S}(V_d)_x$ is the localization of $\mathrm{S}(V_d)$ at x. We consider the decreasing filtration $F \supset \mathfrak{m}_x F \supset \mathfrak{m}_x^2 F \supset \cdots$. By Nakayama's lemma, we know that for $m \in \mathbb{N}$, $\mathfrak{m}_x^m E/\mathfrak{m}_x^{m+1} E = 0$ implies $\mathfrak{m}_x^m F = 0$. It follows that as long as $\mathfrak{m}_x^m F \neq 0$, we have $\dim_k(\mathfrak{m}_x^i F/\mathfrak{m}_x^{i+1} F) \geq 1$ for all $i \in \{0, \dots, m\}$ and it follows that $m + 1 \leq n$. We conclude that $\mathfrak{m}_x^n F = 0$.

This completes the proof of Proposition 6.1 □

There is another construction possibly giving rise to a slightly different spectral cover of $\mathrm{Chow}_n(\mathbb{A}^d)$. We consider the action of $\mathfrak{S}_{n-1}$ on $(\mathbb{A}^d)^n$ permuting $(x_1, \dots, x_{n-1})$ and leaving x_n fixed. The categorical quotient $(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1}$ is a normal scheme equipped with a morphism $(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1} \to (\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_n$ which is finite and generically finite étale of degree n. We also have a morphism

$$\iota : (\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1} = \mathrm{Chow}_{n-1}(\mathbb{A}^d) \times \mathbb{A}^d \to \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$$

given by $([x_1, \dots, x_{n-1}], x_n) \mapsto ([x_1, \dots, x_n], x_n)$.

PROPOSITION 6.2

The morphism $\iota : (\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1} \to \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$ *is a closed embedding. It factors through a universal homeomorphism*

$$(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1} \to \mathrm{Cayley}_n(\mathbb{A}^d), \tag{6.10}$$

which is an isomorphism over $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$.

Proof
We have the following commutative diagram

$$\begin{array}{ccc} (\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1} = \mathrm{Chow}_{n-1}(\mathbb{A}^d) \times \mathbb{A}^d & \xrightarrow{\iota} & \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d \\ \downarrow & & \downarrow \\ \prod_{i=1}^{n-1} \mathrm{S}^i \mathbb{A}^d \times \mathbb{A}^d & \longrightarrow & \prod_{i=1}^{n} \mathrm{S}^i \mathbb{A}^d \times \mathbb{A}^d \end{array}$$

where the vertical arrows are the closed embeddings induced from (6.3) and the lower horizontal arrow is the closed embedding sending $(c_1', \dots, c_{n-1}', x_n)$ to $(c_1, \dots, c_n, x_n)$, where $c_1, \dots, c_n$ are given by (6.1). It follows that ι is a closed embedding.

Let $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$ denote the open subscheme of $\mathrm{Chow}_n(\mathbb{A}^d)$ consisting of multiplicity-free 0-cycles. Let us denote by $(\mathbb{A}^d)^{n,\circ}$ the preimage of B° which is the complement in $(\mathbb{A}^d)^n$ of all diagonals. The morphism $(\mathbb{A}^d)^{n,\circ} \to \mathrm{Chow}_n^\circ(\mathbb{A}^d)$ is finite, étale, and Galois of Galois group $\mathfrak{S}_n$. The morphism $(\mathbb{A}^d)^{n,\circ} \to (\mathbb{A}^d)^{n,\circ} /\!\!/ \mathfrak{S}_{n-1}$ is a finite étale Galois morphism with Galois group $\mathfrak{S}_{n-1}$. It follows that the morphism $(\mathbb{A}^d)^{n,\circ} /\!\!/ \mathfrak{S}_{n-1} \to \mathrm{Chow}_n^\circ(\mathbb{A}^d)$ is finite étale of degree $|\mathfrak{S}_n|/|\mathfrak{S}_{n-1}| = n$.

Over $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$, the morphism $\iota : (\mathbb{A}^d)^{n,\circ} /\!\!/ \mathfrak{S}_{n-1} \to B^\circ \times \mathbb{A}^d$ clearly induces an isomorphism of $(\mathbb{A}^d)^{n,\circ} /\!\!/ \mathfrak{S}_{n-1}$ on $\mathrm{Cayley}_n^\circ(\mathbb{A}^d)$ which is the preimage of $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$ in $\mathrm{Cayley}_n(\mathbb{A}^d)$. Since $(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1}$ is an integral scheme, the function $x^n - c_1 x^{n-1} + \cdots + (-1)^n c_n$ which vanishes over $(\mathbb{A}^d)^{n,\circ} /\!\!/ \mathfrak{S}_{n-1}$ has to vanish on all $(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1}$. It follows that the morphism ι factors through a morphism $(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1} \to \mathrm{Cayley}_n(\mathbb{A}^d)$. This morphism is finite since $(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1}$ is finite over $\mathrm{Chow}_n(\mathbb{A}^d)$. One can check directly that the finite morphism $(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1} \to \mathrm{Cayley}_n(\mathbb{A}^d)$ induces a bijection over the k-points, which implies that it is a universal homeomorphism. □

Remark 6.1
Drinfeld asked the question whether the morphism (6.10) is an isomorphism, as in the case $d = 1$. This is equivalent to saying that $\mathrm{Cayley}_n(\mathbb{A}^d)$ is reduced and normal.

Recall that in the case $G = \mathrm{GL}_n$, the closed subscheme B of A constructed in Theorem 4.1 is $B = \mathrm{Chow}_n(\mathbb{A}^d)$. As the universal spectral cover on B, we will take

$$B^\bullet = \mathrm{Cayley}_n(\mathbb{A}^d)$$

instead of $(\mathbb{A}^d)^n /\!\!/ \mathfrak{S}_{n-1}$. The reason is that, in Proposition 6.1, we have a nice description of the fibers of $B^\bullet$ over B, and a generalization of the Cayley–Hamilton theorem.

For every geometric point $b \in \mathscr{B}_X(k)$, we have a morphism $b : X \to [B/\mathrm{GL}_d]$ lying over the morphism $\tau_X^* : X \to \mathbb{B}\mathrm{GL}_d$ corresponding to the cotangent bundle T_X^*. By forming the Cartesian product

$$\begin{array}{ccc} X_b^\bullet & \longrightarrow & [B^\bullet/\mathrm{GL}_d] \\ \Big\downarrow p_b & & \Big\downarrow \\ X & \xrightarrow{\;b\;} & [B/\mathrm{GL}_d] \end{array} \tag{6.11}$$

we obtain the spectral cover $X_b^\bullet$ of X corresponding to b. Since $B^\bullet \to B$ is a finite morphism, the map $p_b : X_B^\bullet \to X$ is a finite covering. If $b \in \mathscr{B}_X^\heartsuit$, that is, $b(X)$ has nonempty intersection with $[B^\circ/\mathrm{GL}_d]$, then the covering $p_b : X_b^\bullet \to X$ is generically finite étale of degree n.

If X is a curve, and if the spectral curve $X_b^\bullet$ is integral, then, after Beauville, Narasimhan, and Ramanan [3], there is an equivalence of categories between the category of Higgs bundles with spectral datum b and the category of torsion-free $\mathscr{O}_{X_b^\bullet}$'s of generic rank 1. This equivalence can be generalized to the case $d \geq 1$ with the concept of Cohen–Macaulay sheaves.

Let M be a coherent sheaf on a finite type scheme Y. Let $d = \operatorname{codim}(\operatorname{Supp}(M))$. Recall that M is called *Cohen–Macaulay* of codimension d if $\mathrm{H}^i(\mathbb{D}(M)) = 0$ for $i \neq d$. A Cohen-Macaulay sheaf M is called *maximal* if it has codimension 0.

We also recall an important fact about Cohen–Macaulay modules. Suppose that R is a finite A-algebra of degree n with A being a regular ring of pure dimension m. Let M be an R-module of finite type. Then M is a locally free A-module of rank n if and only if M is maximal Cohen–Macaulay of generic rank 1. We refer to [4, Section 2] for a nice discussion on Cohen–Macaulay modules and for further references therein, or to [5] for a comprehensive treatment.

PROPOSITION 6.3
For every $b \in \mathscr{B}_X^\heartsuit(k)$, the fiber $h_X^{-1}(b)$ of the Hitchin morphism is isomorphic to the stack of maximal Cohen–Macaulay sheaves of generic rank 1 *on the spectral cover $X_b^\bullet$.*

Proof
Let $(E, \theta) \in h_X^{-1}(b)$ be a Higgs bundle of rank n whose spectral datum is $b \in \mathscr{B}_X^\heartsuit(k)$. Then $E = p_* F$, where $p : T^*X \to X$ is the projection map and F is a coherent sheaf on the cotangent T_X^*. By the Cayley–Hamilton theorem (see Proposition 6.1),

F is supported by the spectral cover $X_b^\bullet \subset T_X^*$. We have then $E = p_{b*}F$, where $p_b : X_b^\bullet \to X$ is the map in (6.11) and F is a coherent sheaf on $X_b^\bullet$. Since $p_b : X_b^\bullet \to X$ is a finite morphism, and E is a vector bundle over X, F is a maximal Cohen–Macaulay sheaf. Moreover, since p_b is generically finite étale of degree n, F has generic rank 1. Conversely, if F is a maximal Cohen–Macaulay sheaf of generic rank 1 over $X_b^\bullet$, then $E = p_{b*}F$ is a vector bundle of rank n over X. It is naturally equipped with a Higgs field $\theta : E \otimes_{\mathscr{O}_X} \mathscr{T}_X \to E$ as $X_b^\bullet$ is a closed subscheme of T_X^*. □

In spite of the simplicity of the description of $h_X^{-1}(b)$, the proposition above is not of great use. For instance, it does not imply that $h_X^{-1}(b)$ is nonempty. The difficulty is that, in general, the spectral cover $X_b^\bullet$ itself might not be Cohen–Macaulay; equivalently, the map $X_b^\bullet \to X$ might not be flat. Therefore, it is not clear how to construct coherent Cohen–Macaulay sheaves on $X_b^\bullet$. At this point, we see that in order to obtain a useful description of $h_X^{-1}(b)$, we need to construct a finite Cohen–Macaulayfication of $X_b^\bullet$. This can be done in the case of surfaces.

7. Cohen–Macaulay spectral surfaces

In the case of surfaces, for every $b \in \mathscr{B}_X^\heartsuit(k)$, the spectral surface $X_b^\bullet$ admits a canonical finite Cohen–Macaulayfication whose construction relies on the theory of Hilbert schemes of points on surfaces and Serre's theorem on extending vector bundles on smooth surfaces across closed subschemes of codimension 2. We will first recall Serre's theorem on extending locally free sheaves across a closed subscheme of codimension 2 (see [21, Proposition 7]).

THEOREM 7.1

Let X be a smooth surface over k, let Z be a closed subscheme of codimension 2 of X, and let $j : U \to X$ be the open immersion of the complement U of Z in X. Then the functor $V \to j_*V$ is an equivalence of categories between the category of locally free sheaves on U and locally free sheaves on X. Its inverse is the functor j^*.

As we are now considering the case $G = \mathrm{GL}_n$ and $d = 2$, the subscheme B of $A = \mathbb{A}^2 \times \mathrm{S}^2\mathbb{A}^2 \times \cdots \times \mathrm{S}^n\mathbb{A}^2$ is canonically isomorphic to the Chow scheme $\mathrm{Chow}_n(\mathbb{A}^2)$ of 0-cycles of length n on $\mathbb{A}^2$. We recall that a point $b \in \mathscr{B}_X$ is a section $b : X \to [\mathrm{Chow}_n(\mathbb{A}^2)/\mathrm{GL}_2]$ lying over $\tau_X^* : X \to \mathbb{B}\mathrm{GL}_2$ representing the cotangent bundle T_X^*. In other words, b is a section of the relative Chow scheme

$$\mathrm{Chow}_n(T_X^*/X) \to X$$

obtained from $\mathrm{Chow}_n(\mathbb{A}^2)$ by twisting it by the GL_2-torsor attached to the cotangent bundle T_X^* of X.

Recall the open locus $\mathrm{Chow}_n^\circ(\mathbb{A}^2)$ of $\mathrm{Chow}_n(\mathbb{A}^2)$ consisting of multiplicity-free 0-cycles, and Q its complement. Let $\mathrm{Chow}_n^\circ(T_X^*/X)$ be the corresponding open locus in $\mathrm{Chow}_n(T_X^*/X)$, and let $Q(T_X^*/X)$ be its complement. Recall the open locus $\mathscr{B}_X^\heartsuit$ in $\mathscr{B}_X$ consisting of maps $b : X \to [\mathrm{Chow}_n(\mathbb{A}^2)/\mathrm{GL}_2]$ which maps the generic point of X to the open locus $[\mathrm{Chow}_n^\circ(T_X^*/X)/\mathrm{GL}_2]$. In other words,

$$\mathscr{B}_X^\heartsuit = \bigl\{b \in \mathscr{B}_X \bigm| \dim b^{-1}\bigl(Q(T_X^*/X)\bigr) \leq 1\bigr\}. \tag{7.1}$$

We first recall some well-known facts about the Hilbert schemes of 0-dimensional subschemes of a surface (see, e.g., [17]). Let $\mathrm{Hilb}_n(\mathbb{A}^2)$ denote the moduli space of 0-dimensional subschemes of length n of $\mathbb{A}^2$. A point of $\mathrm{Hilb}_n(\mathbb{A}^2)$ is a 0-dimensional subscheme Z of $\mathbb{A}^2$ of length n that will be of the form $Z = \bigsqcup_{\alpha\in\mathbb{A}^2} Z_\alpha$, where Z_α is a local 0-dimensional subscheme of $\mathbb{A}^2$ whose closed point is α. It is known that the Hilbert–Chow morphism

$$\mathrm{HC}_n : \mathrm{Hilb}_n(\mathbb{A}^2) \to \mathrm{Chow}_n(\mathbb{A}^2) \tag{7.2}$$

given by $Z \mapsto \sum_{\alpha\in\mathbb{A}^2} \mathrm{length}(Z_\alpha)\alpha$, where $\mathrm{length}(Z_\alpha)$ is the length of Z_α, is a resolution of singularities of $\mathrm{Chow}_n(\mathbb{A}^2)$. It is clear that HC_n is an isomorphism over $\mathrm{Chow}_n^\circ(\mathbb{A}^2)$.

As the morphism (7.2) is GL_2-equivariant, we can twist it by any GL_2-bundle, in particular, by the GL_2-bundle associated to the cotangent bundle T_X^* over a smooth surface X. By doing so, we obtain

$$\mathrm{HC}_{T_X^*/X} : \mathrm{Hilb}_n(T_X^*/X) \to \mathrm{Chow}_n(T_X^*/X). \tag{7.3}$$

This morphism is a proper morphism and its base change to the open subset $\mathrm{Chow}_n^\circ(T_X^*/X)$ is an isomorphism.

PROPOSITION 7.2
For every $b \in \mathscr{B}_X^\heartsuit(k)$, there exists a unique finite flat covering

$$p_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X \tag{7.4}$$

of degree n, equipped with an X-morphism $\iota : X_b^{\mathrm{CM}} \to T_X^*$ satisfying the following property: there exists an open subset $U \subset X$, whose complement is a closed subset of codimension at least 2, such that ι is a closed embedding over U and for every $x \in U$, the fiber $(p_b^{\mathrm{CM}})^{-1}(x)$ is a point of $\mathrm{Hilb}_n(T_X^*/X)$ *lying over the point $b(x) \in$* $\mathrm{Chow}_n(T_X^*/X)$*. Moreover, the morphism $\iota : X_b^{\mathrm{CM}} \to T_X^*$ factors through the closed subscheme $X_b^\bullet$ of T_X^* and the resulting morphism $q_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X_b^\bullet$ is a finite Cohen–Macaulayfication of $X_b^\bullet$.*

Proof

Let U° be the preimage of $\mathrm{Chow}_n^\circ(T_X^*/X)$ by the section $b : X \to \mathrm{Chow}_n(T_X^*/X)$. By the assumption $b \in \mathscr{B}_X^\heartsuit$, U° is a nonempty open subset of X. As the morphism $\mathrm{HC}_{T_X^*/X}$ of (7.3) is an isomorphism over $\mathrm{Chow}_n(T_X^*/X)$, we have a unique lifting

$$b_{\mathrm{Hilb}}^\circ : U^\circ \to \mathrm{Hilb}_n(T_X^*/X) \times_X U^\circ$$

lying over the restriction $b^\circ = b|U^\circ$.

Since the Hilbert–Chow morphism (7.3) is proper, there exists an open subset $U \subset X$, larger than U°, whose complement $X - U$ is a closed subscheme of codimension at least 2, such that $b' : U^\circ \to \mathrm{Hilb}_n'(T_X^*/X) \times_X U^\circ$ extends to

$$b_{\mathrm{Hilb}}^U : U \to \mathrm{Hilb}_n(T_X^*/X) \times_X U.$$

By pulling back from $\mathrm{Hilb}_n(T_X^*/X)$ the tautological family of subschemes of T_X^*, we get a finite flat morphism $U_b^+ \to U$ of degree n, equipped with a closed embedding $\iota_U : U_b^+ \to T_U^*$.

According to Serre's theorem on extending vector bundles over surfaces, there exists a unique finite flat covering $X_b^{\mathrm{CM}} \to X$ of degree n extending the finite flat covering U_b^+ of U. The closed embedding $\iota_U : U_b^+ \to T_U^*$ extends to a morphism $\iota : X_b^{\mathrm{CM}} \to T_X^*$ which may not be a closed embedding.

By construction, $p_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X$ is a finite flat morphism of degree n, so it follows from smoothness of X that X_b^{CM} is a Cohen–Macaulay surface. Apply the generalized Cayley–Hamilton theorem to the vector bundle $p_{b*}^{\mathrm{CM}} \mathscr{O}_{X_b^{\mathrm{CM}}}$; as an $\mathscr{O}_{T_X^*}$-module over T_X^*, it is supported by $X_b^\bullet$. It follows that the morphism $X_b^{\mathrm{CM}} \to T_X^*$ factors through a map $q_b^{\mathrm{CM}} : X_B^{\mathrm{CM}} \to X_b^\bullet \subset T_X^*$. Since X_b^{CM} is finite over X, it is also finite over $X_b^\bullet$. As $q_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X_b^\bullet$ is an isomorphism over the nonempty open subset U°, it is a finite Cohen–Macaulayfication of $X_b^\bullet$. □

Remark 7.1

Instead of using the Hilbert scheme, we can construct X_b^{CM} over the height 1 points as follows. Let $U^\circ = b^{-1}(\mathrm{Chow}_n^\circ(\mathrm{T}_X^*/X))$, and let Z be the complement of U°. Let z be the generic point of an irreducible component of Z of dimension 1. The localization of X at z is $X_z = \mathrm{Spec}(\mathscr{O}_{X,z})$, where $\mathscr{O}_{X,z}$ is a discrete valuation ring. By restricting $p_{b*}\mathscr{O}_{X_b^\bullet}$ to $\mathscr{O}_{X,z}$ we get a module of finite type which may have torsion. By considering the quotient $\mathrm{Spec}(p_{b*}\mathscr{O}_{X_b^\bullet}/(p_{b*}\mathscr{O}_{X_b^\bullet}^{\mathrm{tors}}))$, we obtain a locally free $\mathscr{O}_{X,z}$-module and thus a section $X_z \to \mathrm{Hilb}_n(\mathrm{T}_X^*/X) \times_X X_z$ over $b|_{X_z}$. By uniqueness of such a section, we have an isomorphism

$$\mathrm{Spec}\big(p_{b*}\mathscr{O}_{X_b^\bullet}/(p_{b*}\mathscr{O}_{X_b^\bullet})^{\mathrm{tors}}\big) \simeq \mathrm{Spec}(p_{b*}^{\mathrm{CM}}\mathscr{O}_{X_b^{\mathrm{CM}}}) \tag{7.5}$$

over the complement of a codimension 2 subscheme of X.

Remark 7.2

We do not know whether the construction of the Cohen–Macaulay spectral surface X_b^{CM} works well in families. The issue is that the construction makes use of the equivalence of categories from Theorem 7.1 which does not work well in families.

THEOREM 7.3

For every $b \in \mathscr{B}_X^{\heartsuit}(k)$, the fiber $h_X^{-1}(b)$ is isomorphic to the stack of Cohen–Macaulay sheaves F of generic rank 1 over the Cohen–Macaulay spectral surface X_b^{CM}. It contains, in particular, the Picard stack $\mathscr{P}_b$ of line bundles on X_b^{CM}. The action of $\mathscr{P}_b$ on itself by translation extends to an action of $\mathscr{P}_b$ on $h_X^{-1}(b)$.

In particular, $h_X^{-1}(b)$ is nonempty.

Proof

Let $(E,\theta) \in \mathscr{M}_X$ be a Higgs bundle over X lying over $b \in \mathscr{B}_X^{\heartsuit}(k)$. The Higgs field $\theta : \mathscr{T}_X \to \mathrm{End}_{\mathscr{O}_X}(E)$ defines a homomorphism $\mathrm{S}(\mathscr{T}_X) \to \mathrm{End}_{\mathscr{O}_X}(E)$ which factors through $p_{a*}\mathscr{O}_{X_b^\bullet}$ by the generalized Cayley–Hamilton theorem (see Proposition 6.1(3)).

Let U° and Z be as in Remark 7.1, and let z be the generic point of an irreducible component of Z of dimension 1. Over X_z we have a homomorphism

$$p_{b*}\mathscr{O}_{X_b^\bullet} \otimes_{\mathscr{O}_X} \mathscr{O}_{X_z} \to \mathrm{End}_{\mathscr{O}_X}(E) \otimes_{\mathscr{O}_X} \mathscr{O}_{X_z}.$$

Since the target is clearly torsion-free, this homomorphism factors through (7.5). Thus, over an open subset $U \subset X$ whose complement is of codimension 2, the above morphism factors through a homomorphism of algebras

$$p_{b*}^{\mathrm{CM}}\mathscr{O}_{X_b^{\mathrm{CM}}} \otimes_{\mathscr{O}_X} \mathscr{O}_U \to \mathrm{End}_{\mathscr{O}_X}(E) \otimes_{\mathscr{O}_X} \mathscr{O}_U.$$

By applying Serre's theorem again, we have a canonical homomorphism $p_{b*}^{\mathrm{CM}}\mathscr{O}_{X_b^{\mathrm{CM}}} \to \mathrm{End}_{\mathscr{O}_X}(E)$. It follows that $E = \tilde{p}_{a*}F$, where F is a Cohen–Macaulay $\mathscr{O}_{X_b^{\mathrm{CM}}}$-module of generic rank 1.

Since $p_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X$ is finite flat, for every line bundle L on X_b^{CM}, $p_{b*}^{\mathrm{CM}}L$ is a vector bundle of rank n carrying a Higgs field. Thus, $h_X^{-1}(b)$ contains $\mathscr{P}_b$. We have an action of $\mathscr{P}_b$ on $h_X^{-1}(b)$ given by $(L,F) \mapsto L \otimes_{\mathscr{O}_{X_b^{\mathrm{CM}}}} F$, where L is a line bundle on X_b^{CM} and F is a Cohen–Macaulay sheaf of generic rank 1. □

Remark 7.3

Let $b \in \mathscr{B}_X^{\heartsuit}(k)$ be such that the Cohen–Macaulay surface X_b^{CM} is integral. Consider the functor associating to a k-scheme Y the set of isomorphism classes of a family of Cohen–Macaulay sheaves of generic rank 1 on X_b^{CM} parameterized by Y. According to [1, Corollary 6.7, Theorem 7.9], the fppf sheafification of this functor is represented by a k-scheme $\mathrm{Pic}(X_b^{\mathrm{CM}})^-$ locally of finite type. In addition, $\mathrm{Pic}(X_b^{\mathrm{CM}})^-$ admits a compactification $\mathrm{Pic}(X_b^{\mathrm{CM}})^=$ whose k-points are given by isomorphism classes of torsion-free rank 1 sheaves on X_b^{CM}.

Definition 7.4
We define $\mathscr{B}_X^{\diamondsuit}(k)$ to be the subset of $\mathscr{B}_X^{\heartsuit}(k)$ consisting of those points b such that the corresponding Cohen–Macaulay spectral surface X_b^{CM} is normal.

LEMMA 7.5
For $b \in \mathscr{B}_X^{\diamondsuit}(k)$, the neutral component $\mathscr{P}_b^0$ of $\mathscr{P}_b$ is a quotient of an Abelian variety by $\mathbb{G}_m$ acting trivially.

Proof
This is a consequence of a theorem of Geisser [10, Theorem 1]. Geisser's theorem states that the multiplicative part of the neutral component P^0 of the Picard variety P of an algebraic variety Y is trivial if and only if $\mathrm{H}^1_{\mathrm{et}}(Y, \mathbb{Z})$ is trivial, whereas the unipotent part is trivial if and only if Y is seminormal. If Y is normal, then $\pi_1(Y)$ is a profinite group, being a quotient of the Galois group of the generic point, and therefore cannot afford a nontrivial continuous homomorphism to $\mathbb{Z}$. It follows that $\mathrm{H}^1_{\mathrm{et}}(Y, \mathbb{Z})$ is trivial. On the other hand, a normal variety is certainly also seminormal. Assume that X_b^{CM} is normal. Then the neutral component P_b^0 of the Picard variety P_b of X_b^{CM} is an Abelian variety. We have $\mathscr{P}_b^0 = [P_b^0/\mathbb{G}_m]$. □

PROPOSITION 7.6
For $b \in \mathscr{B}_X^{\diamondsuit}(k)$, the action of $\mathscr{P}_b$ on the Hitchin fiber $h_X^{-1}(b)$ is free and $h_X^{-1}(b)$ is a disjoint union of $\mathscr{P}_b$-orbits.

Proof
If a line bundle $L \in \mathscr{P}_b$ has a stabilizer $F \in h_X^{-1}(b)$, then, as any such F, regarded as a sheaf on X_b^{CM}, is locally free of rank 1 on the smooth locus U_b of X_b^{CM}, the line bundle L is trivial on U_b. Since X_b^{CM} is normal, and the compliment $X_b^{\mathrm{CM}} \setminus U_b$ is 0-dimensional, it implies that L is trivial, and hence, the action of $\mathscr{P}_b$ is free. We claim that the $\mathscr{P}_b$-orbits on $h_X^{-1}(b)$ are open and closed. The closedness follows from the lemma above. To show that the $\mathscr{P}_b$-orbits are open, we observe that $h_X^{-1}(b)$ is isomorphic to the stack of reflexive sheaves of rank 1 on X_b^{CM} and, for any $F \in h_X^{-1}(b)$, the assignment sending $F' \in h_X^{-1}(b)$ to the reflexive hull of $F' \otimes_{X_b^{\mathrm{CM}}} F$ (i.e., the double dual of $F' \otimes_{X_b^{\mathrm{CM}}} F$) defines an automorphism of $h_X^{-1}(b)$ mapping $\mathscr{P}_b$ isomorphically to the $\mathscr{P}_b$-orbit through F. Since $\mathscr{P}_b$ is open in $h_X^{-1}(b)$ (see [1]), it implies that the $\mathscr{P}_b$-orbits are open in $h_X^{-1}(b)$. The proposition follows. □

We expect that $\mathscr{B}_X^{\diamondsuit}(k)$ is a nonempty open subset of $\mathscr{B}_X(k)$ for most algebraic surfaces. The nonemptiness of $\mathscr{B}_X^{\diamondsuit}(k)$ is closely related to questions on the zero locus of symmetric differentials, of which very little seems to be known in higher dimension.

8. Surfaces fibered over a curve

In this section, we investigate the spectral surfaces $X_b^{\bullet}$ and the Cohen–Macaulay spectral surface X_b^{CM} in the case when X is a fibration over a curve C and apply our findings to ruled and elliptic surfaces.

Let X be a proper smooth surface, and let C be a proper smooth curve. Assume that there is a proper flat surjective map $\pi : X \to C$ such that the generic fiber is a proper smooth curve. We denote by $X^{\circ} \subset X$ the largest open subset such that π is smooth. Consider the cotangent morphism $d\pi : T_C^* \times_C X \to T_X^*$. It induces a map

$$[d\pi] : \mathrm{Chow}_n(T_C^*/C) \times_C X \to \mathrm{Chow}_n(T_X^*/X)$$

on the relative Chow varieties. For every section $b_C : C \to \mathrm{Chow}_n(T_C^*/C)$, the composition

$$b_X : X \simeq C \times_C X \overset{a_C \times \mathrm{id}_X}{\to} \mathrm{Chow}_n(T_C^*/C) \times_C X \overset{[d\pi]}{\to} \mathrm{Chow}_n(T_X^*/X)$$

is a section of $\mathrm{Chow}_n(T_X^*/X) \to X$ and the assignment $b_C \to b_X$ defines a map

$$\iota_\pi : \mathscr{B}_C \to \mathscr{B}_X. \tag{8.1}$$

We claim that the map above is a closed embedding. To see this, we observe that there is a commutative diagram

$$\begin{array}{ccc} \mathscr{B}_C & \xrightarrow{\iota_\pi} & \mathscr{B}_X \\ \downarrow & & \downarrow \\ \mathscr{A}_C & \xrightarrow{j_\pi} & \mathscr{A}_X \end{array} \tag{8.2}$$

where the vertical arrows are the natural embeddings, and the bottom arrow is the embedding

$$j_\pi : \mathscr{A}_C = \bigoplus_{i=1}^{n} \mathrm{H}^0(C, \mathrm{S}^i \Omega_C^1) \longrightarrow \mathscr{A}_X = \bigoplus_{i=1}^{n} \mathrm{H}^0(X, \mathrm{S}^i \Omega_X^1)$$

induced by the injection of vector spaces $\mathrm{H}^0(C, \mathrm{S}^i \Omega_C^1) = \mathrm{H}^0(X, \pi^* \mathrm{S}^i \Omega_C^1) \to \mathrm{H}^0(X, \mathrm{S}^i \Omega_X^1)$. The claim follows. Note that since $\dim C = 1$, the left vertical arrow in (8.2) is in fact an isomorphism. From now on, we will view $\mathscr{B}_C$ as a subspace of $\mathscr{B}_X$. Since the cotangent map $d\pi : T_C^* \times_C X \to T_X^*$ is a closed embedding over the open locus X°, we have

$$\mathscr{B}_C^\heartsuit = \mathscr{B}_C \cap \mathscr{B}_X^\heartsuit.$$

For any $b \in \mathscr{B}_C$, we denote by $C_b^\bullet \to C$ the corresponding spectral curve and we define $X_b^+ = C^\bullet \times_C X$. The natural projection map $p_b^+ : X_b^+ \to X$ is finite flat of degree n. Since X is smooth, it follows that X_b^+ is a Cohen–Macaulay surface.

LEMMA 8.1
There exists a finite X-morphism $q_b^+ : X_b^+ \to X_b^\bullet$ which is a generic isomorphism if $b \in \mathscr{B}_C^\heartsuit$. If the fibration $\pi : X \to C$ has only reduced fibers, then for any $b \in \mathscr{B}_C^\heartsuit$, the map $q_b^+ : X_b^+ \to X_b^\bullet$ is isomorphic to the finite Cohen–Macaulayfication $q_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X_b^\bullet$ in Proposition 7.2 (which is well defined since $b \in \mathscr{B}_X^\heartsuit$).

Proof
Let $i_b^+ : X_b^+ \to T_X^*$ be the restriction of the cotangent morphism $d\pi : T_C^* \times_C X \to T_X^*$ to the closed subscheme $X_b^+ \subset T_C^* \times_C X$. By the Cayley–Hamilton theorem, the map i_b^+ factors through the spectral surface $X_b^\bullet$. Let $q_b^+ : X_b^+ \to X_b^\bullet$ be the resulting map. As X_b^+ is finite over X, the map q_b^+ is finite. In addition, if $b \in \mathscr{B}_C^\heartsuit$, then both X_b^+ and $X_b^\bullet$ are generically étale over X of degree n and it implies that q_b^+ is a generic isomorphism.

Assume that the fibers of π are reduced. Then the smooth locus X° of the map π is open and its complement $X - X^\circ$ is a closed subset of codimension 2. Since the map $i_b^+ : X_b^+ \to T_X^*$ is a closed embedding over X°, Proposition 7.2 implies that the finite flat covering $q_b^+ : X_b^+ \to X_b^\bullet$ is isomorphic to the finite Cohen–Macaulayfication $q_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X_b^\bullet$. □

Definition 8.2
We define $\mathscr{B}_C^\diamondsuit$ to be the open subset of $\mathscr{B}_C^\heartsuit$ consisting of those points b such that the corresponding spectral curve C_b is smooth and irreducible.

COROLLARY 8.3
Assume that the fibration $\pi : X \to C$ has only reduced fibers. Then we have $\mathscr{B}_C^\diamondsuit \subset \mathscr{B}_X^\diamondsuit$, that is, the surface X_b^{CM} is normal for $b \in \mathscr{B}_C^\diamondsuit$.

Proof
Since X_b^{CM} is Cohen–Macaulay, by Serre's criterion for normality, it suffices to show that the $X_b^{\mathrm{CM}} \simeq X_b^+$ is smooth in codimension at most 1. The assumption implies that

the complement $X - X^\circ$ has codimension at least 2. Since C_b is smooth for $b \in \mathscr{B}_C^\diamondsuit$, the open subset $X_b^{+\circ} := \widetilde{C}_a \times_C X^\circ \subset X_b^+$ is smooth (since the map $X_b^{+\circ} \to C_b$ and C_b are smooth) and the complement $X_b^+ - X_b^{+\circ}$ has codimension at least 2. The corollary follows. □

Example 8.1
Consider the case when $X = C \times \mathbb{P}^1$ and $n = 2$. We have $\mathscr{B}_X = \mathscr{B}_C = \mathrm{H}^0(C, \Omega_C^1) \oplus \mathrm{H}^0(C, \mathrm{S}^2\Omega_C^1)$. Let $b = (b_1, b_2) \in \mathscr{B}_C^\heartsuit$, and let $p_b : X_b^\bullet \to X$ be the corresponding spectral surface. Then étale locally over X, the surface $X_b^\bullet$ is isomorphic to the closed subscheme of $\mathrm{Spec}(k[x_1, x_2, t_1, t_2])$ defined by the equations

$$\begin{cases} t_1^2 + b_1 t_1 + b_2 = 0, \\ t_2(2t_1 + b_1) = 0, \\ t_2^2 = 0, \end{cases} \tag{8.3}$$

where x_1, x_2 are local coordinates of C and $\mathbb{P}^1$ and $b_i \in k[x_1]$. Let $\mathrm{discr}_C = (b_1^2 - 4b_2 = 0) \subset C$ be the discriminant divisor for b. From (8.3) we see that $X_b^\bullet$ is an étale cover of degree 2 away from the divisor $\mathrm{discr}_C \times \mathbb{P}^1 \subset X$. Note that the spectral surface $p_b : X_b^\bullet \to X$ is not flat over X as the pushforward $p_{b*}\mathscr{O}_{X_b^\bullet}$ has length 3 over $\mathrm{discr}_C \times \mathbb{P}^1$. The finite Cohen–Macaulayfication $X_b^{\mathrm{CM}} \to X_b^\bullet$ is given by the flat quotient $\mathrm{Spec}(p_{b*}\mathscr{O}_{X_b^\bullet}/(p_{b*}\mathscr{O}_{X_b^\bullet})^{\mathrm{tors}})$ which is isomorphic to $X_b^{\mathrm{CM}} \simeq C_b \times \mathbb{P}^1$. The Hitchin fiber $h_X^{-1}(b)$ is isomorphic to

$$h_X^{-1}(b) \simeq h_C^{-1}(b) \times \mathscr{P}\mathrm{ic}(\mathbb{P}^1).$$

PROPOSITION 8.1
Let X be a smooth projective surface, and let $\pi : X \to C$ be either a ruled surface, or a nonisotrivial elliptic surface with reduced fibers. Then for every n, the pullback map

$$\mathrm{H}^0(C, \mathrm{S}^n\Omega_C^1) \to \mathrm{H}^0(X, \mathrm{S}^n\Omega_X^1)$$

is an isomorphism.

It follows from the proposition above that in the case of ruled surfaces and nonisotrivial elliptic surfaces with reduced fibers, we have $\mathscr{A}_C = \mathscr{A}_X$. Since $\mathscr{B}_C = \mathscr{A}_C$, we have $\mathscr{B}_X = \mathscr{B}_C$ and $\mathscr{B}_X^\diamondsuit$ and $\mathscr{B}_X^\heartsuit$ are open dense in $\mathscr{B}_X$. For every $b \in \mathscr{B}_C$, we have a spectral curve $C_b^\bullet$ which is finite flat of degree n over C. We also have the spectral surface $X_b^\bullet$ which is a finite scheme over X embedded in its cotangent bundle T_X^*. The Cohen–Macaulayfication of $X_b^\bullet$ is $X_b^+ = C_b \times_C X$. In the case of elliptic surfaces, the morphism $X_b^{\mathrm{CM}} \to X_b^\bullet$ may not be an isomorphism, and X_b^{CM} may not be embedded in the cotangent bundle T^*_X. The existence of the Cohen–Macaulay spectral cover guarantees that $h_X^{-1}(b)$ is nonempty.

Proposition 8.1 is obvious for ruled surfaces. Let us investigate it in the case of elliptic surfaces. We assume that there is a proper flat map $\pi : X \to C$ from X to a smooth projective curve C with general fiber a smooth curve of genus 1. We will focus on the case when $\pi : X \to C$ is not isotrivial, relatively minimal, and has reduced fibers (e.g., semistable nonisotrivial elliptic surfaces). Let X° denote the largest open subset of X such that the restriction of π to X° is a smooth morphism $\pi^\circ : X^\circ \to C$. Since the geometric fibers of π are all reduced, the complement of X° in X is a 0-dimensional subscheme. Over X°, we have an exact sequence of tangent bundles

$$0 \to \mathscr{T}_{X^\circ/C} \to \mathscr{T}_{X^\circ} \to (\pi^\circ)^* \mathscr{T}_C \to 0. \tag{8.4}$$

For every $n \in \mathbb{N}$, we have the exact sequence of symmetric powers

$$0 \to \mathrm{S}^{n-1} \mathscr{T}_{X^\circ} \otimes \mathscr{T}_{X^\circ/C} \to \mathrm{S}^n \mathscr{T}_{X^\circ} \to (\pi^\circ)^* \mathrm{S}^n \mathscr{T}_C \to 0. \tag{8.5}$$

Let $\eta \in C$ be the generic point of C with residue field K, and let $X_\eta = X \times_C \eta$, which is an elliptic curve over K. The restriction of (8.4) to X_η is a short exact sequence making the rank 2 vector bundle $\mathscr{T}_X|_{X_\eta}$ a self-extension of the trivial line bundle of X_η. As we assume that the elliptic fibration π is nonisotrivial, that is, the Kodaira–Spencer map is not zero, $\mathscr{T}_X|_{X_\eta}$ is a nontrivial self-extension of the trivial line bundle on X_η. After Atiyah [2], such a nontrivial extension is unique up to isomorphism:

$$0 \to \mathscr{O}_{X_\eta} \to \mathscr{E} \to \mathscr{O}_{X_\eta} \to 0. \tag{8.6}$$

In other words, the restriction of (8.4) to the generic fiber X_η is isomorphic to (8.6).

LEMMA 8.4
The exact sequence of symmetric powers derived from (8.6)

$$0 \to \mathrm{S}^{n-1} \mathscr{E} \to \mathrm{S}^n \mathscr{E} \to \mathscr{O}_{X_\eta} \to 0 \tag{8.7}$$

is not split.

Proof
Indeed, if

$$0 \to \mathscr{L}' \to \mathscr{E} \to \mathscr{L} \to 0 \tag{8.8}$$

is an extension of a line bundle $\mathscr{L}$ by a line bundle $\mathscr{L}'$, then there is a canonical filtration

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_{n-1} \subset \mathscr{F}_n = \mathrm{S}^n \mathscr{E}$$

of $\mathrm{S}^n \mathscr{E}$ such that for every $i \in \{1, \ldots, n\}$ we have $\mathscr{F}_i \simeq \mathrm{S}^i \mathscr{E} \otimes \mathscr{L}'^{\otimes n-i}$ and $\mathscr{F}_i / \mathscr{F}_{i-1} \simeq \mathscr{L}^{\otimes i} \otimes \mathscr{L}'^{\otimes n-i}$. Moreover, the exact sequence

$$0 \to \mathscr{F}_{n-1}/\mathscr{F}_{n-2} \to \mathscr{F}_n/\mathscr{F}_{n-2} \to \mathscr{F}_n/\mathscr{F}_{n-1} \to 0 \tag{8.9}$$

is isomorphic to the sequence (8.8) tensored by $\mathscr{L}^{\otimes(n-1)}$. In particular, if (8.8) is not split, then (8.9) is not split either, and as a consequence, the exact sequence

$$0 \to \mathscr{F}_{n-1} \to \mathscr{F}_n \to \mathscr{L}^{\otimes n} \to 0$$

is not split. Applying above discussion to (8.6), we see that (8.7) is not split. □

LEMMA 8.5

For every $n \in \mathbb{N}$, we have that

(1) $\dim_K \mathrm{Ext}^1(\mathscr{O}_{X_\eta}, \mathrm{S}^n \mathscr{E}) = 1$,

(2) $\dim_K \mathrm{Hom}(\mathrm{S}^n \mathscr{E}, \mathscr{O}_{X_\eta}) = 1$,

(3) *the restriction map* $\mathrm{Hom}(\mathrm{S}^n \mathscr{E}, \mathscr{O}_{X_\eta}) \to \mathrm{Hom}(\mathrm{S}^{n-1} \mathscr{E}, \mathscr{O}_{X_\eta})$ *is zero.*

Proof

It follows from induction on n using the Ext long exact sequences derived from (8.7). □

It follows from the above lemmas that, for every $n \in \mathbb{N}$, $\mathrm{S}^n \mathscr{E}$ is the unique extension of $\mathscr{O}_{X_\eta}$ by $\mathrm{S}^{n-1} \mathscr{E}$, up to isomorphism.

Now we prove that pulling back 1-forms defines an isomorphism

$$\mathrm{H}^0(C, \mathrm{S}^n \Omega_C^1) \simeq \mathrm{H}^0(X, \mathrm{S}^n \Omega_X^1).$$

This map is obviously injective. Let us prove that it is also surjective. A symmetric form $\alpha \in \mathrm{H}^0(X, \mathrm{S}^n \Omega_X^1)$ gives rise to a linear form $\alpha : \mathrm{S}^n \mathscr{T}_X \to \mathscr{O}_X$. By restriction to the generic fiber X_η of the elliptic fibration, we obtain a map $\alpha_\eta : \mathrm{S}^n \mathscr{E} \to \mathscr{O}_{X_\eta}$. By the previous lemma, the restriction of α_η to $\mathrm{S}^{n-1} \mathscr{E}$ is zero. It follows that in the exact sequence (8.5), the restriction of α to $\mathrm{S}^{n-1} \mathscr{T}_{X^\circ} \otimes \mathscr{T}_{X^\circ/C}$ is zero, that is, it factors through $(\pi^\circ)^* \mathrm{S}^n \mathscr{T}_C$. Since the complement of X° in X is 0-dimensional, α factors through $(\pi)^* \mathrm{S}^n \mathscr{T}_C$, that is, it comes from a symmetric form on C. This finishes the proof of Proposition 8.1.

These calculations show that the Hitchin morphism for ruled and elliptic surfaces is closely related to the Hitchin morphism for the base curve. This is compatible with the fact that under the Simpson correspondence in [22], stable Higgs bundles for a smooth projective surface X correspond to irreducible representations of the fundamental group $\pi_1(X)$, and in the case of ruled surfaces and nonisotrivial elliptic surfaces with reduced fibers, we have $\pi_1(X) \simeq \pi_1(C)$, where C is the base curve (see, e.g., [8, Section 7]).

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